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QUASI-TILTED ALGEBRAS OF CANONICAL TYPE

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1. Introduction. This paper deals with the finite-dimensional algebras Σ over an algebraically closed field k whose derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\Sigma))$ of the category $\mathrm{mod}(\Sigma)$ of finite-dimensional modules over Σ is equivalent —as a triangulated category—to the derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{X}))$ of the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves on a (possibly weighted) non-singular projective curve $\mathbb{X}$ over k. As is shown in [13] this only happens if $\mathbb{X}$ is a weighted projective line [5, 6, 17]. Actually each weighted projective line $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ has a tilting bundle [5] whose endomorphism ring is a canonical algebra $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ in the sense of [21], depending on a weight sequence $\boldsymbol{p} = (p_1, \ldots, p_t)$ of positive integers, and a parameter sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise distinct non-zero elements from the projective line over k. In view of the equivalence $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\Lambda)) = \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{X}))$ (see [5]), the finite-dimensional representation theory of Λ is then completely determined by the sheaf theory on the weighted projective line $\mathbb{X}$. In fact the complexity of the classification problem for $\mathrm{coh}(\mathbb{X})$, and hence for $\mathrm{mod}(\Lambda)$, is largely determined by means of a weighted version of Riemann–Roch's theorem by the (virtual) genus

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\Big((t-2)p - \sum_{i=1}^{t} p/p_i\Big)$$

of $\mathbb{X}$, where $p = \mathrm{l.c.m.}(p_1, \ldots, p_t)$ (see [5]).

For $g_{\mathbb{X}} < 1$, the algebra Λ is concealed of extended Dynkin type; accordingly the problem to classify indecomposable objects in $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}(\Lambda)$ is equivalent to the classification of indecomposable modules over a tame hereditary algebra (cf. [5]) or, according to [6], closely related to the classification problem of indecomposable Cohen–Macaulay modules over a simple surface singularity.

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For $g_{\mathbb{X}} = 1$, the algebra Λ is of tubular type, and the theory links its representation theory [21] with the classification problem of indecomposable vector bundles on $\mathbb{X}$ (see [5, 15]) which itself relates to Atiyah's classification [2] of indecomposable vector bundles over an elliptic curve.

For $g_{\mathbb{X}} > 1$, the algebra Λ is wild, and the category of vector bundles on $\mathbb{X}$ is—if k is the field of complex numbers—equivalent to the category of $\mathbb{Z}$-graded Cohen–Macaulay modules over the algebra R of entire automorphic forms attached to a Fuchsian group of signature $(0; p_1, \ldots, p_t; 0)$ (see [12]), accordingly the study of $\operatorname{coh}(\mathbb{X})$ or $\operatorname{mod}(\Lambda)$ (see [17]) relates to the study of Cohen–Macaulay modules over the surface singularity $\widehat{R}$ obtained from R by completion.

In slightly different terminology the canonical algebras and their relatives (concealed-canonical algebras, almost concealed-canonical algebras [16]) are obtained as endomorphism algebras of a tilting object in $\operatorname{coh}(\mathbb{X})$ which is an abelian category with the remarkable feature to be hereditary, meaning that all second extension spaces $\operatorname{Ext}^2(X, Y)$ vanish. Tilting theory, one of whose starting points is marked by [9], has left its traces everywhere in recent representation theory. Tilting is particularly well understood if it takes place in a module category $\operatorname{mod}(\Delta)$ which is hereditary. Here, detailed information is available for the module category over the corresponding tilted algebra (i.e. the endomorphism algebra of the tilting object) including the shape of Auslander–Reiten components [9, 21, 10, 11]. By contrast, only general information is available for those quasi-tilted algebras [8], arising by tilting from a hereditary abelian category if it is not a module category. The main known features concern the existence of a preprojective (resp. preinjective) component [3] and the semiregularity of Auslander–Reiten components if the algebra is quasi-tilted but not tilted [4].

Here, we present a detailed account of the (representation-infinite) quasi-tilted algebras Σ of canonical type, i.e. those where Σ can be realized as the endomorphism ring of a tilting object in a hereditary category $\mathcal{H}$ which is derived-equivalent to a category $\operatorname{coh}(\mathbb{X})$ for a weighted projective line $\mathbb{X}$. We determine the ring structure for such algebras, and as a consequence also the shape of Auslander–Reiten components and the structure of the module category. These results are put into proper context by a conjecture raised in connection with [8] that a hereditary category with a tilting object T is derived-equivalent either to a category $\operatorname{mod}(\Delta)$ with Δ hereditary or else to $\operatorname{coh}(\mathbb{X})$, where $\mathbb{X}$ is a weighted projective line. We are able to confirm the conjecture in the tame case, i.e. if the endomorphism ring of T has tame representation type.

The organization of the paper is as follows: In Section 2 we determine the hereditary abelian k-categories which are derived-equivalent to a category of coherent sheaves on a weighted projective line (Theorem 2.3). Section 3 deals with the structure of the tilting objects for such categories, accordingly with the algebras which are quasi-tilted of canonical type. The main result here is Theorem 3.4 characterizing these algebras—in the representation-infinite case—as semiregular branch enlargements of concealed-canonical algebras, and equivalently by the property that the module category admits a sincere separating family of semiregular standard tubes. Section 4 provides the structure of the module category and investigates the K-theory. Finally, we prove in Theorem 4.7 that a tame quasi-tilted algebra either has canonical type or is tilted from a hereditary algebra.

Modules in this paper are usually finite-dimensional, and $\mathrm{mod}(\Sigma)$ denotes the resulting module category over Σ. We use the notation $\mathcal{A} \vee \mathcal{B}$ for the union of subcategories $\mathcal{A}$, $\mathcal{B}$ to indicate that $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$. If also $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = 0$, we use coproduct notation $\mathcal{A} \amalg \mathcal{B}$ instead; $\mathrm{ind}(\mathcal{A})$ denotes the full subcategory of indecomposables of $\mathcal{A}$.

2. Hereditary categories of canonical type. Let $\mathcal{H}$ be a small abelian hereditary k-category. Our main examples are the categories mod (Δ), Δ hereditary, and $\mathrm{coh}(\mathbb{X})$, $\mathbb{X}$ a weighted projective line. A *k-category* in the present paper is an additive category equipped with morphism and extension spaces that are finite-dimensional vector spaces over k, and where composition is k-bilinear. $\mathcal{H}$ is hereditary if all second extension spaces $\mathrm{Ext}^2(X, Y)$ vanish. By $\mathrm{D^b}(\mathcal{H})$ we denote the derived category (of bounded complexes) of $\mathcal{H}$. Since $\mathcal{H}$ is hereditary, $\mathrm{D^b}(\mathcal{H})$ is the additive closure of $\bigcup_{n \in \mathbb{Z}} \mathcal{H}[n]$, where each $\mathcal{H}[n]$ is a copy of $\mathcal{H}$ with objects denoted $X[n]$, $X \in \mathcal{H}$. Morphisms are determined by $\mathrm{Hom}(X[m], Y[n]) = \mathrm{Ext}^{n-m}_{\mathcal{H}}(X, Y)$, composition is given by the Yoneda composition of Ext, and translation of $\mathrm{D^b}(\mathcal{H})$ acts as $X[n] \mapsto X[n+1]$, where $X \in \mathcal{H}$. Generally, we use the notation $Z \mapsto Z[n]$, $Z \in \mathrm{D^b}(\mathcal{H})$, for the nth iterate of the translation functor.

We say that an abelian k-category $\mathcal{H}$ has *canonical type* if it is derived-equivalent to a category $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X}$, i.e. $\mathrm{D^b}(\mathcal{H}) \cong \mathrm{D^b}(\mathcal{C})$ as triangulated categories. It is equivalent to require that $\mathcal{H}$ is derived-equivalent to $\mathrm{mod}(\Lambda)$ for a canonical algebra Λ. In $\mathrm{D^b}(\mathcal{C}) = \mathrm{D^b}(\mathcal{H})$, Serre duality holds in the form $\mathrm{D}\,\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y[-1], \tau_{\mathbb{X}} X)$, accordingly $\mathrm{D^b}(\mathcal{C})$ has Auslander–Reiten triangles (see [7] for this notion). By $\tau_{\mathbb{X}}$ we denote the Auslander–Reiten translation for $\mathrm{D^b}(\mathcal{C})$, hence for $\mathcal{C}$.

We recall that Σ is called a *tilting object* of $\mathcal{H}$ if $\mathrm{Ext}^i_{\mathcal{H}}(\Sigma, \Sigma) = 0$ for all $i \neq 0$, and moreover Σ generates $\mathrm{D^b}(\mathcal{H})$ as a triangulated category, i.e. only the zero object in $\mathcal{H}$ satisfies $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{H})}(\Sigma[n], X) = 0$ for all integers n. In the following we will always view tilting objects as full subcategories consisting of finitely many pairwise non-isomorphic objects, and identify them with the corresponding endomorphism algebras.

For the rest of the paper $\mathcal{C}$ stands for the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$, and $\mathcal{C}_+$ (resp. $\mathcal{C}_0$) denotes the full subcategory of all vector bundles (resp. sheaves of finite length) on $\mathbb{X}$. We recall from [5] that $\mathcal{C}_0$ decomposes into a coproduct $\coprod_{x\in\mathbb{X}} \mathcal{U}_x$, where $\mathcal{U}_x$ denotes the connected uniserial category of coherent sheaves concentrated at x. We are going to use the existence of a rank (resp. degree) function rk (resp. deg) on the Grothendieck group $\mathrm{K}_0(\mathcal{C}) = \mathrm{K}_0(\mathrm{D^b}(\mathcal{C}))$ such that for an indecomposable $X \in \mathcal{C}$ we have $X \in \mathcal{C}_+$ (resp. $X \in \mathcal{C}_0$) if and only if $\mathrm{rk}X > 0$ (resp. $\mathrm{rk}X = 0$), and further $\deg X > 0$ holds for each non-zero $X \in \mathcal{C}_0$ (see [5, Proposition 4.3]).

Proposition 2.1. *Let $\mathcal{H}$ be a hereditary k-category*, *derived-equivalent to a category $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$. Then $\mathcal{H}$ has a non-zero projective object if and only if $\mathcal{H}$ is equivalent to the category $\mathrm{mod}(\Sigma)$ of finite-dimensional modules over a tame hereditary algebra Σ. In this case the genus of $\mathbb{X}$ is < 1, accordingly the weight type $(p_1, \ldots, p_t)$ of $\mathbb{X}$ (or $\mathcal{C}$) is Dynkin, i.e. satisfies $(t-2) - \sum_{i=1}^{t} 1/p_i < 0$.*

Proof. We note that each indecomposable object Z of $\mathrm{D^b}(\mathcal{H})$ has the form $Z = X[n]$ with $X \in \mathcal{H}$ and $n \in \mathbb{Z}$. For the purpose of the present proof, we call n the *index* of Z, and identify $\mathrm{D^b}(\mathcal{H})$ and $\mathrm{D^b}(\mathcal{C})$. Let X be an indecomposable object from $\mathcal{H}$. Then $\tau_{\mathbb{X}} X = Y[n]$ with Y in $\mathcal{H}$. By means of a translation in $\mathrm{D^b}(\mathcal{H})$ we may assume that $X \in \mathcal{C}$. Because $\mathcal{H}$ is hereditary, we deduce from

$$0 \neq \mathrm{Ext}^1_{\mathcal{C}}(X, \tau_{\mathbb{X}} X) = \mathrm{Hom}_{\mathrm{D^b}(\mathcal{H})}(X, Y[n+1]) = \mathrm{Ext}^{n+1}_{\mathcal{H}}(X, Y)$$

that $n = 0$ or $n = -1$. Therefore, the index is (in direction $n \to +\infty$) decreasing on the $\tau_{\mathbb{X}}$-orbit

$$\ldots, \tau_{\mathbb{X}}^{-n} X, \ldots, \tau_{\mathbb{X}}^{-1} X, X, \tau_{\mathbb{X}} X, \ldots, \tau_{\mathbb{X}}^{n} X, \ldots,$$

and in each step the index jumps by at most one. In particular, on each periodic $\tau_{\mathbb{X}}$-orbit the index is constant. Next, we note that $X \in \mathcal{H}$ is $\mathcal{H}$-projective if and only if for each object $Y \in \mathcal{H}$ we have $0 = \mathrm{D}\,\mathrm{Hom}_{\mathrm{D^b}(\mathcal{H})}(X, Y[1]) = \mathrm{Hom}_{\mathrm{D^b}(\mathcal{C})}(Y, \tau_{\mathbb{X}} X)$. This happens if and only if $\tau_{\mathbb{X}} X \notin \mathcal{H}$, therefore, invoking the preceding argument, if and only if $\tau_{\mathbb{X}} X \in \mathcal{H}[-1]$.

In particular, an Auslander–Reiten component of $\mathrm{D^b}(\mathcal{C})$ which is a tube cannot contain an $\mathcal{H}$-projective indecomposable object. Hence there are no such $\mathcal{H}$-projectives of rank zero. Equally there are no non-zero $\mathcal{H}$-projectives if $\mathbb{X}$ has genus one, i.e. has so-called tubular weight type $(p_1, \ldots, p_t)$ with $t-2 = \sum_{i=1}^{t} 1/p_i$, because in this case $\tau_{\mathbb{X}}$ is periodic of period $p = \mathrm{l.c.m.}(p_1, \ldots, p_t)$. Next, assume that $\mathbb{X}$ has genus > 1, accordingly that $\mathcal{C}$ has wild representation type, and assume the existence of an indecomposable $\mathcal{H}$-projective P of non-zero rank: By the preceding argument we get $\tau_{\mathbb{X}}^{n} P = X_n[-\ell_n]$, where $X_n \in \mathcal{H}$ and $\ell_n \geq 1$ for $n \geq 1$. In view of [17], or invoking formulae (2) and (3) from Section 2, $g_{\mathbb{X}} > 1$ implies that $\mathrm{Hom}(P, \tau_{\mathbb{X}}^{n} P) \neq 0$ for sufficiently large n, hence $\mathrm{Hom}(\mathcal{H}, \mathcal{H}[-\ell_n]) \neq 0$, which is impossible.

We have thus excluded the possibility that $\mathcal{C}$ is tubular or wild, which leaves us with the case of genus < 1, accordingly a tame domestic category of sheaves $\mathcal{C}$. Here, the indecomposable objects of $\mathcal{C}_+$ form an Auslander–Reiten component $\mathcal{D}$ which is the unique component not belonging to the tubular family formed by the indecomposable objects from $\mathcal{C}_0$. Note that $\mathcal{D}$ has a complete slice, and that each complete slice of $\mathcal{D}$ has extended Dynkin type. Let $\mathcal{H}$ contain an indecomposable projective object P. Invoking a translation of $\mathrm{D^b}(\mathcal{C})$, we may assume that P belongs to $\mathcal{D}$. Consider an irreducible map $X \to P$, and the corresponding part of the component $\mathcal{D}$:

$$\begin{array}{ccccccccc} & \cdots & & \cdots & & \cdots & & \cdots & \\ & & \tau_{\mathbb{X}} P & & & & P & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ \tau_{\mathbb{X}} X & & & & X & & & & \tau_{\mathbb{X}}^{-1} X \\ & \cdots & & \cdots & & \cdots & & \cdots & \end{array}$$

By assumption, $\tau_{\mathbb{X}} P \in \mathcal{H}[-1]$, $P \in \mathcal{H}$, and further $\mathrm{Hom}(\tau_{\mathbb{X}} X, \tau_{\mathbb{X}} P) \neq 0$, $\mathrm{Hom}(\tau_{\mathbb{X}} P, X) \neq 0$, $\mathrm{Hom}(X, P) \neq 0$, $\mathrm{Hom}(P, \tau_{\mathbb{X}}^{-1} X) \neq 0$. We thus get $\tau_{\mathbb{X}}^{-1} X \in \mathcal{H}[n]$, $n \geq 0$, $X \in \mathcal{H}$ or $X \in \mathcal{H}[-1]$. Moreover, if $X \in \mathcal{H}$ then $\tau_{\mathbb{X}} X$ belongs to $\mathcal{H}[-1]$. Therefore either X or $\tau_{\mathbb{X}}^{-1} X$ is projective in $\mathcal{H}$, and this projective object is connected to P by an irreducible map. Continuing, and invoking connectedness and stability of $\mathcal{D}$, we obtain a complete slice Σ of $\mathcal{D}$ consisting of $\mathcal{H}$-projective objects. Since Σ generates $\mathrm{D^b}(\mathcal{C}) = \mathrm{D^b}(\mathcal{H})$, we see that Σ is a tilting object for $\mathcal{H}$. Since $\mathcal{H}$ is hereditary, the associated torsion theory splits, and so $\mathrm{mod}(\Sigma)$—viewed as a full subcategory of $\mathrm{D^b}(\mathcal{H})$—is the additive closure of $\mathcal{X} \cup \mathcal{Y}[1]$, where $\mathcal{X} = \{X \in \mathcal{H} \mid \mathrm{Ext}^1_{\mathcal{H}}(\Sigma, X) = 0\}$ and $\mathcal{Y} = \{Y \in \mathcal{H} \mid \mathrm{Hom}_{\mathcal{H}}(\Sigma, Y) = 0\}$. Since Σ is projective in $\mathcal{H}$, we have $\mathcal{X} = \mathcal{H}$, therefore $\mathcal{Y} = 0$. Therefore $\mathrm{mod}(\Sigma) = \mathcal{H}$ as claimed. ∎

By a *cut* in $\mathcal{C}$ we understand a pair $(\mathcal{C}', \mathcal{C}'')$ of extension-closed subcategories $\mathcal{C}'$ and $\mathcal{C}''$ of $\mathcal{C}$ such that $\mathrm{Hom}(\mathcal{C}'', \mathcal{C}') = 0$, and moreover each indecomposable object of $\mathcal{C}$ either belongs to $\mathcal{C}'$ or to $\mathcal{C}''$. (It amounts to the same to require that $(\mathcal{C}'', \mathcal{C}')$ is a splitting torsion pair for $\mathcal{C}$.)

PROPOSITION 2.2. (i) *Let $\mathcal{H}$ be a hereditary abelian k-category and $\mathcal{C} = \mathrm{coh}(\mathbb{X})$. Each equivalence $\mathrm{D^b}(\mathcal{H}) = \mathrm{D^b}(\mathcal{C})$ of triangulated categories produces a cut $(\mathcal{C}', \mathcal{C}'')$ for $\mathcal{C}$ such that $\mathcal{H}$ is equivalent to the additive closure of $\mathcal{C}'' \vee \mathcal{C}'[1]$.*

(ii) *For each cut* $(\mathcal{C}',\mathcal{C}'')$ *in* $\mathcal{C}$, *the additive closure* $\mathcal{H}$ *of* $\mathcal{C}'' \vee \mathcal{C}'[1]$ *in* $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ *is a hereditary abelian* k*-category which is derived-equivalent to* $\mathcal{C}$ *and has a tilting object.*

Proof. (i) If $\mathcal{H}$ contains a non-zero projective, we are done by the preceding proposition. We may hence assume that $\mathrm{ind}(\mathcal{H})$ consists of full Auslander–Reiten components of $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$. We fix a simple sheaf S which is concentrated in an ordinary point, and assume—by means of a translation in $\mathrm{D}^{\mathrm{b}}(\mathcal{H}) = \mathrm{D}^{\mathrm{b}}(\mathcal{C})$—that S belongs to $\mathcal{H}$. Each non-zero F from $\mathcal{C}_+$ satisfies $\mathrm{Hom}(F,S) \neq 0$, therefore heredity of $\mathcal{H}$ implies that $\mathcal{C}_+$ belongs to the additive closure of $\mathcal{H}[-1]\vee\mathcal{H}$. There are two cases to consider: $\mathcal{C}_+ \subseteq \mathcal{H}$ and $\mathcal{C}_+ \cap \mathcal{H}[-1] \neq 0$.

Assume first that $\mathcal{C}_+ \subseteq \mathcal{H}$. Then for each non-zero $X \in \mathcal{C}_0$ there exists a non-zero morphism from (an object of) $\mathcal{H}$ to X which shows that $\mathcal{C}_0$, hence $\mathcal{C}$, belongs to the additive closure of $\mathcal{H} \vee \mathcal{H}[1]$. Next, we deal with the case when there exists a non-zero object F in $\mathcal{C}_+ \cap \mathcal{H}[-1]$. Since each Auslander–Reiten component of $\mathcal{C}_0$ is a stable tube, it contains a $\tau_{\mathbb{X}}$-stable member U. This implies $\mathrm{Hom}(F,U) \neq 0$, hence the component of U belongs to $\mathcal{H}[-1] \vee \mathcal{H}$. Summarizing, we obtain that $\mathcal{C}$ is contained in the additive closure of $\mathcal{H}[-1]\vee\mathcal{H}$ in this case. Invoking a translation we can assume this to be true in general, and put $\mathcal{C}' = \mathcal{C}\cap\mathcal{H}[-1]$, $\mathcal{C}'' = \mathcal{H}\cap\mathcal{C}$. By construction $\mathcal{C}'' \vee \mathcal{C}'[1]$ is contained in $\mathcal{H}$. Conversely, for each indecomposable H of $\mathcal{H}$ there exist $C \in \mathcal{C}$, $n \in \mathbb{Z}$ with $C = H[n]$, implying $n = 0$ (resp. $n = 1$), and accordingly $H \in \mathcal{C}'' \vee \mathcal{C}'[1]$.

(ii) In view of [8, Corollary 1.2.2], for each cut $(\mathcal{C}',\mathcal{C}'')$ in $\mathcal{C}$ the additive closure $\mathcal{H}$ of $\mathcal{C}'' \vee \mathcal{C}'[1]$ is abelian; moreover, $\mathcal{H}$ is hereditary since $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{C})}(X,Y[2]) = 0$ for $X,Y \in \mathcal{H}$. Since, by construction of $\mathcal{H}$, the indecomposable objects of $\bigvee_{n\in\mathbb{Z}}\mathcal{H}[n]$ and $\bigvee_{n\in\mathbb{Z}}\mathcal{C}[n]$ coincide, it follows $\mathrm{D}^{\mathrm{b}}(\mathcal{H}) = \mathrm{D}^{\mathrm{b}}(\mathcal{C})$. It remains to show that $\mathcal{H}$ has a tilting object. Because of Proposition 2.1, this is clear if $\mathcal{H}$ has a non-zero projective object. If $\mathcal{H}$ has no non-zero projectives we claim that $\mathcal{C}''$ is stable under $\tau_{\mathbb{X}}$. Indeed, otherwise there would be indecomposable objects $X' \in \mathcal{C}'$, $X'' \in \mathcal{C}''$ with $\tau_{\mathbb{X}}X'' = X'$, hence because of $\tau_{\mathbb{X}}X'' \in \mathcal{H}[-1]$, X'' would be projective in $\mathcal{H}$, which we have discarded. Let now Σ be any tilting object in $\mathcal{C}$. Writing Σ in the form $\Sigma' \vee \Sigma''$ with $\Sigma' \subseteq \mathcal{C}'$, $\Sigma'' \subseteq \mathcal{C}''$ it follows that $\tau_{\mathbb{X}}\Sigma'' \vee \Sigma'[1]$ is a tilting object in $\mathcal{H}$. ■

Let $\mathbb{X}' \amalg \mathbb{X}''$ be a decomposition of $\mathbb{X}$ into disjoint subsets, and let $\mathcal{C}_0' = \coprod_{x\in\mathbb{X}'}\mathcal{U}_x$, $\mathcal{C}_0'' = \coprod_{x\in\mathbb{X}''}\mathcal{U}_x$. Obviously, $(\mathrm{add}(\mathcal{C}_+ \vee \mathcal{C}_0''),\mathcal{C}_0')$ is a cut of $\mathcal{C}$, accordingly the additive closure $\mathcal{C}(\mathbb{X}',\mathbb{X}'')$ of $\mathcal{C}_0'[-1] \vee \mathcal{C}_+ \vee \mathcal{C}_0''$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ is hereditary abelian with a tilting object and is derived-equivalent to $\mathcal{C}$. Note that $\mathcal{C}(\emptyset,\mathbb{X})$ (resp. $\mathcal{C}(\mathbb{X},\emptyset)$) agrees with $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{op}}$). If both $\mathbb{X}'$ and $\mathbb{X}''$ are non-empty then—in contrast to $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}(\Delta)$—the category $\mathcal{H} =$

$\mathcal{C}(\mathbb{X}',\mathbb{X}'')$ does have non-zero (decomposable) objects whose class vanishes in the Grothendieck group $K_0(\mathcal{H})$: Take $S'[-1]\oplus S''$, where S' (resp. S'') is a simple sheaf concentrated in an ordinary point of $\mathbb{X}'$ (resp. $\mathbb{X}''$).

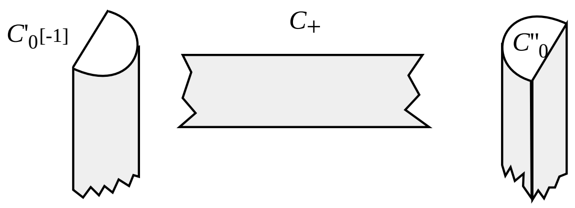


Fig. 1. The shape of $\mathcal{C}(\mathbb{X}',\mathbb{X}'')$

Recall from [5] that for each non-zero $X\in\mathcal{C}$ its *slope* is defined as $\mu X=\deg X/\mathrm{rk} X$. If $\mathbb{X}$ has genus one, hence the sheaf category $\mathcal{C}$ has tubular weight type, then for each $q\in\mathbb{Q}\cup\{\infty\}$, the additive closure $\mathcal{C}^{(q)}$ of indecomposable objects of $\mathcal{C}$ of slope q is a uniserial category whose indecomposables form a tubular family, parametrized by $\mathbb{X}$ (see [15]). Note that $\mathcal{C}^{(\infty)}$ agrees with $\mathcal{C}_0$. Let r denote an irrational number, and let $\mathcal{C}'_r$ (resp. $\mathcal{C}''_r$) be the additive closure of all $\mathcal{C}^{(q)}$ with $q<r$ (resp. $r<q\le\infty$), then $(\mathcal{C}'_r,\mathcal{C}''_r)$ is a cut in $\mathcal{C}$, and the additive closure $\mathcal{C}\langle r\rangle$ of $\mathcal{C}''_r\cup\mathcal{C}'_r[1]$ in $\mathrm{D^b}(\mathcal{C})$ is abelian hereditary with a tilting object and is derived-equivalent to $\mathcal{C}$.

Recall that the *Euler form* on $K_0(\mathcal{C})$ is given on (classes of) objects from $\mathcal{C}$ by

$$\langle X,Y\rangle=\dim_k\mathrm{Hom}(X,Y)-\dim_k\mathrm{Ext}^1(X,Y).$$

The weighted form of Riemann–Roch's theorem [14] then states

$$\frac{1}{p}\langle\langle X,Y\rangle\rangle=(1-g_{\mathbb{X}})\mathrm{rk}X\,\mathrm{rk}Y+\frac{1}{p}\begin{vmatrix}\mathrm{rk}X & \mathrm{rk}Y\\ \deg X & \deg Y\end{vmatrix}\tag{1}$$

$$=\mathrm{rk}X\,\mathrm{rk}Y\left(1-g_{\mathbb{X}}+\frac{1}{p}(\mu Y-\mu X)\right),\tag{2}$$

where $p=\mathrm{l.c.m.}(p_1,\ldots,p_t)$, $\langle\langle X,Y\rangle\rangle=\sum_{j=0}^{p-1}\langle\tau_{\mathbb{X}}^{j}X,Y\rangle$, and $g_{\mathbb{X}}=1+(p/2)\delta_{\mathbb{X}}$ with $\delta_{\mathbb{X}}=(t-2)-\sum_{i=1}^{t}1/p_i$. Note, moreover, that

$$\mu(\tau_{\mathbb{X}}F)=\mu F+p\delta_{\mathbb{X}}.\tag{3}$$

THEOREM 2.3. *Let $\mathcal{H}$ be a hereditary k-category derived equivalent to $\mathcal{C}=\mathrm{coh}(\mathbb{X})$, for a weighted projective line $\mathbb{X}$. Then $\mathcal{H}$ is equivalent to exactly one of the following three types*:

(i) $\mathrm{mod}(\Delta)$, *where Δ is tame hereditary*,
(ii) $\mathcal{C}(\mathbb{X}',\mathbb{X}'')$ *for some decomposition $\mathbb{X}=\mathbb{X}'\amalg\mathbb{X}''$ of the set $\mathbb{X}$*,
(iii) $\mathcal{C}\langle r\rangle$, *where r is an irrational number, and $\mathcal{C}$ is of tubular type.*

Proof. By Proposition 2.1, we only need to deal with the case when $\mathcal{H}$ does not have any non-zero projective object, therefore $\operatorname{ind}(\mathcal{H})$ is a union of full Auslander–Reiten components of $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$. Next, we fix an equivalence $\mathrm{D}^{\mathrm{b}}(\mathcal{H}) = \mathrm{D}^{\mathrm{b}}(\mathcal{C})$, and consider a cut $(\mathcal{C}', \mathcal{C}'')$ of $\mathcal{C}$ such that $\mathcal{H}$ equals the additive closure of $\mathcal{C}'' \vee \mathcal{C}'[1]$. By assumption, $\operatorname{ind}(\mathcal{C}')$ and $\operatorname{ind}(\mathcal{C}'')$ consist of full Auslander–Reiten components from $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$.

Case 1: $g_{\mathbb{X}} < 1$. Here, the indecomposable objects from $\mathcal{C}_+$ form a single Auslander–Reiten component, thus $\mathcal{C}_+$ lies in $\mathcal{C}'$ or in $\mathcal{C}''$. If $\mathcal{C}_+ \subseteq \mathcal{C}''$, then $\operatorname{Hom}(\mathcal{C}'', X) \neq 0$ for each $X \in \mathcal{C}_0$, hence $\mathcal{C}_0$ belongs to $\mathcal{C}''$, and $\mathcal{C} = \mathcal{C}''$ follows. Otherwise $\mathcal{C}_+$ is contained in $\mathcal{C}'$; we then denote by $\mathbb{X}'$ (resp. $\mathbb{X}''$) the set of all $x \in \mathbb{X}$ such that $\mathcal{U}_x \subseteq \mathcal{C}''$ (resp. $\mathcal{U}_x \subseteq \mathcal{C}'$). It follows that $\mathcal{C}' = \operatorname{add}(\mathcal{C}_+ \vee \mathcal{C}''_0)$, $\mathcal{C}'' = \operatorname{add}(\mathcal{C}'_0)$, and hence $\mathcal{H} = \mathcal{C}(\mathbb{X}', \mathbb{X}'')$.

Case 2: $g_{\mathbb{X}} = 1$. Assume first that there exists $q \in \mathbb{Q} \cup \{\infty\}$ such that $\mathcal{C}^{(q)} \cap \mathcal{C}' \neq 0$ and $\mathcal{C}^{(q)} \cap \mathcal{C}'' \neq 0$. By means of an automorphism of $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ (a sequence of mutations [15]), we may assume $q = \infty$. This shows that $\mathcal{C}_+ \subseteq \mathcal{C}'$, and we continue as in case 1 to show that $\mathcal{H}$ is of the form $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$.

Next, assume that for each $q \in \mathbb{Q} \cup \{\infty\}$ we have $\mathcal{C}^{(q)} \subseteq \mathcal{C}'$ or $\mathcal{C}^{(q)} \subseteq \mathcal{C}''$. If all $\mathcal{C}^{(q)}$ belong to $\mathcal{C}''$ then $\mathcal{H} = \mathcal{C}$, otherwise we define r as the supremum (in $\mathbb{R} \cup \{\infty\}$) of all q with $\mathcal{C}^{(q)} \subseteq \mathcal{C}'$. If $r \in \mathbb{Q} \cup \{\infty\}$, then by an automorphism of $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ we can assume $r = \infty$, resulting in the cut $(\mathcal{C}_+, \mathcal{C}_0)$ (resp. $(\mathcal{C}, 0)$), hence in $\mathcal{H} = \mathcal{C}(\emptyset, \mathbb{X})$ (resp. $\mathcal{C}(\mathbb{X}, \emptyset)$). We may thus assume that r is irrational, where in view of (2) we get $\mathcal{C}' = \mathcal{C}'_r$, $\mathcal{C}'' = \mathcal{C}''_r$, thus $\mathcal{H} = \mathcal{C}\langle r \rangle$.

Case 3: $g_{\mathbb{X}} > 1$. By (2) and (3), for any two components $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{C}$ there exist non-zero morphisms from $\mathcal{A}$ to $\mathcal{B}$ (and from $\mathcal{B}$ to $\mathcal{A}$). Hence $\mathcal{C}_+$ is either contained in $\mathcal{C}'$ or in $\mathcal{C}''$. As in case 1 this implies that $\mathcal{H}$ is of type $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$. ∎

3. Quasi-tilted algebras of canonical type. Following [16], a k-algebra Σ is called *quasi-canonical* if it is derived-equivalent to a canonical algebra. A special case are the *concealed-canonical* (resp. *almost concealed-canonical*) algebras defined as the endomorphism algebras of tilting bundles (resp. tilting sheaves) on a weighted projective line $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ or equivalently as the endomorphism algebras of tilting modules T over a canonical algebra $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$, where T is built from indecomposable modules of strictly positive rank (resp. of modules of non-negative rank). In [18] the concealed-canonical algebras are further characterized as the connected algebras whose module category admits a sincere separating family of standard stable tubes. The property of being quasi-canonical (resp. concealed-canonical) is preserved when passing from Σ to its opposite algebra Σ^{op}, whereas the corresponding statement holds for an almost concealed-canonical algebra only if it is already concealed-canonical. We note that the representation-infinite algebras derived-equivalent to tame canonical algebras are described completely in [1].

Definition 3.1. An algebra Σ that can be realized as the endomorphism ring of a tilting object for a hereditary category $\mathcal{H}$ of canonical type will be said to be *quasi-tilted of canonical type*.

It amounts to the same to say that Σ is quasi-tilted and quasi-canonical. In particular, Σ is quasi-tilted of canonical type if and only if Σ^{op} is.

We need a result on tilting objects in a category of type $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$ as defined in the preceding section.

Proposition 3.2. *Let* $\mathcal{C} = \mathrm{coh}(\mathbb{X})$, *and let* Σ_+, Σ_0', Σ_0'' *be full subcategories of* $\mathcal{C}_+$, $\mathcal{C}_0'$ *and* $\mathcal{C}_0''$, *respectively*, *consisting of pairwise non-isomorphic indecomposable objects given by a decomposition* $\mathbb{X} = \mathbb{X}' \amalg \mathbb{X}''$. *The following assertions are equivalent*:

(i) $\Sigma_0'[-1] \vee \Sigma_+ \vee \Sigma_0''$ *is a tilting object in* $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$.

(ii) Σ_+, Σ_0', Σ_0'' *are partial tilting objects in* $\mathcal{C}_+$, $\mathcal{C}_0'$ *and* $\mathcal{C}_0''$, *respectively*, Σ_+ *belongs to the the category* ${}^{\perp}\Sigma_0' \cap \Sigma_0''^{\perp}$, *and the number of indecomposable objects of* $\Sigma_0' \cup \Sigma_0'' \cup \Sigma_+$ *equals the rank of* $\mathrm{K}_0(\mathbb{X})$.

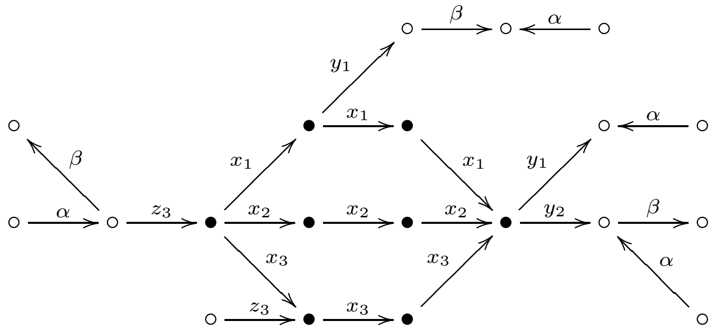
(iii) $\Sigma_+ \vee (\tau_{\mathbb{X}}^{-1} \Sigma_0' \amalg \Sigma_0'')$ *is a tilting sheaf in* $\mathcal{C} = \mathrm{coh}(\mathbb{X})$.

Proof. (i)$\Leftrightarrow$(ii). Since there are no extensions between $\mathcal{C}_0'$ and $\mathcal{C}_0''$, and $\mathrm{Ext}^1_{\mathcal{H}}(\mathcal{C}_+, \mathcal{C}_0) = 0$, the condition $\mathrm{Ext}^1_{\mathcal{H}}(\Sigma, \Sigma) = 0$ is equivalent to the empty condition $0 = \mathrm{Ext}^1_{\mathcal{H}}(\Sigma_0'[-1], \Sigma_+) = \mathrm{Ext}^2_{\mathcal{C}}(\Sigma_0', \Sigma_+)$, and the two additional conditions $0 = \mathrm{Ext}^1_{\mathcal{H}}(\Sigma_+, \Sigma_0'[-1]) = \mathrm{Hom}_{\mathcal{C}}(\Sigma_+, \Sigma_0')$ and $0 = \mathrm{Ext}^1_{\mathcal{H}}(\Sigma_0'', \Sigma_+) = \mathrm{Ext}^1_{\mathcal{C}}(\Sigma_0'', \Sigma_+)$. In view of $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}_+, \mathcal{C}_0) = 0 = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_0, \mathcal{C}_+)$ the two conditions can be expressed as $\Sigma_+ \in \Sigma_0''^{\perp}$ and $\Sigma_+ \in {}^{\perp}\Sigma_0'$.

(ii)$\Leftrightarrow$(iii). Because of Serre duality, $\mathrm{Hom}_{\mathcal{C}}(\Sigma_+, \Sigma_0') = 0$ if and only if $\mathrm{Ext}^1_{\mathcal{C}}(\tau_{\mathbb{X}}^{-1} \Sigma_0', \Sigma_+) = 0$. ∎

We recall from [17] that a concealed canonical algebra Σ can be characterized by the existence of a trisection of $\mathrm{mod}(\Sigma)$ into extension- and $\tau^{\pm}$-closed subcategories $\mathrm{mod}_+(\Sigma) \vee \mathrm{mod}_0(\Sigma) \vee \mathrm{mod}_-(\Sigma)$ such that $\mathrm{mod}_0(\Sigma)$ is an exact abelian subcategory of $\mathrm{mod}(\Sigma)$, projectives (resp. injectives) belong to $\mathrm{mod}_+(\Sigma)$ (resp. $\mathrm{mod}_-(\Sigma)$) and moreover for each $M \neq 0$ in $\mathrm{mod}_+(\Sigma)$ (resp. $N \neq 0$ in $\mathrm{mod}_-(\Sigma)$) there exists a non-zero morphism from M to $\mathrm{mod}_0(\Sigma)$ (resp. from $\mathrm{mod}_0(\Sigma)$ to N). It is equivalent to assert the existence of a separating tubular family $(\mathcal{T}_x)_{x \in \mathbb{X}}$ of standard stable tubes [17] (see also [23] for a related treatment).

Next we are going to deal with so-called semiregular branch enlargements of a concealed-canonical algebra. An example is given below by means of the quiver



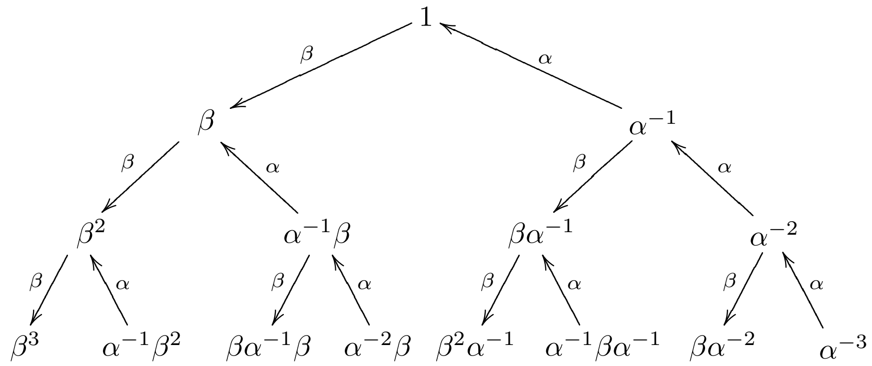
where we require that in addition to the relations $x_1^3 + x_2^3 + x_3^3 = 0$, $y_i x_i = 0 = x_i z_i$ $(i = 1, 2, 3)$ all possible relations $\beta\alpha = 0$ do hold.

By definition, a *semiregular tube* is an Auslander–Reiten component containing an oriented cycle but not both a projective and an injective object. As is shown in [19], the semiregular tubes are just the tubes arising from a stable tube by either ray insertions or else by coray insertions. It is not difficult to see (compare [21]) that it is equivalent to deal with the components arising from a stable tube by either coray deletions or ray deletions. We are going to deal with the topic in more detail in the case when the tubes in question are additionally standard [21].

Let thus $\mathcal{T}$ be a standard stable tube of rank p, accordingly $\mathcal{U} = \mathrm{add}(\mathcal{T})$ be a connected uniserial subcategory where the simples form an Auslander–Reiten orbit $(\tau^j S)_{j\in\mathbb{Z}_p}$ of period p. A proper subset $\tau^a S, \tau^{a+1} S, \ldots, \tau^b S$ of this orbit is called a *segment*, said to be of *length* $\ell = b - a + 1$. Two segments are called *non-adjacent* if their union is not a segment and is not the whole orbit. We select pairwise non-adjacent segments $\mathcal{S}(j)$, $j = 1, \ldots, h$, each consisting of $\ell(j)$ simple objects $T(j), \tau T(j), \ldots, \tau^{\ell(j)-1}T(j)$. Note that $\sum_{j=1}^{h} \ell(j) < p$, further that the Ext-spaces between simple objects from non-adjacent segments vanish. Since $\ell(j) < p$, we further have for $0 \le a, b \le \ell(j) - 1$ that $\mathrm{Ext}^1_{\mathcal{U}}(\tau^a T(j), \tau^b T(j)) = k$ if $b = a + 1$, and 0 otherwise. Hence the full subcategory $\mathcal{C}(j)$ of $\mathcal{U}$ consisting of all objects having a finite filtration with factors from the segment $\mathcal{S}(j)$ is uniserial, in fact equivalent to the category of k-linear representations of the quiver $\circ \to \circ \to \ldots \to \circ \to \circ$ of type $\mathbb{A}_n$, $n = \ell(j)$. Accordingly the Auslander–Reiten quiver of $\mathcal{C}(j)$ is a wing.

A tilting object $B(j)$ in $\mathcal{C}(j)$ consisting of indecomposable objects is known [21, p. 205] to be a branch consisting of $\ell(j)$ elements. Recall that a *complete branch* is the category given by the infinite quiver whose objects are the words in the two letters α^{-1}, β. There are two types of arrows $\beta : w \to \beta w$ and $\alpha : \alpha^{-1} w \to w$; moreover, we require the relations $\beta \circ \alpha = 0$.

The empty word 1 is called the root of the complete branch:



A finite connected full subquiver B of ℓ vertices (and the corresponding subcategory) containing 1 is called a *branch of length* ℓ rooted in 1.

By a *system of branches* in a standard stable tube $\mathcal{T}$ we understand a system $(B(j))_{j=1,\dots,h}$ as described before. If $B = \bigcup_{j=1}^{h} B(j)$, then the subcategory $\mathcal{U}' = B^{\perp}$ of $\mathcal{U}$ right perpendicular to B (see [6]), agreeing with the category right perpendicular to $\bigcup_{j=1}^{h} \mathcal{S}(j)$, viewed as a full subcategory of $\mathcal{U}$ is again connected uniserial, accordingly $\mathcal{T}' = \mathcal{U}' \cap \mathcal{T}$ is a standard stable tube with $p' = p - \sum_{j=1}^{h} \ell(j)$ simple objects. Since ${}^{\perp}B = (\tau^{-1}B)^{\perp}$, a corresponding assertion holds for the left perpendicular category $\mathcal{U}'' = {}^{\perp}B$ with corresponding stable tube $\mathcal{T}'' = \mathcal{U}'' \cap \mathcal{T}$. Note that the condition $\mathrm{Ext}^1(B,X) = 0$ (resp. $\mathrm{Hom}(B,X) = 0$) amounts to coray deletions (resp. ray deletions) in $\mathcal{T}$, and defines a semiregular standard tube $\overline{\mathcal{T}}'$ (resp. $\overline{\mathcal{T}}''$) where B (resp. τB) consists of the Ext-projectives (resp. the Ext-injectives) of $\overline{\mathcal{T}}'$ (resp. $\overline{\mathcal{T}}''$) which hence are branches. Alternatively, $\overline{\mathcal{T}}'$ (resp. $\overline{\mathcal{T}}''$) arises from the standard stable tube $\mathcal{T}'$ (resp. $\mathcal{T}''$) by ray (resp. coray) insertions.

Let Σ_+ be a concealed-canonical algebra, realized as a tilting bundle on a weighted projective line $\mathbb{Y}$. We fix a sequence $\lambda_1', \dots, \lambda_r', \lambda_1'', \dots, \lambda_s''$, of pairwise distinct points of $\mathbb{Y}$, containing all the exceptional points of $\mathbb{Y}$ but possibly also ordinary points, and let p_i (resp. q_j) denote the weight of λ_i' (resp. λ_j''). For each $i = 1, \dots, r$ (resp. $j = 1, \dots, s$) we select a possibly empty sequence $(S_i'(a))$, $a = 1, \dots, h_i'$, (resp. $(S_j''(b))$, $b = 1, \dots, h_j''$) of pairwise non-isomorphic simple objects from $\mathrm{mod}_0(\Sigma_+) = \mathrm{coh}_0(\mathbb{Y})$, concentrated at λ_i' (resp. λ_j'') and abstract branches $B_i'(a)$, $a = 1, \dots, h_i'$ (resp. $B_j''(b)$, $b = 1, \dots, h_j''$). Note that we allow that p_i or q_j equals one, accordingly that $S_i'(1)$ (resp. $S_j''(1)$) is an ordinary simple object from $\mathrm{mod}_0(\Sigma_+) = \mathrm{coh}_0(\mathbb{Y})$. The algebra Σ obtained from Σ_+ by first forming the multi-point extension-coextension

$$ {}^{a=1,\dots,h_i'}_{i=1,\dots,r}[\mathrm{D}\, S_i'(a)]\Sigma_+[S_j''(b)]^{b=1,\dots,h_j''}_{j=1,\dots,s}, $$

and then rooting each branch $B_i'(a)$ (resp. $B_j''(b)$) in $\mathrm{D}\,S_i'(a)$ (resp. in $S_j''(b)$) is said to be obtained from Σ_+ by *semiregular branch enlargement*. Here, D denotes the standard duality $\mathrm{Hom}_k(-,k)$. Let $\ell_i'(a)$ (resp. $\ell_j''(b)$) denote the number of points of $B_i'(a)$ (resp. $B_j''(b)$), and put

$$(4)\quad \overline{p}_i = p_i + \sum_{a=1}^{h_i'} \ell_i'(a),\ i=1,\dots,r \text{ and } \overline{q}_j = q_j + \sum_{b=1}^{h_j''} \ell_j''(b),\ j=1,\dots,s.$$

Moreover, we let $\mathbb{X}$ be the weighted projective line which attaches the weights $\overline{p}_1,\dots,\overline{p}_r,\overline{q}_1,\dots,\overline{q}_s$ to the points $\lambda_1',\dots,\lambda_r'$, $\lambda_1'',\dots,\lambda_s''$. Note that, here, we are identifying the sets underlying $\mathbb{X}$ and $\mathbb{Y}$ with the projective line over k.

PROPOSITION 3.3. *Let Σ be an algebra obtained from a concealed-canonical algebra Σ_+ by means of a semiregular branch enlargement. With the above notations consider a decomposition $\mathbb{X}' \amalg \mathbb{X}''$ of $\mathbb{X}$ such that each λ_i' belongs to $\mathbb{X}'$ and each λ_i'' belongs to $\mathbb{X}''$. Then Σ can be realized as a tilting object of the category $\mathcal{C}(\mathbb{X}',\mathbb{X}'')$.*

Proof. For each $i=1,\dots,r$ (resp. $j=1,\dots,s$) we arrange the $S_i'(a)$ (resp. the $S_j''(b)$) in such a way that $S_i'(a) = \tau_{\mathbb{Y}}^{-m_i'(a)} S_i'$ (resp. $S_j''(b) = \tau_{\mathbb{Y}}^{-m_j''(b)} S_j''$), where S_i' (resp. S_j'') are simple sheaves on $\mathbb{Y}$ concentrated at λ_i' (resp. λ_j'') and where

$$(5)\qquad m_i'(1) < m_i'(2) < \dots < m_i'(h_i) \le p_i,$$

$$(6)\qquad 0 < m_j''(1) < m_j''(2) < \dots < m_j''(h_j) \le q_j.$$

We further put

$$\overline{m}_i'(a) = m_i'(a) + \sum_{c=1}^{a} \ell_i'(c), \qquad \overline{m}_j''(b) = m_j''(b) + \sum_{d=1}^{b} \ell_j''(d).$$

In view of (4) we can select mutually non-adjacent segments

$$\mathcal{S}_i'(a) = \{T_i'(a), \tau_{\mathbb{X}} T_i'(a), \dots, \tau_{\mathbb{X}}^{\ell_i'(a)-1} T_i'(a)\}, \qquad a=1,\dots,h_i',$$

$$\mathcal{S}_j''(b) = \{T_j''(b), \tau_{\mathbb{X}} T_j''(b), \dots, \tau_{\mathbb{X}}^{\ell_j''(b)-1} T_j''(b)\}, \qquad b=1,\dots,h_j'',$$

where $T_i'(a) = \tau_{\mathbb{X}}^{-\bar{m}_i'(a)} \overline{S}_i'$ (resp. $T_j'(b) = \tau_{\mathbb{X}}^{-\bar{m}_j''(b)} \overline{S}_j''$) for some simple sheaf $\overline{S}_i'$ (resp. $\overline{S}_j''$) on $\mathbb{X}$ concentrated at λ_i' (resp. at λ_j''). Since the subcategory $\mathcal{C}_i'(a)$ (resp. $\mathcal{C}_j''(b)$) generated by $\mathcal{S}_i(a)$ (resp. $\mathcal{S}_j''(b)$) has $\ell_i'(a)$ (resp. $\ell_j''(b)$) simple objects, the branch $B_i'(a)$ (resp. $B_j''(b)$) can be realized as a tilting object $\mathcal{B}_i'(a)$ in $\mathcal{C}_i'(a)$ (resp. $\mathcal{B}_j''(b)$ in $\mathcal{C}''(b)$) with root $T_i'(a)^{[\ell_i'(a)]}$ (resp. $T_j''(b)^{[\ell_j''(b)]}$). Here, $S^{[n]}$ denotes the unique indecomposable sheaf of length n with simple top S.

It follows from [6] that the category

$$(7)\qquad {}^{\perp}\Big(\bigcup_{i=1,a=1}^{r,h'_i} \mathcal{S}'_i(a)\Big) \cap \Big(\bigcup_{j=1,b=1}^{s,h''_j} \mathcal{S}'_j(b)\Big)^{\perp}$$

is equivalent to the category of coherent sheaves on a weighted projective line with the weight (resp. parameter) data $p_1, \ldots, p_r$, $q_1, \ldots, q_s$ (resp. $\lambda'_1, \ldots, \lambda'_r$), $\lambda''_1, \ldots, \lambda''_s$, hence to $\mathrm{coh}(\mathbb{Y})$. Accordingly Σ_+ can be realized as a tilting bundle on $\mathbb{Y}$. From now on we will identify $\mathrm{coh}(\mathbb{Y})$ with the subcategory (7) of $\mathrm{coh}(\mathbb{X})$.

By construction,

$$\Omega = \Big(\coprod_{i=1,a=1}^{r,h'_i} \mathcal{B}'_i(a)[-1]\Big) \vee \Sigma_+ \vee \Big(\coprod_{j=1,b=1}^{s,h''_j} \mathcal{B}''_j(b)\Big)$$

belongs to $\mathcal{H} = \mathcal{C}(\mathbb{X}', \mathbb{X}'')$, further there are no extensions between any two members of Ω. Moreover, Ω consists of pairwise non-isomorphic indecomposable objects whose number agrees with the rank of the Grothendieck group $\mathrm{K}_0(\mathcal{H}) = \mathrm{K}_0(\mathcal{C})$. In view of Proposition 3.2, Ω is a tilting object in $\mathcal{H}$. We need to show that Ω is isomorphic to Σ. This follows from the definition of a branch enlargement (resp. branch coenlargement) as in [16] in view of the following facts:

(a) The object $T''_j(b)^{[\ell+1]}$, $\ell = \ell''_j(b)$, belongs to $\mathrm{coh}(\mathbb{Y})$, and agrees with the simple sheaf $S''_j(b)$ from $\mathrm{coh}(\mathbb{Y})$.

(b) The natural epimorphism $T''_j(b)^{[\ell+1]} \to T_j(b)^{[\ell]}$ with kernel $\tau^{\ell}_{\mathbb{X}} T''_j(b)$ induces an isomorphism $\mathrm{Hom}_{\mathbb{X}}(F, T''_j(b)^{[\ell+1]}) \to \mathrm{Hom}_{\mathbb{X}}(F, T_j(b)^{[\ell]})$ for each vector bundle F on $\mathbb{Y}$, i.e. $S''_j(b) = r(T''^{[\ell]}_j(b))$, where r is the right adjoint to inclusion $\mathrm{coh}(\mathbb{Y}) \hookrightarrow \mathrm{coh}(\mathbb{X})$ (see [6]).

Assertion (b) follows because for each bundle F on $\mathbb{Y}$ we have $\mathrm{Ext}^1_{\mathbb{X}}(T, F) = 0 = \mathrm{Hom}_{\mathbb{X}}(F, \tau_{\mathbb{X}} T)$ and further $\mathrm{Ext}^1_{\mathbb{X}}(F, \tau_{\mathbb{X}} T) = 0$ for each $T \in \mathcal{C}''_j(b)$. The argument for the branch enlargement part of Ω is dual. ∎

According to [8, Corollary 2.3.6] any representation-finite, quasi-tilted algebra Σ is tilted, and those of canonical type are obtained through tilting from extended Dynkin type [9, 21]. To characterize the quasi-tilted algebras of canonical type we may thus restrict to the representation-infinite case.

Theorem 3.4. *The following assertions are equivalent for a k-algebra Σ.*

(i) *Σ is representation-infinite and quasi-tilted of canonical type.*

(ii) *Σ is isomorphic to the endomorphism ring of a tilting object in a category $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$.*

(iii) *Σ is a semiregular branch enlargement of a concealed-canonical algebra.*

(vi) *The category* mod(Σ) *admits a sincere separating family of semiregular standard tubes.*

Proof. (i)⇒(ii). Assume Σ is realized as a tilting object for a hereditary category $\mathcal{H}$ which is derived-equivalent to coh($\mathbb{X}$) for some weighted projective line $\mathbb{X}$. According to Theorem 2.3, besides being of shape $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$, there are two further possibilities for $\mathcal{H}$: $\mathcal{H} \cong \mathrm{mod}(\Delta)$ or $\mathcal{H} \cong \mathcal{C}\langle r \rangle$.

In the first case, Δ is hereditary of extended Dynkin type. Accordingly Σ is either representation-finite tilted, which we have excluded, or else Σ is tame concealed, hence has a realization as a tilting bundle for a category of type coh($\mathbb{X}$). Next, we assume that $\mathcal{C}$ has tubular type, and $\mathcal{H} \cong \mathcal{C}\langle r \rangle$ for some irrational number r. Recall $\mathcal{C}\langle r \rangle$ is the additive closure of $\mathcal{C}''_r \vee \mathcal{C}'_r[1]$, therefore a suitable shift moves Σ to $\mathcal{C}''_r$, hence realizes Σ as a tilting object of coh($\mathbb{X}$). Summarizing, we may in each case assume that Σ is a tilting object in a category of type $\mathcal{C}(\mathbb{X}', \mathbb{X}'')$.

(ii)⇒(iii). As in Proposition 3.2, Σ has the form $\Sigma'_0[-1] \vee \Sigma_+ \vee \Sigma''_0$, and $\Sigma_+ \vee \Sigma''_0$ is a tilting object in the category ${}^\perp\Sigma'_0$ which is a category of coherent sheaves on a weighted projective line $\mathbb{X}_r$. According to [16], $\Sigma_r = \Sigma_+ \vee \Sigma''_0$ is a branch coenlargement of Σ_+ by Σ''_0. Dually, $\Sigma_\ell = \Sigma'_0[-1] \vee \Sigma_+$ is a branch enlargement of Σ_+ by Σ'_0, and by construction no tube is involved in both a branch enlargement and a branch coenlargement.

(iii)⇒(i). See Proposition 3.3.

(ii)⇒(iv). See Proposition 4.3.

(iv)⇒(iii). Let B' (resp. B'') denote the full subcategories consisting of a representative system of indecomposable projectives (resp. injectives) from the separating family; in view of semiregularity, B' and B'' are branches. Since, moreover, inclusion from $\mathcal{M} = B'^\perp \cap {}^\perp B''$ into mod(Σ) has a left adjoint [6], $\mathcal{M}$ is equivalent to the module category mod(Σ_+) for a finite-dimensional algebra Σ_+. Moreover, the hypothesis implies that the intersection of $\mathcal{M}$ with the separating family of Σ is a sincere separating tubular family consisting of standard stable tubes for mod (Σ_+). In view of [18], this implies that Σ_+ is concealed-canonical, accordingly that Σ is a semiregular branch enlargement of Σ_+. ■

Corollary 3.5. *The following are equivalent for a k-algebra Σ:*

(i) *Σ is almost concealed-canonical.*

(ii) *The category* mod(Σ) *admits a sincere separating family of standard tubes not containing injectives.*

Proof. (i)⇒(ii) follows from [16]. For (ii)⇒(i) we represent Σ as above in the form $\Sigma'_0[-1] \vee \Sigma_+ \vee \Sigma''_0$. The absence of injectives in the separating tubular family constructed in the proof of Proposition 4.3 implies that Σ_0' is empty, hence $\Sigma = \Sigma_+ \vee \Sigma_0''$ is actually a tilting sheaf, accordingly Σ is almost concealed-canonical. ■

4. Module category and K-theory. Let Σ be a connected, representation-infinite, quasi-tilted algebra of canonical type. According to Theorem 3.4 we fix a realization of Σ as a tilting object $\Sigma_0'[-1] \vee \Sigma_+ \vee \Sigma_0''$ in a category $\mathcal{H} = \mathcal{C}(\mathbb{X}', \mathbb{X}'')$, and identify $\mathrm{mod}(\Sigma)$ with the full subcategory of $\mathrm{D^b}(\mathcal{H}) = \mathrm{D^b}(\mathcal{C})$ consisting of all objects X satisfying $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{C})}(\Sigma, X[n]) = 0$ for all integers $n \neq 0$. Because $\mathcal{H}$ is hereditary, $\mathrm{mod}(\Sigma)$ lies in the additive closure of $\mathcal{H} \vee \mathcal{H}[1]$, accordingly in the additive closure of $\mathcal{C}'[-1] \vee \mathcal{C}_+ \vee \mathcal{C}_0 \vee \mathcal{C}_+[1] \vee \mathcal{C}_0''[1]$.

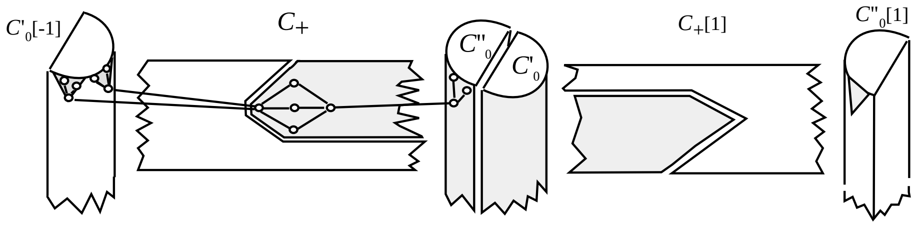


Fig. 2. The position of Σ and $\mathrm{mod}(\Sigma)$ in $\mathrm{D^b}(\mathcal{C})$

PROPOSITION 4.1. *Each indecomposable Σ-module M belongs to one of the subcategories*

(a) $\mathrm{mod}_0^\ell(\Sigma)$, *consisting of all $X[-1]$ where $X \in \mathcal{C}_0'$ satisfies* $\mathrm{Hom}_{\mathcal{C}}(\Sigma_+, X) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(\Sigma_0', X)$,

(b) $\mathrm{mod}_+(\Sigma)$, *consisting of all X from ${\Sigma_0''}^{\perp} \cap \mathcal{C}_+$ satisfying* $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma_+, X) = 0$,

(c) $\mathrm{mod}_0^c(\Sigma)$, *consisting of all X from $\mathcal{C}_0$ satisfying* $\mathrm{Hom}_{\mathcal{C}}(\Sigma_0', X) = 0$ *and* $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma_0'', X) = 0$,

(d) $\mathrm{mod}_-(\Sigma)$, *consisting of all $Z[1]$ with $Z \in {\Sigma_0'}^{\perp} \cap \mathcal{C}_+$ satisfying* $\mathrm{Hom}_{\mathcal{C}}(\Sigma_+, Z) = 0$,

(e) $\mathrm{mod}_0^r(\Sigma)$, *consisting of all $Z[1]$ with $Z \in \mathcal{C}_0$ satisfying* $\mathrm{Hom}_{\mathcal{C}}(\Sigma_+, X) = 0 = \mathrm{Hom}_{\mathcal{C}}(\Sigma_0'', X)$.

The objects from the additive closure of Σ (resp. $\tau_{\mathbb{X}}\Sigma[1]$) are the projective (resp. injective) Σ-modules; moreover, the following equivalences hold:

(a) $M \in \mathrm{mod}_0^\ell(\Sigma) \Leftrightarrow \mathrm{rk} M = 0$, $\deg M < 0$, *and* $\langle M, Y\rangle > 0$ *for some* $Y \in \mathrm{mod}_-(\Sigma)$,

(b) $M \in \mathrm{mod}_+(\Sigma) \Leftrightarrow \mathrm{rk} M > 0$,

(c) $M \in \mathrm{mod}_0^c(\Sigma) \Leftrightarrow \mathrm{rk} M = 0$ *and* $\deg M > 0$,

(d) $M \in \mathrm{mod}_-(\Sigma) \Leftrightarrow \mathrm{rk} M < 0$,

(e) $M \in \mathrm{mod}_0^r(\Sigma) \Leftrightarrow \mathrm{rk}\, M = 0$, $\deg M < 0$, *and* $\langle X, M\rangle > 0$ *for some* $X \in \mathrm{mod}_+(\Sigma)$.

Further, in the ordering $\mathrm{mod}_0^\ell(\Sigma)$, $\mathrm{mod}_+(\Sigma)$, $\mathrm{mod}_0^c(\Sigma)$, $\mathrm{mod}_-(\Sigma)$, $\mathrm{mod}_0^r(\Sigma)$ *there are no non-zero morphisms from right to left.*

Proof. An indecomposable object X of $\mathrm{D^b}(\mathcal{H}) = \mathrm{D^b}(\mathcal{C})$ belongs to $\mathrm{mod}(\Sigma)$ if and only if

(8) $$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{H})}(\Sigma[n], X) = 0 \quad \text{for each non-zero integer } n.$$

Since $\mathcal{H}$ is hereditary, only indecomposable objects X from $\mathcal{H}$ or $\mathcal{H}[1]$, i.e. from

$$\mathcal{C}_0'[-1] \vee \mathcal{C}_+ \vee \mathcal{C}_0 \vee \mathcal{C}_+[1] \vee \mathcal{C}_0''$$

can satisfy this condition. It is then straightforward to check that (8) amounts to the alternatives listed above.

The isolation of the parts through K-theoretic properties then easily follows from the fact that for an indecomposable member of $\mathcal{C}$ the rank function is > 0 (resp. 0) on indecomposable members from $\mathcal{C}_+$ (resp. $\mathcal{C}_0$) and that the degree function is strictly positive on indecomposable members from $\mathcal{C}_0$. The last assertion is obvious. ■

Note that $\Sigma_\ell = \Sigma_0'[-1] \vee \Sigma_+$ (resp. $\Sigma_r = \Sigma_+ \vee \Sigma_0''$) are full convex subcategories of Σ. Moreover, Σ_r is a tilting object in the subcategory ${}^\perp(\tau_{\mathbb{X}}\Sigma_0')$ of $\mathrm{coh}(\mathbb{X})$ left perpendicular to $\tau_{\mathbb{X}}\Sigma_0'$ which itself is a category of the form $\mathrm{coh}(\mathbb{X}_r)$ for a weighted projective line $\mathbb{X}_r$. Accordingly Σ_r and $\Sigma_\ell^{\mathrm{op}}$ are almost concealed-canonical.

PROPOSITION 4.2. *Let* Σ *be a representation-infinite quasi-tilted algebra of canonical type.*

(i) *The support of an indecomposable* Σ*-module either belongs to* Σ_ℓ *or else to* Σ_r.

(ii) Σ *is tame if and only if both* Σ_ℓ *and* Σ_r *are tame.*

Proof. (i) In more detail we read from the position of Σ and $\mathrm{mod}(\Sigma)$ in $\mathrm{D^b}(\mathcal{C})$ that:

1. $M \in \mathrm{mod}_0^\ell(\Sigma) \Rightarrow \mathrm{supp}(M) \subseteq \Sigma_0'[-1]$,
2. $M \in \mathrm{mod}_+(\Sigma) \Rightarrow \mathrm{supp}(M) \subseteq \Sigma_\ell$,
3. $M \in \mathrm{mod}_0^c(\Sigma) \Rightarrow \begin{cases} \mathrm{supp}(M) \subseteq \Sigma_r & \text{if } M \in \mathcal{C}_0'', \\ \mathrm{supp}(M) \subseteq \Sigma_\ell & \text{if } M \in \mathcal{C}_0', \end{cases}$
4. $M \in \mathrm{mod}_-(\Sigma) \Rightarrow \mathrm{supp}(M) \subseteq \Sigma_r$,
5. $M \in \mathrm{mod}_0^r(\Sigma) \Rightarrow \mathrm{supp}(M) \subseteq \Sigma_0''$.

Assertion (ii) is an immediate consequence of (i). ■

Let $\mathrm{mod}^\ell(\Sigma)$ (resp. $\mathrm{mod}^r(\Sigma)$) denote the additive closure of $\mathrm{mod}_0^\ell(\Sigma) \vee \mathrm{mod}_+(\Sigma)$ (resp. $\mathrm{mod}_-(\Sigma) \vee \mathrm{mod}_0^r(\Sigma)$). Note that each indecomposable

Σ-module belongs to exactly one of the subcategories $\mathrm{mod}^{\ell}(\Sigma)$, $\mathrm{mod}_0^c(\Sigma)$, $\mathrm{mod}^r(\Sigma)$; moreover, in this ordering there are no non-zero morphisms from right to left.

PROPOSITION 4.3. *Let Σ be a representation-infinite quasi-tilted algebra of canonical type.*

(i) $\mathrm{ind}_0^c(\Sigma)$ *decomposes into a tubular family of semiregular tubes, indexed by the projective line over k.*

(ii) *Each morphism from a module M of* $\mathrm{mod}^{\ell}(\Sigma)$ *to a module N from* $\mathrm{mod}^r(\Sigma)$ *factors through a module U from* $\mathrm{mod}_0^c(\Sigma)$.

(iii) *Each Auslander–Reiten component of* $\mathrm{mod}(\Sigma)$ *has support in Σ_r or Σ_ℓ.*

(iv) $\mathrm{mod}(\Sigma)$ *has a unique preprojective* (*resp. preinjective*) *component, agreeing with the preprojective component of* $\mathrm{mod}(\Sigma_\ell)$ (*resp. the preinjective component of* $\mathrm{mod}(\Sigma_r)$).

(v) *The categories* $\mathrm{mod}_0^{\ell}(\Sigma)$ (*resp.* $\mathrm{mod}_0^r(\Sigma)$) *have only finitely many* (*non-isomorphic*) *indecomposable objects belonging to the preprojective* (*resp. preinjective*) *component.*

(vi) *If Σ_ℓ* (*resp. Σ_r*) *is tame, each component in* $\mathrm{mod}^{\ell}(\Sigma)$ (*resp.* $\mathrm{mod}^r(\Sigma)$) *different from the preprojective* (*resp. preinjective*) *component is a stable tube or obtained from a stable tube by ray* (*resp. coray*) *insertions.*

(vii) *If Σ_ℓ* (*resp. Σ_r*) *is wild, each component in* $\mathrm{mod}^{\ell}(\Sigma)$ (*resp.* $\mathrm{mod}^r(\Sigma)$) *different from the preprojective* (*resp. preinjective*) *component is of type $\mathbb{Z}\mathbb{A}_\infty$ or obtained from $\mathbb{Z}\mathbb{A}_\infty$ by ray* (*resp. coray*) *insertions.*

Proof. (i) is shown in the preceding section.

(ii) Let $\mathcal{C}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$ and $\mathcal{T}_x = \mathrm{ind}(\mathcal{U})_x$ denote the tube corresponding to $\mathcal{U}_x$. For x in $\mathbb{X}'$ (resp. $\mathbb{X}''$) the intersection $\mathcal{T}'_x = \mathcal{T}_x \cap \mathrm{mod}_0^c(\Sigma)$ is a semiregular standard tube without projectives (resp. injectives). Moreover, the family $(\mathcal{T}'_x)_{x\in\mathbb{X}}$ is sincere and separating: Let $M \in \mathrm{mod}^{\ell}(\Sigma)$, $N \in \mathrm{mod}^r(\Sigma)$ be indecomposable modules. If $f : M \to N$ is a non-zero morphism, the assertion on supports in Proposition 4.2 implies that M (resp. N) has support in Σ_ℓ (resp. Σ_r). Accordingly, f reduces to a morphism between modules having support in $\mathrm{mod}_+(\Sigma)$, hence for any $x \in \mathbb{X}$ factors through a module from $\mathcal{U}_x$ having support in Σ_+.

(iii) follows from the description of supports in the proof of Proposition 4.2 and the fact that $\mathrm{ind}_0(\Sigma)$ is a union of full components.

(iv) follows from the arguments of [24] (compare also [16, 20]).

(v) Each module in $\mathrm{mod}_0^{\ell}(\Sigma)$ has support in the representation-finite algebra $\Sigma'[-1]$; the proof for $\mathrm{mod}_0^r(\Sigma)$ is dual.

(vi) In view of (iii) the assertion reduces to the case when Σ is almost concealed-canonical, accordingly Σ' is empty. Since Σ is tame, we have to deal with the cases $g_{\mathbb{X}} < 1$, where Σ is tame concealed, and $g_{\mathbb{X}} = 1$, where the assertion follows from [15, 21].

(vii) As in (vi) we reduce to the case of an almost concealed-canonical algebra where the assertion follows from [16, 20]. ■

We refer to [16] for the notion of the weight type of a quasi-canonical algebra. For the algebras $\Sigma_+, \Sigma_\ell, \Sigma_r, \Sigma$ associated with a representation-infinite quasi-tilted algebra Σ of canonical type these weights are given by the following data:

$$\begin{aligned}
\Sigma_+ :&\quad p_1, \ldots, p_r, q_1, \ldots, q_s\\
\Sigma_\ell :&\quad \overline{p}_1, \ldots, \overline{p}_r, q_1, \ldots, q_s\\
\Sigma_r :&\quad p_1, \ldots, p_r, \overline{q}_1, \ldots, \overline{q}_s\\
\Sigma :&\quad \overline{p}_1, \ldots, \overline{p}_r, \overline{q}_1, \ldots, \overline{q}_s.
\end{aligned}$$

In each case the weight data above can be recovered from a separating tubular family as follows: For a tube without injectives (resp. projectives) $\overline{p}_i$ (resp. $\overline{q}_j$) equals the number of rays (resp. corays) in the tube, whereas p_i (resp. q_j) equals $\overline{p}_i$ (resp. $\overline{q}_j$) minus the number of projectives (resp. injectives) in the tube and agrees with the number of corays (resp. rays). Accordingly, we form the invariants deciding on the genus of weighted projective lines associated with Σ_+, Σ_ℓ, Σ_r and Σ:

$$\begin{aligned}
\delta(\Sigma_+) &= (r+s-2) - \Big(\sum_{i=1}^{r} \frac{1}{p_i} + \sum_{j=1}^{s} \frac{1}{q_j}\Big),\\
\delta(\Sigma_\ell) &= (r+s-2) - \Big(\sum_{i=1}^{r} \frac{1}{\overline{p}_i} + \sum_{j=1}^{s} \frac{1}{q_j}\Big),\\
\delta(\Sigma_r) &= (r+s-2) - \Big(\sum_{i=1}^{r} \frac{1}{p_i} + \sum_{j=1}^{s} \frac{1}{\overline{q}_j}\Big),\\
\delta(\Sigma) &= (r+s-2) - \Big(\sum_{i=1}^{r} \frac{1}{\overline{p}_i} + \sum_{j=1}^{s} \frac{1}{\overline{q}_j}\Big).
\end{aligned}$$

Proposition 4.4. *Let Σ be a representation-infinite quasi-tilted algebra of canonical type.*

(i) *The Grothendieck group $\mathrm{K}_0(\Sigma)$ with the attached Euler form $\langle -, - \rangle_\Sigma$, and the attached quadratic form q_Σ, is uniquely determined—up to isomorphism—by the weight sequence $\overline{p}_1, \ldots, \overline{p}_r$, $\overline{q}_1, \ldots, \overline{q}_s$ of Σ attached to a sincere separating family of semiregular standard tubes.*

(ii) *The characteristic polynomial of the Coxeter transformation for* Σ *is given as*

$$\chi_\Sigma(T) = (T-1)^2 \prod_{i=1}^{r} \frac{T^{\bar{p}_i}-1}{T-1} \prod_{j=1}^{s} \frac{T^{\bar{q}_j}-1}{T-1}.$$

Accordingly all roots of χ_Σ *are roots of unity, and the spectral radius of the Coxeter transformation equals one.*

(iii) *The radical of* q_Σ *has rank one* (*resp. two*) *according as* $\delta(\Sigma) \neq 0$ (*resp.* $\delta(\Sigma) = 0$).

(iv) q_Σ *is positive semi-definite* (*resp. indefinite*) *if and only if* $\delta(\Sigma) \leq 0$ (*resp.* $\delta(\Sigma) > 0$).

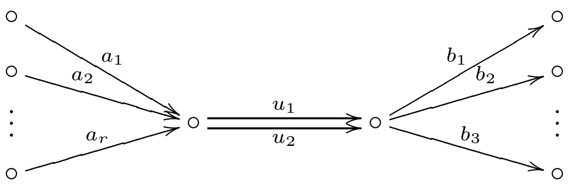
(v) q_Σ *is weakly non-negative, i.e.* $\mathrm{q}_\Sigma([M]) \geq 0$ *for each* $M \in \operatorname{mod}(\Sigma)$ *if and only if* $\delta(\Sigma_\ell) \leq 0$ *and* $\delta(\Sigma_r) \leq 0$.

P r o o f. Assertions (i)–(iv) deal with invariants under derived equivalence, and are known for categories of coherent sheaves on weighted projective lines [14, 17]. (v) follows from Proposition 4.2 together with the fact that an almost concealed-canonical algebra is tame if and only if its quadratic form is non-negative. ∎

COROLLARY 4.5. *Assume that* Σ *is a tame quasi-tilted algebra of canonical type* $\mathbb{X}$. *Then the number of exceptional points of* $\mathbb{X}$ *is bounded by* 8. ∎

We present some examples of quasi-tilted algebras of canonical type with unusual K-theoretic properties.

EXAMPLE 4.6. Consider the algebra $\Sigma = \Sigma(r,s)$, given by the quiver



with relations

$$(u_2 - \lambda'_i u_1) \circ a_i = 0, \quad b_i \circ (u_2 - \lambda''_j u_1) = 0, \quad i = 1, \ldots, r,\ j = 1, \ldots, s,$$

where $\lambda'_1, \ldots, \lambda'_r$, $\lambda''_1, \ldots, \lambda''_s$ are pairwise distinct elements of k. Note that $\Sigma(r,s)$ is quasi-canonical of weight type $(2,2,\ldots,2)$ ($r+s$ entries). Note that $\delta(\Sigma_\ell) = r/2-2$, $\delta(\Sigma_r) = s/2-2$, and $\delta(\Sigma) = (r+s)/2-2$. Accordingly, Σ is tame if and only if $r, s \leq 4$.

(i) The algebra $\Sigma(2,2)$ is tame domestic, its Auslander–Reiten quiver consists of just one tubular family and two further components, a preprojective component and a preinjective component, each having a slice of extended Dynkin type $\widetilde{\mathbb{D}}_4$. Since $\Sigma(2,2)$ is quasi-canonical of weight type $(2,2,2,2)$, the quadratic form q_Σ is non-negative, and the radical of q_Σ

has rank two. Accordingly $\Sigma(2,2)$ is not derived-equivalent to a hereditary algebra.

(ii) The algebra $\Sigma = \Sigma(4,4)$ is tame, since Σ_ℓ and Σ_r are both tubular algebras of type $(2,2,2,2)$. All but a finite number of components of $\operatorname{ind}(\Sigma)$ are stable tubes, and there are an infinite number of tubular families. The quadratic form q_Σ is weakly non-negative, but indefinite. Despite the presence of an infinite number of tubular families, the radical of q_Σ has rank one. Note that the restriction of q_Σ to the two convex subcategories Σ_ℓ and Σ_r results in quadratic forms q_{Σ_ℓ} and q_{Σ_r} having radicals of rank two.

It follows from [22] that a quasi-tilted algebra Σ of tame representation type which is not tilted, is a semiregular branch enlargement of a tame concealed algebra. Invoking the theorem of the previous section, Σ is hence realizable as a tilting object in a hereditary category of type $\mathcal{C}(\mathbb{X}',\mathbb{X}'')$, and is quasi-canonical. The following result solves the tame case of a problem arising in connection with [8].

THEOREM 4.7. *Let $\mathcal{H}$ be a hereditary category with a tilting object Σ whose endomorphism ring is tame. Then $\mathcal{H}$ is derived-equivalent to a category of type* $\operatorname{mod}(\Delta)$, *$\Delta$ a hereditary algebra*, *or of type* $\operatorname{coh}(\mathbb{X})$, *$\mathbb{X}$ a weighted projective line.*

Proof. If Σ is tilted then it can be realized as a tilting object on a module category $\operatorname{mod}(\Delta)$ where Δ is a hereditary algebra. Otherwise the quoted result [22] shows that Σ is a semiregular branch of a concealed-canonical algebra, hence in view of Theorem 3.4 is quasi-tilted of canonical type. ∎

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