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Quasi-Unitary Algebras Attached to Temperature States in Statistical Mechanics. A Comment on the Work of Haag, Hugenholtz and Winnink

D. Kastler
University of Aix-Marseille

J. C. T. Pool
Applied Mathematics Division, Argonne National Laboratory
and
II. Institut für Theoretische Physik, Universität Hamburg

E. Thue Poulsen
Matematisk Institut, Aarhus Universitet

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Abstract. We show that the *-algebra of "analytic elements" with respect to time translations which plays a central role in Haag, Hugenholtz and Winnink's formulation of the Kubo-Martin-Schwinger boundary condition, is a quasi-unitary algebra in the sense of Dixmier. The commutant theorem proved by Haag, Hugenholtz and Winnink is thus reduced to Dixmier's commutant theorem for quasi-unitary algebras.

1. Introduction

In a very interesting paper [1] (referred to below as HHW), Haag, Hugenholtz and Winnink describe general features of the equilibrium states of quantum statistical mechanics at finite temperature. A state is viewed as normalized positive linear functional ω on a C^*-algebra $\mathfrak{A}$ of quasi-local observables. Time evolution is described by a one-parameter group, $t \to \alpha_t$, of automorphisms of $\mathfrak{A}$. An algebraic formulation of the Kubo-Martin-Schwinger [2, 3] boundary condition is given as a property of equilibrium states with respect to the time-development automorphisms. Furthermore, it is shown that, in contrast to the zero temperature situation, the representation of $\mathfrak{A}$ obtained from an equilibrium state ω by means of the Gelfand-Segal construction is reducible, the corresponding weak closure being one-to-one with its commutant.

The main mathematical tool in HHW is a norm-dense *-subalgebra $\tilde{\mathfrak{A}}$ of "analytic elements" of $\mathfrak{A}$. The purpose of the present note is two-fold. First, we fix some points of rigor in HHW using the necessary amount of vectorial distributions: to each C^*-algebra $\mathfrak{A}$ with an abelian

n-parameter group of *-automorphisms $t \in R \to \alpha_t$, such that $\alpha_t(A)$ is norm-continuous in t for each $A \in \mathfrak{A}$, is associated a norm-dense α-invariant *-subalgebra $\tilde{\mathfrak{A}}$ equipped with an n-parameter group of automorphisms, $\beta \in R \to j_\beta$ with the properties stated in Proposition 2 below (formally, $j_\beta = \alpha_{-i\beta/2}$). Second, we point out the relation of the formalism described in HHW with the notion of quasi-unitary algebras as developed by DIXMIER [4]. We show that if one assumes the Kubo-Martin-Schwinger boundary condition as formulated in HHW [(4.2) or (4.3) of [1]; (32) below] for an α-invariant state ω, a condition physically cogent in the finite temperature equilibrium situation, then the *-algebra $\tilde{\mathfrak{A}}$ may be given the structure of a quasi-unitary algebra. The commutant theorem proved in HHW (Theorem 4 in [1]) then merges into DIXMIER's commutant theorem for quasi-unitary algebras (Theorem 1 of [4]).

2. The Sub-*-algebra $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$

Our general frame of work is that of a C^*-algebra $\mathfrak{A}$ acted upon by a one-parameter strongly continuous group of automorphisms: $t \in R \to \alpha_t$ is a homomorphic mapping of the additive group of the reals into the automorphism group of $\mathfrak{A}$ such that $\alpha_t(A)$ is norm-continuous in t for each $A \in \mathfrak{A}$ (one can equivalently require continuity of all numerical functions $t \to \Phi(\alpha_t(A))$ for all $A \in \mathfrak{A}$ and states Φ of $\mathfrak{A}$, cf. [5], 10.2 Corollary and [6], 2.6.4).

A special role will be played in the sequel by the set $\mathfrak{A}^{(\infty)}$ of "infinitely differentiable elements" of $\mathfrak{A}$. We remind that the infinitesimal operator D of the one-parameter group α is defined by the property

$$\left\| \frac{\alpha_h(A) - A}{h} - DA \right\| \xrightarrow[h=0]{} 0 \tag{1}$$

on the subset $\mathfrak{A}^{(1)}$ of elements $A \in \mathfrak{A}$ for which $h^{-1}[\alpha_h(A) - A]$ tends to a limit in the norm for $h = 0$. One checks immediately that $\mathfrak{A}^{(1)}$ is a linear subset of $\mathfrak{A}$ and that D is linear; and furthermore that for each $t \in R$ $\mathfrak{A}^{(1)}$ is invariant under α_t, each $A \in \mathfrak{A}^{(1)}$ being such that

$$\underset{h=0}{\text{norm-lim}} \frac{\alpha_{t+h}(A) - \alpha_t(A)}{h} = \alpha_t(DA) = D\,\alpha_t(A)\,. \tag{2}$$

Thus $\mathfrak{A}^{(1)}$ consists of the $A \in \mathfrak{A}$ such that the function

$$X_A : t \in R \to \alpha_t(A) \in \mathfrak{A} \tag{3}$$

(the orbit of A under α) is differentiable.

We now define $\mathfrak{A}^{(p)}$ as the domain of the $p - th$ power D^p of D, p positive integer and set

$$\mathfrak{A}^{(\infty)} = \bigcap_{p=1}^{\infty} \mathfrak{A}^{(p)}\,. \tag{4}$$

Using (2) recursively, one sees that, for $A \in \mathfrak{A}^{(p)}$

$$\left(\frac{\partial X_A}{\partial s^p}\right)_{s=t} = \alpha_t(D^p A) = D^p \alpha_t(A) , \tag{5}$$

the derivative being defined in the norm-sense; thus $\mathfrak{A}^{(p)}$ ($\mathfrak{A}^{(\infty)}$) consists of the $A \in \mathfrak{A}$ for which the function X_A is C^p (C^∞).

Lemma 1. *$\mathfrak{A}^{(p)}$, $p = 1, 2, \ldots, \infty$ is a norm-dense, α-invariant sub-*-algebra of $\mathfrak{A}$.*

Proof. $\mathfrak{A}^{(p)}$ is evidently α-invariant. For finite p and $A, B \in \mathfrak{A}^{(p)}$ we have, as one easily checks

$$D^p(A^*) = (D^p A)^* , \tag{6}$$

$$D^p(A B) = \sum_{k=0}^{p} \binom{p}{k} D^{p-k} A \cdot D^k B \tag{7}$$

(where we set $D^0 A = A$). Thus $\mathfrak{A}^{(p)}$ is a sub-*-algebra of $\mathfrak{A}$ and the same holds of $\mathfrak{A}^{(\infty)}$ due to (4). On the other hand $\mathfrak{A}^{(\infty)}$ has been shown by GÅRDING [7] to be norm dense in $\mathfrak{A}$.

As mentioned in the Introduction, our aim is to define the operator $\alpha_{-i\beta/2}$, β real, on appropriate elements of $\mathfrak{A}$. To this end we notice that α_t acts on the functions X_A as a shift of the argument, i.e. as a convolution with a Dirac measure. Looked at on Fourier transforms, this becomes a multiplication times the function $\xi \to e^{it\xi}$, so that we will obtain the desired definition of $\alpha_{-i\beta/2}$ as a multiplication (allowed under appropriate circumstances) times the function $\xi \to e^{\frac{1}{2}\beta\xi}$. Our first task will be to introduce the Fourier transforms $\hat{X}_A$ in a precise manner.

Let $\mathscr{C}(R, \mathfrak{A})$ be the linear space of continuous norm-bounded functions from R to $\mathfrak{A}$. $\mathscr{C}(R, \mathfrak{A})$ is a normed *-algebra under the following definitions

$$X Y(t) = X(t)\, Y(t) , \tag{8}$$

$$\overline{X}(t) = X(t)^* , \qquad X, Y \in \mathscr{C}(R, \mathfrak{A}) , \quad t \in R , \tag{9}$$

$$\|X\| = \operatorname*{Sup}_{t \in R} \|X(t)\| . \tag{10}$$

On the other hand $\mathscr{C}(R, \mathfrak{A})$ can be embedded in the set $\mathscr{S}'(R, \mathfrak{A})$ of tempered $\mathfrak{A}$-valued distributions on R by setting

$$\langle X, f \rangle = \int f(t)\, X(t)\, dt , \quad f \in \mathscr{S}(R) , \tag{11}$$

where the integral on the right-hand side is a well defined Bochner integral and $\mathscr{S}(R)$ denotes the set of C^∞ functions on R with rapid decrease. Let $\hat{R}$ be the dual real line with the corresponding sets $\mathscr{S}(\hat{R})$ and $\mathscr{S}'(\hat{R}, \mathfrak{A})$ of functions with rapid decrease and tempered $\mathfrak{A}$-valued distributions. It is known that the Fourier transform mapping of tempered vectorial distributions

$$T \in \mathscr{S}'(R, \mathfrak{A}) \to \hat{T} \in \mathscr{S}'(\hat{R}, \mathfrak{A}) , \tag{12}$$

defined by

$$\langle \hat{T}, f \rangle = \langle T, \check{f} \rangle , \quad f \in \mathscr{S}(\hat{R}) \tag{12a}$$

where

$$f \in \mathscr{S}(\hat{R}) \to \check{f} \in \mathscr{S}(R)$$
$$\check{f}(s) = \frac{1}{2\pi} \int e^{-is\xi} f(\xi)\, d\xi , \quad s \in R , \tag{13}$$

is the usual Fourier transform of $\mathscr{S}$-functions, is a one-to-one bicontinuous mapping of $\mathscr{S}'(R, \mathfrak{A})$ onto $\mathscr{S}'(\hat{R}, \mathfrak{A})$. We now state

Lemma 2. *The correspondance $A \in \mathfrak{A} \to X_A \in \mathscr{C}(R, \mathfrak{A})$ defined by* (3) *is an isometric homomorphism of *-algebras, mapping $\mathfrak{A}$ onto the subset of $X \in \mathscr{C}(R, \mathfrak{A})$ characterized by the property*

$$\alpha_t(\langle X, \check{f} \rangle) = \langle X, \check{f}_t \rangle , \quad \check{f} \in \mathscr{S}(R) , \quad t \in R , \tag{14}$$

or else

$$\alpha_t(\langle \hat{X}, f \rangle) = \langle \hat{X}, f^t \rangle , \quad f \in \mathscr{S}(\hat{R}) , \quad t \in R , \tag{14a}$$

where

$$\check{f}_t(s) = \check{f}(s-t) \quad \text{and} \quad f^t(\xi) = e^{it\xi} f(\xi) , \quad s \in R , \quad \xi \in \hat{R} .$$

Proof. $A \to X_A$ is obviously linear, isometric due to (10) and the known fact that $\|\alpha_t(A)\| = \|A\|$, $t \in R$, [4; 1.8.3], and *-homomorphic because $\alpha_t(A)^* = \alpha_t(A^*)$ and $\alpha_t(AB) = \alpha_t(A) \cdot \alpha_t(B)$. On the other hand the vectorial distribution X_A is such that

$$\langle X_A, \check{f} \rangle = \int dt\, \check{f}(t)\, X_A(t) = \int dt\, \check{f}(t)\, \alpha_t(A) = \alpha(\check{f})\, A , \quad \check{f} \in \mathscr{S}(R) \tag{15}$$

(see Appendix A), therefore by Eq. (A 5) there, we have

$$\alpha_t(\langle X_A, \check{f} \rangle) = \alpha(\delta_t)\, \alpha(\check{f})\, A = \alpha(\delta_t * \check{f})\, A$$
$$= \alpha(\check{f}_t)\, A = \langle X_A, \check{f}_t \rangle .$$

Conversely take an $X \in \mathscr{C}(R, \mathfrak{A})$ fulfilling (14). We have, exchanging the continuous α_t with the Bochner integral

$$\alpha_t\{\int ds\, \check{f}(s)\, X(s)\} = \int ds\, \check{f}(s)\, \alpha_t\{X(s)\}$$
$$= \int ds\, \check{f}_t(s)\, X(s) = \int ds\, \check{f}(s)\, X(s+t) ,$$

therefore $X(s+t) = \alpha_t\{X(s)\}$ for s, $t \in R$ and in particular $X(t) = \alpha_t\{X(0)\}$; q.e.d. Property (14a) results immediately by Fourier transform.

The subset $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$, which we now define, is the central object of this paper.

Definition 1. *We denote by $\tilde{\mathfrak{A}}$ the subset of $\mathfrak{A}^{(\infty)}$ consisting of the elements A such that the $\mathfrak{A}$-valued distribution $\hat{X}_A$ has compact support in $\hat{R}$. $\tilde{\mathfrak{A}}$ will alternatively be considered, by means of the injective mapping $A \to X_A$, as a subset of $\mathscr{C}(R, \mathfrak{A}) \subset \mathscr{S}'(R, \mathfrak{A})$.*

Proposition 1. *$\tilde{\mathfrak{A}}$ is a norm dense, α-invariant sub-*-algebra of $\mathfrak{A}$.*

Proof. Since $\tilde{\mathfrak{A}}$ is linear and $\mathfrak{A}^{(\infty)}$ is dense in $\mathfrak{A}$, it suffices to check that each $A \in \mathfrak{A}^{(\infty)}$ is weakly adherent to $\tilde{\mathfrak{A}}$, for then it follows from Hahn-Banach's theorem that $\tilde{\mathfrak{A}}$ is norm-dense in $\mathfrak{A}$. Let $A \in \mathfrak{A}^{(\infty)}$. Using the notation of the Appendix A it is clear that

$$\widehat{\langle \alpha(\hat{f})\, A, \varphi\rangle} = \langle \alpha(\hat{f})\, A, \hat{\varphi}\rangle = \langle \alpha(\hat{f} * \hat{\varphi})\, A = \alpha(\widehat{f\varphi})\, A$$

for $f, \varphi \in \mathscr{D}(\hat{R})$ (the set of C^∞-functions on $\hat{R}$ with compact supports), and hence that $\alpha(\hat{f})\, A \in \tilde{\mathfrak{A}}$ if f belongs to $\mathscr{D}(\hat{R})$. If we arbitrarily preassign $\varepsilon > 0$ and the $\Phi_k \in \mathfrak{A}^*$, $k = 1, 2, \ldots, n$, our proof thus reduces to find an $f \in \mathscr{D}(\hat{R})$ realizing the condition

$$|\Phi_k(\alpha(\hat{f}) - A)| \leqq \varepsilon\,.$$

Using the interchangeability of the continuous Φ_k with the Bochner integral, we have

$$\Phi_k(\alpha(\hat{f}) - A) = \int \Phi_k(\alpha_g(A))\,(d\hat{f}(g) - d\delta(g))\,,$$

where δ denotes the Dirac measure at the origin; the last condition will thus be fulfilled by approximating δ by elements of $\mathscr{D}(\hat{R})$ in the weak topology of measures with respect to C^∞ functions, an elementary procedure.

The fact that $A \in \tilde{\mathfrak{A}}$ implies $A^* \in \tilde{\mathfrak{A}}$ results obviously from Eq. (9), from the fact that the distributions $\hat{X}$ and $(\hat{X}^*)$ have symmetric supports in $\hat{R}$, and from the *-symmetric character of $\mathfrak{A}^{(\infty)}$. The α-invariance of $\tilde{\mathfrak{A}}$ follows immediately from that of $\mathfrak{A}^{(\infty)}$ and property (14). In order to complete our proof it remains to establish the multiplicative character of $\tilde{\mathfrak{A}}$. Since $\mathfrak{A}^{(\infty)}$ is multiplicative (Lemma 1), we have to prove that if $A, B \in \mathfrak{A}^{(\infty)}$ are such that $\hat{X}_A$ and $\hat{X}_B$ have compact supports in $\hat{R}$, the same holds of $\hat{X}_{AB} = \widehat{X_A X_B}$. This results from property (18) in the following.

Lemma 3. *Let us denote by $(\hat{X}_A, \hat{X}_B) \to \hat{X}_A * \hat{X}_B$ and $\hat{X}_A \to \hat{X}_A^*$ the operations on $\tilde{\mathfrak{A}}$ obtained by transporting the *-algebraic operations* (8), (9) *in the Fourier transform* (12):

$$\hat{X}_A * \hat{X}_B = \widehat{X_A X_B} = \hat{X}_{AB}\,, \tag{16}$$

$$\hat{X}_A^* = \hat{\bar{X}}_A = \hat{X}_{A^*}\,. \tag{17}$$

These operations can be directly defined on the distributions $\hat{X}_A \in \mathscr{S}'(\hat{R})$: one has

$$\langle \hat{X}_A * \hat{X}_B, f\rangle = \langle (\hat{X}_A)_\xi \otimes (\hat{X}_B)_\eta, f(\xi + \eta)\rangle\,, \quad f \in \mathscr{S}(\hat{R})\,, \tag{18}$$

where $(\hat{X}_A)_\xi \otimes (\hat{X}_B)_\eta$ *is the unique vectorial distribution on* $\hat{R} \times \hat{R}$ *such that*

$$\langle (\hat{X}_A)_\xi \otimes (\hat{X}_B)_\eta, u(\xi)\, v(\eta)\rangle = \langle \hat{X}_A, u\rangle \langle \hat{X}_B, v\rangle \tag{19}$$

$$u, v \in \mathcal{S}(\hat{R})$$

and

$$\langle \hat{X}_A^*, f\rangle = \langle \hat{X}_A, f^*\rangle\,, \quad f \in \mathcal{S}(\hat{R})\,, \tag{20}$$

where

$$f^*(\xi) = \overline{f(-\xi)}\,, \quad \xi \in \hat{R}\,. \tag{21}$$

The proof of this lemma, which is slightly technical, is given in Appendix B.

3. The Operators j_β on $\tilde{\mathfrak{A}}$

We denote by e_β, $\beta \in R$, the numerical function $\xi \in \hat{R} \to \exp\left\{\frac{1}{2}\beta\xi\right\}$. Since e_β is a C^∞-function we can multiply by e_β the vectorial distribution $\hat{X}_A$, $A \in \tilde{\mathfrak{A}}$, thus obtaining a distribution $e_\beta \hat{X}_A \in \mathcal{S}'(\hat{R}, \mathfrak{A})$ with the same compact support as $\hat{X}_A$.

Lemma 4. *For each* $A \in \tilde{\mathfrak{A}}$ *the* $\mathfrak{A}$*-valued distribution* $e_\beta \hat{X}_A$ *is of the form* $\hat{X}_{j_\beta(A)}$ *with* $j_\beta(A)$ *an element of* $\tilde{\mathfrak{A}}$ *determined by*

$$\langle X_{j_\beta(A)}, \hat{f}\rangle = \langle X_A, \widehat{e_\beta f}\rangle = \alpha(\widehat{e_\beta f})\, A\,, \quad f \in \mathcal{D}(\hat{R})\,. \tag{22}$$

Proof. Let $K \in \hat{R}$ be the support of $\hat{X}_A$ and take $u = v e_\beta$ with $v \in \mathcal{S}(\hat{R})$ and $v = 1$ on K. We then have $u \in \mathcal{S}(\hat{R})$ and $e_\beta \hat{X}_A = u\hat{X}_A$. Therefore, for each $f \in \mathcal{S}(\hat{R})$, using known properties of the Bochner integral

$$\begin{aligned}\langle e_\beta \hat{X}_A, f\rangle &= \langle u\hat{X}_A, f\rangle = \langle \hat{X}_A, uf\rangle = \langle X_A, \widehat{uf}\rangle \\ &= \langle X_A, \hat{u} * \hat{f}\rangle = \int \alpha_t(A)\, [\int \hat{u}(t-s)\, \hat{f}(s)\, ds]\, dt \\ &= \int [\int \hat{u}(t-s)\, \alpha_t(A)\, dt]\, \hat{f}(s)\, ds = \int \alpha_s(\alpha(\hat{u})\, A)\, \hat{f}(s)\, ds \\ &= \langle X_{\alpha(\hat{u})A}, \hat{f}\rangle = \langle \hat{X}_{\alpha(\hat{u})A}, f\rangle\,.\end{aligned}$$

Thus $e_\beta \hat{X}_A = \hat{X}_{\alpha(\hat{u})A}$. Since $\alpha(\hat{u})\, A \in \mathfrak{A}^{(\infty)}$ (see Lemma (b) of Appendix A) and since $e_\beta \hat{X}_A$ has compact support, $\alpha(\hat{u})\, A$ belongs to $\tilde{\mathfrak{A}}$ and can be denoted $j_\beta(A)$ since it depends only upon A and β. If f has compact support the beginning of the previous calculation with $u = e_\beta$ leads to (22).

Proposition 2. *The linear operators* j_β, $\beta \in R$, *of* $\tilde{\mathfrak{A}}$ *have the following properties:*

$$j_\beta(AB) = j_\beta(A)\, j_\beta(B)\,, \tag{23}$$

$$j_\beta(A^*) = j_{-\beta}(A)^*\,, \quad A, B \in \tilde{\mathfrak{A}}\,, \quad \beta, \beta_1, \beta_2, t \in R, \tag{24}$$

$$j_{\beta_1}(j_{\beta_2}(A)) = j_{\beta_1+\beta_2}(A)\,, \tag{25}$$

$$j_0(A) = A\,, \tag{26}$$

$$j_\beta(\alpha_t(A)) = \alpha_t(j_\beta(A))\,, \tag{27}$$

and

$$j_\beta(\alpha(\hat{f})\,A) = \alpha(\widehat{e_\beta f})\,A\,, \qquad A \in \mathfrak{A}\,, \qquad f \in \mathscr{D}(\hat{R})\,. \tag{28}$$

If the state ω of $\mathfrak{A}$ is α-invariant, that is if $\omega(\alpha_t(A)) = \omega(A)$ for all $t \in R$ and $A \in \mathfrak{A}$, the restriction of ω to $\tilde{\mathfrak{A}}$ is j_β-invariant i.e. $\omega(j_\beta(A)) = \omega(A)$ for all $\beta \in R$ and $A \in \mathfrak{A}$.

Proof. The operators j_β are obviously linear. In order to prove properties (23) through (26) we have, according to (16), (17) and the Lemmas 2 and 3, to check that

$$e_\beta(\hat{X}_A * \hat{X}_B) = (e_\beta \hat{X}_A) * (e_\beta \hat{X}_B)\,, \tag{23a}$$

$$(e_\beta \hat{X}_A)^* = e_{-\beta} \hat{X}_A^*\,, \tag{24a}$$

$$e_{\beta_1} e_{\beta_2} \hat{X}_A = e_{\beta_1 + \beta_2} \hat{X}_A\,, \tag{25a}$$

$$e_0 \hat{X}_A = \hat{X}_A\,. \tag{26a}$$

We have, for $f \in \mathscr{S}(\hat{R})$,

$$\begin{aligned}
\langle e_\beta(\hat{X}_A * \hat{X}_B), f\rangle &= \langle \hat{X}_A * \hat{X}_B, e_\beta f\rangle \\
&= \langle (\hat{X}_A)_\xi \otimes (\hat{X}_B)_\eta, e_\beta(\xi + \eta)\, f(\xi + \eta)\rangle \\
&= \langle (e_\beta \hat{X}_A)_\xi \otimes (e_\beta \hat{X}_B)_\eta, f(\xi + \eta)\rangle \\
&= \langle (e_\beta \hat{X}_A) * (e_\beta \hat{X}_B), f\rangle
\end{aligned}$$

whence (23a); and

$$\begin{aligned}
\langle (e_\beta \hat{X}_A)^*, f\rangle &= \langle e_\beta \hat{X}_A, f^*\rangle^* = \langle \hat{X}_A, e_\beta f^*\rangle^* \\
&= \langle \hat{X}_A, (e_{-\beta} f)^*\rangle^* = \langle \hat{X}_A^*, e_{-\beta} f\rangle \\
&= \langle e_{-\beta} \hat{X}_A^*, f\rangle
\end{aligned}$$

whence (24a); (25a) and (26a) are evident from the definition of e_β. Property (27) further results from (22) and from the fact that due to (A 5) of Appendix A the α_t and $\alpha(\hat{f})$ commute.

Assume now that the state ω of $\mathfrak{A}$ is α-invariant. Select an $f \in \mathscr{S}(\hat{R})$ such that $e_\beta f \in \mathscr{S}(\hat{R})$ and $f(0) \neq 0$. With $A \in \tilde{\mathfrak{A}}$, (22) gives

$$\int \alpha_t(j_\beta(A))\, \hat{f}(t) = \int \alpha_t(A)\, \widehat{e_\beta f}(t)\, dt\,.$$

Applying ω on both sides, exchanging it with the Bochner integrals and taking account of the α-invariance of ω we have

$$\begin{aligned}
\omega(j_\beta(A)) \int \hat{f}(t)\, dt = \omega(j_\beta(A))\, f(0) &= \omega(A) \int \widehat{e_\beta f}(t)\, dt \\
&= \omega(A)\, e_\beta(0)\, f(0)
\end{aligned}$$

whence the j_β-invariance of ω since $e_\beta(0) = 1$ and $f(0) \neq 0$.

It remains us to prove (28). We first recall that for $A \in \mathfrak{A}$ and $\hat{f} \in \mathscr{D}(\hat{R})$, $\alpha(\hat{f})\,A \in \tilde{\mathfrak{A}}$. Then we have by (22), for $g \in \mathscr{S}(R)$,

$$\begin{aligned}\langle X_{j_\beta(\alpha(\hat{f})A)}, \hat{g}\rangle &= \alpha(\widehat{e_\beta g})\,\alpha(\hat{f})\,A = \alpha(\widehat{e_\beta g} * \hat{f})\,A\\ &= \alpha(\widehat{e_\beta g f})\,A = \alpha(\widehat{e_\beta f} * \hat{g})\,A = \alpha(\hat{g})\,\alpha(\widehat{e_\beta f})\,A\\ &= \langle X_{\alpha(\widehat{e_\beta f})}, \hat{g}\rangle\end{aligned}$$

where we used (A 5) of Appendix A and (15). (28) follows by comparison.

4. The Kubo-Martin-Schwinger Condition

The purpose of this section is to review the algebraic formulation of the Kubo-Martin-Schwinger boundary condition presented in [1]. If T is fixed, k is Boltzman's constant and $\beta = 1/kT$ and if ω is a state of $\mathfrak{A}$, then the Kubo-Martin-Schwinger boundary condition for ω (if ω is to describe equilibrium at temperature T) may be expressed as follows (see [1], p. 225).

If $A \in \tilde{\mathfrak{A}}$ and $B \in \mathfrak{A}$ then the two functions

$$\left.\begin{aligned} t \in R \to F_{AB}(t) &= \omega(B\,\alpha_t(A))\,, \quad &(29\text{a})\\ t \in R \to G_{AB}(t) &= \omega(\alpha_t(A)\,B) \quad &(29\text{b})\end{aligned}\right\}$$

are bounded C^∞ functions due to (5) and the property $\|\alpha_t(A)\| = \|A\|$, $A \in \mathfrak{A}$. Considering them in the usual way as belonging to the set $\mathscr{S}'(R)$ of tempered distributions on R we can write, for $\hat{f} \in \mathscr{S}(R)$, exchanging ω and multiplication by B with Bochner integrals and using (15)

$$\left.\begin{aligned}\langle F_{AB}, \hat{f}\rangle &= \int F_{AB}(t)\,\hat{f}(t)\,dt = \omega(B\cdot\alpha(\hat{f})\,A) = \omega(B\,\langle X_A, \hat{f}\rangle)\\ \langle G_{AB}, \hat{f}\rangle &= \int G_{AB}(t)\,\hat{f}(t)\,dt = \omega(\alpha(\hat{f})\,A\cdot B) = \omega(\langle X_A, \hat{f}\rangle\,B)\,,\end{aligned}\right\} \quad (30)$$

or, in terms of Fourier transforms

$$\left.\begin{aligned}\langle \hat{F}_{AB}, f\rangle &= \omega(B\,\langle \hat{X}_A, f\rangle)\\ \langle \hat{G}_{AB}, f\rangle &= \omega(\langle \hat{X}_A, f\rangle\,B)\end{aligned}\right\}, \quad f \in \mathscr{S}(\hat{R})\,. \quad (30\text{a})$$

Lemma 5. *For $A \in \tilde{\mathfrak{A}}$ and $B \in \mathfrak{A}$ the functions F_{AB} and G_{AB} defined in* (29a, b) *are extendable for complex values to entire analytic functions of exponential type. Moreover one has*

$$\begin{aligned}\hat{F}_{j_\beta(A)B} &= e_\beta \hat{F}_{AB}\\ \hat{G}_{j_\beta(A)B} &= e_\beta \hat{G}_{AB}\end{aligned} \quad (31\text{a})$$

and

$$\begin{aligned}F_{AB}\left(t - i\frac{\beta}{2}\right) &= F_{j_\beta(A)B}(t) = \omega\big(B\,\alpha_t(j_\beta(A))\big)\\ G_{AB}\left(t - i\frac{\beta}{2}\right) &= G_{j_\beta(A)B}(t) = \omega\big(\alpha_t(j_\beta(A))\,B\big)\end{aligned} \quad (31\text{b})$$

for all $A \in \tilde{\mathfrak{A}}$, $B \in \mathfrak{A}$ and real t and β.

Proof. For $A \in \tilde{\mathfrak{A}}$, $\hat{X}_A$, and thus $\hat{F}_{AB}$ and $\hat{G}_{AB}$ have compact support. Therefore the first assertion is a consequence of the Paley-Wiener theorem [9, Théorème XVI, p. 128]. On the other hand, we have, by (31) and Lemma 4

$$\langle \hat{F}_{j_\beta(A)B}, f\rangle = \omega(B\langle \hat{X}_A, f\rangle) = \omega(B\langle e_\beta \hat{X}_A, f\rangle) = \langle e_\beta \hat{F}_{AB}, f\rangle$$

for all $f \in \mathscr{S}(\hat{R})$ and analogously for $G_{j_\beta(A)B}$, whence (31a). The properties (31b) then immediately result by Fourier transform.

Definition 2. *A state ω of $\mathfrak{A}$ fulfills the Kubo-Martin-Schwinger (KMS) condition for the temperature β ($\beta \in R$) whenever, for all $A \in \tilde{\mathfrak{A}}$ and $B \in \mathfrak{A}$ the entire functions F_{AB} and G_{AB} defined in* (29a, b) *are related by*

$$F_{AB}(t + i\beta) = G_{AB}(t), \quad t \in R. \tag{32}$$

States with this property will be called (temperature β) KMS-states.

Lemma 6. *Either of the following conditions is necessary and sufficient for a state ω of $\mathfrak{A}$ to be a KMS-state*:

(a) *for all $A \in \mathfrak{A}$ and $B \in \mathfrak{A}$*

$$\hat{F}_{AB} = e_{2\beta}\hat{G}_{AB}, \tag{33a}$$

(b) *for all $A \in \tilde{\mathfrak{A}}$ and $B \in \mathfrak{A}$*

$$\omega(j_{2\beta}(A)\cdot B) = \omega(BA), \tag{33b}$$

(c) *for all $A \in \tilde{\mathfrak{A}}$ and $B \in \mathfrak{A}$*

$$\omega(B\cdot j_{-2\beta}(A)) = \omega(AB). \tag{33c}$$

Moreover if ω is α-invariant the following statement is equivalent to (a) *or* (b);

(d) *for all $A, B \in \tilde{\mathfrak{A}}$*

$$\omega(j_{-\beta}(A)^* j_{-\beta}(B)) = \omega(BA^*). \tag{33d}$$

Proof. (a), (b) and (c) are obviously each equivalent to (32) by virtue of (31a) and (31b). If ω is α-invariant, ω restricted to $\tilde{\mathfrak{A}}$ is j_β-invariant, thus (33c) can be written using (23)

$$\omega(j_\beta(A)\cdot j_{-\beta}(B)) = \omega(BA)$$

whence (33d) by replacing A by A^* and using (24).

5. The Quasi-Unitary Algebra Associated with an Invariant KMS-State of $\mathfrak{A}$

We now discuss the relation of the HHW-formulation of the KMS boundary condition with the notion of quasi-unitary algebra introduced by Dixmier in [4]. We consider a fixed $\beta \in R$ (to be interpreted as $1/kT$) and a fixed normalized positive linear functional ω on $\mathfrak{A}$ (to be interpreted as the equilibrium state of temperature T) which is α-invariant,

$$\omega(\alpha_t(A)) = \omega(A)\,, \quad A \in \mathfrak{A}\,, \quad t \in R \tag{34}$$

and satisfies the algebraic formulation of the KMS boundary condition:

$$\omega(j_{-\beta}(A)^* j_{-\beta}(B)) = \omega(BA^*) \tag{33d}$$

for all $A, B \in \tilde{\mathfrak{A}}$. *We define a positive sesquilinear form* $(\cdot,\cdot)$ *on* $\tilde{\mathfrak{A}}$ *by*

$$(A, B) = \omega(A^* B)\,, \quad A, B \in \tilde{\mathfrak{A}} \tag{35}$$

and two mappings $A \to A^j$ *and* $A \to A^s$ *of* $\tilde{\mathfrak{A}}$ *into* $\tilde{\mathfrak{A}}$ *by*

$$A^j = j_\beta(A)\,, \quad A \in \tilde{\mathfrak{A}}\,, \tag{36a}$$

$$A^s = j_{-\beta}(A^*) = j_\beta(A)^*\,, \quad A \in \tilde{\mathfrak{A}} \tag{36b}$$

[see (24)]. Utilizing these notations, we have the following lemma.

Lemma 7. *$A \to A^j$ and $A \to A^s$ are one-to-one mappings of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}$ such that*

$$(aA + bB)^j = aA^j + bB^j\,, \tag{37a}$$

$$(aA + bB)^s = \bar{a}A^s + \bar{b}B^s\,, \tag{37b}$$

$$(AB)^j = A^j B^j\,, \tag{38a}$$

$$(AB)^s = B^s A^s\,, \tag{38b}$$

$$A^s = A^{j*} = A^{jsj}\,, \tag{39}$$

$$A^{ss} = A\,, \tag{40}$$

$$A^{sj} = A^*\,, \tag{41}$$

$$(A^s, B^s) = (B, A) = \overline{(A, B)}\,, \tag{42}$$

$$(A^j, A) \geqq 0\,, \tag{43}$$

$$(A^j, B) = (A, B^j)\,, \tag{44}$$

$$(XA, B) = (A, X^* B) = (A, X^{sj} B)\,, \tag{45a}$$

$$(AX, B) = (A, BX^{js})\,, \tag{45b}$$

for all $A, B, X \in \tilde{\mathfrak{A}}$ and complex numbers a, b.

Proof. (37a) and (38a) are a restatement of properties of j_β from Proposition 2. By the definition of $A \to A^s$, we have by (24)

$$A^s = j_{-\beta}(A^*) = j_\beta(A)^* = A^{j*}\,, \quad A \in \mathfrak{A}\,,$$

which establishes the first equality of (39). The mapping $A \to A^s$ is the composition of the linear mapping $A \to A^j$ and the conjugate-linear mapping $A \to A^*$; therefore, (37b) follows. By (39) and (38a),

$$(AB)^s = (AB)^{j*} = (A^j B^j)^* = B^{j*} A^{j*} = B^s A^s\,, \quad A, B \in \tilde{\mathfrak{A}}\,;$$

hence, (38b). By (39), (25), (26)

$$A^{ss} = j_{-\beta}((A^s)^*) = j_{-\beta}((A^{j*})^*) = j_{-\beta}(j_\beta(A)) = A\,, \qquad A \in \tilde{\mathfrak{A}}\,;$$

hence, (40). Replacing A by A^j in (41), which follows immediately from (36a, b), we obtain

$$(A^j)^{sj} = (A^j)^* = A^s\,;$$

hence, the remainder of (39) is established. If A and B in (32) are replaced by A^* and B^*, respectively, we obtain

$$(A^s, B^s) = \omega((A^s)^* B^s) = \omega(B^* A^{**}) = (B, A)$$

whence (42), since $\overline{(A, B)} = (B, A)$ by the positivity of ω. If $A \in \tilde{\mathfrak{A}}$, then

$$(A^j, A) = \omega(j_{-\beta}(A^*)\,A)\,.$$

Since ω is α-invariant, ω is invariant under $j_{\beta/2}$ according to Proposition 2; hence, using (24), (25)

$$\begin{aligned}(A^j, A) &= \omega\big(j_{\beta/2}(j_{-\beta}(A^*)\,A)\big)\\ &= \omega(j_{-\beta/2}(A^*)\,j_{\beta/2}(A))\\ &= \omega\big(j_{\beta/2}(A)^*\,j_{\beta/2}(A)\big)\end{aligned}$$

which is positive since ω is positive and (43) is valid. $A, B \to (A^j, B)$ and $A, B \to (A, B^j)$ are two sesqui-linear forms on $\tilde{\mathfrak{A}}$ such that

$$(A^j, A) = (A, A^j)$$

[because of (42) and (43)]; hence, these two sesqui-linear forms are equal which establishes (44). The first equality of (45a) follows immediately from the definition of $(\cdot,\cdot)$ and the second equality follows from (41). (45b) follows from (45a) by using (42) and (39). Q.E.D.

Lemma 8. *The null left ideal of ω in $\tilde{\mathfrak{A}}$*

$$\tilde{N}_\omega = \{A \in \tilde{\mathfrak{A}} : \omega(A^* A) = 0\} \tag{46}$$

is invariant under $A \to A^s$; hence, $\tilde{N}_\omega$ is a two-sided ideal in $\tilde{\mathfrak{A}}$ and its norm-closure N_ω in $\mathfrak{A}$ is a two-sided ideal in $\mathfrak{A}$. The scalar product (34) *is positive definite if and only if $\tilde{N}_\omega = \{0\}$; if the C^*-algebra $\mathfrak{A}$ is simple, $\tilde{N}_\omega = \{0\}$.*

Proof. $\tilde{N}_\omega$ is a left ideal since ω is positive and $\tilde{N}_\omega$ is invariant under $A \to A^s$ by (35) and (42), thus, by (38b), it is a two-sided ideal of $\tilde{\mathfrak{A}}$. Since $\tilde{\mathfrak{A}}$ is norm-dense in $\mathfrak{A}$, the norm-closure of any two-sided ideal in $\tilde{\mathfrak{A}}$ is a two-sided ideal in $\mathfrak{A}$. If $A \in \tilde{\mathfrak{A}}$, then $\omega(A^*A) = (A, A)$; consequently, $(\cdot,\cdot)$ is positive definite if and only if $\tilde{N}_\omega = \{0\}$. If $\mathfrak{A}$ is simple, then $\{0\}$ is the only two-sided ideal in $\mathfrak{A}$. Q.E.D.

Lemma 9. *The set*

$$\{AB : A, B \in \tilde{\mathfrak{A}}\}$$

is dense in $\tilde{\mathfrak{A}}$ *for the topology defined by the scalar product* (35). *If, moreover,* $\tilde{N}_\omega = \{0\}$, *the set*

$$\{A B + A^j B^j : A, B \in \tilde{\mathfrak{A}}\}$$

is dense in $\tilde{\mathfrak{A}}$ *for the topology defined by the pre-hilbertian scalar product* (35).

Proof. If $X \in \tilde{\mathfrak{A}} \subset \mathfrak{A}$, then there exist $Y, Z \in \mathfrak{A}$ such that $X = YZ$ by the spectral theorem. Since $\tilde{\mathfrak{A}}$ is norm-dense in $\mathfrak{A}$, we see that X can be approximated by AB in the norm of $\mathfrak{A}$ by approximating Y by $A \in \tilde{\mathfrak{A}}$ and Z by $B \in \tilde{\mathfrak{A}}$ in the norm of A. Since

$$(A, A) = \omega(A^* A) \leqq \|A\|^2 ,$$

it follows that the set $\{AB : A, B \in \tilde{\mathfrak{A}}\}$ is dense in $\mathfrak{A}$ for the norm $\sqrt{(A, A)}$. Assume $\tilde{N}_\omega = \{0\}$. Let $X \in \tilde{\mathfrak{A}}$ be such that

$$(X, AB + A^j B^j) = 0$$

for all $A, B \in \tilde{\mathfrak{A}}$. Due to (38a) and (44), this is equivalent to the condition

$$(X + X^j, AB) = 0$$

for all $A, B \in \tilde{\mathfrak{A}}$. By the density of $\{AB : A, B \in \tilde{\mathfrak{A}}\}$ in $\mathfrak{A}$ for the pre-hilbertian topology, $X + X^j = 0$. Using (44) again, this implies

$$(X, Y + Y^j) = 0$$

for all $Y \in \tilde{\mathfrak{A}}$. The proof that $X = 0$ is, therefore, reduced to showing the density of the set

$$\{Y + Y^j : Y \in \tilde{\mathfrak{A}}\}$$

in the prehilbert space $\tilde{\mathfrak{A}}$ which follows from the density of this set in the C^*-algebra $\mathfrak{A}$. The latter follows easily by noticing that, for $f_1 \in \mathscr{D}(\hat{R})$ and $A \in \mathfrak{A}$, we have due to (28)

$$\alpha(\hat{f}_1) A + \{\alpha(\hat{f}_1) A\}^j = \alpha(\widehat{(1 + e_\beta) f_1}) A .$$

The last expression can be made arbitrarily close to A in norm by adequately choosing $f = (1 + e_\beta) f_1$ [f runs through $\mathscr{D}(\hat{R})$ as f_1 does].

Proposition 3. *If* ω *is an* α*-invariant normalized positive linear functional on* $\mathfrak{A}$, *if* ω *satisfies the KMS boundary condition for* $\beta \in R$, *i.e.*

$$\omega(j_{-\beta}(A)^* j_{-\beta}(B)) = \omega(BA^*) , \quad A, B \in \mathfrak{A} ,$$

and if $\tilde{N}_\omega = \{A \in \tilde{\mathfrak{A}} : \omega(A^* A) = 0\} = \{0\}$, *then* $\tilde{\mathfrak{A}}$ *equipped with the pre-hilbertian scalar product* $(\cdot,\cdot)$ *defined in* (35) *and the mappings* $A \to A^j$ *and* $A \to A^s$ *defined in* (36) *is a quasi-unitary algebra.*

Proof. A quasi-unitary algebra is an algebra $\tilde{\mathfrak{A}}$ with a pre-hilbertian scalar product $(\cdot,\cdot)$, a mapping $A \to A^j$ and a mapping $A \to A^s$ which satisfy (37), (38), (40), (42), (43), (45a), the final assertion of Lemma 9 and the continuity of $A \to BA$ with respect to the pre-hilbertian topology for every $B \in \tilde{\mathfrak{A}}$ (see Definition 1 of [4]).

6. The Representation of $\mathfrak{A}$ Defined by ω

We consider now the representation of $\mathfrak{A}$ defined, via the Gelfand-Segal construction, by an α-invariant KMS-state ω of $\mathfrak{A}$ which satisfies in addition the condition

$$\tilde{N}_\omega = \{A \in \tilde{\mathfrak{A}} : \omega(A^* A) = 0\} = \{0\}, \tag{47}$$

and describe features resulting from the quasi-unitary character of $\tilde{\mathfrak{A}}$. Let λ be this representation, with $\mathscr{H}$ and Ω the corresponding Hilbert space and cyclic vector:

$$\omega(A) = (\Omega, \lambda(A)\,\Omega), \quad A \in \mathfrak{A}. \tag{48}$$

Since α is strongly continuous and ω is α-invariant we know [4; 2.12.11] that $\mathscr{H}$ carries a strongly continuous representation U of R implementing the α-automorphisms and leaving Ω invariant:

$$\lambda(\alpha_t(A)) = U(t)\,\lambda(A)\,U(t)^{-1}, \quad A \in \mathfrak{A}, \quad t \in R, \tag{49}$$

$$U(t)\,\Omega = \Omega. \tag{50}$$

The Hilbert space $\mathscr{H}$ is the completion of the quotient $\mathfrak{A}/N_\omega$, where

$$N_\omega = \{A \in \mathfrak{A} : \omega(A^* A) = 0\}, \tag{51}$$

with respect to the scalar product

$$(A + N_\omega, B + N_\omega) = \omega(A^* B), \quad A, B \in \mathfrak{A}. \tag{52}$$

If $A, B \in \tilde{\mathfrak{A}}$ are such that $A = B \bmod N_\omega$, or else $A - B \in N_\omega \cap \tilde{\mathfrak{A}} = \tilde{N}_\omega$, we have $A = B$ by (47). Thus the mapping $A \in \tilde{\mathfrak{A}} \to A + N_\omega = \pi(A)\,\Omega \in \mathscr{H}$ is injective and allows us to consider $\tilde{\mathfrak{A}}$ as a linear subset of $\mathscr{H}$, on which the scalar products (52) and (35) moreover coincide. Furthermore, since $\tilde{\mathfrak{A}}$ is dense in $\mathfrak{A}$ (Proposition 1) and since $(A, A)^{1/2} = \omega(A^* A)^{1/2} \leqq \|A\|$, $\tilde{\mathfrak{A}}$ is dense in $\mathfrak{A}/N_\omega$ for the $\mathscr{H}$-norm. $\mathscr{H}$ can thus be obtained as the completion of $\tilde{\mathfrak{A}}$ with respect to its prehilbertian scalar product (35). The representations λ and U can then be obtained by continuous extension from the formulae

$$\lambda(A)\,B = AB, \quad A \in \tilde{\mathfrak{A}} \subset \mathfrak{A}, \quad B \in \tilde{\mathfrak{A}} \subset \mathscr{H}, \quad t \in R. \tag{53}$$

$$U(t)\,B = \alpha_t(B) \tag{54}$$

Analogously, a conjugate-linear representation ϱ of $\mathfrak{A}$ can be obtained by continuous extension from the definition

$$\varrho(A)\,B = BA^s, \quad A \in \tilde{\mathfrak{A}} \subset \mathfrak{A}, \quad B \in \tilde{\mathfrak{A}} \subset \mathscr{H} \tag{55}$$

since, using (42), (38b) and (40)

$$(BA^s, BA^s)^{1/2} = (AB^s, AB^s)^{1/2} \leqq \|A\|\,(B^s, B^s)^{1/2} = \|A\|\,(B, B)^{1/2};$$

ϱ is conjugate-linear by (37a), multiplicative by (38b) and such that $\varrho(A^*) = \varrho(A)^*$ for all $A \in \mathfrak{A}$, since, for $A, B_1, B_2 \in \tilde{\mathfrak{A}}$, using (45b) and (41)

$$(\varrho(A) B_1, B_2) = (B_1 A^s, B_2) = (B_1, B_2 A^{sjs}) = (B_1, B_2 A^{*s})$$
$$= (B_1, \varrho(A^*) B_2) .$$

Further, the involutive conjugate-linear mapping $A \to A^s$ of $\tilde{\mathfrak{A}}$, [cf. (37a), (40)], isometric for the $\mathscr{H}$-norm by (42), extends continuously to a conjugation S of $\mathscr{H}$. S has the properties

$$S\lambda(A) S = \varrho(A) , \quad A \in \mathfrak{A} , \tag{56}$$

$$U(t) S = S U(t) , \quad t \in R , \tag{57}$$

$$S\Omega = \Omega . \tag{58}$$

One has, namely, for $A, B \in \tilde{\mathfrak{A}}$

$$S\lambda(A) SB = S\lambda(A) B^s = S(AB^s) = BA^s = \varrho(A) B$$

where we used (38b), whence (56); and

$$U(t) SB = \alpha_t(B^s) = \alpha_t(B^{j*}) = \alpha_t(B)^{j*} = \alpha_t(B)^s = SU(t) B$$

where we used (36b) and (27), whence (57); and finally using the fact that

$$(\Omega, A) = (\Omega, \pi(A) \Omega) = \omega(A) , \quad A \in \tilde{\mathfrak{A}} \subset \mathscr{H} , \tag{59}$$

we have, taking account of the j-invariance of ω (see Proposition 2)

$$(S\Omega, A) = \overline{(\Omega, A^s)} = \overline{\omega(A^s)} = \omega(A) ,$$

whence (58). Note that (58) entails the properties

$$\varrho(A) \Omega = SA = A^s , \quad A \in \tilde{\mathfrak{A}} \subset \mathscr{H} , \tag{60}$$

and

$$(\Omega, \varrho(A) \Omega) = \overline{(\Omega, \pi(A) \Omega)} = (\pi(A) \Omega, \Omega) , \quad A \in \mathfrak{A} . \tag{61}$$

We now see that the quasi-unitary character of $\tilde{\mathfrak{A}}$ allows to derive Theorem 4 of [1], namely the fact that the weak closures of $\lambda(\mathfrak{A})$ and $\varrho(\mathfrak{A})$ are commutant of one another:

$$\lambda(\mathfrak{A})'' = \varrho(\mathfrak{A})' , \tag{62}$$

from Theorem 1 in [4]. Since Ω is cyclic for ϱ by (60), we conclude that it is cyclic and separating for both $\lambda(\mathfrak{A})''$ and $\varrho(\mathfrak{A})''$. The argument in the last paragraph of p. 278 in [4] shows that the operators $A \to A^j = j_\beta(A)$ and $A \to j_{-\beta}(A)$ in $\tilde{\mathfrak{A}}$ have respective minimal closed extensions J and J^{-1} on $\mathscr{H}$ (with domains D_J and $D_{J^{-1}}$) which are self-adjoint, inverse of each other, and such that

$$D_{J^{-1}} = SD_J, D_J = SD_{J^{-1}}, \tag{63a}$$

$$J = SJ^{-1}S, J^{-1} = SJS . \tag{63b}$$

Set L be the spectral measure associated by the Stone theorem to the representation U

$$U(t) = \int e^{it\xi}\, dE(\xi)\,, \tag{64}$$

with H the corresponding infinitesimal generator

$$H = \int \xi\, dE(\xi)\,. \tag{65}$$

For $A, B \in \tilde{\mathfrak{A}} \subset \mathscr{H}$ we have $U(t)\,A = \alpha_t(A)$, and thus

$$(B, U(t)\,A) = \omega(B^* \alpha_t(A)) = F_{AB^*}(t) = \int e^{it\xi}(B, dE(\xi)\,A)\,.$$

Thus in this case $\hat{F}_{AB^*}$ is a measure such that

$$d\hat{F}_{AB^*}(\xi) = (B, dE(\xi)\,A)\,. \tag{66}$$

From the fact that $\hat{F}_{AB^*}$ has compact support we conclude that $\tilde{\mathfrak{A}}$ is contained in the domain of each continuous function of H.

In particular, for $\beta \in R$, using (31a) and (31b)

$$\left(B, e^{-\frac{1}{2}\beta H} A\right) = \int e^{-\frac{1}{2}\beta\xi}(B, dE(\xi)\,A) = \langle \hat{F}_{AB^*}, e_{-\beta}\rangle = \langle e_{-\beta}\hat{F}_{AB^*}, 1\rangle$$
$$= \langle \hat{F}_{j_{-\beta}(A)\,B^*}, 1\rangle = F_{j_{-\beta}(A)\,B^*}(0) = \omega(B^*, j_{-\beta}(A)) = (B, J^{-1}A)\,.$$

Thus $e^{-\frac{1}{2}\beta H}$ and J^{-1} coincide on $\tilde{\mathfrak{A}}$ and we can conclude following [1; end of p. 234] from the fact that $\tilde{\mathfrak{A}}$ is a dense set of analytic vectors for $e^{-\frac{1}{2}\beta H}$ in $\mathscr{H}$ that

$$J^{-1} = e^{-\frac{1}{2}\beta H}\,. \tag{67}$$

Appendix A

Extension of the Mapping α to Bounded Measures

Let $\mathfrak{A}$ be a C^*-algebra and $\alpha : g \in G \to \alpha_g$ a strongly continuous homomorphic mapping of the locally compact abelian group G into the automorphism group of $\mathfrak{A}$. We denote by $\mathscr{C}_0(G)$ the set of continuous functions on G vanishing at infinity. $\mathscr{C}_0(G)$ is a Banach space for the norm $\|f\|_\infty = \operatorname{Sup}_{g \in G} |f(g)|$. The strong dual of $\mathscr{C}_0(G)$ is the set $M_1(G)$ of bounded measures on G with the norm $\|\mu\|_1 = \operatorname{Sup}_{\substack{f \in \mathscr{C}_0(G)\\ \|f\|_\infty = 1}} |\langle \mu, f\rangle|$. $M_1(G)$ is a *-Banach algebra for the convolution product $\mu * \nu$ and the adjunction $\mu^* = \check{\bar{\mu}}$ of measures.

Lemma (a). *For arbitrary $A \in \mathfrak{A}$ and $\mu \in M_1(G)$ the Bochner integral*

$$\alpha(\mu)\,A = \int \alpha_g(A)\, d\mu(g) \tag{A 1}$$

exists and defines an element $\alpha(\mu) A$ *with the properties*

$$\|\alpha(\mu) A\| \leqq \|\mu\|_1 \|A\| , \tag{A 2}$$

$$\alpha(\mu)(aA + bB) = a\alpha(\mu) A + b\alpha(\mu) B , \tag{A 3}$$

$\mu, \nu \in M_1(G)$, $A, B \in \mathfrak{A}$, a, b *complex numbers*

$$\{\alpha(\mu) A\}^* = \alpha(\bar{\mu}) A^* , \tag{A 4}$$

$$\alpha(\mu)\alpha(\nu) = \alpha(\mu * \nu) . \tag{A 5}$$

Proof. The existence of the Bochner integral (A 1) is assured by the continuity of the function $g \to \alpha_g(A)$ and the fact that $\|\alpha_g(A)\| = \|A\|$ which also entails (A 2). (A 3) is obvious and (A 4) stems from $(\alpha_g(A))^* = \alpha_g(A^*)$. On the other hand we can write, using the interchangeability of Bochner integrals with continuous linear mappings

$$\begin{aligned}\alpha(\mu)\alpha(\nu) A &= \int d\mu(g)\, \alpha_g\left(\int d\nu(g')\, \alpha_{g'}(A)\right) \\ &= \int\int d\mu(g)\, d\nu(g')\, \alpha_{g+g'}(A)\end{aligned}$$

whence (A 5).

Lemma (b). *Let* $f \in \mathcal{D}(R)$*, the set of infinitely differentiable functions on* R *with compact supports. Then* $\alpha(f) A$ *is contained in the domain of* D^p*, the* p^{th} *power of the infinitesimal generator of the one-parameter group* α*, for all* $A \in \mathfrak{A}$ *and all positive integers* p*. Furthermore the set* $\{\alpha(f) A : A \in \mathfrak{A}, f \in \mathcal{D}(R)\}$ *is norm-dense in* α*.*

This Lemma is an immediate corollary of the Theorem in [7].

Appendix B

Direct Characterization of the Convolution in the Fourier Transform of $\tilde{\mathfrak{A}}$

The space of rapidly decreasing $\mathfrak{A}$-valued C^∞ functions on R is denoted by $\mathcal{S}(R, \mathfrak{A})$. It is well-known that this space can be identified with the complete tensor product $\mathcal{S}(R) \hat{\otimes} \mathfrak{A}$ just as $\mathcal{S}'(R, \mathfrak{A}) = \mathcal{S}'(R) \hat{\otimes} \mathfrak{A}$. Those results about tensor products which we shall use below can be found in, e.g. [8].

If we define a Fourier transform of $\mathcal{S}(\hat{R}, \mathfrak{A})$ into $\mathcal{S}(\hat{R}, \mathfrak{A})$ by (13), then this transform is of the form $\mathcal{F} \otimes I$, where $\mathcal{F}$ denotes the usual Fourier transform of $\mathcal{S}(\hat{R})$ into $\mathcal{S}(R)$.

Lemma (c). *Let* T *denote a tempered* $\mathfrak{A}$*-valued distribution. Then there exists a unique continuous linear mapping* T' *from* $\mathcal{S}(R, \mathfrak{A})$ *into* $\mathfrak{A}$ *such that*

$$\langle T', f \cdot A \rangle = \langle T, f \rangle \cdot A , \quad f \in \mathcal{S}(R) , \quad A \in \mathfrak{A} .$$

In other words, $T' = T \otimes I$.

Proof. The mapping

$$(f, A) \in \mathcal{S}(R) \times \mathfrak{A} \to \langle T, f \rangle A \in \mathfrak{A}$$

is clearly bilinear and separately continuous, and therefore continuous by the uniform boundedness theorem. But this implies that the linear mapping $T' = T \otimes I$ from the algebraic tensor product $\mathscr{S}(R) \otimes \mathfrak{A}$ to $\mathfrak{A}$ is continuous w.r.t. the projective topology on $\mathscr{S}(R) \otimes \mathfrak{A}$. Since $\mathscr{S}(R, \mathfrak{A})$ is the completion of $\mathscr{S}(R) \otimes_\pi \mathfrak{A}$, the result follows.

From this lemma it is clear that if $T \in \mathscr{S}'(R, \mathfrak{A})$, $F \in \mathscr{S}(\hat{R}, \mathfrak{A})$, and $\hat{F}$ denotes the Fourier transform of F, then

$$\langle \hat{T}', F \rangle = \langle T', \hat{F} \rangle . \tag{B 1}$$

Now consider $A \in \mathfrak{A}^{(\infty)}$, then by assumption $X_A \in C^\infty(R, \mathfrak{A})$, and since all derivatives of X_A are bounded in view of formula (5), it is clear that $X_A \hat{f} \in \mathscr{S}(R, \mathfrak{A})$ for all $\hat{f} \in \mathscr{S}(R)$ [in the terminology of [9], we have $X_A \in \mathscr{O}_M(R, \mathfrak{A})$]. Therefore, if $T \in \mathscr{S}'(R, \mathfrak{A})$, we may define $T \cdot X_A \in \mathscr{S}'(R, \mathfrak{A})$ by putting

$$\langle T \cdot X_A, \hat{f} \rangle = \langle T', X_A \hat{f} \rangle , \quad \hat{f} \in \mathscr{S}(R) . \tag{B 2}$$

Lemma (d). *Let $f \in \mathscr{S}(\hat{R})$ and $A \in \mathfrak{A}^{(\infty)}$, and define the function $F : \hat{R} \to \mathfrak{A}$ by*

$$F(\xi) = \langle \hat{X}_A, f_{-\xi} \rangle , \quad \xi \in \hat{R} . \tag{B 3}$$

Then $F \in \mathscr{S}(\hat{R}, \mathfrak{A})$.

Proof. We have

$$\begin{aligned} F(\xi) &= \langle X_A, \hat{f}_{-\xi} \rangle \\ &= \int X_A(s)\, e^{is\xi} \hat{f}(s)\, ds , \end{aligned}$$

from which it is seen that F is the inverse Fourier transform of the function $X_A \hat{f} \in \mathscr{S}(R, \mathfrak{A})$.

Lemma (e). *Let $T \in \mathscr{S}'(R, \mathfrak{A})$ and $A \in \mathfrak{A}^{(\infty)}$. Then*

$$\langle \widehat{T X_A}, f \rangle = \langle \hat{T}', F \rangle , \quad f \in \mathscr{S}(\hat{R}) , \tag{B 4}$$

where $\hat{T}'$ is defined as in Lemma (c) and F is defined by (B 3).

Proof. We have

$$\begin{aligned} \langle \widehat{T X_A}, f \rangle &= \langle T X_A, \hat{f} \rangle && \text{(by definition)} \\ &= \langle T', X_A \hat{f} \rangle && \text{[by (B 2)]} \\ &= \langle \hat{T}', F \rangle , \end{aligned}$$

where the last equality is a consequence of (B 1) and the observation made in the proof of Lemma (d) that $X_A \hat{f} = \hat{F}$.

Lemma (e) obviously implies that the convolutions $\hat{X}_A * \hat{X}_B$ defined by the Eq. (16) and (18) agree for $A, B \in \mathfrak{A}^{(\infty)}$. The proof of the Eq. (17) is straightforward.

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Note added in proof: F. ROCCA and M. SIRUGUE informed us that they proved that a KMS functional is automatically α-invariant (a result already contained in [1a], as we learned from S. DOPLICHER). In view of this, the specifications of some of our statements are redundant.

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D. KASTLER
Centre de Physique Théorique
31, Chemin Joseph Aiguier
13-Marseille 9e, France

J. C. T. POOL
Applied Mathematics Division
Argonne National Laboratory
Argonne, Illinois 60439, USA

E. THUE POULSEN
Matematisk Institut
Aarhus Universitet
Universitetsparken
Aarhus C, Danmark