TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 268, Number 2, December 1981

QUASI-COHERENT SHEAVES OVER AFFINE HENSEL SCHEMES

BY

SILVIO GRECO AND ROSARIO STRANO

ABSTRACT. The following two theorems concerning affine Hensel schemes are proved.

THEOREM A. *Every quasi-coherent sheaf over an affine Hensel scheme is generated by its global sections.*

THEOREM B. $H^p(X, F) = 0$ *for all positive* p *and all quasi-coherent sheaves* F *over an affine Hensel scheme* X.

Introduction. The first one to consider the "Henselian structure" of an algebraic variety along a closed subvariety was Hironaka [17], but the theory of Henselian schemes was developed systematically a few years later by Kurke in his doctoral thesis, now included in the book [18] by Kurke, Pfister and Roczen. Some results were obtained independently in [11] and [12], while morphisms and fiber products were studied by Mora [19].

Henselian schemes are similar to formal schemes ([22], [7]) and provide a good notion of "algebraic tubular neighborhood" of a subvariety, which has the advantage, with respect to the widely used formal neighborhoods, to be "closer" to the algebraic situation; this idea, included in the above paper by Hironaka, was developed by Cox [3], [4].

In this paper we show that quasi-coherent sheaves over an affine Hensel scheme behave as they are expected to; namely, they are generated by global sections (Theorem A, see 1.11) and their cohomology is trivial (Theorem B, see 1.12). These results were announced in [14]; applications are given by Roczen [21].

The paper is divided into 4 sections. In the first one we recall some facts from the theory of Hensel couples, and we give the main results, along with some obvious corollaries. We give also, as a consequence of Theorem A, a particular case of the fundamental theorem of affine morphisms (see 1.22).

§2 contains some technical results. We study the canonical homomorphism ϕ: ${}^hA_f \otimes_A {}^hA_g \to {}^hA_{fg}$, where ^{h} denotes Henselization, and $(f, g) = (1)$. The main facts are Theorem 2.4 (ϕ is a suitable localization), Theorem 2.5 (ϕ is surjective). Our proofs are based on some nice properties of the absolutely integrally closed rings (already used by M. Artin [1]), and on a result of Gruson [13] concerning étale coverings over Hensel couples.

Received by the editors May 8, 1980 and, in revised form, November 25, 1980.

1980 *Mathematics Subject Classification*. Primary 14A20, 14F20, 13J15; Secondary 14B25, 14E20, 14F05, 13B20.

In §3 we study Ker ϕ. This allows us to prove Theorem A. Note that by **[12**, §6, Theorem 1], Theorem A implies Gruson's Theorem **[13]**.

The proof of Theorem B is given in §4, as a consequence of Theorem A.

The authors wish to thank M. Hochster and D. Buchsbaum for some helpful discussions on the subject of this paper.

1. Preliminaries and main results. We recall some known facts concerning Henselian couples and Henselian schemes, and we state the main results of this paper, along with some corollaries.

A. *Hensel couples and Henselization.*

1.1. A *couple* $(A, \underline{a})$ consists of a ring A (commutative with 1) and of an ideal $\underline{a} \subset A$. A morphism of couples $f\colon (A, \underline{a}) \to (B, \underline{b})$ is a ring homomorphism $f\colon A \to B$ such that $f(\underline{a}) \subset \underline{b}$.

1.2. An *N-polynomial* over the couple $(A, \underline{a})$ is a monic polynomial $a_0 + a_1X + \cdots + X^n \in A[X]$ such that $a_0 \in \underline{a}$, and a_1 is a unit modulo $\underline{a}$.

The couple $(A, \underline{a})$ is said to be a *Hensel couple* (shortly H-couple) if (i) $\underline{a} \subset \operatorname{rad} A$, (ii) every N-polynomial has a root in $\underline{a}$.

A local ring A with maximal ideal $\underline{m}$ is Henselian if and only if $(A, \underline{m})$ is a Hensel couple [**20**, p. 76, Proposition 3].

For more details on Hensel couples we refer to **[10]**, **[20]**, **[18]**. Here we list some properties we shall use freely throughout this paper. See **[10]** for indications on the proofs.

1.3. $(A, \underline{a})$ is an H-couple if and only if $\underline{a} \subset \operatorname{red} A$, and for any finite A-algebra B the canonical map $B \to B/\underline{a}B$ induces a bijection between the sets of idempotents.

1.4. If $(A, \underline{a})$ is an H-couple and B is an A-algebra integral over A, then $(B, \underline{a}B)$ is an H-couple.

1.5. Let $(A, \underline{a})$ be a couple and let $\underline{b} \subset \underline{a}$ be an ideal. Then $(A, \underline{a})$ is an H-couple if and only if $(A/\underline{b}, \underline{a}/\underline{b})$ and $(A, \underline{b})$ are H-couples. It follows that $(A, \underline{a})$ is an H-couple if and only if $(A, \sqrt{\underline{a}}\,)$ is such, if and only if $(A_{\mathrm{red}}, \underline{a}A_{\mathrm{red}})$ is such.

1.6. To every couple $(A, \underline{a})$ one can attach an H-couple $(B, \underline{b})$ along with a morphism $(A, \underline{a}) \to (B, \underline{b})$ such that for any H-couple $(B', \underline{b}')$ the canonical map

$$\operatorname{Hom}[(B, \underline{b}), (B', \underline{b}')] \to \operatorname{Hom}[(A, \underline{a}), (B', \underline{b}')]$$

is bijective. This couple is called the Henselization of $(A, \underline{a})$ and is denoted by ${}^h(A, \underline{a})$. We often write ${}^hA$ in place of B, and we call it the Henselization of A with respect to $\underline{a}$. If C is an A-algebra we often write ${}^hC$ for the Henselization of C with respect to $\underline{a}C$.

We summarize some properties of the Henselization we shall need later:

1.7. Let $(A, \underline{a})$ be a couple. Then:

(i) ${}^h(A, \underline{a})$ exists and is unique up to canonical isomorphism.

(ii) ${}^hA/\underline{a}{}^hA = A/\underline{a}$ and the $\underline{a}$-adic completions of A and ${}^hA$ coincide.

(iii) ${}^h(A, \underline{a})$ is the direct limit of the set of all local étale (L.E.) neighborhoods of $(A, \underline{a})$ (see [**20**, Theorem 2, Chapter XI]). In particular ${}^hA$ is a direct limit of étale A-algebras, and depends only on $\sqrt{\underline{a}}$.

(iv) ${}^hA$ is A-flat, and is faithfully flat if and only if $\underline{a} \subset \operatorname{rad} A$.

(v) ${}^hA = {}^h(A_{1+\underline{a}})$. Hence the kernel of the canonical map $A \to {}^hA$ coincides with the kernel of $A \to A_{1+\underline{a}}$. Thus if $\underline{a} \neq A$ and A is a domain, then $A \to {}^hA$ is injective.

(vi) If A is noetherian (resp. normal, regular, excellent) the same holds for ${}^hA$.

(vii) If $(A, \underline{a}) = \varinjlim (A_i, \underline{a}_i)$, then ${}^h(A, \underline{a}) = \varinjlim {}^h(A_i, \underline{a}_i)$.

(viii) If B is an A-algebra, integral over B, then ${}^hB = B \otimes_A {}^hA$. In particular ${}^h(A/I) = ({}^hA)/I\,{}^hA$ for every ideal $I \subset A$.

B. *Henselian schemes.*

1.8. Let $(A, \underline{a})$ be a Hensel couple and put $X = \operatorname{Spec} A/\underline{a}$. For each $f \in A$ put $X_f = D(f) \cap X$, and $\mathcal{O}_X(X_f) = {}^hA_f$. This defines a presheaf of rings over X, which is actually a sheaf (whence $\Gamma(X, \mathcal{O}_X) = A$).

More generally to any A-module M one can associate the presheaf $\tilde{M}$ defined by $\tilde{M}(X_f) = {}^hA_f \otimes_A M$. It turns out that $\tilde{M}$ is a sheaf over X (the above claims are proved in [**18**, 7.1.3]; another proof is sketched in [**11**]).

1.9. The ringed space $(X, \mathcal{O}_X)$ is called the *Henselian spectrum* of $(A, \underline{a})$ and is denoted by $\operatorname{Sph}(A, \underline{a})$ or $\operatorname{Sph} A$ if $\underline{a}$ is understood. An *affine Henselian scheme* is a ringed space isomorphic to $\operatorname{Sph}(A, \underline{a})$ for some H-couple $(A, \underline{a})$. A *Henselian scheme* is defined accordingly, in the obvious way.

An important example of Henselian scheme is the Henselization of a scheme along a closed subscheme (see [**17**], [**11**], [**18**]).

1.10. Let $X = \operatorname{Sph}(A, \underline{a})$ be an affine Henselian scheme. Then:

(i) If $x \in X$ corresponds to $\underline{p} \in \operatorname{Spec} A$, then $\mathcal{O}_{X,x} = {}^hA_{\underline{p}}$ (see 1.7(vii)).

(ii) $\operatorname{Sph}(A, \underline{a})$ depends only on $\sqrt{\underline{a}}$ (see 1.7(iii)).

(iii) The functor $M \mapsto \tilde{M}$, from (A-modules) to ($\mathcal{O}_X$-modules), is exact and fully faithful, and commutes with direct limits. Hence $\tilde{M}$ is always quasi-coherent, and is coherent if A is noetherian and M is finitely generated (apply 1.7(iv) and (vi)).

C. *Main results of this paper and corollaries.*

1.11. THEOREM (THEOREM A). *Let X be an affine Hensel scheme, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

(i) *$\mathcal{F} = \tilde{M}$ where $M = \Gamma(X, \mathcal{F})$, or equivalently*

(ii) *$\mathcal{F}$ is generated by its global sections.*

1.12. THEOREM (THEOREM B). *Let X, $\mathcal{F}$ be as in* 1.11. *Then $H^p(X, \mathcal{F}) = 0$ for all* $p > 0$.

The proofs of 1.11 and 1.12 will be given in §§3 and 4 respectively. Here we give some corollaries.

1.13. COROLLARY. *If $X = \operatorname{Sph}(A, \underline{a})$ is an affine Henselian scheme, then the functor $M \mapsto \tilde{M}$ is an equivalence between the categories of A-modules and of quasi-coherent $\mathcal{O}_X$-modules. If A is noetherian it induces an equivalence between the categories of finitely generated A-modules and of coherent $\mathcal{O}_X$-modules.*

PROOF. Immediate from 1.11 and 1.10(iii).

1.14. Corollary. *Let* $X = \mathrm{Sph}(A, \underline{a})$ *be an affine Henselian scheme, with* A *noetherian. Then any quasi-coherent* $\mathcal{O}_X$*-module is the direct limit of the family of its coherent submodules.*

Proof. Apply 1.13 and 1.10(iii).

1.15. Corollary. *Let* X *be a Hensel scheme, and let* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *be an exact sequence of* $\mathcal{O}_X$*-modules. If any two of them are quasi-coherent, so is the third.*

Proof. It follows from 1.11, by the same argument used for ordinary schemes (see [7, 1.4.7]).

1.16. Corollary. *Let* X *be a Henselian scheme, and let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Let* $\mathcal{U}$ *be an affine covering of* X*. Then for all* $p \geqslant 0$ *we have.*

$$H^p(X, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$$

Proof. It follows from 1.12, by general facts on cohomology (see e.g. [**6**]).

1.17. Remark. Theorems 1.11 and 1.12 are well known for ordinary schemes (see [7], [**9**]). Moreover they are true for coherent sheaves over a noetherian affine formal scheme ([7, 10.10.2] for 1.11; [**15**, Proposition 4.1] for 1.12). A general theory of quasi-coherent sheaves over a formal scheme is not known, and very likely it cannot be as well behaved as in the Henselian case.

Application of "Theorem B" to the equivalence of singularities is given by Roczen [**21**].

D. *Application to integral morphisms*. An important fact in the theory of ordinary or formal schemes is that if $X \to Y$ is an affine morphism, and Y is affine, then X is also affine. We shall prove this fact for a class of morphisms of Henselian schemes; so far we are not able to prove the general case.

We recall first some facts on morphisms of Henselian schemes. For details see [**19**].

1.18. Let X be a Henselian scheme. An *ideal of definition* of X is a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ with the following property: there is an affine open covering $U_i = \mathrm{Sph}(A_i, \underline{a}_i)$ of X such that the ideals $\Gamma(U_i, \mathcal{I})$ and $\underline{a}_i$ of A_i have the same radical for all i. If $\mathcal{I}$ is an ideal of definition of X then $(X, \mathcal{O}_X/\mathcal{I})$ is an ordinary scheme having X as underlying topological space.

One can show that there is a unique maximal ideal of definition of X, characterized by the property that $\mathcal{O}_X/\mathcal{I}$ is a reduced sheaf of rings. If this is the case we say that $\mathcal{I}$ is the *canonical ideal* of X.

1.19. Let X, Y be two Hensel schemes. A *morphism* $f\colon X \to Y$ is a morphism of ringed spaces such that for all $x \in X$ the induced homomorphism $f_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local.

A morphism of couples $(A, \underline{a}) \to (B, \underline{b})$ induces a morphism of the corresponding Henselian schemes, and conversely if $A/\underline{a}$ and $B/\underline{b}$ are reduced.

1.20. A morphism $f\colon X \to Y$ of Henselian schemes is said to be *adic* if for any ideal of definition $\mathcal{I}$ of Y, $\mathcal{I} \cdot \mathcal{O}_X$ is an ideal of definition of X. One can show that it is sufficient to check this property for one arbitrarily chosen ideal of definition of X, e.g. the canonical ideal.

1.21. Definition. A morphism $f\colon X \to Y$ of Henselian schemes is said to be

(i) *affine*, if there is an open affine covering $\{U_i\}$ of Y such that $f^{-1}(U_i)$ is affine for all i,

(ii) *integral* (resp. *finite*) if it is adic and affine, and if moreover the covering of (i) can be chosen so that $\Gamma(f^{-1}(U_i), \mathcal{O}_X)$ is integral (resp. finite) over $\Gamma(U_i, \mathcal{O}_Y)$ for all i.

1.22. Proposition. *Let $f\colon X \to Y$ be an integral morphism of Henselian schemes. Then*

(i) *$f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module,*

(ii) *if $U \subset X$ is affine, then $f^{-1}(U)$ is affine.*

Proof. (i) The question being local, we may assume $X = \mathrm{Sph}(B, \underline{b})$, $Y = \mathrm{Sph}(A, \underline{a})$ where B is integral over A and $\underline{b} = \underline{a}B$. Then if $g \in A$ we have ${}^hB_g = B \otimes_A {}^hA_g$ (by 1.7(viii)) and hence $f_*\mathcal{O}_X = \tilde{B}$.

(ii) It is sufficient to show that if $Y = \mathrm{Sph}(A, \underline{a})$ then X is affine. Now by (i) and 1.11 we have that $f_*\mathcal{O}_X = \tilde{B}$, where $B = \Gamma(Y, f_*\mathcal{O}_X)$. We can assume that the open cover given in 1.21(ii) is of the form $U_i = Y_{f_i}$, $i = 1, \ldots, n$, where $f_1, \ldots, f_n \in A$ are such that $(f_1, \ldots, f_n) = A$, and $f^{-1}(Y_{f_i}) = \mathrm{Sph}(B_i, \underline{b}_i)$ for suitable H-couples $(B_i, \underline{b}_i)$, with B_i integral over ${}^hA_{f_i}$. Hence we have

$$ {}^hA_{f_i} \otimes_A B = \Gamma\big(Y_{f_i}, f_*\mathcal{O}_X\big) = \Gamma\big(f^{-1}(Y_{f_i}), \mathcal{O}_X\big) = B_i \quad \text{for each } i. $$

Put $C = \bigoplus {}^hA_{f_i}$. Then C is f.flat over A by 1.7(iv), and by the above we have that $B \otimes_A C$ is integral over C, and hence B is integral over A. It follows that $(B, \underline{a}B)$ is an H-couple (see 1.4) and a direct computation shows that $\mathrm{Sph}(B, \underline{a}B) = X$. This completes the proof.

1.23. Remarks. (i) By the same proof as in 1.22(i) one can show that if $f\colon X \to Y$ is a finite morphism of Henselian schemes and Y is locally noetherian, then $f_*\mathcal{O}_X$ is coherent.

(ii) One is tempted to make the following three conjectures, which are probably equivalent to each other:

Conjecture A. Let $f\colon X \to Y$ be an affine morphism of Henselian schemes. Then for any affine $U \subset X$, $f^{-1}(U)$ is affine.

This is true for ordinary schemes [7, 9.1.10] and for locally noetherian formal schemes [7, 10.16.4]. Our method used in 1.22 does not apply, because $f_*\mathcal{O}_X$ is not quasi-coherent in general.

Conjecture B. Let X be a Henselian scheme, and let $\mathcal{J}$ be an ideal of definition of X. If the usual scheme $(X, \mathcal{O}_X/\mathcal{J})$ is affine, then X is affine.

This conjecture is true for locally noetherian formal schemes (see [7, 10.6.3 and 2.3.5]).

Conjecture C (*Serre's criterion*). A Hensel scheme X is affine if (and only if by 1.12) $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

When X is a locally noetherian formal scheme this follows from Conjecture B and Serre's criterion for ordinary schemes [**8**, 5.2.1].

2. The homomorphism ${}^hA_f \otimes_A {}^hA_f \to {}^hA_{fg}$. In this section we prove some technical results which will be essential later. The most important are Theorems 2.4 and 2.5 below.

A. *Flatness of the homomorphism ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$.* Let $(A, \underline{a})$ be a couple. If B is an A-algebra, we denote by B_{Zar} the *Zariskification* of B with respect to $\underline{a}B$, that is $B_{\mathrm{Zar}} = B_{1+\underline{a}B}$. If $B_{\mathrm{Zar}} = B$, B is said to be a *Zariski* ring. Moreover, if B is an A-algebra, we denote simply by ${}^hB$ the Henselization of B with respect to $\underline{a}B$.

2.1. PROPOSITION. *Let $(A, \underline{a})$ be an H-couple and let $S, T \subset A$ be multiplicative sets; then ${}^hA_{ST}$ is the Henselization of ${}^hA_S \otimes_A {}^hA_T$; in particular the homomorphism ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$ is flat and $({}^hA_S \otimes_A {}^hA_T)_{\mathrm{Zar}} \to {}^hA_{ST}$ is faithfully flat.*

Before proving the above proposition we need the following.

2.2. PROPOSITION. *Let $(A, \underline{a})$ be a couple and let $A \to B \to C$ be ring homomorphisms. Then*

(i) *if B, C are local étale (L.E. for short) neighborhoods* [**10**, *Definition* 8] *of A, then C is a L.E. neighborhood of B;*

(ii) *if B, C are direct limits of L.E. neighborhoods of A, then C is a direct limit of L.E. neighborhoods of B.*

PROOF. Consider the commutative diagram

$$\begin{tikzcd} & B \arrow[dd, "\phi"] \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & B \otimes_A C \\ & C \arrow[ur] & \end{tikzcd}$$

it is easy to see that $(B \otimes_A C)_{\mathrm{Zar}}$ is a L.E. neighborhood of B. Now we show that the canonical homomorphism $\sigma\colon C \to (B \otimes_A C)_{\mathrm{Zar}}$ is an isomorphism; to this end let $\tau\colon B \otimes_A C \to C$ be defined by $\tau(b \otimes c) = \phi(b) \cdot c$; τ induces a homomorphism $\rho\colon (B \otimes_A C)_{\mathrm{Zar}} \to C$ since $(C, \underline{a}C)$ is a Zariski couple, i.e. $\underline{a}C \subset \operatorname{rad} C$. Let us consider $\sigma \circ \rho\colon (B \otimes_A C)_{\mathrm{Zar}} \to (B \otimes_A C)_{\mathrm{Zar}}$ and $\rho \circ \sigma\colon C \to C$; both homomorphisms induce the identity mod $\underline{a}$. In order to prove they are the identity, we need the following lemma, after observing that C, $(B \otimes_A C)_{\mathrm{Zar}}$ are of the type $D_{1+\underline{a}D}$, with D étale over A and $\operatorname{Hom}_A(D_{1+\underline{a}D}, E) = \operatorname{Hom}_A(D, E)$ for every A-algebra E such that $\underline{a}E \subset \operatorname{rad} E$.

2.3. LEMMA. *Let D, E be A-algebras with D unramified over A and $\underline{a}E \subset \operatorname{rad} E$; then the canonical map*

$$\operatorname{Hom}_A(D, E) \to \operatorname{Hom}_{A/\underline{a}}(D/\underline{a}D, E/\underline{a}E)$$

is injective.

PROOF. Let $\psi, \psi'\colon D \rightrightarrows E$ be two homomorphisms such that $\bar{\psi} = \bar{\psi}'$, where $\bar{\psi}, \bar{\psi}'$ are the compositions $D \rightrightarrows E \to E/\underline{a}E$; we show that $\psi(a) = \psi'(a)$ for every $a \in D$. In order to do that it is enough to show that for every $\underline{p} \in \max(E)$ it is $\psi(a) = \psi'(a)$ in $E_{\underline{p}}$; so we may assume E local; this follows from [**20**, Chapter VIII, Lemma 2], since the two homomorphisms $D \rightrightarrows E/\underline{a}E \to E/\underline{p}$ are equal.

Now we prove (ii) of 2.2. The proof is similar. In this case C (and $(B \otimes_A C)_{\text{Zar}}$) are of the form $D = \varinjlim (D_i)_{1+\underline{a}D_i}$ with D_i étale over A. Moreover

$$\operatorname{Hom}_A(D, D) = \varprojlim \operatorname{Hom}_A\big((D_i)_{1+\underline{a}D_i}, D\big) = \varprojlim \operatorname{Hom}_A(D_i, D).$$

The hypotheses of 2.3 are satisfied, so that the map $\operatorname{Hom}_A(D_i, D) \to \operatorname{Hom}_{A/\underline{a}}(D_i/\underline{a}D_i, D/\underline{a}D)$ is injective; hence the map

$$\operatorname{Hom}_A(D, D) \to \operatorname{Hom}_{A/\underline{a}}(D/\underline{a}D, D/\underline{a}D)$$

is injective. Proposition 2.2 is now completely proved.

Now we are able to prove Proposition 2.1. Put $B = ({}^hA_S \otimes_A {}^hA_T)_{\text{Zar}}$; we want to show that ${}^hB = {}^hA_{ST}$. B is a direct limit of L.E. neighborhoods of A_{ST} and the same is true for ${}^hA_{ST}$, so by Proposition 2.2 the homomorphism $B \to {}^hA_{ST}$ is a limit of L.E. neighborhoods.

By the definition of Henselization [20, Theorem 2, Chapter XI] ${}^hB$ is the limit of all L.E. neighborhoods of B; hence we have a commutative diagram

$$\begin{tikzcd} & {}^hA_{ST} \arrow[dd, "\phi"] \\ B \arrow[ur] \arrow[dr] & \\ & {}^hB \end{tikzcd}$$

On the other hand since ${}^hA_{ST}$ is Henselian, we have a commutative diagram

$$\begin{tikzcd} & {}^hB \arrow[dd, "\psi"] \\ B \arrow[ur] \arrow[dr] & \\ & {}^hA_{ST} \end{tikzcd}$$

From the commutative diagrams

$$\begin{tikzcd} & {}^hB \arrow[dd, "\phi \circ \psi"] \\ B \arrow[ur] \arrow[dr] & \\ & {}^hB \end{tikzcd} \qquad \begin{tikzcd} & {}^hA_{ST} \arrow[dd, "\psi \circ \phi"] \\ A_{ST} \arrow[ur] \arrow[dr] & \\ & {}^hA_{ST} \end{tikzcd}$$

we get $\phi \circ \psi = 1_{{}^hB}$, $\psi \circ \phi = 1_{{}^hA_{ST}}$.

B. *Statement of the main results and reduction to absolutely integrally closed rings.* Now we state the main results of this section.

2.4. THEOREM. *Let $(A, \underline{a})$ be an H-couple and let $f, g \in A$ such that $(f, g) = (1)$. Then there is a canonical isomorphism $({}^hA_f \otimes_A {}^hA_g)_{\text{Zar}} \simeq {}^hA_{fg}$.*

2.5. THEOREM. *Let $(A, \underline{a})$ be an H-couple and let $f, g \in A$ such that $(f, g) = (1)$. Then the canonical homomorphism ${}^hA_f \otimes_A {}^hA_g \to {}^hA_{fg}$ is surjective.*

In this subsection we study absolutely integrally closed rings (AIC for short) and we reduce the proof of Theorems 2.4, 2.5 to the case when A is an AIC ring.

2.6. DEFINITION. A ring A is said to be *absolutely integrally closed* (AIC for short) if every monic polynomial $f(X) \in A[X]$ splits into a product of linear factors.

The following facts are easily verified.

(i) If A is a domain with field of fractions K, then A is AIC if and only if A is integrally closed and K is algebraically closed.

(ii) Every homomorphic image of an AIC ring is AIC.

2.7. LEMMA. *Let A be an AIC ring and let S be a multiplicative subset of A. Then A_S is an AIC ring.*

PROOF. Let $f(X) \in A_S[X]$ be a monic polynomial and write $f(X) = a_0/s + a_1/s\, X + \cdots + X^n$. Then we have

$$\begin{aligned} s^n f(X) &= a_0 s^{n-1} + a_1 s^{n-2}(sX) + \cdots + (sX)^n \\ &= (sX - b_1)(sX - b_2) \cdots (sX - b_n) \end{aligned}$$

whence $f(X) = (X - b_1/s)(X - b_2/s) \cdots (X - b_n/s)$.

2.8. LEMMA. *Let A be an AIC ring and $\underline{a} \subset A$ be an ideal such that $A/\underline{a}$ is connected. Then ${}^hA = A_{\mathrm{Zar}}$.*

PROOF. By 2.7 A_{Zar} is an AIC ring and then we may assume $\underline{a} \subset \operatorname{rad} A$. By 1.2, it is sufficient to show that any N-polynomial $F(X)$ has a root in $\underline{a}$. For this it is sufficient to show that if $f(X) = (X - a_1) \cdots (X - a_n)$ is the reduction of $F \bmod \underline{a}$, then $a_i = 0$ for some i.

We know that $f(0) = 0$ and hence $\prod a_i = 0$. Moreover if $s_j = \prod_{i \neq j} a_i$ we have that $\sum s_j = f'(0)$ is invertible. Let j_0 be the first index such that $s_{j_0} \neq 0$. Put $e_1 = s_{j_0}$ and $e_2 = s_{j_0+1} + \cdots + s_n$. If $e_2 = 0$ we have $a_{j_0} = 0$. Otherwise we have $e_1 e_2 = 0$ and $e_1 + e_2$ invertible which implies that $A/\underline{a}$ is disconnected, a contradiction.

2.9. LEMMA. *Let A be a ring. Then there is a faithfully flat A-algebra B, integral over A, which is an AIC ring.*

We will call such an A-algebra B a *f.flat absolutely integral closure of A* (f.flat A.I.closure for short).

PROOF. Let $\{f_i\}_{i \in I}$ be the set of all monic polynomials in $A[X]$ and let $A_i = A[X]/(f_i)$. Put $A_1 = \bigotimes_I A_i$. Then put $A_n = (A_{n-1})_1$ and let $B = \varinjlim A_n$. It is easy to verify that B has the required properties.

2.10. LEMMA. *Let $(A, \underline{a})$ be a couple and let B, C be A-algebras with B integral over A. Then there is a canonical isomorphism $(B \otimes_A C)_{\mathrm{Zar}} \xrightarrow{\sim} B \otimes_A C_{\mathrm{Zar}}$.*

PROOF. Put $R = B \otimes_A C_{\mathrm{Zar}}$. There is a natural homomorphism τ: $R \to (B \otimes_A C)_{\mathrm{Zar}}$. Since R is integral over C_{Zar}, we have $\underline{a}R \subset \operatorname{rad}(C_{\mathrm{Zar}})R \subset \operatorname{rad} R$. Thus $1 + \underline{a}(B \otimes_A C)$ maps to invertible elements of R, so that we get a homomorphism $(B \otimes_A C)_{\mathrm{Zar}} \to R$, which is easily seen to be the inverse of τ.

2.11. PROPOSITION. *Assume that Theorem 2.4 is true when A is an AIC domain. Then it is true for any ring.*

PROOF. Let $(A, \underline{a})$ be an H-couple, and let $f, g \in A$ such that $(f, g) = (1)$. Put $R = ({}^hA_f \otimes_A {}^hA_g)$. We want to show that $R_{\text{Zar}} = {}^hA_{fg}$ provided this is true whenever A is an AIC domain. We do this in three steps.

Step 1. Assume A is a domain. Let K be the field of fractions of A and let B be the integral closure of A in $\bar{K}$, the algebraic closure of K. Then B is an AIC domain and by assumption the conclusion holds for B. Let Q be the cokernel of ϕ: $R \to {}^hA_{fg}$. Since Henselization and Zariskification commute with integral base change (1.7(viii) and 2.10) we have $Q \otimes_A A = 0$. On the other hand ϕ is faithfully flat by 2.6, so that Q is R-flat [2, I.3.5], and hence A-flat. Thus $Q \to Q \otimes_A B$ is injective and $Q = 0$.

Step 2. Assume $A = C/I$ where $C = {}^h(\mathbf{Z}[X_1, \ldots, X_n], \underline{b})$ for some ideal $\underline{b}$. Since Henselization, Zariskification and surjectivity are preserved by passing to quotients we may assume $I = 0$. Then A is noetherian and normal [**10**, Theorem 7]; hence $A = A_1 \times \cdots \times A_n$ where each A_i is a normal domain. By Step 1 the result is true for each A_i and it is easy to see that it holds for A as well.

Step 3. *General case.* We have $A = \varinjlim A_i$ where $\{A_i\}_{i \in I}$ is the direct set of all finitely generated $\mathbf{Z}$-subalgebras of A. Put $B_i = {}^h(A_i, \underline{a} \cap A_i)$; since Henselization commutes with direct limits we have $A = \varinjlim B_i$. If $a \cdot f + b \cdot g = 1$ we may assume that a, b, f, g come from all B_i's and the conclusion follows from Step 2, since everything involved commutes with direct limits.

2.12. PROPOSITION. *Assume that Theorem* 2.5 *is true when A is a f.flat A.I. closure of a normal domain A', with $f, g \in A'$. Then it is true for any ring.*

PROOF. As in the proof of 2.11 Steps 2 and 3 we reduce to the case when A is a normal domain. Let B be a f.flat A.I. closure of A and let Q be the cokernel of the homomorphism ${}^hA_f \otimes_A {}^hA_g \to {}^hA_{fg}$. Since $Q \otimes_A B = 0$ by f.flatness we have $Q = 0$.

C. *Connected components of affine schemes.* In this subsection we prove some basic facts about connected components of affine schemes. The principal result is Proposition 2.18 which will be used later to reduce the proof of 2.4 and 2.5 to connected components.

2.13. DEFINITION. Let $X = \operatorname{Spec} A$ be an affine scheme and let $\underline{p} \in X$. The *connected component* of X containing $\underline{p}$ is the closed subscheme Y of X, defined by $Y = \operatorname{Spec} A/\underline{c}$ where $\underline{c}$ is the ideal of A generated by all the idempotents $e \in \underline{p}$.

2.14. PROPOSITION. *Let $X = \operatorname{Spec} A$, $\underline{p} \in X$ and Y be the connected component of X containing $\underline{p}$. Then*

(i) *Y is the largest connected closed subscheme of X containing $\underline{p}$.*

(ii) *Y is the intersection of all the open-closed subschemes of X containing $\underline{p}$, that is $Y = \operatorname{Spec} A_S$ where S is the multiplicative set of A generated by all the idempotents $1 - e$, $e \in \underline{p}$.*

PROOF. First we show that $\operatorname{Spec} A/\underline{c}$ is connected; in fact if $e \in A$ is such that $e^2 - e \in \underline{c} \subset \underline{p}$ then either $e \in \underline{p}$ or $1 - e \in \underline{p}$. Hence either $e \in \underline{c}$ or $1 - e \in \underline{c}$. Now let $\underline{b} \subset \underline{p} \subset A$ be an ideal such that $\operatorname{Spec} A/\underline{b}$ is connected and let $e \in \underline{p}$ be an idempotent; since $\operatorname{Spec} A/\underline{b}$ is connected we have either $e \in \underline{b}$ or $1 - e \in \underline{b}$, but this second case is impossible; hence $e \in \underline{b}$ and $\underline{c} \subset \underline{b}$. Finally we have

$$A/\underline{c} = \varinjlim_{e \in \underline{p}} A/eA = \varinjlim_{e \in \underline{p}} A_{1-e} = A_S.$$

2.15. Remarks. (i) Proposition 2.14 asserts that the connected components of an affine scheme coincide with its quasi-components. This is false for a general topological space [**5**, p. 118, §3, Example 5].

(ii) Proposition 2.14(ii) is trivial when the connected components of X are open. But this is not the case in general (see e.g. [**2**, Chapter I, p. 173, Example 16]).

Now let $(A, \underline{a})$ be an H-couple. Since there is a 1-1 correspondence between idempotents in A and in $A/\underline{a}$ we have a 1-1 correspondence between connected components of Spec A and of Spec $A/\underline{a}$ given by

$$\operatorname{Spec} A/\underline{c} = \operatorname{Spec} A_S \mapsto \operatorname{Spec} A/\underline{a} + \underline{c} = \operatorname{Spec}(A/\underline{a})_S.$$

In the following we often identify affine schemes with the corresponding rings and we will talk of a connected component as a ring.

2.16. Lemma. *Let $(A, \underline{a})$ be an H-couple, and let $f \in A$. Then*

(i) *Any open-closed subset of* $\operatorname{Spec}({}^hA_f)$ *is of the form* $\operatorname{Spec}({}^hA_g)$ *for a suitable $g \in A$.*

(ii) *Every decomposition of* $\operatorname{Spec}({}^hA_f)$ *into the disjoint union of two open subsets comes from a decomposition ${}^hA_f = {}^hA_{f_\alpha} \times {}^hA_{f_\beta}$, where $f_\alpha, f_\beta \in A$ are such that $f_\alpha + f_\beta \equiv f^n \bmod \underline{a}$ for some n, and conversely.*

(iii) *Every connected component of* $\operatorname{Spec}({}^hA_f)$ *is of the form* $\operatorname{Spec}({}^hA_S)$ *where S is a suitable multiplicative subset of A.*

Proof. By 1.3 the map $U \mapsto \overline{U} = U \cap \operatorname{Spec}(A/\underline{a})_f$ is a bijection between the sets of open-closed subset of $\operatorname{Spec}({}^hA_f)$ and of $\operatorname{Spec}(A/\underline{a})_f$ respectively. Moreover if $U \subset \operatorname{Spec}({}^hA_f)$ is open-closed we have $\overline{U} = \operatorname{Spec}(A/\underline{a})_g$ for a suitable $g \in A$ and it is easy to see that indeed one has $U = \operatorname{Spec}({}^hA_g)$. This proves (i).

Clearly (i) implies (ii), and since Henselization commutes with direct limits, we see that (i) and 2.14 imply (iii).

The following proposition is essential later.

2.17. Proposition. *Let $(A, \underline{a})$ be an H-couple with A an AIC ring. Let $f, g \in A$ with $(f, g) = (1)$. Let C (resp. D) be a connected component of $(A/\underline{a})_f$ (resp. $(A/\underline{a})_g$) which, via* 1.3 *and* 2.16(iii), *corresponds to the connected component ${}^hA_S$ (resp. ${}^hA_T$) of ${}^hA_f$ (resp. ${}^hA_g$). Suppose moreover that the image of the homomorphism ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$ is A-flat. Then there is an A-algebra $A \xrightarrow{\phi} B$ such that*

(i) *$(B, \underline{a}B)$ is an H-couple.*

(ii) *The induced morphism* $\operatorname{Spec}(B/\underline{a}B) \to \operatorname{Spec}(A/\underline{a})$ *is injective and its image is $C \cup D$.*

(iii) *${}^hB_S$, ${}^hB_T$ are connected components of ${}^hB_f$, ${}^hB_g$ respectively and ϕ induces isomorphisms ${}^hA_S \simeq {}^hB_S$ and ${}^hA_T \simeq {}^hB_T$.*

(iv) *If A is a domain and ${}^hA_{ST} \neq 0$, then B is an AIC domain.*

Before proving Proposition 2.17 we prove a lemma.

2.18. LEMMA. *Let* $(A, \underline{a})$ *be an H-couple,* $f, g \in A$ *such that* $(f, g) = (1)$. *Suppose*

$$ {}^hA_f = {}^hA_{f_\alpha} \times {}^hA_{f'_\alpha}, \qquad {}^hA_g = {}^hA_{g_\beta} \times {}^hA_{g'_\beta} $$

with $f_\alpha, f'_\alpha, g_\beta, g'_\beta \in A$ *such that* $f_\alpha + f'_\alpha \equiv f^n \bmod \underline{a}$ *and* $g_\beta + g'_\beta \equiv g^m \bmod \underline{a}$ *and let* $B_{\alpha\beta}$ *be the kernel*

$$ 0 \to B_{\alpha\beta} \to {}^hA_{f_\alpha} \times {}^hA_{g_\beta} \xrightarrow{\sigma} {}^hA_{f_\alpha g_\beta} $$

with $\sigma = \phi_1 - \phi_2$, ϕ_1, ϕ_2 *being the canonical homomorphisms* $\phi_1\colon {}^hA_{f_\alpha} \to {}^hA_{f_\alpha g_\beta}$, $\phi_2\colon {}^hA_{g_\beta} \to {}^hA_{f_\alpha g_\beta}$. *Then is A-flat. Then there is an A-algebra* $A \to^{\Phi} B$ *such that*

PROOF. It is easy to prove that $B_{\alpha\beta}$ is a commutative A-algebra with 1. Put $C = {}^hA_{f_\alpha} \times {}^hA_{g_\beta}$, D = image of C in ${}^hA_{f_\alpha g_\beta}$ under σ. We note that D is an A-module and $f'_\alpha \cdot d = 0$ for every $d \in D$ since $f'_\alpha = 0$ in ${}^hA_{f_\alpha g_\beta}$. From the exact sequence $0 \to B_{\alpha\beta} \to C \to D \to 0$ we get the exact sequence

$$ 0 \to B_{\alpha\beta}/\underline{a}C \cap B_{\alpha\beta} \to (A/\underline{a})_{f_\alpha} \times (A/\underline{a})_{g_\beta} \to D/\underline{a}D \to 0. $$

We have $\underline{a}C \cap B_{\alpha\beta} \subset \operatorname{rad} B_{\alpha\beta}$; in fact an element of $B_{\alpha\beta}$ of the form $1 + a$, $a \in \underline{a}C$, is invertible in C, hence in $B_{\alpha\beta}$.

Now we tensor with $(A/\underline{a})_{f'_\alpha}$ and note that $(A/\underline{a})_{f_\alpha f'_\alpha} = 0$ and $(D/\underline{a}D)_{f'_\alpha} = 0$; in fact for every $\bar{d} \in (D/\underline{a}D)_{f'_\alpha}$ we have

$$ \bar{d} = (1/f'_\alpha \cdot f'_\alpha)\bar{d} = 1/f'_\alpha\big(f'_\alpha \cdot \bar{d}\big) = 0. $$

Hence we have

$$ \big(B_{\alpha\beta}/\underline{a}C \cap B_{\alpha\beta}\big)_{f'_\alpha} \simeq (A/\underline{a})_{g_\beta f'_\alpha}. $$

Hence, if $\underline{m}$ is a maximal ideal of $B_{\alpha\beta}$ containing f_α and g_α, then $\underline{m}_{f'_\alpha} = (B_{\alpha\beta})_{f'_\alpha}$ since g'_β is invertible in $(B_{\alpha\beta}/\underline{a}C \cap B_{\alpha\beta})_{f'_\alpha}$ and $\underline{a}C \cap B_{\alpha\beta} \subset \underline{m}$. It follows that $f'_\alpha \in \underline{m}$. Since $\underline{a}B_{\alpha\beta} \subset \underline{m}$, we get $f^n \in \underline{m}$. In the same way, $g^m \in \underline{m}$, which shows that $\underline{m} = B_{\alpha\beta}$, a contradiction. This proves the lemma.

Now we prove Proposition 2.17. Let B be the kernel

$$ 0 \to B \to {}^hA_S \times {}^hA_T \to {}^hA_{ST}. \tag{2.1} $$

According to the notation of 2.18 we have $A_S = \varinjlim A_{f_\alpha}$, $A_T = \varinjlim A_{g_\beta}$ (see 2.14 and 2.16) and hence $B = \varinjlim B_{\alpha\beta}$. Moreover by 2.18 we have $(s, t)B = B$ whenever $s \in S$, $t \in T$. Consider the map ${}^hA_S \times {}^hA_T \xrightarrow{\tau} {}^hA_S \otimes_A {}^hA_T$, where $\tau = \phi_1 - \phi_2$ and $\phi_1\colon {}^hA_S \to {}^hA_S \otimes_A {}^hA_T$, $\phi_2\colon {}^hA_T \to {}^hA_S \otimes_A {}^hA_T$ are the canonical homomorphisms. We prove that τ is surjective; in fact we know that ${}^hA_S = (A_S)_{\mathrm{Zar}} = A_{S+\underline{a}}$, ${}^hA_T = (A_T)_{\mathrm{Zar}} = A_{T+\underline{a}}$. Let $s' = s + a_1 \in S + \underline{a}$, $t' = t + a_2 \in T + \underline{a}$; since $(s, \bar{t})B = B$ and $\underline{a}B \subset \operatorname{rad} \bar{B}$ we can write $1 = as' + b \cdot t' \in B$. Hence $1/s' \cdot t' = a/t' + b/s'$ with $a/t' \in {}^hA_T$, $b/s' \in {}^hA_S$. It follows that if R = image of ${}^hA_S \times {}^hA_T$ in ${}^hA_{ST}$, R is also the image of ${}^hA_S \otimes_A {}^hA_T$ in ${}^hA_{ST}$; hence R is A-flat by hypothesis.

From the exact sequence

$$ 0 \to B \to {}^hA_S \times {}^hA_T \to R \to 0 \tag{2.2} $$

since R is A-flat, tensoring over A by $A/\underline{a}$ we get the exact sequence

$$ 0 \to B/\underline{a}B \to (A/\underline{a})_S \times (A/\underline{a})_T \to R/\underline{a}R \to 0. $$

Now we show that $R/\underline{a}R \simeq (A/\underline{a})_{ST}$. In fact from ${}^hA_S \otimes_A {}^hA_T \to R \to {}^hA_{ST}$ we get homomorphisms $(A/\underline{a})_{ST} \to R/\underline{a}R \to (A/\underline{a})_{ST}$ such that the composition is the identity; on the other hand ${}^hA_S \otimes_A {}^hA_T \to R$ is surjective, so that $(A/\underline{a})_{ST} \to R/\underline{a}R$ is surjective, hence an isomorphism.

From the exact sequence

$$(2.3) \qquad 0 \to B/\underline{a}B \to (A/\underline{a})_S \times (A/\underline{a})_T \to (A/\underline{a})_{ST} \to 0$$

tensoring over $A/\underline{a}$ by $(A/\underline{a})_S$ we get the exact sequence

$$0 \to (B/\underline{a}B)_S \to (A/\underline{a})_S \times (A/\underline{a})_{ST} \to (A/\underline{a})_{ST} \to 0.$$

Hence $(B/\underline{a}B)_S \simeq (A/\underline{a})_S$; likewise $(B/\underline{a}B)_T \simeq (A/\underline{a})_T$.

From the exact sequence (2.2) we see that B is A-flat since both ${}^hA_S \times {}^hA_T$ and R are. Hence ${}^hA_S = A_{S'} \to B_{S'} \to {}^hB_S$ is flat where $S' = S + \underline{a}$, and since $(A/\underline{a})_S \xrightarrow{\sim} (B/\underline{a}B)_S$, ${}^hA_S \to {}^hB_S$ is faithfully flat.

From the commutative diagram

$$\begin{tikzcd} & {}^hA_S \times {}^hA_T \arrow[d] \\ B \arrow[ur] \arrow[dr] & \\ & {}^hB_S \times {}^hB_T \end{tikzcd}$$

since $B \to {}^hB_S \times {}^hB_T$, ${}^hA_S \times {}^hA_T \to {}^hB_S \times {}^hB_T$ are both f.flat homomorphisms, $B \to {}^hA_S \times {}^hA_T$ is also f.flat. Hence $B \to {}^hA_S$ is flat and thus $(B_S)_{\mathrm{Zar}} \to {}^hA_S = (A_S)_{\mathrm{Zar}}$ is f.flat, hence injective. But this map is clearly surjective because B is an A-algebra. Thus $(B_S)_{\mathrm{Zar}} \xrightarrow{\sim} {}^hA_S$ and we get ${}^hA_S \simeq {}^hB_S$. Likewise ${}^hA_T \simeq {}^hB_T$. It follows then by 2.1 that ${}^hA_{ST} \simeq {}^hB_{ST}$.

The rest of the proposition is easy; in fact from the exact sequence (2.1), we see that $(B, \underline{a}({}^hA_S \times {}^hA_T) \cap B)$ is an H-couple (this uses Lemma 2 of [**10**]), and then $(B, \underline{a}B)$ is also an H-couple.

From the exact sequence (2.3), we see easily that Spec $B/\underline{a}B$ is the union in Spec $A/\underline{a}$ of Spec$(A/\underline{a})_S$ and Spec$(A/\underline{a})_T$.

Finally if A is an AIC domain and ${}^hA_{ST} \neq 0$ then both ${}^hA_S$ and ${}^hA_T$ are subrings of ${}^hA_{ST}$; indeed by 2.1 we have ${}^hA_{ST} = {}^h({}^hA_S \otimes_A {}^hA_T)$; on the other hand ${}^hA_S \otimes_A {}^hA_T$ is a ring of fractions of A by 2.8 and hence it is a domain. Thus ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$ is injective, and by flatness we have also that ${}^hA_S \to {}^hA_S \otimes_A {}^hA_T$ is injective, hence ${}^hA_S \to {}^hA_{ST}$ is injective. Thus $B = {}^hA_S \cap {}^hA_T$, and using 2.7 and 2.8, we easily see that B is an AIC domain.

Before proving the next proposition we need a lemma.

2.19. LEMMA. *Let $(A, \underline{a})$ be a couple with A a normal domain and let $f, g \in A$. Let B, C be N-existensions of A_f, A_g respectively and let D be the image of $B \otimes_A C$ in ${}^hA_{fg}$ under the homomorphism*

$$B \otimes_A C \to {}^hA_f \otimes_A {}^hA_g \to {}^hA_{fg}.$$

Then D is a direct factor of $B \otimes_A C$ as an A-algebra. In particular D is A-flat and passing to the direct limit the image of ${}^hA_f \otimes_A {}^hA_g$ in ${}^hA_{fg}$ is A-flat.

PROOF. By 2.1 we see easily that ${}^h(B \otimes_A C) \simeq {}^hA_{fg}$. Moreover $B \otimes_A C$ is an ind-étale A-algebra [10, Lemma 3]; hence it is normal [20, p. 75, Proposition 2]. Moreover it is easy to see that $B \otimes_A C$ is a localization of a finite A-algebra, hence has a finite number of minimal primes; thus $B \otimes_A C = D_1 \times \cdots \times D_n$ where D_i is a normal domain. Suppose $\underline{a}D_i \neq D_i$ for $1 \leqslant i \leqslant r$; then ${}^h(B \otimes_A C) = {}^hD_1 \times \cdots \times {}^hD_r$ and $D_i \hookrightarrow {}^hD_i$ for $1 \leqslant i \leqslant r$ because D_i is a domain. Hence $D = D_1 \times \cdots \times D_r$.

2.20. PROPOSITION. *Let $(A, \underline{a})$ be an H-couple with A an AIC ring. Let $f, g \in A$, and let ${}^hA_S$, ${}^hA_T$ be connected components of ${}^hA_f$, ${}^hA_g$ respectively. Then the image of the homomorphism ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$ is A-flat in each of the following two cases*:

(i) *A is an AIC domain*;

(ii) *A is the f.flat A.I. closure of a normal domain A', with $f, g \in A'$.*

PROOF. Since flatness and images are preserved by direct limits it is enough to show that the image of ${}^hA_{f_\alpha} \otimes_A {}^hA_{g_\beta} \to {}^hA_{f_\alpha g_\beta}$ is A-flat. This follows from 2.19 in case (i). In case (ii) by 2.19 the image of ${}^hA'_f \otimes_{A'} {}^hA'_g \to {}^hA'_{fg}$ is A'-flat. By tensoring by A over A' (using 1.7(viii)) and localizing by a suitable element in ${}^hA_f \otimes_A {}^hA_g$, we get that the image of ${}^hA_{f_\alpha} \otimes_A {}^hA_{g_\beta} \to {}^hA_{f_\alpha g_\beta}$ is A-flat.

D. *Simply connected schemes*. In this subsection we give some facts on simply connected schemes, which allow us to give a connectedness criterion (Proposition 2.23) which is essential later.

2.21. DEFINITION. A connected scheme X is *simply connected* if every étale covering $Z \to X$ with Z connected is an isomorphism. Recall that an étale covering is a finite étale morphism.

2.22. LEMMA. *Let $X = \operatorname{Spec} A/\underline{a}$ be a connected scheme, with A an AIC domain; then X is simply connected.*

PROOF. Let $A/\underline{a} \hookrightarrow \overline{C}$ be a connected étale covering; by [13, Theorem 1] since $(A_{\mathrm{Zar}}, \underline{a}A_{\mathrm{Zar}})$ is an H-couple there exists an étale covering $A_{\mathrm{Zar}} \hookrightarrow C$ such that $C/\underline{a}C \simeq \overline{C}$. C is a normal ring [20, Proposition 2, p. 75] and since it has a finite number of minimal primes we can write $C = C_1 \times \cdots \times C_r$ with C_i normal domains; but since C is connected, C is a normal domain. But C is integral over A_{Zar} which is an AIC domain, so that $A_{\mathrm{Zar}} \simeq C$, and hence $A/\underline{a} \simeq \overline{C}$.

The main result of this section is the following.

2.2.3. PROPOSITION. *Let $X = \operatorname{Spec} A$ be an affine scheme and let C, D be connected components of U, V respectively, where $X = U \cup V$ is an affine open covering. Then*

(i) *if $X = C \cup D$ and X is simply connected, then $C \cap D$ is connected*;

(ii) *if A is a quotient of an AIC domain and U, V are basic, i.e. $U = \operatorname{Spec} A_f$, $V = \operatorname{Spec} A_g$, $f, g \in A$, then $C \cap D$ is connected.*

PROOF. In order to prove (i) we assume that $C \cap D$ is not connected and we construct an étale connected covering $Z \to X$ of degree 2. If $C \cap D = E$ is not connected, there are open subsets W_i of X such that $F_i = W_i \cap E \neq \varnothing$ $(i = 1, 2)$

and $E = F_1 \amalg F_2$. We show first that we may assume the following

(a) $W_1 \cap W_2 = \varnothing$,

(b) $W_1 \cup W_2 = U \cap V$.

Clearly we may assume $W_i \subset U \cap V$. Now E is closed in $U \cap V$ and hence is compact. Thus we may assume that W_i is a finite union of open affine subsets of X. Then also $W = W_1 \cap W_2$ is a finite union of open affines and hence it is compact. Let now $\{U_i\}$ (resp. $\{V_j\}$) be the family of all the open-closed subsets of U (resp. V) containing C (resp. D). By 2.14 we have $E = \bigcap_{i,j}(U_i \cap V_j)$ and hence $\bigcap_{i,j}(W \cap (U_i \cap V_j)) = \varnothing$. But $U_i \cap V_j$ is closed in $U \cap V$ and hence $W \cap U_i \cap V_j = \varnothing$ for suitable i, j. Thus after replacing U, V by U_i, V_j we may assume (a).

By the same argument applied to $T = U \cap V - W_1 \cup W_2$ we see that we may assume (b) as well. Now we can construct Z. For this let U_1, U_2 be two disjoint copies of U and V_1, V_2 be two disjoint copies of V and glue them along the W_i's as follows:

U_1 and V_1 along the image of W_1,

V_1 and U_2 along the image of W_2,

U_2 and V_2 along the image of W_1,

V_2 and U_1 along the image of W_2.

This is possible because of (a) above. We obtain an X-scheme $Z \xrightarrow{f} X$. By (b) we have $f^{-1}(U) = U_1 \amalg U_2$ and $f^{-1}(V) = V_1 \amalg V_2$; thus f is an étale covering of degree 2. Finally we have $f^{-1}(C) = C_1 \amalg C_2$ and $f^{-1}(D) = D_1 \amalg D_2$ where f: $C_i \to C$ and f: $D_i \to D$ are homeomorphisms. Thus C_i and D_i are connected; moreover $C_1 \cap D_1$ contains the image of F_1 and then it is nonempty; likewise $D_1 \cap C_2$, $D_2 \cap C_2$ are not empty. Finally since $X = C \cup D$ we have $Z = C_1 \cup C_2 \cup D_1 \cup D_2$ whence Z is connected. This proves (i).

To prove (ii) write $A = R/I$ where R is an AIC domain. If $C \cap D = \varnothing$ there is nothing to prove; otherwise $C \cup D$ is connected, and hence by 2.14 and 2.7 we may assume that A is connected. Thus by 2.7 and 2.8 we may also assume that (R, I) is an H-couple. Let f', $g' \in R$ be liftings of f, g such that $(f', g')R = R$. Then C, D lift canonically to connected components of ${}^hR_{f'}$ and ${}^hR_{g'}$ (see 2.24 below). Now we can apply 2.17 and 2.20 to find an AIC domain R' such that (R', IR') is an H-couple, and Spec R'/IR' is canonically homeomorphic to $C \cup D$. The conclusion follows then by (i) and 2.22.

Observe that Proposition 2.23 can be proved, as pointed out by the referee, by using the Mayer-Vietoris sequence for the étale cohomology. This would replace the explicit construction of the étale cover in the previous proof.

E. *Proof of Theorem* 2.4. We need two more lemmas.

2.24. LEMMA. *Let* $(A, \underline{a})$ *be a couple. Then the following are equivalent*:

(a) $(A, \underline{a})$ *is a Hensel couple*.

(b) *The following two conditions hold*:

(i) *if* $C = A/\underline{c}$ *is a connected component of* A *then* $(C, \underline{a}C)$ *is a Hensel couple*;

(ii) *the map* $C = A/\underline{c} \mapsto \overline{C} = A/\underline{c} + \underline{a}$ *induces a bijection between the set* $\mathcal{C}(A)$ *of connected components of* A *and the set* $\mathcal{C}(A/\underline{a})$ *of connected components of* $A/\underline{a}$.

PROOF. By [10, Corollary 2] (a) $\Rightarrow$ (b)(i); moreover if $(A, \underline{a})$ is an H-couple the map $A \to A/\underline{a}$ induces a 1-1 correspondence between the set of idempotents in A and idempotents in $A/\underline{a}$. Hence (a) $\Rightarrow$ (b)(ii) by 2.14.

Conversely let $(B, \underline{b}) = {}^h(A, \underline{a})$ and let $\phi\colon A \to B$ be the canonical homomorphism. Since $A/\underline{a} = B/\underline{b}$ by (b)(ii), $\phi\colon A \to B$ induces a bijection between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ given by

$$C = A/\underline{c} \mapsto B/\underline{c}B = C \otimes_A B.$$

Now since $(C, \underline{a}C)$ is an H-couple we have $C = {}^hC = C \otimes_A B$. But we have also $C = A_S$ (2.14); hence $C \otimes_A B = B_S = A_S$. Hence for every $\underline{p} \in \operatorname{Spec} A$ we have $A_{\underline{p}} = B_{\underline{p}}$; then $\phi\colon A \to B$ is an isomorphism and $(A, \underline{a})$ is Hensel.

2.25. LEMMA. *Let $(A, \underline{a})$ be an H-couple with A an AIC domain. Let $f, g \in A$. Then every connected component of ${}^hA_f \otimes_A {}^hA_g$ is of the form ${}^hA_S \otimes_A {}^hA_T$ where ${}^hA_S$, ${}^hA_T$ are connected components of ${}^hA_f$, ${}^hA_g$ respectively.*

PROOF. By 2.8 ${}^hA_S \otimes_A {}^hA_T$ is a domain, hence connected; the conclusion follows easily.

2.26. PROOF OF THEOREM 2.4 (CONCLUDED). By 2.11 we may assume A an AIC domain. Put $R = ({}^hA_f \otimes_A {}^hA_g)_{\mathrm{Zar}}$ and we want to show that $R = {}^hA_{fg}$. By 2.1 it is enough to show that $(R, \underline{a}R)$ is an H-couple. By 2.24 we have to prove that $\mathcal{C}(R) = \mathcal{C}(A_{fg}/\underline{a}A_{fg})$ and that if C is a connected component of R, then $(C, \underline{a}C)$ is Hensel. Let C be a connected component of R. Then C lies in a connected component B of ${}^hA_f \otimes_A {}^hA_g$. By 2.25 we can write $B = {}^hA_S \otimes_A {}^hA_T$, with ${}^hA_S$, ${}^hA_T$ connected components of ${}^hA_f$, ${}^hA_g$ respectively. Then, using 2.8, we get

$$B_{\mathrm{Zar}} = ((A_S)_{\mathrm{Zar}} \otimes_A (A_T)_{\mathrm{Zar}})_{\mathrm{Zar}} = (A_S \otimes_A A_T)_{\mathrm{Zar}} = (A_{ST})_{\mathrm{Zar}}.$$

If we apply 2.23 to $\operatorname{Spec}(A/\underline{a})_S$, $\operatorname{Spec}(A/\underline{a})_T$, we see that $(A/\underline{a})_{ST}$ is connected. Thus B_{Zar} is Hensel by 2.8. In particular, it is connected, and hence must be equal to C. It remains to show that $\mathcal{C}(R) = \mathcal{C}(R/\underline{a}R)$ but by the above argument, if B_{Zar} is a connected component of R, $B/\underline{a}B$ is connected. Since moreover $\underline{a}R \subset \operatorname{rad} R$ it is immediate to see that the map $\mathcal{C}(R) \to \mathcal{C}(R/\underline{a}R)$ is bijective. By 2.24 the theorem is proved.

F. *Proof of Theorem* 2.5. First we prove a proposition.

2.27. PROPOSITION. *Let $(A, \underline{a})$ be an H-couple and suppose that A is the f.flat A.I. closure of a normal domain. Let $f, g \in A$ with $(f, g) = (1)$ and let $C = \operatorname{Spec}({}^hA_S)$, $D = \operatorname{Spec}({}^hA_T)$ be connected components of ${}^hA_f$, ${}^hA_g$ respectively such that $C \cap D \neq \varnothing$. Then ${}^hA_S \otimes_A {}^hA_T$ is a Zariski ring.*

PROOF. Let B be the A-algebra such that $\operatorname{Spec}(B/\underline{a}B) = C \cup D$ (see 2.17 and 2.20). We have by 2.8

$${}^hA_S \otimes_A {}^hA_T = (B_S)_{\mathrm{Zar}} \otimes_A (B_T)_{\mathrm{Zar}} = B_{S_1T_1}$$

where $S_1 = S + \underline{a}B$, $T_1 = T + \underline{a}B$.

It is enough to show that if $\underline{p} \subset B$ is maximal with respect to the multiplicative set S_1T_1 then $\underline{p} \supset \underline{a}$. Suppose $\underline{p} + \underline{a}B$ intersects ST; then putting $B' = B/\underline{p} + \underline{a}B$ we have $B'_{ST} = 0$. Since $(B/\underline{p}, \underline{a}(B/\underline{p}))$ is Hensel and B is connected, we have

$B' = B/\underline{p} + \underline{a}B$ connected; likewise B'_S, B'_T are connected. Moreover since $\underline{p} \cap S_1 = \varnothing$ and $\underline{a}B_{S_1} \subset \operatorname{rad} B_{S_1}$ we have $(\underline{p} + \underline{a}B)B_{S_1} \neq B_{S_1}$; hence $B'_S \neq 0$; likewise $B'_T \neq 0$.

If $B'_{ST} = 0$, we would have $B'_{st} = 0$ for some $s \in S$, $t \in T$; but $\operatorname{Spec} B' = \operatorname{Spec} B'_S \cup \operatorname{Spec} B'_T \subset \operatorname{Spec} B'_s \amalg \operatorname{Spec} B'_t$, a contradiction since B' is connected.

2.28. COROLLARY. *Under the hypotheses of* 2.27 *we have* ${}^hA_S \otimes_A {}^hA_T \simeq {}^hA_{ST}$.

PROOF. From 2.4 passing to the direct limit we get

$$({}^hA_S \otimes_A {}^hA_T)_{\mathrm{Zar}} \simeq {}^hA_{ST}$$

and by 2.27 ${}^hA_S \otimes_A {}^hA_T$ is a Zariski ring.

2.29. PROOF OF THEOREM 2.5 (CONCLUDED). By 2.12 we may assume that A is the f.flat A.I. closure of a normal domain. To prove the surjectivity of ${}^hA_f \otimes_A {}^hA_g \to {}^hA_{fg}$ it is sufficient to prove that, for connected components ${}^hA_S$ of ${}^hA_f$ and ${}^hA_T$ of ${}^hA_g$, the map ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$ is surjective, since every connected component of ${}^hA_f \otimes_A {}^hA_g$ is a connected component of some ${}^hA_S \otimes_A {}^hA_T$. But two cases are possible: either ${}^hA_{ST} = 0$ or ${}^hA_{ST} \neq 0$ in which case by 2.28 is ${}^hA_S \otimes_A {}^hA_T = {}^hA_{ST}$. Theorem 2.5 is completely proved.

2.30. COROLLARY. *Let $(A, \underline{a})$ be an H-couple with A an AIC ring. Let $f, g \in A$ with $(f, g) = (1)$ and let $C = \operatorname{Spec}({}^hA_S)$, $D = \operatorname{Spec}({}^hA_T)$ be connected components of ${}^hA_f$, ${}^hA_g$ respectively such that $C \cap D \neq \varnothing$. Then ${}^hA_S \otimes_A {}^hA_T \simeq {}^hA_{ST}$.*

PROOF. By 2.4 passing to the direct limit we have $({}^hA_S \otimes_A {}^hA_T)_{\mathrm{Zar}} \simeq {}^hA_{ST}$; on the other hand by 2.5 passing to the limit we have that ${}^hA_S \otimes_A {}^hA_T \to {}^hA_{ST}$ is surjective. So we can apply 2.17 and prove, as in 2.27, that ${}^hA_S \otimes_A {}^hA_T$ is a Zariski ring.

3. Proof of Theorem A. We need some preliminaries. Let $(A, \underline{a})$ be an H-couple, and put $X = \operatorname{Sph}(A, \underline{a})$; let $f_1, f_2 \in A$, $R = {}^hA_{f_1} \otimes_A {}^hA_{f_2}$, and $R' = {}^hA_{f_1f_2}$. Let ϕ: $R \to R'$ be the canonical homomorphism. Our aim is to study $\operatorname{Ker} \phi$. For this we need:

3.1. DEFINITION. An idempotent $e \in R$ is said to be *admissible* if there are:

(a) decompositions

$$X_{f_i} = \coprod_{j=1}^{r_i} X_{f_{ij}} \quad \text{for } i = 1, 2 \text{ and } r_i > 0,$$

(b) a subset $I \subset \{1, \ldots, r_1\} \times \{1, \ldots, r_2\}$ such that

$$Re = \prod_{(i,j) \in I} {}^hA_{f_{1i}} \otimes_A {}^hA_{f_{2j}}.$$

We say that e is represented by I and the f_{ij}'s, and we write $e \sim (f_{ij}, I)$. The proof of the following two lemmas is straightforward.

3.2. LEMMA. *Let $e \sim (f_{ij}, I)$ be an admissible idempotent of R. Then $e \in \operatorname{Ker} \phi$ if and only if $X_{f_{1i}} \cap X_{f_{2j}} = \varnothing$, for all $(i, j) \in I$.*

If $e, e' \in R$ are idempotents we put as usual $e' \leqslant e$ if and only if $e' = e' \cdot e$ (that is, if Re' is a direct factor of Re).

3.3. Lemma. *The situation being as above, we have*:

(i) *Any two admissible idempotents can be represented by the same* f_{ij}*'s* (*with different* I*'s if they are different*).

(ii) *If* $e \sim (f_{ij}, I)$, $e' \sim (f_{ij}, I')$ *then* $e' \leqslant e$ *if and only if* $I' \subset I$.

(iii) *The set of the admissible idempotents contained in* $\operatorname{Ker} \phi$ *is directed with respect to* $\leqslant$.

3.4. Lemma. *With the above notations, assume further that* A *is AIC and that either* (i) $f_1 = f_2$, *or* (ii) $(f_1, f_2)A = A$. *Then* ϕ *induces an isomorphism* $R/ER \simeq R'$, *where* E *is the set of admissible idempotents contained in* $\operatorname{Ker} \phi$.

Proof. Observe first that ϕ is surjective: this is obvious in case (i) and follows from 2.5 in case (ii). Next we see that $R' = R_{\mathrm{Zar}}$: this follows from 1.7(v), the surjectivity of ϕ, and 2.1 in case (i), and by 2.4 in case (ii). The map ϕ factors through R/ER, and to show $R/ER \simeq R' = R_{\mathrm{Zar}}$, we need only show that every element of $1 + \underline{a}R$ is invertible in R/ER. Thus it suffices to show that for any $\underline{p} \in \operatorname{Spec} R$ such that $\underline{p} \supset E$ we have $\underline{p}R' \neq R'$. Let $\underline{p}_i$ be the contraction of $\underline{p}$ to ${}^hA_{f_i}$, and let Y_i be the connected component of $\operatorname{Spec} {}^hA_{f_i}$ containing $\underline{p}_i$. Put $X_i = Y_i \cap X_{f_i}$, and note that by 2.24 it is a connected component of X_{f_i}. Since A is AIC, by 2.8 and 2.16 there is a multiplicative subset $S_i \subset A$ such that $(A_{S_i}, \underline{a}S_{S_i})$ is an H-couple, and $X_i = \operatorname{Spec}(A/\underline{a})_{S_i}$.

Define $\pi\colon R \to A_{S_1S_2}$ by $\pi = \pi_1 \otimes \pi_2$, where $\pi_i\colon {}^hA_{f_i} \to A_{S_i}$ are the canonical maps. Note that $\underline{p}_iA_{S_i} \neq A_{S_i}$, and hence if $S = S_1S_2$ we have

(1) $$\underline{p}A_S \neq A_S.$$

Now we claim that if $X_{f_1} \cap X_{f_2} \neq \varnothing$, then

(2) $$X_1 \cap X_2 \neq \varnothing.$$

Indeed by 2.14(ii) and 2.16 we have $X_i = \bigcap_j X_{f_{ij}}$ where $X_{f_i} = X_{f_{ij}} \amalg X_{g_{ij}}$. Hence if $X_1 \cap X_2 = \varnothing$, by the compactness of $X_{f_1} \cap X_{f_2}$ we have $X_{f_{1j_1}} \cap X_{f_{2j_2}} = \varnothing$ for suitable j_1, j_2. Put

$$Re = {}^hA_{f_{1j_1}} \otimes_A {}^hA_{f_{2j_2}}.$$

Then e is an admissible idempotent contained in $\operatorname{Ker} \phi$ (see 3.1 and 3.2), whence $e \in \underline{p}$, and $\underline{p}Re = Re$. This easily implies $\underline{p}A_S = A_S$, contrary to (1). Thus $X_1 \cap X_2 \neq \varnothing$. Now by 2.1 we have $R' = {}^hR$ and by (2) and 2.30 we have ${}^hA_S = A_S$; hence there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\pi"'] & R' \\ A_S \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where σ comes from the universal property of Henselization. The conclusion follows then from (1).

3.5. Corollary. *Under the assumptions of* 3.4 *we have* $R' = \varinjlim_{e \in E} R/Re$.

Proof. Apply 3.3, 3.4 and 2.5.

Now we can prove Theorem 1.11. Recall that we are given an affine Hensel scheme $X = \mathrm{Sph}(A, \underline{a})$, and a quasi-coherent sheaf $\mathcal{F}$ over X, and we want to prove that $\mathcal{F} = \tilde{M}$, where $M = \Gamma(X, \mathcal{F})$. We give the proof in several steps. By assumptions there is a covering $X = X_{f_1} \cup \cdots \cup X_{f_n}$ such that $\Gamma(X_{f_i}, \mathcal{F})$ generates the sheaf $\mathcal{F}/X_{f_i}$ for $i = 1, \ldots, n$.

3.6. *Step* 1. *The conclusion is true if $n = 2$ and A is AIC.*

PROOF. Put $M_i = \Gamma(X_{f_i}, \mathcal{F})$. We want to show that the canonical homomorphisms ${}^hA_{f_i} \otimes_A M \to M_i$ are surjective. Put $f = f_1$ and $g = f_2$. Let

$$\tau: {}^hA_{fg} \otimes_{{}^hA_f} M_1 \to {}^hA_{fg} \otimes_{{}^hA_g} M_2$$

be the isomorphism of ${}^hA_{fg}$-modules induced by the restriction of M_1 and M_2 to X_{fg}, and let u_1 be the composition of τ with the canonical homomorphism $M_1 \to {}^hA_{fg} \otimes_{{}^hA_f} M_1$. Let u_2: $M_2 \to {}^hA_{fg} \otimes_{{}^hA_g} M_2$ be the canonical homomorphism.

Then we have the exact sequence of A-modules

$$0 \to M \xrightarrow{t} M_1 \times M_2 \xrightarrow{u} {}^hA_{fg} \otimes_{{}^hA_g} M_2$$

where t is induced by the restrictions, and $u(m_1, m_2) = u_1(m_1) - u_2(m_2)$.

Since ${}^hA_f$ is A-flat and M_1 (resp. M_2) is a module over ${}^hA_f$ (resp. over ${}^hA_g$), tensoring the above exact sequence with ${}^hA_f$ gives the exact sequence:

$$0 \to {}^hA_f \otimes_A M \to {}^hA_f \otimes_A {}^hA_f \otimes_{{}^hA_f} M_1 \times {}^hA_f \otimes_A {}^hA_g \otimes_{{}^hA_g} M_2 \to {}^hA_f \otimes_A {}^hA_{fg} \otimes_{{}^hA_g} M_2.$$

Put $R = {}^hA_f \otimes_A {}^hA_g$, $S = {}^hA_f \otimes_A {}^hA_{fg}$, $T = {}^hA_f \otimes_A {}^hA_f$, and let ϕ_1: ${}^hA_f \to {}^hA_{fg}$, ϕ_2: ${}^hA_g \to {}^hA_{fg}$ be the canonical homomorphisms. Define ψ: $R \to S$, χ: $T \to S$ by

$$\psi(x \otimes y) = x \otimes \phi_2(y), \qquad \chi(x \otimes y) = x \otimes \phi_1(y).$$

Let ϕ: $R \to R' = {}^hA_{fg}$ be the canonical homomorphism, and let $e \in R$ be an admissible idempotent such that $\phi(e) = 1$. This means that we have decompositions

$$X_f = X_{f_1} \amalg \cdots \amalg X_{f_r}, \qquad X_g = X_{g_1} \amalg \cdots \amalg X_{g_s}$$

and a subset $I \subset \{1, \ldots, r\} \times \{1, \ldots, s\}$ such that

$$Re = \prod_{(i,j) \notin I} {}^hA_{f_i} \otimes_A {}^hA_{g_j} \quad \text{(see 3.1)}$$

and $X_{f_i} \cap X_{g_j} = \varnothing$ for $(i, j) \in I$, that is ${}^hA_{f_ig_j} = 0$ for $(i, j) \in I$. Let $Te' = \prod_{i=1,\ldots,r} {}^hA_{f_i} \otimes_A {}^hA_{f_i}$. Thus

$$\chi(Te') = \prod_{i=1,\ldots,r} {}^hA_{f_i} \otimes_A {}^hA_{f_ig} = \prod_{(i,j) \notin I} {}^hA_{f_i} \otimes_A {}^hA_{f_ig_j}$$

$$\subset \prod_{(i,j) \notin I} {}^hA_{f_i} \otimes_A {}^hA_{fg_j} = Se''$$

where $e'' = \psi(e)$. It follows that $u_1(Te' \otimes_{{}^hA_f} M_1) \subset Se'' \otimes_{{}^hA_g} M_2$ and hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & {}^hA_f \otimes_A M & \to & T \otimes_{{}^hA_f} M_1 & \times & R \otimes_{{}^hA_g} M_2 & \xrightarrow{u} & S \otimes_{{}^hA_g} M_2 \\
 & & \downarrow \alpha_e & & & \downarrow \beta_e & & & \downarrow \gamma_e \\
0 & \to & K_e & \to & Te' \otimes_{{}^hA_f} M_1 & \times & Re \otimes_{{}^hA_g} M_2 & \xrightarrow{\bar{u}} & Se'' \otimes_{{}^hA_g} M_2
\end{array}$$

where $\bar{u}$ is induced by u, and K_e is the kernel of $\bar{u}$.

Now it is clear that the obvious inverse maps of β_e and γ_e render the right-hand square commutative, and an easy diagram chase shows that α_e is surjective. Now it is clear from 3.5 that as e varies the second rows of the above diagrams form a direct system, whose limit is

$$0 \to K \to M_1 \times {}^hA_{fg} \otimes_{{}^hA_g} M_2 \to (S/IS) \otimes_{{}^hA_g} M_2$$

where $I = \operatorname{Ker} \phi$, and $K = \varinjlim K_e$. Since all the maps ${}^hA_f \otimes_A M \to K_e$ are surjective, we have that ${}^hA_f \otimes_A \vec{M} \to K$ is also surjective. On the other hand the homomorphism ${}^hA_{fg} = R/I \to S/IS$ induced by ψ has a left inverse ψ': $S/IS \to R/I$ defined by $\psi'(\overline{a \otimes b}) = \phi_1(a) \cdot b$. Hence the canonical map

$${}^hA_{fg} \otimes_{{}^hA_g} M_2 \to (S/IS) \otimes_{{}^hA_g} M_2$$

is injective; this easily implies $K = M_1$ and the proof is complete.

3.7. *Step* 2. *The conclusion is true for $n = 2$ and arbitrary A.*

PROOF. By 2.9 we can embed A into a f.flat A-algebra B which is integral over A and AIC, and the conclusion follows easily by 3.7 and 1.4.

3.8. *Step* 3. *The conclusion is true in general.*

PROOF. Let $I = \{f \in A$: $\mathfrak{F}/X_f$ is generated by $\Gamma(X_f, \mathfrak{F})\}$. We want to prove that $1 \in I$, and for this it is sufficient to show that I is an ideal of A. The only problem is to show that if $f, g \in I$ then $f + g \in I$. Now by 3.7 the result is true for ${}^hA_{f+g}$ and $\mathfrak{F}/X_{f+g}$ and the conclusion follows.

4. Proof of Theorem B.

4.1. LEMMA. *Let X be a topological space and let $\mathfrak{F}$ be an abelian sheaf over X. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of X, and assume that $H^1(U_{i_0, \ldots, i_p}, \mathfrak{F}) = 0$ for all p and all $i_0, \ldots, i_p \in I$, where $U_{i_0, \ldots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Then*

$$\check{H}^1(\mathfrak{U}, \mathfrak{F}) = H^1(X, \mathfrak{F}).$$

PROOF. The proof of 4.5 of [**16**, p. 222] applies, with the modification that one need only assume that $\mathcal{G}$ is flasque in order to prove that $\check{H}^1(\mathfrak{U}, \mathfrak{F}) = H^1(X, \mathfrak{F})$.

4.2. PROPOSITION. *Let X be an affine Hensel scheme, and let $\mathfrak{F}$ be a quasi-coherent sheaf on X. Then for any basic open covering $\mathfrak{U}$ of X we have*

$$\check{H}^1(\mathfrak{U}, \mathfrak{F}) = H^1(X, \mathfrak{F}) = 0.$$

PROOF. From Theorem A it follows that $H^1(X, \mathfrak{F}) = 0$ (same proof as in the usual case, see [7, Proposition 1.4.6]). Hence we have also $H^1(X_f, \mathfrak{F}/X_f) = 0$ for any open basic $X_f \subset X$. And since the intersection of basic opens is basic, the conclusion follows from 4.1.

Now we give a lemma about coverings of affine schemes.

4.3. DEFINITION. Let $X = \operatorname{Spec} B$ be an affine scheme and let $\mathfrak{U} = \{X_{f_1}, \ldots, X_{f_n}\}, f_i \in B$, a covering by basic open subsets. We say that $\mathfrak{U}$ is *special* if for every $i = 1, \ldots, n$ the open $X_{f_i} \cup X_{f_{i+1}} \cup \cdots \cup X_{f_n}$ is a basic open X_{h_i}, $h_i \in B$.

4.4. LEMMA. *Let $X = \operatorname{Spec} B$ and let $\mathfrak{U} = \{X_{f_1}, \ldots, X_{f_n}\}, f_i \in B$, be a covering by basic opens. Then there exists a refinement of $\mathfrak{U}$, $\mathfrak{V} = \{X_{g_1}, \ldots, X_{g_n}\}$, $g_i \in B$, such that $g_i = a_i \cdot f_i$, $a_i \in B$, and $\mathfrak{V}$ is special.*

PROOF. Suppose, by induction, that $\{X_{f_1}, \ldots, X_{f_n}\}$ is a covering such that $X_{f_i} \cup \cdots \cup X_{f_n} = X_{h_i}$ for $i = 1, \ldots, j$. We must find a covering $\{X_{g_1}, \ldots, X_{g_n}\}$ with $g_i = a_i \cdot f_i$ and $X_{g_i} \cup \cdots \cup X_{g_n} = X_{h_i}$ for $i = 1, \ldots, j+1$. In fact from $X_{h_j} = X_{f_j} \cup \cdots \cup X_{f_n}$ we have $h_j^r = b_j f_j + \cdots + b_n f_n$. Let $h_{j+1} = b_{j+1} f_{j+1} + \cdots + b_n f_n$ and put $g_1 = f_1, \ldots, g_j = f_j, g_{j+1} = h_{j+1} f_{j+1}, \ldots, g_n = h_{j+1} f_n$. We claim that $X_{g_i} \cup \cdots \cup X_{g_n} = X_{h_i}$ for $i = 1, \ldots, j+1$.

For $i = j+1$ this is easy since $h_{j+1}^2 = b_{j+1} g_{j+1} + \cdots + b_n f_n$ and $X_{g_{j+1}} \subset X_{h_{j+1}}, \ldots, X_{g_n} \subset X_{h_{j+1}}$.

For $i = 1, \ldots, j$ we have $X_{g_i} = X_{f_i} \subset X_{h_i}, \ldots, X_{g_j} = X_{f_j} \subset X_{h_i}$, $X_{g_{j+1}} \subset X_{f_{j+1}} \subset X_{h_i}, \ldots, X_{g_n} \subset X_{f_n} \subset X_{h_i}$. On the other hand if $p \in \operatorname{Spec} B$ contains $g_i, \ldots, g_j$, $g_{j+1}, \ldots, g_n$ it contains h_{j+1}, hence h_j by $h_j^r = b_j g_j + h_{j+1}$. Thus

$$X_{g_i} \cup \cdots \cup X_{g_n} \supset X_{f_i} \cup \cdots \cup X_{f_{j-1}} \cup X_{h_j}$$
$$= X_{f_i} \cup \cdots \cup X_{f_{j-1}} \cup X_{f_j} \cup \cdots \cup X_{f_n} = X_{h_i}.$$

The following proposition is essential in order to prove Theorem B.

4.5. PROPOSITION. *Let X be an affine Hensel scheme and let $\mathfrak{U} = \{U_0, \ldots, U_n\}$ be a special open covering of X. Then $\check{H}^p(\mathfrak{U}, \mathfrak{F}) = 0$ for all $p > 0$ and all quasi-coherent sheaves $\mathfrak{F}$ on X.*

PROOF. If $n = 1$ the conclusion follows from 4.2. If $n > 1$ we proceed by induction. For this let $Y = U_1 \cup \cdots \cup U_n$, and consider the coverings of Y: $\mathfrak{V} = \{U_0 \cap Y, U_1, \ldots, U_n\}$ and $\mathfrak{V}' = \{U_1, \ldots, U_n\}$. Define $C^\cdot$ by the exact sequence:

$$0 \to C^\cdot \to C^\cdot(\mathfrak{V}, \mathfrak{F}/Y) \xrightarrow{\phi} C^\cdot(\mathfrak{V}', \mathfrak{F}/Y) \to 0$$

where ϕ is the obvious map of Čech complexes. The induction hypothesis implies that $H^p(C^\cdot) \simeq \check{H}^p(\mathfrak{V}, \mathfrak{F}/Y)$ for $p > 1$. However, for $p > 0$,

$$C^p = \bigoplus_{i_1 < \cdots < i_p} \Gamma(U_{0,i_1,\ldots,i_p} \cap Y, \mathfrak{F}).$$

Let $\mathfrak{V}''$ be the covering $\{Y \cap U_0 \cap U_1, \ldots, Y \cap U_0 \cap U_n\}$ of the affine Hensel scheme $Y \cap U_0$. Then we see that $C^p = C^{p-1}(\mathfrak{V}'', \mathfrak{F}/Y \cap U_0)$ for $p > 0$, so that $H^p(C^\cdot) \simeq H^{p-1}(\mathfrak{V}'', \mathfrak{F}/Y \cap U_0)$. By induction, this last group is zero for $p > 1$. Since $C^p(\mathfrak{U}, \mathfrak{F}) = C^p(\mathfrak{V}, \mathfrak{F}/Y)$ for $p > 0$, we see that $H^p(\mathfrak{U}, \mathfrak{F}) = 0$ for $p > 1$. To complete the proof, apply 4.2 again.

PROOF OF THEOREM B (THEOREM 1.12). By 4.4 and 4.5 we have $\check{H}^p(X, \mathfrak{F}) = 0$ for all $p > 0$. But this applies also to X_f and $\mathfrak{F}/X_f$, for every basic open $X_f \subset X$, and the conclusion follows from a theorem of Cartán (see [**6**, p. 227, 5.9.2]).

REFERENCES

1. M. Artin, *On the join of Hensel rings*, Adv. in Math. **7** (1971), 282–296.
2. N. Bourbaki, *Algèbre commutative*, Chapters I, II, Hermann, Paris, 1961.
3. D. Cox, *Algebraic tubular neighborhoods*. I, Math. Scand. **42** (1978), 211–228.
4. ______, *Algebraic tubular neighborhoods*. II, Math. Scand. **42** (1978), 229–243.
5. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1967.
6. R. Godement, *Théorie des faisceaux*, Hermann, Paris, 1958.

7. A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique*. I, Springer-Verlag, Berlin, 1971.

8. ______, *Eléments de géométrie algébrique*. II, Inst. Hautes Étude Sci. Publ. Math., no. 8, 1961.

9. ______, *Eléments de géométrie algébrique*. III, Inst. Hautes Étude Sci. Publ. Math., no. 11, 1961.

10. S. Greco, *Anelli Henseliani*, CIME, Cremonese, Roma, 1973.

11. ______, *Sulla nozione di preschema henseliano*, Atti Acc. Naz. Lincei **50** (1971), 78–81.

12. ______, *Sul sollevamento dei rivestimenti étale*, Sympos. Math. **11** (1973), 67–80.

13. L. Gruson, *Une propriété des couples henséliennes*, Coll. d'Algèbre Comm., Exp. 10, Univ. de Rennes, 1972.

14. S. Greco and R. Strano, *Theorems* A *and* B *for henselian schemes*, Atti Acc. Naz. Lincei **67** (1979), 95–98.

15. R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. of Math. **88** (1968), 403–450.

16. ______, *Algebraic geometry*, Springer-Verlag, Berlin, 1977.

17. H. Hironaka, *Formal line bundles along exceptional loci*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fundamental Res., Bombay, 1968), Oxford Univ. Press, London, 1969.

18. H. Kurke, G. Pfister and M. Roczen, *Henselsche Ringe und algebraische Geometrie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975.

19. F. Mora, *Ideali di definizioni e morfismi di schemi henseliani*, Ann. Mat. Pura Appl. **105** (1975), 191–204.

20. M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Math., vol. 169, Springer-Verlag, Berlin, 1970.

21. M. Roczen, *Henselsche äquivalenz von singularitäten*, Wiss. Beitr. Martin-Luther-Univ. Halle-Wittenberg M 4 (1975), 29–46.

22. O. Zariski, *Theory and application of holomorphic function on algebraic varieties over arbitrary ground field*, Mem. Amer. Math. Soc., no. 5, Amer. Math. Soc., Providence, R. I., 1971.

Istituto Matematico del Politecnico, Torino, Italy

Bhaskaracharya Pratishthana, Poona, India

Seminario Matematico dell' Università, Catania, Italy