QUASIFLATS IN CAT(0) COMPLEXES

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ABSTRACT. We show that if X is a piecewise Euclidean 2-complex with a cocompact isometry group, then every 2-quasiflat in X is at finite Hausdorff distance from a subset Q which is locally flat outside a compact set, and asymptotically conical.

1. INTRODUCTION

In a number of rigidity theorems for quasi-isometries, an important step is to determine the structure of individual quasi-flats; this is then used to restrict the behavior of quasi-isometries, often by exploiting the pattern of asymptotic incidence of the quasiflats [Mos73, KL97a, KL97b, EF97, Esk98, BKMM12]. In this paper, we study 2-quasiflats in CAT(0) 2-complexes, and show that they have a very simple asymptotic structure:

Theorem 1.1. *Let X be a proper, piecewise Euclidean,* CAT(0) *2-complex with a cocompact isometry group. Then every 2-quasiflat $Q \subset X$ lies at finite Hausdorff distance from a subset $Q' \subset X$ which is locally flat (i.e. locally isometric to $\mathbf{R}^2$) outside a compact set.*

This result, and more refined statements appearing in later sections, are applied to 2-dimensional right-angled Artin groups in [BKS08]. The main application is to show that if X, X' are the standard CAT(0) complexes of 2-dimensional right-angled Artin groups, then any quasi-isometry $X \to X'$ between them must map flats to within finite Hausdorff distance of flats.

The strategy for proving Theorem 1.1 is to replace the quasiflat Q with a canonical object that has more rigid structure. To that end, we first associate an element $[Q]$ of the locally finite homology group $H_2^{\mathrm{lf}}(X)$, and then show that the support set $\mathrm{supp}([Q])$ of $[Q]$ – the set of points $x \in X$ such that the induced homomorphism

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$H_2^{\mathrm{lf}}(X) \to H_2(X, X \setminus \{x\})$ is nontrivial on $[Q]$ – is at bounded Hausdorff distance from Q. The support set $Q' := \mathrm{supp}([Q])$ behaves much like a minimizing locally finite cycle, and this leads to asymptotically rigid behavior, in particular asymptotic flatness.

Remarks.

(1) Support sets were used implicitly in [KL97b], and also in [Xie05].
(2) The paper [KL], which may be viewed as a more sophisticated version of the results presented here, exploits similar geometric ideas in asymptotic cones, to study k-quasiflats in CAT(0) spaces which have no $(k+1)$-quasiflats.
(3) Many of the results of this paper (though not Theorem 1.1 itself) can be adapted to n-quasiflats in n-dimensional CAT(0) complexes.
(4) One may use the results in this paper to give a new proof that quasi-isometries between Euclidean buildings map flats to within uniform Hausdorff distance of flats [KL97b]. This then leads to a (partly) different proof of rigidity of quasi-isometries between Euclidean buildings.

Contents

2. Preliminaries

2.1. CAT(κ) **spaces.** We recall some standard facts, and fix notation. We refer the reader to [BH99, KL97b] for more detail. Our notation and conventions are consistent with [KL97b].

Let X be a CAT(0) space.

If $x, y \in X$, then $\overline{xy} \subset X$ denotes the geodesic segment with endpoints x, y. If $p \in X$, we let $\angle_p(x, y)$ denote the angle between x and

y at p. This induces a pseudo-distance on $X \setminus \{p\}$. By collapsing subsets of zero diameter and completing, we obtain the space of directions $\Sigma_p X$, which is a CAT(1) space. The quotient map yields the logarithm $\log_p : X \setminus \{p\} \to \Sigma_p X$; it associates to $x \in X \setminus \{p\}$ the direction at p of the geodesic segment $\overline{px}$. The tangent cone at p, denoted $C_p X$, is a CAT(0) space isometric to the cone over $\Sigma_p X$.

Given two constant (not necessarily unit) speed rays $\gamma_1, \gamma_2 : [0, \infty) \to X$, their distance is defined to be

$$\lim_{t \to \infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t}.$$

This defines a pseudo-distance on the set of constant speed rays in X; the metric space obtained by collapsing zero diameter subsets is the *Tits cone* of X, denoted $C_T X$. The Tits cone is isometric to the Euclidean cone over the Tits boundary $\partial_T X$. For every $p \in X$, there are natural logarithm maps:

$$\log_p : X \to C_p X, \quad \log_p : C_T X \to X,$$
$$\log_p : X \setminus \{p\} \to \Sigma_p X, \quad \log_p : \partial_T X \to \Sigma_p X.$$

Definition 2.1. If Z is a CAT(1) space, $Y \subset Z$, and $z \in Z$, then the *antipodal set of z in Y*, is

$$\mathrm{Ant}(z, Y) := \{y \in Y \mid d(z, y) = \pi\}.$$

Recall that by our definition, every CAT(1) space has diameter $\leq \pi$.

If X is a CAT(0) complex and $p, x \in X$ are distinct points, $Y \subset \Sigma_x X$, then the antipodal set $\mathrm{Ant}(\log_x p, Y)$ is the set of directions in Y which are tangent to extensions of the geodesic segment $\overline{px}$ beyond x.

2.2. **Locally finite homology.** Let Z be a topological space. We recall that the k^{th} locally finite (singular) chain group $C_k^{\mathrm{lf}}(Z)$ is the collection of (possibly infinite) formal sums of singular k-simplices, such that for every compact subset $Y \subset Z$, only finitely many nonzero terms are contributed by singular simplices whose image intersects Y. The usual boundary operator yields a well-defined chain complex $C_*^{\mathrm{lf}}(Z)$; its homology is the *locally finite homology of Z*.

Suppose K is a simplicial complex. Then there is a simplicial version of the locally finite chain complex – the locally finite simplicial chain complex – defined by taking (possibly infinite) formal linear combinations of oriented simplices of K, where every simplex σ of K touches only finitely many simplices with nonzero coefficients. The usual proof that simplicial homology is isomorphic to singular homology gives an isomorphism between the locally finite simplicial homology of K, and the locally finite homology of its geometric realization $|K|$, when K is locally finite [Hat02, 3.H, Exercise 6].

The *support set* of $\sigma \in H_k^{\mathrm{lf}}(Z)$ is the subset $\mathrm{supp}(\sigma) \subset Z$ consisting of the points $z \in Z$ for which the inclusion homomorphism

$$H_k^{\mathrm{lf}}(Z) \to H_k(Z, Z \setminus \{z\})$$

is nonzero on σ. This is a closed subset when Z is Hausdorff.

Now suppose K is an n-dimensional locally finite simplicial complex, with polyhedron Z. Then the simplicial chain groups $C_k^{\mathrm{lf}}(K)$ vanish for $k > n$, and hence $H_n^{\mathrm{lf}}(Z)$ is isomorphic to the group of locally finite simpicial n-cycles $Z_n^{\mathrm{lf}}(K)$. The support set of a locally finite simplicial n-cycle $\sigma \in Z_n^{\mathrm{lf}}(Z)$ is the union of the closed n-simplices having nonzero coefficient in σ, as follows from excision.

3. Locally finite homology and support sets

The key results in this section are the geodesic extension property of Lemma 3.1, and the asymptotic conicality result for support sets with quadratic area growth, in Theorem 3.11. We remark that most of the statements (and proofs) in this section extend with minor modifications to supports of n-dimensional locally finite homology classes in n-dimensional CAT(0) complexes.

In this section X will be a proper, piecewise Euclidean, CAT(0) 2-complex.

3.1. **The geodesic extension property and metric monotonicity.**

The fundamental property of support sets is the extendability of geodesics:

Lemma 3.1. *Suppose $\sigma \in H_2^{\mathrm{lf}}(X)$, and let $S := \mathrm{supp}(\sigma) \subset X$ be the support of σ. If $p \in X$, and $x \in S$, the geodesic segment $\overline{px}$ may be prolonged to a ray in S: there is a ray $\overline{x\xi} \subset S$ with fits together with $\overline{px}$ to form a ray $\overline{p\xi}$.*

Proof. Let $\gamma : [0, L] \to X$ be the unit speed parametrization of $\overline{px}$, and let $\hat{\gamma} : I \to X$ be a maximal extension of γ such that $\hat{\gamma}(I \setminus [0, L]) \subset S$, where I is an interval contained in $[0, \infty)$. Since S is a closed subset of the complete space X, either $I = [0, R]$ for some $R < \infty$, or $I = [0, \infty)$.

Suppose $I = [0, R]$ for $R < \infty$, and let $y := \hat{\gamma}(R)$. Consider the closed ball $B := \overline{B(y, r)}$, where r is small enough that B is isometric to the r-ball in the tangent cone $C_y X$. Note that this implies that $S \cap B$ is also a cone. Let $\sigma = [\sigma_B + \tau]$, where $\sigma_B \in C_2^{\mathrm{lf}}(X)$ is carried by B (and is therefore a finite 2-chain), $\tau \in C_2^{\mathrm{lf}}(X)$ is carried by $X \setminus B(y, r)$, and $\partial \sigma_B = -\partial \tau$ is carried by $\partial B \cap S$. Consider the singular chain μ obtained by coning off $\partial \sigma_B$ at p. Then $\partial \mu = \partial \sigma_B$, so the contractibility of X implies that μ is homologous to σ_B relative to $\partial \mu$. Thus $\mu + \tau$ belongs to the homology class of σ. Therefore y lies in the carrier of μ, for otherwise $\mu + \tau$ would be carried by $X \setminus \{y\}$, contradicting the fact that $y \in \mathrm{supp}(\sigma)$. Thus there is a point $z \in \partial B \cap S$ such that the segment $\overline{pz}$ passes through y. Since $B \cap S$ is a cone, we have $\overline{yz} \subset S$. This implies that $\hat{\gamma}$ is not a maximal extension, which is a contradiction.

Another way to argue the last part of the proof is to observe that σ_B projects under $\log_y : X \setminus \{y\} \to \Sigma_y X$ to a nontrivial 1-cycle η in $\Sigma_y X$. Therefore, there must be a direction $v \in \Sigma_y S$ making an angle π with $\log_y p$, since otherwise η would lie in the open ball of radius π centered at $\log_y p$, which is contractible. Then $\hat{\gamma}$ may be extended in the direction v, which contradicts the maximality of $\hat{\gamma}$. □

Remark 3.2. The geodesic extension property has a flavor similar to convexity, but note that support sets need not be convex. To obtain an example, let Z be the union of two disjoint circles Y_1, Y_2 of length 2π with a geodesic segment of length $< \pi$ (so Z is a "pair of glasses"), and let X be the Euclidean cone over Z. Then cone over $Y_1 \cup Y_2$ is a support set, but is not convex.

Corollary 3.3 (Monotonicity and lower density bound)**.** *Suppose* $\sigma \in H_2^{\mathrm{lf}}(X)$ *and* $S := \mathrm{supp}(\sigma)$.

1. *(Metric monotonicity) For all* $0 < r \leq R$, $p \in X$, *if* $\Phi : X \to X$ *is the map which contracts points toward* p *by the factor* $\frac{r}{R}$, *then*

$$B(p, r) \cap S \ \subset \ \Phi(B(p, R) \cap S). \tag{3.4}$$

2. *(Monotonicity of density) For all* $0 \leq r \leq R$,

$$\frac{\mathrm{Area}(B(p, r) \cap S)}{r^2} \leq \frac{\mathrm{Area}(B(p, R) \cap S)}{R^2}. \tag{3.5}$$

3. *(Lower density bound) For all* $p \in S$, $r > 0$,

$$\mathrm{Area}(B(p, r) \cap S) \geq \pi\, r^2, \tag{3.6}$$

with equality only if $B(p, r) \cap S$ *is isometric to an* r*-ball in* $\mathbf{R}^2$.

Here Area(Y) *refers to* 2*-dimensional Hausdorff measure, which is the same as Lebesgue measure (computed by summing over the intersections with* 2*-dimensional faces).*

Remark 3.7. Since the map Φ in assertion 1 has Lipschitz constant $\frac{r}{R}$, the inclusion (3.4) can be viewed as a much stronger version of the usual monotonicity formula for minimal submanifolds in nonpositively curved spaces, which corresponds to (3.5).

Proof of Corollary 3.3. (3.4) follows from Lemma 3.1.

Assertion 2 follows from assertion 1 and the fact that Φ has Lipschitz constant $\frac{r}{R}$.

If $p \in S$, then σ determines a nonzero class $\Sigma_p\sigma \in H_1(\Sigma_p X)$, by the composition

$$H_2(X, X \setminus \{p\}) \xrightarrow{\partial} H_1(X \setminus \{p\}) \xrightarrow{\log_{\Sigma_p X}} H_1(\Sigma_p X).$$

Since $\Sigma_p X$ is a CAT(1) graph, $\mathrm{supp}(\Sigma_p\sigma)$ contains a cycle of length at least 2π. If $r > 0$ is small, then $B(p, r) \cap S$ is isometric to a cone of radius r over $\mathrm{supp}(\Sigma_p\sigma)$, and therefore has area at least πr^2. Now (3.5) implies (3.6). Equality in (3.6) implies that $\mathrm{supp}(\Sigma_p\sigma)$ is a circle of length 2π, $B(p, r_0) \cap S$ is isometric to an r_0-ball in $\mathbf{R}^2$ for small $r_0 > 0$, and that the contraction map Φ is similarity. This implies 3. ☐

The corollary implies that the ratio

$$\frac{\mathrm{Area}(B(p, r) \cap S)}{r^2}$$

has a (possibly infinite) limit $\bar{A}$ as $r \to \infty$, which is clearly independent of the basepoint. When it is finite we say that σ has *quadratic growth.* In this case, Corollary 3.3 implies that

$$\frac{\mathrm{Area}(B(p, r) \cap S)}{r^2} \le \bar{A} \tag{3.8}$$

for all $p \in X$, $r > 0$.

3.2. **Asymptotic conicality.** We will use Lemma 3.1 and Corollary 3.3 to see that quadratic growth support sets are asymptotically conical, provided the CAT(0) 2-complex X satisfies a mild additional condition. To see why an additional assumption is needed, consider a piecewise Euclidean CAT(0) 2-complex X homeomorphic to $\mathbf{R}^2$, whose singular set consists of a sequence of cone points $\{p_i\}$ tending to infinity, where $\Sigma_{p_i}X$ is a circle of length $2\pi + \theta_i$, and $\sum_i \theta_i < \infty$. Then X is the support set of the locally finite fundamental class $[X]$ of the 2-manifold X, but it is not locally flat outside any compact subset of X.

To exclude this kind of behavior, one would like to know, for instance, that the cone angle 2π is isolated among the set of cone angles of points in X. When dealing with general CAT(0) 2-complexes, one needs to know that if $p \in X$ and $v \in \Sigma_p X$ is a direction whose antipodal set $\mathrm{Ant}(v, \mathrm{supp}(\tau))$ in a 1-cycle $\tau \in Z_1(\Sigma_p X)$ has small diameter, then v is close to a suspension point of τ. This condition will hold automatically if X admits a cocompact group of isometries. The precise condition we need is:

Definition 3.9. A family $\mathcal{F}$ of CAT(1) graphs has *isolated suspensions* if for every $\alpha > 0$ there is a $\beta > 0$ such that if $\Gamma \in \mathcal{F}$, $\tau \in Z_1(\Gamma)$ is a 1-cycle, $v \in \Gamma$, and

$$\mathrm{diam}(\mathrm{Ant}(v, \mathrm{supp}(\tau)) < \beta,$$

then $\mathrm{supp}(\tau)$ is a metric suspension and v lies at distance $< \alpha$ from a pole (i.e. suspension point) of $\mathrm{supp}(\tau)$). A CAT(0) 2-complex X has *isolated suspensions* if the collection of spaces of directions $\{\Sigma_x X\}_{x\in X}$ has isolated suspensions.

Remark 3.10. It follows from a compactness argument that any finite collection of CAT(1) graphs has the isolated suspensions property. In particular, any CAT(0) 2-complex with a cocompact isometry group has the isolated suspension property.

For the remainder of this section X will be a piecewise Euclidean, proper CAT(0) 2-complex with isolated suspensions.

Theorem 3.11. *Suppose $\sigma \in H_2^{\mathrm{lf}}(X)$ has quadratic area growth, and $S := \mathrm{supp}(\sigma)$. Then for all $p \in X$ there is an $r_0 < \infty$ such that*

1. If $x \in S \setminus B(p, r_0)$, then S is locally isometric to a product of the form $\mathbf{R} \times W$ near x, where W is an i-pod (i.e. a cone over a finite set). In particular S is locally convex near x.

2. The map $S \setminus B(p, r_0) \to [r_0, \infty)$ given by the distance function from p is a fibration with fiber homeomorphic to a finite graph with all vertices of valence ≥ 2.

3. S is asymptotically conical, in the following sense. For every $p \in X$ and every $\epsilon > 0$, there is an $r < \infty$ such that if $x \in S \setminus B(p, r)$, then the angle (at x) between the geodesic segment $\overline{xp}$ and the $\mathbf{R}$-factor of some local product splitting of S is $< \epsilon$.

4. If the area growth of S is Euclidean, i.e.

$$\frac{\text{Area}(B(p,r) \cap S)}{\pi r^2} \to 1$$

as $r \to \infty$, then S is a 2-flat.

Before entering into the proof of this theorem, we point out that the proof is driven by the following observation. The locally finite cycle σ is an area minimizing object in the strongest possible sense: any compact piece τ solves the Plateau problem with boundary condition $\partial\tau$ (i.e. filling $\partial\tau$ with a least area chain); in fact, because of the dimension assumption, there is only one way to fill $\partial\tau$ with a chain. Then we adapt the standard monotonicity formula from minimal surface theory to see that the support set is asymptotically conical. Roughly speaking the idea is that the ratio

$$\frac{\text{Area}(B(p,r) \cap \text{supp}(\sigma))}{r^2}$$

is nondecreasing and bounded above, and hence has limit as $r \to \infty$. For large r, one concludes that the monotonicity inequality is nearly an equality, which leads to 2 of Theorem 3.11.

Proof of Theorem 3.11. We begin with a packing estimate.

Lemma 3.12. *For all $\epsilon > 0$ there is an N such that for all $r \geq 0$, the intersection $B(p, r) \cap S$ does not contain an ϵr-separated subset of cardinality greater than N.*

Proof. Take $\epsilon < 1$, and suppose the points

$$x_1, \ldots, x_k \in B(p, r) \cap S$$

are ϵr-separated. Then the collection

$$\left\{ B\left(x_i, \frac{\epsilon r}{2}\right) \cap S \right\}_{1 \leq i \leq k}$$

is disjoint, is contained in $B(p, 2r) \cap S$, and by assertion 2 of Corollary 3.3 it has area at least $k\pi(\frac{\epsilon r}{2})^2$. Thus (3.8) implies the lemma. □

Lemma 3.13. *For all $\beta > 0$ there is an $r < \infty$ such that if $x \in S \setminus B(p,r)$, then*

$$\operatorname{diam}(\operatorname{Ant}(\log_x p\ ,\ \Sigma_x S)) < \beta. \tag{3.14}$$

Proof. The idea is that quadratic area growth bounds the complexity of the support set from above, which implies that on sufficiently large scales, it looks very much like a metric cone. On the other hand, failure of (3.14) implies that there is a pair of rays leaving p which coincide until x, and then branch apart with an angle at least β; when x is far enough from p, this will contradict the approximately conical structure of S at large scales.

Pick $\delta, \mu > 0$, to be determined later.

By Lemma 3.12 there is finite upper bound on the cardinality of an δr-separated subset sitting in $B(p,r) \cap S$, where r ranges over $[1,\infty)$. Let N be the maximal such cardinality, which will be attained by some δr_0-separated subset $\{x_1,\ldots,x_N\} \subset B(p,r_0) \cap S$, for some r_0. Applying Lemma 3.1, let $\gamma_1,\ldots,\gamma_N : [0,\infty) \to X$ be constant speed geodesics emanating from p, such that $\gamma_i(r_0) = x_i$, and $\gamma_i(t) \in S$ for all $t \in [r_0,\infty)$, $1 \le i \le N$. The functions

$$t \mapsto \frac{d(\gamma_i(t),\gamma_j(t))}{t} \tag{3.15}$$

are nondecreasing, and hence for all $r \in [r_0,\infty)$ the collection

$$\gamma_1(r),\ldots,\gamma_N(r)$$

is δr-separated, and by maximality, it is therefore a δr-net in $B(p,r)\cap S$ as well. Using the monotonicity (3.15) again, we may find $r_1 \in [r_0,\infty)$ such that for all $1 \le i,j \le N$, and every $r \in [r_1,\infty)$,

$$\frac{d(\gamma_i(r),\gamma_j(r))}{r} + \mu > \lim_{t\to\infty} \frac{d(\gamma_i(t),\gamma_j(t))}{t}\ . \tag{3.16}$$

Now suppose $x \in S \setminus B(p,r_1)$, and $v_1, v_2 \in \operatorname{Ant}(\log_x p, \Sigma_x S)$ satisfy $\angle_x(v_1,v_2) \ge \beta$. The idea of the rest of the proof is to invoke Lemma 3.1 to produce two rays emanating from p which agree until they reach x, but then "diverge at angle at least β"; since both rays will be well-approximated by one of the γ_i's, their separation behavior will contradict (3.16).

Let $r_2 := d(p,x)$. By Lemma 3.1 we may prolong the segment $\overline{px}$ into two rays $\overline{p\xi_1}, \overline{p\xi_2}$, such that $\log_{\Sigma_x} \xi_i = v_i$, and $\overline{p\xi_i} \setminus B(p,r_2) \subset S$. Let

η_1, η_2 be the unit speed parametrizations of $\overline{p\xi_1}$ and $\overline{p\xi_2}$ respectively. Applying triangle comparison, we may choose an $r_3 \geq r_2$ such that

$$d(\eta_1(r_3), \eta_2(r_3)) > r_3 \cos\frac{\beta}{2}. \tag{3.17}$$

Pick i, j such that

$$d(\gamma_i(r_3), \eta_1(r_3)) < \delta r_3 \quad \text{and} \quad d(\gamma_j(r_3), \eta_2(r_3)) < \delta r_3.$$

By triangle comparison, we have

$$d(\gamma_i(r_3), \gamma_j(r_3)) \geq d(\eta_1(r_3), \eta_2(r_3)) - 2\delta r_3 > r_3 \cos\frac{\beta}{2} - 2\delta r_3$$

while

$$\begin{aligned} d(\gamma_i(r_2), \gamma_j(r_2)) &\leq d(\gamma_i(r_2), \eta_1(r_2)) + d(\eta_1(r_2), \eta_2(r_2)) + d(\eta_2(r_2), \gamma_j(r_2)) \\ &\leq 2\delta r_2, \end{aligned}$$

since $d(\eta_1(r_2), \eta_2(r_2)) = 0$. On the other hand, by (3.16)

$$\begin{aligned} \mu &> \frac{d(\gamma_i(r_3), \gamma_j(r_3))}{r_3} - \frac{d(\gamma_i(r_2), \gamma_j(r_2))}{r_2} \\ &\geq \cos\frac{\beta}{2} - 4\delta. \end{aligned}$$

When $\mu + 4\delta < \cos\frac{\beta}{2}$ this gives a contradiction. □

The Lemma together with the definition of isolated suspensions implies parts 1 and 3 of Theorem 3.11. Part 4 follows from Lemma 3.3.

To prove 2 of Theorem 3.11, we apply the definition of isolated suspensions with $\alpha_0 = \frac{\pi}{4}$ and let $\beta_0 > 0$ be the corresponding constant; then we apply Lemma 3.13 with $\beta = \beta_0$, and let r_0 be the resulting radius. For each $x \in X \setminus B(p, r_0)$, the space of directions $\Sigma_x S$ is a metric suspension, and the direction $\log_x p \in \Sigma_x X$ makes an angle at most $\frac{\pi}{4}$ from a pole of $\Sigma_x S$.

We call a point $x \in S \setminus B(p, r_0)$ *singular* if its tangent cone is not isometric to $\mathbf{R}^2$; thus singular points in $S \setminus B(p, r_0)$ have tangent cones of the form $\mathbf{R} \times W$, where W is an i-pod with $i > 2$, and the set of regular points forms an open subset which carries the structure of a flat Riemannian manifold. Using a partition of unity, we may construct a smooth vector field ξ on the regular part of $S \setminus B(p, r_0)$ such that

- $\xi(x)$ makes an angle at least $\frac{3\pi}{4}$ with $\log_x p$ at every regular point x.

• For each singular point $x \in S \setminus B(p, r_0)$ whose space of directions is the metric suspension of an i-pod, if we decompose a small neighborhood $B(x, \rho) \cap S$ into a union

$$C_1 \cup \ldots \cup C_i,$$

where the C_j's are Euclidean half-disks of radius ρ which intersect along a segment η of length 2ρ, then the restriction of ξ to C_j extends to a smooth vector field ξ_j on the manifold with boundary C_j, and $\xi_j(y)$ is a unit vector tangent to $\eta = \partial C_j$ for every $y \in \eta$.

Now a standard Morse theory argument using a reparametrization of the flow of ξ implies that

$$d_p : S \setminus B(p, r_0) \to [r_0, \infty)$$

is a fibration, and that the fiber is locally homeomorphic to an i-pod near each point $x \in S \setminus B(p, r_0)$ whose space of directions is the metric suspension of an i-pod. Here $i \geq 2$. □

3.3. **Asymptotic branch points.** The next result will be used when we consider support sets associated with quasiflats.

Lemma 3.18. *Let $\sigma \in H_2^{\mathrm{lf}}(X)$ be a quadratic growth class with support S, pick $p \in X$, and let*

$$d_p : S \setminus B(p, r_0) \to [r_0, \infty)$$

be the fibration as in 2 of Theorem 3.11. If the fiber has a branch point, then for all $R < \infty$, the support set S contains an isometrically embedded copy of an R-ball

$$B_R := B(z, R) \subset \mathbf{R} \times W, \tag{3.19}$$

where W is an infinite tripod, and $z \in \mathbf{R} \times W$ lies on the singular line.

Proof. Let $\pi : Y \to S \setminus B(p, r_0)$ be the universal covering map. Since $S \setminus B(p, r_0)$ is homeomorphic to $\mathcal{G} \times [0, \infty)$, the covering map π is equivalent to the product of the universal covering $\tilde{\mathcal{G}} \to \mathcal{G}$ with the identity map $[0, \infty) \to [0, \infty)$. Since $\mathcal{G}$ contains a branch point, we may find a proper embedding $\phi : V \to \tilde{\mathcal{G}}$ of a tripod V into $\tilde{\mathcal{G}}$. Consider the map ψ given by the composition

$$V \times [0, \infty) \longrightarrow \tilde{\mathcal{G}} \times [0, \infty) \longrightarrow \mathcal{G} \times [0, \infty) \simeq S \setminus B(p, r_0).$$

We may put a locally CAT(0) metric on $V \times (0, \infty)$ by pulling back the metric from $S \setminus B(p, r_0)$. For each of the three "rays" $\gamma_i \subset V$ whose union is V, the metric on $\gamma_i \times (0, \infty)$ is locally isometric to a flat metric with geodesic boundary. It follows from a standard argument that if $y \in V \times (0, \infty)$ lies on the singular locus and $\psi(y)$ lies outside $B(p, r_0 + R)$, then the R-ball in $V \times (0, \infty)$ is isometric to B_R as in (3.19). Since ψ is a locally isometric map of a CAT(0) space into a CAT(0) space, it is an isometric embedding. □

4. Quasi-flats in 2-complexes

In this section, X will denote a piecewise flat, proper CAT(0) 2-complex with isolated suspensions.

Theorem 4.1. *Let $Q \subset X$ be an (L, A)-quasiflat. Then there is a nontrivial quadratic growth, locally finite homology class $\sigma \in H_2^{\mathrm{lf}}(X)$ whose support set $S \subset X$ is at Hausdorff distance at most $D = D(L, A)$ from Q, with the following property.*

1. For every $p \in X$, there is an $r_0 \in [0, \infty)$ such that $S \setminus \overline{B(p, r_0)}$ is locally isometric to $\mathbf{R}^2$.

2. S is asymptotically conical, in the following sense. For every $p \in X$ and every $\epsilon > 0$, there is an $r_1 \in [r_0, \infty)$ such that if $x \in S \setminus B(p, r_1)$, then the angle at x between the geodesic segment $\overline{xp}$ and S is $< \epsilon$, and the map $S \setminus B(p, r_1) \to [r_0, \infty)$ given by the distance function from p is a fibration with circle fiber.

3. If the area growth of S is Euclidean, i.e.

$$\frac{\mathrm{Area}(B(p, r) \cap S)}{\pi r^2} \to 1$$

as $r \to \infty$, then S is a 2-flat.

Proof. Using a standard argument, we may assume without loss of generality (and at the cost of some deterioration in quasi-isometry constants which will be suppressed), that Q is the image of a C-Lipschitz (L, A)-quasi-isometric embedding $f : \mathbf{R}^2 \to X$, where $C = C(L, A)$. The mapping f is proper, and hence induces a homomorphism $f_* : H_2^{\mathrm{lf}}(\mathbf{R}^2) \to H_2^{\mathrm{lf}}(X)$ of locally finite homology groups. We define S to be the support set of the image of the fundamental class of $\mathbf{R}^2$ under f_*:

$$S := \mathrm{supp}\,(f_*([\mathbf{R}^2])) \subset \mathrm{Im}(f) = Q. \tag{4.2}$$

Lemma 4.3. *There are constants $D = D(L, A)$ and $a = a(L, A)$ such that such that:*

1. The Hausdorff distance between S and Q is at most D.

2. For every $p \in X$, the area of $B(p, r) \cap S$ is at most $a\,(1 + r)^2$.

Proof. Using the uniform contractibility of $\mathbf{R}^2$, one may construct a proper map $g : Q \to \mathbf{R}^2$ such that $d(g \circ f, \mathrm{id}_{\mathbf{R}^2})$ is bounded by a function of (L, A). In particular, the composition of proper maps

$$\mathbf{R}^2 \xrightarrow{f} Q \xrightarrow{g} \mathbf{R}^2$$

is properly homotopic to $\mathrm{id}_{\mathbf{R}^2}$. Hence $(g \circ f)_*([\mathbf{R}^2]) = [\mathbf{R}^2]$, so $\mathrm{supp}((g \circ f)_*([\mathbf{R}^2])) = \mathbf{R}^2$. On the other hand

$$\mathrm{supp}((g \circ f)_*([\mathbf{R}^2])) \subset\ g(S),$$

which implies that $Q = \mathrm{Im}(f)$ is contained in a controlled neighborhood of S.

The last assertion follows from the fact that $S \subset Q$ and Q has quadratic area growth, being the image of a Lipschitz quasi-isometric embedding. □

Therefore Theorem 3.11 applies to S, and by part 2, we get a fibration

$$d_p : S \setminus B(p, r_0) \to [r_0, \infty)$$

whose fiber is homeomorphic to a finite graph $\mathcal{G}$ all of whose vertices have valence ≥ 2. If $\mathcal{G}$ had a branch point, we could apply Lemma 3.18, contradicting the fact that S is a quasi-flat. Thus S is locally isometric to $\mathbf{R}^2$ outside $B(p, r_0)$. □

5. Square complexes

In this section X will be a locally finite CAT(0) square complex with isolated suspensions.

Remark 5.1. It is not difficult to show that if $\mathcal{F}$ is the collection of CAT(1) graphs Γ all of whose edges have length $\frac{\pi}{2}$, then $\mathcal{F}$ has isolated suspensions. In particular, any CAT(0) square complex has isolated suspensions. However, we will not need this fact for our primary applications, so we omit the proof.

Theorem 5.2. *Let $\sigma \in H_2^{\mathrm{lf}}(X)$ be a quadratic growth locally finite homology class whose support set S is a quasiflat. Then there is a finite collection $\{H_1, \ldots, H_k\}$ of half-plane subcomplexes contained in S, and a finite subcomplex $W \subset S$ such that*

$$S\ =\ W \cup (\cup_i\ H_i)\ .$$

Proof. Pick $p \in X$ and $\epsilon \in (0, \frac{\pi}{2})$. Let r_1 be as in Theorem 4.1, and set

$$Y_1 := S \setminus B(p, r_1).$$

Then Y_1 is a complete flat Riemannian surface with concave boundary $\partial Y_1 = S(p, r_0) \cap Y_1$.

Now pick $\alpha \in (0, \frac{\pi}{8})$, $r_2 \in [r_1, \infty)$, and let $Y_2 := S \setminus B(p, r_2)$.

Lemma 5.3. *Provided r_2 is sufficiently large (depending on α), for every $x \in Y_2$, and every semi-circle $\tau \subset \Sigma_x S$ such that*

$$d(\tau, \log_x p) > \alpha,$$

there is a subset $Z \subset S$ isometric to a Euclidean half-plane, such that $\Sigma_x Z = \tau$.

Proof. First suppose $y \in Y_2$, and $v \in \Sigma_y S$ is a tangent vector such that $\angle_y(v, \log_y p) > \alpha$. Provided $r_2 \sin \alpha > r_1$, there will be a unique geodesic ray $\gamma_v \subset S$ starting at y with direction v; this follows from a continuity argument, since triangle comparison implies that any geodesic segment with initial direction v remains outside $B(p, r_1)$.

If $\tau \subset \Sigma_x S$ is a semi-circle (i.e. a geodesic segment of length π), and $\angle_x(\tau, \log_x p) > \alpha$, then the union of the rays γ_v, for $v \in \tau$, will form a subset of S isometric to a Euclidean half-plane. □

Proof of Theorem 5.2 continued. We now assume that r_2 is large enough that Lemma 5.3 applies.

Our next step is to construct a finite collection of half-planes in S.

Consider the boundary ∂Y_2. This is the frontier of the set $K := S \cap \overline{B(p, r_2)}$ in S. Since K is locally convex near $\partial K = \partial Y_2$, it follows that for each $x \in \partial Y_2$, there is a well-defined space of directions $\Sigma_x K$, which consists of the directions $v \in \Sigma_x S$ such that $\angle_x(v, \log_x p) \leq \frac{\pi}{2}$. Also, there is a normal space $\nu_x K \subset \Sigma_x S$ consisting of the directions $v \in \Sigma_x S$ making an angle at least $\frac{\pi}{2}$ with $\Sigma_x K$. When ϵ is small, the angle $\angle_x(\log_x p, \Sigma_x S)$ is small, and hence $\pi - \angle_x(v, \log_x p)$ will be small for every $v \in \nu_x K$. In particular, when ϵ is small, for every $v \in \nu_v K$ there will be a semi-circle $\tau_v \subset \Sigma_x S$ such that

1. τ_v makes an angle at least $\frac{\pi}{8}$ with $\log_x p$.

2. If $Z_v \subset S$ is the subset obtained by applying Lemma 5.3 to τ_v, then the boundary of Z_v is parallel to one of the sides of a square $P \subset S$ which contains x.

3. The angle between ∂Z_v and v is at least $\frac{\pi}{8}$.

We let $H_v \subset Z_v$ be the largest half-plane subcomplex of Z_v. It follows from property 2 that H_v may be obtained from Z_v by removing a strip of thickness < 1 around ∂Z_v.

Now let $\mathcal{H}$ be the collection of all half-planes obtained this way, where x ranges over ∂Y_2, and $v \in \nu_x K$. Observe that this is a finite collection, since each $H \in \mathcal{H}$ has a boundary square lying in $B(p, 1 + r_2)$, and two half-planes $H, H' \in \mathcal{H}$ sharing a boundary square must be the same.

We now claim that

$$S \setminus \cup_{H \in \mathcal{H}} H$$

is contained in $\overline{B(p, r_2 + \sec \frac{\pi}{8})}$. To see this note that if $y \in Y_2$, then there is a shortest path in S from y to K. Since S is locally convex, this path will be a geodesic segment $\overline{yx}$ in X, where $x \in \partial Y_2$. Let $v := \log_x y \in \Sigma_x S$. Then $\overline{yx}$ is contained in Z_v, and in view of condition 3 above, all but an initial segment of length at most $\sec \frac{\pi}{8}$ will be contained in $H_v \subset Z_v$. The claim follows. $\square$

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