Dept. of Math. University of Oslo
Pure Mathematics No. 31
ISSN 0806–2439 October 2004

QUASILINEAR ANISOTROPIC DEGENERATE PARABOLIC EQUATIONS WITH TIME-SPACE DEPENDENT DIFFUSION COEFFICIENTS

GUI-QIANG CHEN AND KENNETH H. KARLSEN

Abstract. We study the well-posedness of discontinuous entropy solutions to quasilinear anisotropic degenerate parabolic equations with explicit (t, x)–dependence:

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i} f_i(u, t, x) = \sum_{i,j=1}^{d} \partial_{x_j} \left(a_{ij}(u, t, x)\partial_{x_i} u\right),$$

where $a(u, t, x) = (a_{ij}(u, t, x)) = \sigma^a(u, t, x)\sigma^a(u, t, x)^\top$ is nonnegative definite and each $x \mapsto f_i(u, t, x)$ is Lipschitz continuous. We establish a well-posedness theory for the Cauchy problem for such degenerate parabolic equations via Kružkov's device of doubling variables, provided $\sigma^a(u, t, \cdot) \in W^{2,\infty}$ for the general case and the weaker condition $\sigma^a(u, t, \cdot) \in W^{1,\infty}$ for the case that a is a diagonal matrix. We also establish a continuous dependence estimate for perturbations of the diffusion and convection functions.

1. Introduction

We are concerned with quasilinear anisotropic degenerate parabolic equations with explicit (t, x)–dependence:

$$\partial_t u + \mathrm{div} f(u, t, x) = \mathrm{div}\left(a(u, t, x)\nabla u\right) + s(u, t, x), \tag{1.1}$$

where $(t, x) \in Q_T := (0, T) \times \mathbf{R}^d$ and $T > 0$ is a fixed final time. Equation (1.1) is subject to initial data

$$u(0, x) = u_0(x), \tag{1.2}$$

where, for example, u_0 belongs to $L^1 \cap L^\infty(\mathbf{R}^d)$. In (1.1), $f = (f_1, \dots, f_d)$ is a given vector–valued flux function, s is a given scalar source/sink term, and $a = (a_{ij})$ is a given symmetric matrix–valued diffusion function of the form

$$\begin{cases} a(u, t, x) = \sigma^a(u, t, x)\sigma^a(u, t, x)^\top \geq 0, \\ \sigma^a(u, t, x) \in \mathbf{R}^{d\times K}, \quad 1 \leq K \leq d. \end{cases} \tag{1.3}$$

More explicitly, the components of a read

$$a_{ij}(u, t, x) = \sum_{k=1}^{K} \sigma^a_{ik}(u, t, x)\sigma^a_{jk}(u, t, x), \qquad i, j = 1, \dots, d.$$

Nonnegativity of the matrix $a(u, t, x)$ is interpreted in the usual sense:

$$\sum_{i,j=1}^{d} a_{ij}(u, t, x)\lambda_i\lambda_j \geq 0, \qquad \forall\ \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbf{R}^d$$

for each $u \in \mathbf{R}$ and each $(t, x) \in Q_T$. Precise regularity conditions on the functions f, s, σ^a are given in Section 3.

Since (1.1) is allowed to be strongly degenerate, solutions are not necessarily smooth and weak solutions must be sought. As $a(\cdot, t, x)$ is allowed to be zero on an interval of solution values, weak solutions can be discontinuous and are not uniquely determined by their initial data. Thus one

Date: today.
1991 *Mathematics Subject Classification.* 35K65, 35B35, 35G25, 35D99.
Key words and phrases. degenerate parabolic equations, quasilinear, entropy solutions, stability, uniqueness, continuous dependence, existence.

needs to work within a suitable framework of entropy solutions. Although the existence problem for entropy solutions in the BV class was largely settled by the work of Vol'pert and Hudjaev [36], the uniqueness problem for such solutions of (1.1) remained open.

In this respect, the isotropic diffusion case has received much attention in recent years. The term isotropic means that σ^a takes the form

$$\sigma^a(u,t,x) = \sigma(u,t,x)I$$

for some scalar function $\sigma(u,t,x)$, where I denotes the identity matrix in $\mathbf{R}^{d\times d}$. In the isotropic diffusion case, some general uniqueness results for entropy solutions were proved in the one-dimensional context by Wu and Yin [37] (cf. [38]) and Bénilan and Touré [4]. In the multidimensional (isotropic) context, a general uniqueness result for the homogeneous Dirichlet problem is due to Carrillo [12, 11], who used Kružkov's device of doubling variables [27]. By now, it is clear that one needs to take into account some form of parabolic dissipation when attempting to apply Kružkov's method to second order equations. In the isotropic case, this can be achieved in two ways: One way is to recover a proper form of parabolic dissipation from the Kružkov entropy inequality (or simply the weak form of the equation), which was developed by Carrillo [12, 11]. The second way is to explicitly include a proper form of parabolic dissipation on the right-hand side of the entropy inequality, which was introduced in Chen and DiBenedetto [14]. In [24, 26], Carrillo's approach was adapted to the Cauchy problem and several results on the uniqueness, L^1 contraction, and continuous dependence were proved for more general equations. To mention just a few examples, other extensions can be found in [8, 23, 29, 30, 31, 35]. The question of convergence of numerical schemes in the isotropic diffusion case has also been addressed in a series of papers by different authors; see [1, 5, 19, 20, 23, 25, 31, 32] and the references cited therein.

The general anisotropic diffusion case was first treated by Chen and Perthame [16], who developed a notion of (entropy/kinetic) solutions containing a proper form of parabolic dissipation and a certain chain rule property (which is not needed when the diffusion matrix a is a diagonal matrix). In the anisotropic case, in fact even when a is a diagonal matrix with different components on the diagonal (see (1.4) below), it seems necessary to explicitly include this form in the notion of solutions. The uniqueness of entropy solutions in L^1 was proved in [16] by developing a kinetic formulation and using the regularization by convolution. One advantage of the kinetic formulation is that an L^1 theory for possibly unbounded solutions can be constructed. A macroscopic understanding of the Chen-Perthame "pure" L^1 theory has been developed by Bendahmane and Karlsen [2] based on the Kružkov device of doubling variables and a notion of renormalized entropy solutions. In this paper we will follow the Kružkov approach developed in [2].

There are also some other recent papers dealing with the anisotropic diffusion case. In [33], the relation between dissipative solutions and entropy solutions is studied, and the convergence of certain relaxation approximations is established. In [15], we introduced a kinetic framework for deriving explicit continuous dependence estimates and convergence rates for approximate entropy solutions (see Section 3 for a further discussion of such estimates).

Quasilinear parabolic problems containing the combined effects of nonlinear convection, degenerate diffusion, and nonlinear reaction occur in a broad spectrum of applications, including flow in porous media (see the discussion and references in [19]) and sedimentation-consolidation processes [9]. In these applications, however, the convective and diffusive terms typically depend explicitly on the spatial position x. They often take the form

$$\partial_t u + \mathrm{div} f(u,t,x) = \sum_{i=1}^{d} \partial_{x_i}\left(a_{ii}(u,t,x)\partial_{x_i} u\right) + s(u,t,x), \tag{1.4}$$

where

$$\sigma^a = \mathrm{diag}(\sigma^a_{11},\dots,\sigma^a_{dd}) \geq 0, \qquad a_{ii}(u,t,x) = (\sigma^a_{ii}(u,t,x))^2.$$

We refer to (1.4) as the quasi-isotropic case. Actually, as we will see later, (1.4) constitutes an important special case of (1.1) for which the uniqueness results can be obtained under regularity conditions on σ^a that are weaker than those needed for the anisotropic case (1.1).

Most of the results mentioned above require that the diffusion coefficients do not explicitly depend on the spatial variable x, which greatly restricts the range of applications. An exception is the work of Karlsen and Ohlberger [24], which treats (1.4) with

$$a_{ii}(u,t,x) = K_i(t,x)\beta(u), \quad 0 \le \beta \in L^\infty_{\mathrm{loc}}(\mathbf{R}), \quad i = 1,\dots,d,$$

where $K(x) = \mathrm{diag}(K_1(x),\dots,K_d(x))$ is bounded away from zero and satisfies some strong integrability and regularity conditions ensuring in particular that $K(x)\beta(u)\nabla u$ belongs to $L^1(Q_T;\mathbf{R}^d)$. The method of proof in [24] follows the approach of Carrillo [12]. As this approach attempts to recover the parabolic dissipation term from Kružkov's entropy inequality, it seems difficult to remove the restrictive assumptions made on $a_{ii}(u,t,x)$. See also [13] for a discussion of the difficulties of implementing this approach of doubling variables in the context of elliptic-parabolic problems with x–dependent second order terms. In summary, the applicability of Kružkov's method in the case of x–dependent second order operators has not been entirely clear. This fact has motivated the present paper.

In this paper, by a careful use of the device of doubling variables, along the lines of [2], we prove the uniqueness of entropy solutions for degenerate parabolic equations with fairly general x–dependent diffusion coefficients. To prove the uniqueness, we employ an approximation of the symmetric Kružkov entropies $|\cdot - c|$, $c \in \mathbf{R}$, and the following entropy inequality (see Sections 3 and 4 for precise statements):

$$\begin{aligned} &\partial_t |u-c| + \sum_{i=1}^d \partial_{x_i}\Big(\mathrm{sign}(u-c)\left(f_i(u,t,x) - f_i(c,t,x)\right)\Big) \\ &- \sum_{i,j=1}^d \partial^2_{x_i x_j}\Big(\mathrm{sign}(u-c)\left(A_{ij}(u,t,x) - A_{ij}(c,t,x)\right)\Big) \\ &- \mathrm{sign}(u-c)s(u,t,x) + \sum_{i=1}^d \mathrm{sign}(u-c) f_{i,x_i}(u,t,x) \\ &+ \sum_{i,j=1}^d \partial_{x_j}\Big(\mathrm{sign}(u-v)\left(A_{ij,x_i}(u,t,x) - A_{ij,x_i}(c,t,x)\right)\Big) \\ &\le -\mathrm{sign}'(u-c)\sum_{i,j=1}^d a_{ij}(u,t,x)\partial_{x_i}u\partial_{x_j}u \qquad \text{in } \mathcal{D}(Q_T), \end{aligned} \tag{1.5}$$

where

$$A_{ij,u}(u,t,x) = a_{ij}(u,t,x). \tag{1.6}$$

Notice that there is a parabolic dissipation term explicitly included in (1.5). Even in the isotropic case, in contrast to [12, 24, 26], we insist on using (1.5). As a consequence, our results are significantly more general than those in [24]. In the anisotropic case, we include a chain rule property in the notion of entropy solutions, as was done first in [16].

Although some of the underlying ideas, such as using the parabolic dissipation term, are already present in the proofs of similar results when the coefficients are independent of (t,x), which have been developed in recent years, the proofs of our main theorems are technically cumbersome and require some new techniques to carry through. Also, it is important to have a quotable theorem with precise conditions and proofs for the (t,x)–dependent case. Furthermore, as we explain below, the conditions for uniqueness are different in the quasi-isotropic case (1.4) and the anisotropic case (1.1). In addition to the uniqueness and existence result, we also provide an explicit estimate for continuous dependence on the nonlinearities in (1.4) with solutions having BV regularity in the spatial variable. This result is relevant to the theory of flow through porous media.

We conclude this introduction by providing more precise conditions under which our main uniqueness result holds. Roughly speaking, the basic condition is that the functions f, s, and σ^a are globally Lipschitz continuous in x. At least for the quasi-isotropic case (1.4), we are able to prove the uniqueness of entropy solutions under this condition. However, in the full anisotropic case (1.1), we need more regularity on $x \mapsto \sigma^a(u,t,x)$ than just $W^{1,\infty}$. Namely, we need

$$x \mapsto \sigma^a(u,t,x) \in W^{2,\infty}.$$

To explain briefly the reason why different requirements are needed for uniqueness in the quasi-isotropic case (1.4) and the anisotropic case (1.1), we introduce the following smoothness index, which occurs naturally in the proofs:

$$\begin{aligned} &\epsilon_{ij}(\xi,t,x,s,y) \\ &:= \sum_{k=1}^{K}\left\{\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk}(\xi,t,x) - 2\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk}(\xi,s,y) + \sigma^a_{ik}(\xi,s,y)\sigma^a_{jk}(\xi,s,y)\right\} \\ &= a_{ij}(\xi,t,x) - 2\sum_{k=1}^{K}\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk}(\xi,s,y) + a_{ij}(\xi,s,y) \end{aligned} \tag{1.7}$$

for $\xi \in \mathbf{R}$, $(t,x) \in Q_T$, $(s,y) \in Q_T$, and $1 \le i,j \le d$. With our method of proof, the uniqueness follows from the mere assumption that $\sigma^a(u,t,\cdot) \in W^{1,\infty}$ if the components of the matrix

$$\epsilon(\xi,t,x,t,y) = (\epsilon_{i,j}(\xi,t,x,t,y))$$

behave like $|x-y|^2$ and their first order partial derivatives with respect to the spatial variables behave like $|x-y|$. However, in general, this is only true for the diagonal elements of ϵ, since

$$\epsilon_{ii}(\xi,t,x,s,y) = \sum_{k=1}^{K}\left(\sigma^a_{ik}(\xi,t,x) - \sigma^a_{ik}(\xi,s,y)\right)^2.$$

This explains in a nutshell why the uniqueness follows in the quasi-isotropic case (1.4) if σ^a is globally Lipschitz continuous in x.

In the anisotropic case (1.1), we have at our disposal only the weaker fact

$$\begin{aligned} &\sum_{i,j=1}^{d}\partial^2_{x_iy_j}\Psi(x-y)\epsilon_{ij}(\xi,t,x,s,y) \\ &= \sum_{k=1}^{K}\sum_{i,j=1}^{d}\partial^2_{x_iy_j}\Psi(x-y)\Big(\sigma^a_{ik}(\xi,t,x) - \sigma^a_{ik}(\xi,s,y)\Big)\Big(\sigma^a_{jk}(\xi,t,x) - \sigma^a_{jk}(\xi,s,y)\Big) \end{aligned}$$

for any C^2 function $\Psi : \mathbf{R}^d \to \mathbf{R}$, which is still sufficient to produce the required $|x-y|^2$ behavior. Notice that the above fact follows from the symmetry property (in i and j):

$$\partial^2_{x_iy_j}\Psi(x-y) = \partial^2_{x_jy_i}\Psi(x-y).$$

However, an analogous fact does not hold for the first order partial derivatives of ϵ with respect to the spatial variables, which would have produced the required $|x-y|$ behavior. To compensate for this, we need to assume more regularity of $x \mapsto \sigma^a(u,t,x)$. More precisely, we need that

$$\partial_{x_j}\epsilon_{ji}(\xi,t,x,t,y) - \partial_{y_j}\epsilon_{ij}(\xi,t,x,t,y), \qquad i \ne j,$$

behaves like $|x-y|$, and this follows if $x \mapsto \sigma^a(u,t,x)$ is $W^{2,\infty}$ regular.

The remaining part of this paper is organized as follows: We first introduce the notion of entropy solutions in Section 2. We state the main theorems (Theorems 3.1 and 3.2) of this paper in Section 3. We prove Theorem 3.1 in Section 4 and Theorem 3.2 in Section 5. In Section 6, we state a theorem about the existence of entropy solutions. Finally, we discuss a weak formulation of the initial condition in Section 7.

2. Entropy Solutions

For $i = 1, \dots, d$ and $k = 1, \dots, K$, we let

$$\zeta^a_{ik}(u, t, x) = \int_0^u \sigma^a_{ik}(w, t, x)\, dw$$

and

$$\zeta^{a,\psi}_{ik}(u, t, x) = \int_0^u \psi(w)\sigma^a_{ik}(w, t, x)\, dw \qquad \text{for } \psi \in C(\mathbf{R}).$$

Given any convex C^2 entropy function $\eta : \mathbf{R} \to \mathbf{R}$, we define the entropy fluxes

$$q = (q_i) : \mathbf{R} \to \mathbf{R}^d, \qquad r = (r_{ij}) : \mathbf{R} \to \mathbf{R}^{d\times d}$$

by

$$q_u(u, t, x) = \eta'(u) f_u(u, t, x), \qquad r_u(u, t, x) = \eta'(u) a(u, t, x).$$

We refer to (η, q, r) as an *entropy-entropy flux triple.*

We use the following definition of entropy solutions.

Definition 2.1 (Entropy Solutions). *An entropy solution of* (1.1) *is a measurable function $u : Q_T \to \mathbf{R}$ satisfying the following conditions:*

($\mathbf{D}$.1) *Weak Regularity: $u \in L^\infty(Q_T)$, $u - u_0 \in L^\infty(0, T; L^1(\mathbf{R}^d))$, and*

$$\sum_{i=1}^d \Big(\partial_{x_i} \zeta^a_{ik}(u, t, x) - \zeta^a_{ik,x_i}(u, t, x)\Big) \in L^2(Q_T), \qquad k = 1, \dots, K.$$

($\mathbf{D}$.2) *Chain Rule: For $k = 1, \dots, K$,*

$$\begin{aligned}\sum_{i=1}^d &\Big(\partial_{x_i} \zeta^{a,\psi}_{ik}(u, t, x) - \zeta^{a,\psi}_{ik,x_i}(u, t, x)\Big) \\ &= \psi(u) \sum_{i=1}^d \Big(\partial_{x_i} \zeta^a_{ik}(u, t, x) - \zeta^a_{ik,x_i}(u, t, x)\Big)\end{aligned}$$

a.e. in Q_T and in $L^2(Q_T)$, for any $\psi \in C(\mathbf{R})$.

($\mathbf{D}$.3) *Entropy Inequality: For any entropy-entropy flux triple (η, q, r),*

$$\begin{aligned}\partial_t \eta(u) &+ \sum_{i=1}^d \partial_{x_i} q_i(u, t, x) - \sum_{i,j=1}^d \partial^2_{x_i x_j} r_{ij}(u, t, x) - \eta'(u) s(u, t, x) \\ &+ \sum_{i=1}^d \Big(\eta'(u) f_{i,x_i}(u, t, x) - q_{i,x_i}(u, t, x)\Big) + \sum_{i,j=1}^d \partial_{x_j} r_{ij,x_i}(u, t, x) \\ &\le -\eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \Big(\partial_{x_i} \zeta^a_{ik}(u, t, x) - \zeta^a_{ik,x_i}(u, t, x)\Big)\right)^2 \quad \text{in } \mathcal{D}'(Q_T).\end{aligned} \tag{2.1}$$

($\mathbf{D}$.4) *Initial Condition: The initial condition is assumed in the following strong L^1 sense:* $\operatorname{ess\,lim}_{t\downarrow 0} \|u(t,\cdot) - u_0\|_{L^1(\mathbf{R}^d)} = 0$.

Remark 2.1. In [16], it was pointed out that the chain rule ($\mathbf{D}$.2) should be included in the definition of entropy solutions in the anisotropic diffusion case. It is, however, automatically fulfilled when a is a diagonal matrix, as argued in [16]. This applies to (1.4), in which case ($\mathbf{D}$.2) can be deleted from Definition 2.1.

Remark 2.2. Regarding ($\mathbf{D}$.4), we refer to Section 7 for a discussion of a weaker formulation of the initial condition.

Remark 2.3. A usual assumption on the initial function u_0 is that it belongs to $L^1 \cap L^\infty(\mathbf{R}^d)$. However, for our well-posedness theory, the L^1 requirement on u_0 can be replaced by more general conditions so that the corresponding solution $u(t,x)$ satisfies $u - u_0 \in L^\infty(0,T;L^1(\mathbf{R}^d))$. Thus, we include this condition in (**D**.1), which traditionally would read $u \in L^\infty(0,T;L^1(\mathbf{R}^d)) \cap L^\infty(Q_T)$.

3. Main Theorems

Throughout this paper, we suppose that the entropy solutions in question take values in a fixed closed bounded interval $I \subset \mathbf{R}$. The existence of such an interval is ensured by the existence theory in Section 6.

Let us detail the conditions that are imposed on f, s, σ^a in (1.1) and (1.3) to ensure the uniqueness. The vector–valued flux function $f = (f_i) : \mathbf{R} \times Q_T \to \mathbf{R}^d$ satisfies

$$\begin{aligned} &f(u,\cdot,\cdot) \in L^\infty(Q_T;\mathbf{R}^d) \qquad \text{(uniformly in } u \in I),\\ &f(\cdot,t,x) \in W^{1,\infty}(I;\mathbf{R}^d) \qquad \text{(uniformly in } (t,x) \in Q_T),\\ &f(u,t,\cdot) \in W^{1,\infty}(\mathbf{R};\mathbf{R}^d) \qquad \text{(uniformly in } u \in I, t \in [0,T]). \end{aligned} \tag{3.1}$$

The source/sink term $s : \mathbf{R} \times Q_T \to \mathbf{R}$ satisfies

$$\begin{aligned} &s(u,\cdot,\cdot) \in L^\infty(Q_T) \qquad \text{(uniformly in } u \in I),\\ &s(\cdot,t,x) \in W^{1,\infty}(I) \qquad \text{(uniformly in } (t,x) \in Q_T). \end{aligned} \tag{3.2}$$

The diffusion matrix $a = (a_{ij}) : \mathbf{R} \times Q_T \to \mathbf{R}^{d\times d}$ is defined via the matrix-valued function $\sigma^a = (\sigma^a_{ik}) : \mathbf{R} \times Q_T \to \mathbf{R}^{d\times K}$, which satisfies

$$\begin{aligned} &\sigma^a(\cdot,t,x) \in L^\infty(I;\mathbf{R}^{d\times K}) \qquad \text{(uniformly in } (t,x) \in Q_T),\\ &\sigma^a(u,\cdot,\cdot) \in L^\infty(Q_T;\mathbf{R}^{d\times K}) \qquad \text{(uniformly in } u \in I),\\ &\sigma^a(u,t,\cdot) \in W^{2,\infty}(\mathbf{R};\mathbf{R}^{d\times K}) \qquad \text{(uniformly in } u \in I, t \in [0,T]). \end{aligned} \tag{3.3}$$

The Lipschitz regularity of $x \mapsto f(u,t,x)$ has become standard in the context of scalar conservation laws, and it was used in [24] for degenerate parabolic equations. In fact, a weaker one-sided Lipschitz condition on $x \mapsto f(u,t,x)$ is sufficient for the uniqueness: see [10] for scalar conservation laws, [6, 34] for linear transport equations, and [26] for degenerate parabolic equations. We do not pursue this here.

The main new point is the third condition in (3.3) regarding the space regularity of $\sigma^a(u,t,x)$. We mention that, in the quasi-isotropic case (1.4), we may replace this condition by the weaker condition

$$\sigma^a(u,t,\cdot) \in W^{1,\infty}(\mathbf{R};\mathbf{R}^{d\times d}) \qquad \text{(uniformly in } u \in I, t \in [0,T]). \tag{3.4}$$

In Section 4, we prove the following theorem:

Theorem 3.1 (Uniqueness in L^∞)**.** *Suppose* (1.3) *and* (3.1)–(3.3) *hold. Let u, v be entropy solutions of* (1.1) *with initial data $u_0, v_0 \in L^\infty(\mathbf{R}^d)$, respectively, so that $u_0 - v_0 \in L^1(\mathbf{R}^d)$. Then there exists a constant C, depending on T, such that*

$$\|u - v\|_{L^\infty(0,T;L^1(\mathbf{R}^d))} \le C\|u_0 - v_0\|_{L^1(\mathbf{R}^d)}. \tag{3.5}$$

In the quasi-isotropic case (1.4)*, this result holds when the third condition in* (3.3) *is replaced by* (3.4).

We next restrict our attention to the equations of the form

$$\partial_t u + \operatorname{div}(k(x)f(u)) = \sum_{i=1}^d \partial_{x_i}\left(a_{ii}(u,x)\partial_{x_i} u\right), \tag{3.6}$$

where $k = (k_i) : \mathbf{R}^d \to \mathbf{R}^d$, $f : \mathbf{R} \to \mathbf{R}$, and

$$\sigma^a = \operatorname{diag}(\sigma^a_{11}, \dots, \sigma^a_{dd}) \ge 0, \qquad a_{ii}(u,x) = (\sigma^a_{ii}(u,x))^2. \tag{3.7}$$

Equations of this type occur frequently in the theory of flow through porous media. As an example, we consider immiscible two-phase flow of water and oil in a reservoir. Then $k = k(x)$ is a given velocity field coming from the Darcy law, and the x-dependency of $a_{ii}(x,u)$ describes the spatial flow properties (permeability) of the reservoir (see [19] and the references given therein).

We will establish explicit estimates revealing the continuous dependence on the coefficients k, f, and σ^a_{ii}. Therefore, let us also introduce the equation

$$\partial_t v + \operatorname{div}(l(x)g(v)) = \sum_{i=1}^{d} \partial_{x_i}\left(b_{ii}(v,x)\partial_{x_i} v\right), \tag{3.8}$$

where $l = (l_i) : \mathbf{R}^d \to \mathbf{R}^d$, $g : \mathbf{R} \to \mathbf{R}$, and

$$\sigma^b = \operatorname{diag}(\sigma^b_{11}, \dots, \sigma^b_{dd}) \geq 0, \qquad b_{ii}(v,x) = (\sigma^b_{ii}(v,x))^2. \tag{3.9}$$

We want to estimate the L^1 difference between a BV entropy solution $u(t,\cdot)$ of (3.6) and a BV entropy solution $v(t,\cdot)$ of (3.8). To this end, we assume that

$$f, g \in W^{1,\infty}(I); \qquad k, l \in \left(L^\infty \cap W^{1,\infty} \cap W^{1,1}\right)(\mathbf{R}^d; \mathbf{R}^d), \tag{3.10}$$

and

$$\begin{aligned} &\sigma^a(\cdot,x), \sigma^b(\cdot,x) \in L^\infty(I; \mathbf{R}^{d\times K}) \qquad (\text{uniformly in } x \in \mathbf{R}^d),\\ &\sigma^a(u,\cdot), \sigma^b(u,\cdot) \in W^{1,\infty}(\mathbf{R}; \mathbf{R}^{d\times d}) \qquad (\text{uniformly in } u \in I). \end{aligned} \tag{3.11}$$

In Section 5, we prove the following continuous dependence result.

Theorem 3.2 (Continuous Dependence in BV)**.** *Suppose* (3.7), (3.9), (3.10), *and* (3.11) *hold. Let $u, v \in L^\infty(0,T; BV(\mathbf{R}^d))$ be entropy solutions of* (3.6), (3.8), *respectively, with initial data*

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \qquad u_0, v_0 \in L^\infty \cap BV(\mathbf{R}^d),\ u_0 - v_0 \in L^1(\mathbf{R}^d).$$

With u, v taking values in the closed interval $I \subset \mathbf{R}$, we have, for any $t \in (0,T)$,

$$\begin{aligned} &\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbf{R}^d)}\\ &\leq \|u_0 - v_0\|_{L^1(\mathbf{R}^d)} + C_1\, t \Bigg(\max_{i=1,\dots,d} \|k_i - l_i\|_{L^\infty(\mathbf{R}^d)} + \max_{i=1,\dots,d} |k_i - l_i|_{W^{1,1}(\mathbf{R}^d)}\\ &\qquad\qquad + \|f - g\|_{L^\infty(I)} + |f - g|_{W^{1,\infty}(I)} \Bigg)\\ &\quad + C_2\sqrt{t}\, \left\| \sigma^a(\cdot,\cdot,\cdot) - \sigma^b(\cdot,\cdot,\cdot) \right\|_{L^\infty(I\times Q_T; \mathbf{R}^{d\times d})}, \end{aligned} \tag{3.12}$$

for some constants C_1 and C_2 that depend on, among other quantities, the minimum value of $\|u\|_{L^\infty(0,T;BV(\mathbf{R}^d))}$ and $\|v\|_{L^\infty(0,T;BV(\mathbf{R}^d))}$.

If the right-hand sides of (3.6) *and* (3.8) *are replaced respectively by the anisotropic (t,x)–independent operators*

$$\sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij}(u)\partial_{x_i} u\right), \qquad a(u) = (a_{ij}(u)) = \sigma^a(u)\sigma^a(u)^\top \geq 0$$

with $\sigma^a \in L^\infty(I; \mathbf{R}^{d\times K})$, $1 \leq K \leq d$, and

$$\sum_{i,j=1}^{d} \partial_{x_j}\left(b_{ij}(v)\partial_{x_i} v\right), \qquad b(v) = (b_{ij}(v)) = \sigma^b(v)\sigma^b(v)^\top \geq 0$$

with $\sigma^b \in L^\infty(I; \mathbf{R}^{d\times K})$, $1 \leq K \leq d$, then estimate (3.12) *still holds, except that the last term is changed to*

$$C_2\sqrt{t}\sqrt{\left\| \left(\sigma^a - \sigma^b\right)\left(\sigma^a - \sigma^b\right)^\top \right\|_{L^\infty(I;\mathbf{R}^{d\times d})}}.$$

Remark 3.1. As σ^a and σ^b are diagonal matrices in the first part of Theorem 3.2,

$$\left\| \sigma^a(\cdot,\cdot,\cdot) - \sigma^b(\cdot,\cdot,\cdot) \right\|_{L^\infty(I\times Q_T;\mathbf{R}^{d\times d})} = \max_{i=1,\dots,d} \left\| \sigma^a_{ii}(\cdot,\cdot,\cdot) - \sigma^b_{ii}(\cdot,\cdot,\cdot) \right\|_{L^\infty(I\times Q_T)}.$$

Remark 3.2. Although we do not bother to do so here, the precise (optimal) form of the constants C_1 and C_2 can be traced from the proof in Section 5.

Remark 3.3. Compared with the existing literature on explicit continuous dependence estimates, the main new point in the first part of Theorem 3.2 is that the diffusion coefficients σ^a_{ii} and σ^b_{ii} depend on the spatial variable x. We recall that results on continuous dependence on the flux function in scalar conservation laws ($k, l \equiv 1$, $a_{ii}, b_{ii} \equiv 0$) have been obtained in [28, 7]. In the isotropic diffusion case ($k, l \equiv 1$ and $a_{ii} \equiv a$, $b_{ii} \equiv b$ for some scalar functions $a, b \geq 0$), explicit continuous dependence estimates were first obtained in [17] in the framework of semigroup solutions (see [3] for an earlier but non-explicit result when there is no convection term). Still in the isotropic diffusion case, generalizations to the situation where k, l vary with x can be found in [26, 21]. Finally, continuous dependence estimates for the anisotropic diffusion case with (t, x)–independent coefficients were proved recently in [15]. The second part of Theorem 3.2 generalizes the result in that paper to the situation where the convective flux depends on the spatial variable x. Moreover, in [15], we used the kinetic formulation and regularization by convolution, whereas herein we use the method of doubling variables. A novelty of the proof in [15] (and also the one herein) is that it works directly with the entropy solutions, rather than some approximations (like semigroup or vanishing viscosity).

Remark 3.4. Theorem 3.2 with $l = k$, $g = f$, and $\sigma^b = \sigma^a$ implies that an L^1 *contraction property* holds for entropy solutions in the class $L^\infty(0, T; BV(\mathbf{R}^d))$, which should be compared with Theorem 3.1.

4. Proof of Theorem 3.1

We need a C^1 approximation of $\operatorname{sign}(\cdot)$ and thus a C^2 approximation of the Kružkov entropy flux $|\cdot - c|$, $c \in \mathbf{R}$.

For $\varepsilon > 0$, set

$$\operatorname{sign}_\varepsilon(\xi) = \begin{cases} -1, & \xi < -\varepsilon, \\ \sin\left(\frac{\pi}{2\varepsilon}\xi\right), & |\xi| \leq \varepsilon, \\ 1, & \xi > \varepsilon. \end{cases} \tag{4.1}$$

For each $c \in \mathbf{R}$, the corresponding entropy function

$$u \mapsto \eta_\varepsilon(u, c) = \int_c^u \operatorname{sign}_\varepsilon(\xi - c)\, d\xi$$

is convex, belongs to $C^2(\mathbf{R})$, and $\eta''_\varepsilon \in C_c(\mathbf{R})$. Moreover, η_ε is symmetric in the sense that $\eta_\varepsilon(u, c) = \eta_\varepsilon(c, u)$ and

$$\eta_\varepsilon(u, c) \to \eta(u, c) := |u - c| \qquad \text{as } \varepsilon \downarrow 0, \text{ for all } u \in \mathbf{R}.$$

For each $c \in \mathbf{R}$ and $1 \leq i, j \leq d$, we define the entropy flux functions

$$u \mapsto q^\varepsilon_i(u, c, t, x) = \int_c^u \operatorname{sign}_\varepsilon(\xi - c)\, f'_i(\xi, t, x)\, d\xi,$$

$$u \mapsto r^\varepsilon_{ij}(u, c, t, x) = \int_c^u \operatorname{sign}_\varepsilon(\xi - c)\, a_{ij}(\xi, t, x)\, d\xi.$$

Then, as $\varepsilon \downarrow 0$,

$$\begin{aligned} q^\varepsilon_i(u, c, t, x) &\to q_i(u, c) := \operatorname{sign}(u - c)\left(f_i(u, t, x) - f_i(c, t, x)\right), \\ r^\varepsilon_{ij}(u, c, t, x) &\to r_{ij}(u, c, t, x) := \operatorname{sign}(u - c)\left(A_{ij}(u, t, x) - A_{ij}(c, t, x)\right), \end{aligned} \tag{4.2}$$

where A_{ij} is defined in (1.6). Let $q^\varepsilon = (q^\varepsilon_i)$, $r^\varepsilon = \left(r^\varepsilon_{ij}\right)$, and similarly for q, r.

Following [2], we use Kružkov's method of doubling variables. Since our main new point is the x–dependency in the diffusion coefficients, we set $s \equiv 0$, $f \equiv 0$ (and consequently $q^\varepsilon \equiv 0$). These terms can be included into the analysis by copying the arguments from [24, 26] (we leave the details to the reader).

From the entropy inequality for $u = u(t, x)$,

$$
\begin{aligned}
\int \Bigg(& \eta_\varepsilon(u, c)\partial_t \phi \\
& - \sum_{i,j=1}^{d} \operatorname{sign}_\varepsilon(u - c) \Big(\partial_{x_i} A_{ij}(u, t, x) - A_{ij,x_i}(u, t, x)\Big)\partial_{x_j}\phi \Bigg)\, dx\, dt \\
\geq \int & \operatorname{sign}'_\varepsilon(u - c) \sum_{k=1}^{K} \left(\sum_{i=1}^{d} \Big(\partial_{x_i}\zeta^a_{ik}(u, t, x) - \zeta^a_{ik,x_i}(u, t, x)\Big)\right)^2 \phi \, dx\, dt
\end{aligned}
\tag{4.3}
$$

for any $c \in \mathbf{R}$ and any $\phi = \phi(t, x) \in \mathcal{D}(Q_T)$ with $\phi \geq 0$. When we find it notationally convenient, we drop the domain of integration, as we did in (4.3).

From the entropy inequality for $u = u(s, y)$,

$$
\begin{aligned}
\int \Bigg(& \eta_\varepsilon(v, c)\partial_s \phi \\
& - \sum_{i,j=1}^{d} \operatorname{sign}_\varepsilon(v - c) \Big(\partial_{y_i} A_{ij}(v, s, y) - A_{ij,y_i}(v, s, y)\Big)\partial_{y_j}\phi \Bigg)\, dx\, dt \\
\geq \int & \operatorname{sign}'_\varepsilon(v - c) \sum_{k=1}^{K} \left(\sum_{j=1}^{d} \Big(\partial_{y_j}\zeta^a_{jk}(v, s, y) - \zeta^a_{jk,y_j}(v, s, y)\Big)\right)^2 \phi \, dy\, ds
\end{aligned}
\tag{4.4}
$$

for any $c \in \mathbf{R}$ and any $\phi = \phi(s, y) \in \mathcal{D}(Q_T)$ with $\phi \geq 0$.

Choose $c = v(s, y)$ in (4.3) and integrate over (s, y). Choose $c = u(t, x)$ in (4.4) and integrate over (t, x). Adding the two resulting inequalities and using the basic inequality $a^2 + b^2 \geq 2ab$ for any two real numbers a, b yields

$$
\begin{aligned}
\int \Bigg(& \eta_\varepsilon(u, v)\,(\partial_t + \partial_s)\,\phi \\
& - \sum_{i,j=1}^{d} \Big[\operatorname{sign}_\varepsilon(u - v) \Big(\partial_{x_i} A_{ij}(u, t, x) - A_{ij,x_i}(u, t, x)\Big)\partial_{x_j}\phi \\
& \quad + \operatorname{sign}_\varepsilon(v - u) \Big(\partial_{y_i} A_{ij}(v, s, y) - A_{ij,y_i}(v, s, y)\Big)\partial_{y_j}\phi \Big] \Bigg)\, dx\, dt\, dy\, ds \\
\geq \int & 2 \operatorname{sign}'_\varepsilon(u - v) \sum_{k=1}^{K} \sum_{i,j=1}^{d} \Big(\partial_{x_i}\zeta^a_{ik}(u, t, x) - \zeta^a_{ik,x_i}(u, t, x)\Big) \\
& \times \Big(\partial_{y_j}\zeta^a_{jk}(v, s, y) - \zeta^a_{jk,y_j}(v, s, y)\Big)\phi \, dx\, dt\, ds\, dy,
\end{aligned}
\tag{4.5}
$$

where $\phi = \phi(t, x, s, y)$ is any nonnegative function in $\mathcal{D}(Q_T \times Q_T)$.

Observe that

$$
\begin{aligned}
& - \operatorname{sign}_\varepsilon(u - v) \Big(\partial_{x_i} A_{ij}(u, t, x) - A_{ij,x_i}(u, t, x)\Big)\partial_{x_j}\phi \\
& \quad + \operatorname{sign}_\varepsilon(u - v) \Big(\partial_{y_i} A_{ij}(v, s, y) - A_{ij,y_i}(v, s, y)\Big)\partial_{y_j}\phi \\
& = -\operatorname{sign}_\varepsilon(u - v) \Big(\partial_{x_i} A_{ij}(u, t, x) - A_{ij,x_i}(u, t, x)\Big)(\partial_{x_j} + \partial_{y_j})\phi \\
& \quad + \operatorname{sign}_\varepsilon(u - v) \Big(\partial_{y_i} A_{ij}(v, s, y) - A_{ij,y_i}(v, s, y)\Big)(\partial_{x_j} + \partial_{y_j})\phi \\
& \quad + \operatorname{sign}_\varepsilon(u - v) \Big(\partial_{x_i} A_{ij}(u, t, x) - A_{ij,x_i}(u, t, x)\Big)\partial_{y_j}\phi \\
& \quad - \operatorname{sign}_\varepsilon(u - v) \Big(\partial_{y_i} A_{ij}(v, s, y) - A_{ij,y_i}(v, s, y)\Big)\partial_{x_j}\phi.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\int \Bigg(& \eta_\varepsilon(u,v)(\partial_t+\partial_s)\phi \\
& - \sum_{i,j=1}^{d} \operatorname{sign}_\varepsilon(u-v) \Big[\Big(\partial_{x_i} A_{ij}(u,t,x) - A_{ij,x_i}(u,t,x) \Big) \\
& \qquad - \Big(\partial_{y_i} A_{ij}(v,s,y) - A_{ij,y_i}(v,s,y) \Big) \Big] (\partial_{x_j}+\partial_{y_j})\phi \Bigg)\, dx\, dt\, dy\, ds \\
\geq {} & E_1(\varepsilon)+E_2(\varepsilon)+E_3(\varepsilon),
\end{aligned} \tag{4.6}
$$

where $E_\ell(\varepsilon) = \int I_\ell(\varepsilon)\, dx\, dt\, dy\, ds$, $l = 1, 2, 3$, with

$$
\begin{aligned}
I_1(\varepsilon) &= 2 \operatorname{sign}'_\varepsilon(u-v) \sum_{k=1}^{K} \sum_{i,j=1}^{d} \Big(\partial_{x_i} \zeta^a_{ik}(u,t,x) - \zeta^a_{ik,x_i}(u,t,x) \Big) \\
&\qquad \times \Big(\partial_{y_j} \zeta^a_{jk}(v,s,y) - \zeta^a_{jk,y_j}(v,s,y) \Big) \phi, \\
I_2(\varepsilon) &= - \sum_{i,j=1}^{d} \operatorname{sign}_\varepsilon(u-v) \Big(\partial_{x_i} A_{ij}(u,t,x) - A_{ij,x_i}(u,t,x) \Big) \partial_{y_j}\phi, \\
I_3(\varepsilon) &= \sum_{i,j=1}^{d} \operatorname{sign}_\varepsilon(u-v) \Big(\partial_{y_i} A_{ij}(v,s,y) - A_{ij,y_i}(v,s,y) \Big) \partial_{x_j}\phi.
\end{aligned}
$$

Pick a function $\delta \in \mathcal{D}(\mathbf{R})$ that satisfies $\delta \geq 0$, $\delta(\sigma) = \delta(-\sigma)$, $\delta(\sigma) = 0$ for $|\sigma| \geq 1$, and $\int_{\mathbf{R}} \delta(\sigma)\, d\sigma = 1$. For $\rho > 0$ and $x \in \mathbf{R}^d$, let $\omega_\rho(x) = \frac{1}{\rho^d} \delta\left(\frac{x_1}{\rho}\right) \cdots \delta\left(\frac{x_d}{\rho}\right)$. With $\rho_0, \rho > 0$, we take our test function $\phi = \phi(t,x,s,y)$ to be of the form

$$
\phi(t,x,s,y) = \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \delta_{\rho_0}\left(\frac{t-s}{2}\right) \omega_\rho\left(\frac{x-y}{2}\right),
$$

where $0 \leq \varphi \in \mathcal{D}(Q_T)$. To simplify the notation, we will often drop the arguments from the displayed functions and for example write $\phi = \varphi\, \delta_{\rho_0} \omega_\rho$. The following cancellation properties will be used repeatedly: $(\partial_t + \partial_s)\, \phi = (\partial_t + \partial_s)\, \varphi\, \delta_{\rho_0} \omega_\rho$ and $(\nabla_x + \nabla_y)\, \phi = (\nabla_x + \nabla_y)\, \varphi\, \delta_{\rho_0} \omega_\rho$.

By the chain rule (**D**.2) in Definition 2.1, followed by a couple of integrations by parts, we have

$$
\begin{aligned}
& - \sum_{i,j=1}^{d} \int \operatorname{sign}_\varepsilon(u-v) \Big[\Big(\partial_{x_i} A_{ij}(u,t,x) - A_{ij,x_i}(u,t,x) \Big) \\
& \qquad - \Big(\partial_{y_i} A_{ij}(v,s,y) - A_{ij,y_i}(v,s,y) \Big) \Big] (\partial_{x_j}+\partial_{y_j})\phi\, dx\, dt\, dy\, ds \\
& = - \sum_{i,j=1}^{d} \int \Bigg[\partial_{x_i} \int_v^u \operatorname{sign}_\varepsilon(\xi - v)\, a_{ij}(\xi,t,x)\, d\xi \\
& \qquad - \int_v^u \operatorname{sign}(\xi - v) a_{ij,x_i}(\xi,t,x)\, d\xi \Bigg] (\partial_{x_j}+\partial_{y_j}) \varphi\, \delta_{\rho_0} \omega_\rho\, dx\, dt\, dy\, ds \\
& \quad - \sum_{i,j=1}^{d} \int \Bigg[\partial_{y_i} \int_u^v \operatorname{sign}_\varepsilon(u - \xi)\, a_{ij}(\xi,s,y)\, d\xi \\
& \qquad - \int_u^v \operatorname{sign}(u - \xi) a_{ij,y_i}(\xi,s,y)\, d\xi \Bigg] (\partial_{x_j}+\partial_{y_j}) \varphi\, \delta_{\rho_0} \omega_\rho\, dx\, dt\, dy\, ds \\
& = \sum_{i,j=1}^{d} \int r^\varepsilon_{ij}(u,v,t,x)
\end{aligned}
$$

$$\times \Big[(\partial^2_{x_j x_i} + \partial^2_{y_j x_i})\varphi\, \delta_{\rho_0} \omega_\rho + (\partial_{x_j} + \partial_{y_j})\varphi\, \delta_{\rho_0} \partial_{x_i} \omega_\rho\Big]\, dx\, dt\, dy\, ds$$
$$+ \sum_{i,j=1}^d \int r^\varepsilon_{ij,x_i}(u,v,t,x)(\partial_{x_j} + \partial_{y_j})\varphi\, \delta_{\rho_0}\omega_\rho\, dx\, dt\, dy\, ds$$
$$+ \sum_{i,j=1}^d \int r^\varepsilon_{ij}(v,u,s,y)$$
$$\times \Big[(\partial^2_{x_j y_i} + \partial^2_{y_j y_i})\varphi\, \delta_{\rho_0} \omega_\rho + (\partial_{x_j} + \partial_{y_j})\varphi\, \delta_{\rho_0} \partial_{y_i} \omega_\rho\Big]\, dx\, dt\, dy\, ds$$
$$+ \sum_{i,j=1}^d \int r^\varepsilon_{ij,y_i}(v,u,s,y)(\partial_{x_j} + \partial_{y_j})\varphi \delta_{\rho_0}\omega_\rho\, dx\, dt\, dy\, ds.$$

Using $r_{ij}(v,u,s,y) = r_{ij}(u,v,s,y)$, $r_{ij,y_i}(v,u,s,y) = r_{ij,y_i}(u,v,s,y)$, and also the property $\partial_{y_i}\omega_\rho = -\partial_{x_i}\omega_\rho$, we deduce

$$\begin{aligned}
&- \lim_{\varepsilon\downarrow 0} \sum_{i,j=1}^d \int \operatorname{sign}_\varepsilon(u-v) \Big[\Big(\partial_{x_i} A_{ij}(u,t,x) - A_{ij,x_i}(u,t,x)\Big) \\
&\qquad - \Big(\partial_{y_i} A_{ij}(v,s,y) - A_{ij,y_i}(v,s,y)\Big)\Big](\partial_{x_j} + \partial_{y_j})\phi\, dx\, dt\, dy\, ds \\
&= \sum_{i,j=1}^d \int \Bigg(r_{ij}(u,v,t,x)(\partial^2_{x_j x_i} + \partial^2_{y_j x_i})\varphi \\
&\qquad + r_{ij}(u,v,s,y)(\partial^2_{x_j y_i} + \partial^2_{y_j y_i})\varphi \Bigg) \delta_{\rho_0}\omega_\rho\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^d \int \Big(r_{ij,x_i}(u,v,t,x) + r_{ij,y_i}(u,v,s,y)\Big) \\
&\qquad \times (\partial_{x_j} + \partial_{y_j})\varphi\, \delta_{\rho_0}\omega_\rho\, dx\, dt\, dy\, ds + R_0(\rho_0,\rho),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
R_0(\rho_0,\rho) = \sum_{i,j=1}^d \int \Big(r_{ij}(u,v,t,x) - r_{ij}(u,v,s,y)\Big)& \\
\times (\partial_{x_j} + \partial_{y_j})\varphi\, \delta_{\rho_0}\partial_{x_i}\omega_\rho\, dx\, dt\, dy\, ds&.
\end{aligned} \tag{4.8}$$

Sending $\varepsilon \downarrow 0$ in (4.6) and using (4.2) and (4.7) yields

$$\begin{aligned}
&\int \Bigg(\eta(u,v)(\partial_t + \partial_s)\varphi \\
&\quad + \sum_{i,j=1}^d \Big(r_{ij}(u,v,t,x)(\partial^2_{x_j x_i} + \partial^2_{y_j x_i})\varphi + r_{ij}(u,v,s,y)(\partial^2_{x_j y_i} + \partial^2_{y_j y_i})\varphi\Big) \\
&\quad + \sum_{i,j=1}^d \Big(r_{ij,x_i}(u,v,t,x) + r_{ij,y_i}(u,v,s,y)\Big)(\partial_{x_j} + \partial_{y_j})\varphi \Bigg) \delta_{\rho_0}\omega_\rho\, dx\, dt\, dy\, ds \\
&\qquad \geq \lim_{\varepsilon\downarrow 0} \Big(E_1(\varepsilon) + E_2(\varepsilon) + E_3(\varepsilon)\Big) + R_0(\rho_0,\rho).
\end{aligned} \tag{4.9}$$

The first goal now is to study the right-hand side of (4.9). By the chain rule (**D**.2) in Definition 2.1 and an integration by parts, we get

$$E_1(\varepsilon) = 2 \sum_{k=1}^K \sum_{i,j=1}^d \int \Big(\partial_{x_i} \zeta^a_{ik}(u,t,x) - \zeta^a_{ik,x_i}(u,t,x)\Big)$$

$$
\begin{aligned}
&\times \Bigg(\partial_{y_j} \int_u^v \operatorname{sign}'_\varepsilon (u-\eta)\, \sigma^a_{jk}(\eta,s,y)\, d\eta \\
&\qquad - \int_u^v \operatorname{sign}'_\varepsilon (u-\eta)\, \sigma^a_{jk,y_j}(\eta,s,y)\, d\eta \Bigg) \phi \, dx\, dt\, ds\, dy \\
&= -2 \sum_{k=1}^K \sum_{i,j=1}^d \int \Big(\partial_{x_i} \zeta^a_{ik}(u,t,x) - \zeta^a_{ik,x_i}(u,t,x) \Big) \psi^\varepsilon_{jk}(u,s,y) \\
&\qquad\qquad \times \partial_{y_j} \phi \, dx\, dt\, ds\, dy \\
&\quad - 2 \sum_{k=1}^K \sum_{i,j=1}^d \int \Big(\partial_{x_i} \zeta^a_{ik}(u,t,x) - \zeta^a_{ik,x_i}(u,t,x) \Big) \psi^\varepsilon_{jk,y_j}(u,s,y) \\
&\qquad\qquad \times \phi \, dx\, dt\, ds\, dy \\
&=: E_{1,1}(\varepsilon) + E_{1,2}(\varepsilon),
\end{aligned}
$$

where, for $1 \le k \le K$, $1 \le j \le d$, we have introduced the functions

$$
\psi^\varepsilon_{jk}(\xi,s,y) = \int_\xi^v \operatorname{sign}'_\varepsilon (\xi-\eta)\, \sigma^a_{jk}(\eta,s,y)\, d\eta,
$$

and

$$
\psi^\varepsilon_{jk,y_j}(\xi,s,y) = \int_\xi^v \operatorname{sign}'_\varepsilon (\xi-\eta)\, \sigma^a_{jk,y_j}(\eta,s,y)\, d\eta.
$$

Since $\operatorname{sign}'_\varepsilon(\cdot) \in C(\mathbf{R})$ and $\sigma^a_{jk}(\cdot,s,y), \sigma^a_{jk,y_j}(\cdot,s,y) \in L^\infty_{\mathrm{loc}}(\mathbf{R})$, we find that, for each fixed $(s,y) \in Q_T$,

$$
\psi^\varepsilon_{jk}(\cdot,s,y),\ \psi^\varepsilon_{jk,y_j}(\cdot,s,y) \in C(\mathbf{R}),
$$

so that the chain rule can be used. By the chain rule ($\mathbf{D}$.2) and an integration by parts, we deduce

$$
\begin{aligned}
E_{1,1}(\varepsilon) &= 2 \sum_{k=1}^K \sum_{i,j=1}^d \int \int_v^u \psi^\varepsilon_{jk}(\xi,s,y) \sigma^a_{ik}(\xi,t,x)\, d\xi\, \partial^2_{x_i y_j} \phi \, dx\, dt\, ds\, dy \\
&\quad + 2 \sum_{k=1}^K \sum_{i,j=1}^d \int \int_v^u \psi^\varepsilon_{jk}(\xi,s,y) \sigma^a_{ik,x_i}(\xi,t,x)\, d\xi\, \partial_{y_j} \phi \, dx\, dt\, ds\, dy \\
&=: E_{1,1,1}(\varepsilon) + E_{1,1,2}(\varepsilon).
\end{aligned}
$$

Let $h : \mathbf{R} \to \mathbf{R}$ be any locally integrable function. Then we will use repeatedly the following fact that holds for each fixed $b \in \mathbf{R}$:

$$
\lim_{\varepsilon \downarrow 0} \int_a^b \operatorname{sign}'_\varepsilon (\eta - a)\, h(\eta)\, d\eta = \operatorname{sign}(b-a) h(a) \qquad \text{for a.e. } a \in \mathbf{R},
$$

which is a consequence of the general theory of mollifiers (here $\operatorname{sign}'_\varepsilon(\cdot)$ acts as a C^1 mollifier). Hence, for a.e. $\xi \in \mathbf{R}$,

$$
\lim_{\varepsilon \downarrow 0} \psi^\varepsilon_{jk}(\xi) = -\operatorname{sign}(\xi - v) \sigma^a_{jk}(\xi,s,y),
$$

so that, by the dominated convergence theorem, as $\varepsilon \downarrow 0$,

$$
\int_v^u \psi^\varepsilon_{jk}(\eta,s,y) \sigma^a_{ik}(\eta,t,x)\, d\eta \to - \int_v^u \operatorname{sign}(\xi - v) \sigma^a_{ik}(\xi,t,x) \sigma^a_{jk}(\xi,s,y)\, d\xi
$$

for a.e. $(t,x), (s,y) \in Q_T$.

Consequently,

$$
\begin{aligned}
&\lim_{\varepsilon \downarrow 0} E_{1,1,1}(\varepsilon) \\
&\quad = -2 \sum_{k=1}^K \sum_{i,j=1}^d \int \int_v^u \operatorname{sign}(\xi - v) \sigma^a_{ik}(\xi,t,x) \sigma^a_{jk}(\xi,s,y)\, d\xi\, \partial^2_{x_i y_j} \phi \, dx\, dt\, ds\, dy.
\end{aligned}
$$

Similarly,

$$\lim_{\varepsilon\downarrow 0} E_{1,1,2}(\varepsilon) = -2\int\int_v^u \operatorname{sign}(\xi-v)\sigma^a_{ik,x_i}(\xi,t,x)\sigma^a_{jk}(\xi,s,y)d\xi\,\partial_{y_j}\phi\,dx\,dt\,ds\,dy.$$

Next, again by the chain rule and an integration by parts, we deduce

$$\begin{aligned} E_{1,2}(\varepsilon) &= 2\sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \psi^\varepsilon_{jk,y_j}(\xi,s,y)\sigma^a_{ik}(\xi,t,x)\,d\xi\,\partial_{x_i}\phi\,dx\,dt\,ds\,dy \\ &\quad + 2\sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \psi^\varepsilon_{jk,y_j}(\xi,s,y)\sigma^a_{ik,x_i}(\xi,t,x)\,d\xi\,\phi\,dx\,dt\,ds\,dy \\ &=: E_{1,2,1}+E_{1,2,2}. \end{aligned}$$

It follows as above that

$$\lim_{\varepsilon\downarrow 0} E_{1,2,1} = -2\sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-v)\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y)\,d\xi\,\partial_{x_i}\phi\,dx\,dt\,ds\,dy$$

and

$$\lim_{\varepsilon\downarrow 0} E_{1,2,2} = -2\sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-v)\sigma^a_{ik,x_i}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y)\,d\xi\,\phi\,dx\,dt\,ds\,dy.$$

Again, by the chain rule and an integration by parts, it follows as before that

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} E_2(\varepsilon) &= \sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-v)\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk}(\xi,t,x)\,d\xi\,\partial^2_{x_iy_j}\phi\,dx\,dt\,dy\,ds \\ &\quad + \sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-v)a_{ij,x_i}(\xi,t,x)\,d\xi\,\partial_{y_j}\phi\,dx\,dt\,dy\,ds \end{aligned}$$

and

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} E_3(\varepsilon) &= \sum_{k=1}^K\sum_{i,j=1}^d \int\int_u^v \operatorname{sign}(\xi-u)\sigma^a_{ik}(\xi,s,y)\sigma^a_{jk}(\xi,s,y)\,d\xi\,\partial^2_{x_iy_j}\phi\,dx\,dt\,dy\,ds \\ &\quad + \sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-u)a_{ij,y_i}(\xi,s,y)\,d\xi\,\partial_{x_j}\phi\,dx\,dt\,dy\,ds, \end{aligned}$$

or, by symmetry of the Kružkov entropies,

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} E_2(\varepsilon) &= \sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-v)\sigma^a_{ik}(\xi,s,y)\sigma^a_{jk}(\xi,s,y)\,d\xi\,\partial^2_{x_iy_j}\phi\,dx\,dt\,dy\,ds \\ &\quad + \sum_{i,j=1}^d \int\int_u^v \operatorname{sign}(\xi-v)a_{ij,y_i}(\xi,s,y)\,d\xi\,\partial_{x_j}\phi\,dx\,dt\,dy\,ds. \end{aligned}$$

In summary, we have

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\Big(E_1(\varepsilon)+E_2(\varepsilon)+E_3(\varepsilon)\Big) \\ &= \sum_{k=1}^K\sum_{i,j=1}^d \int\int_v^u \operatorname{sign}(\xi-v)\bigg\{\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk}(\xi,t,x) \end{aligned}$$

$$
\begin{aligned}
&\qquad - 2\sigma^a_{ik}(\xi,t,x)\sigma^a_{jk}(\xi,s,y) + \sigma^a_{ik}(\xi,s,y)\sigma^a_{jk}(\xi,s,y)\Big\}\, d\xi \\
&\qquad \times \partial^2_{x_i y_j}\phi \, dx\, dt\, dy\, ds \\
&+ \sum_{k=1}^{K}\sum_{i,j=1}^{d} \int\int_v^u \operatorname{sign}(\xi - v)\Big\{ a_{ij,y_j}(\xi,s,y) \\
&\qquad - 2\sum_{k=1}^{K} \sigma^a_{ik}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y)\Big\}\, d\xi \partial_{x_i}\phi \, dx\, dt\, dy\, ds \\
&+ \sum_{k=1}^{K}\sum_{i,j=1}^{d} \int\int_v^u \operatorname{sign}(\xi - v)\Big\{ a_{ij,x_i}(\xi,s,y) \\
&\qquad - 2\sum_{k=1}^{K} \sigma^a_{ik,x_i}(\xi,t,x)\sigma^a_{jk}(\xi,s,y)\Big\}\, d\xi \partial_{y_j}\phi \, dx\, dt\, dy\, ds \\
&- 2\sum_{k=1}^{K}\sum_{i,j=1}^{d} \int\int_v^u \operatorname{sign}(\xi - v)\sigma^a_{ik,x_i}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y)\, d\xi \\
&\qquad \times \phi \, dx\, dt\, dy\, ds.
\end{aligned}
$$

Recalling the definition of the functions ϵ_{ij} in (1.7), we observe that

$$
a_{ij,y_j}(\xi,s,y) - 2\sum_{k=1}^{K} \sigma^a_{ik}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y) = \partial_{y_j}\epsilon_{ij}(\xi,t,x,s,y),
$$

$$
a_{ij,x_i}(\xi,s,y) - 2\sum_{k=1}^{K} \sigma^a_{ik,x_i}(\xi,t,x)\sigma^a_{jk}(\xi,s,y) = \partial_{x_i}\epsilon_{ij}(\xi,t,x,s,y),
$$

and

$$
-2\sum_{k=1}^{K} \sigma^a_{ik,x_i}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y) = \partial^2_{x_i y_j}\epsilon_{ij}(\xi,t,x,s,y),
$$

so that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\Big(E_1(\varepsilon) + E_2(\varepsilon) + E_3(\varepsilon)\Big) \\
&= \sum_{i,j=1}^{d} \int \partial^2_{x_i y_j}\phi \int_v^u \operatorname{sign}(\xi - v)\epsilon_{ij}(\xi,t,x,s,y)\, d\xi\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^{d} \int \partial_{x_i}\phi \int_v^u \operatorname{sign}(\xi - v)\partial_{y_j}\epsilon_{ij}(\xi,t,x,s,y)\, d\xi\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^{d} \int \partial_{y_j}\phi \int_v^u \operatorname{sign}(\xi - v)\partial_{x_i}\epsilon_{ij}(\xi,t,x,s,y)\, d\xi\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^{d} \int \phi \int_v^u \operatorname{sign}(\xi - v)\partial^2_{x_i y_j}\epsilon_{ij}(\xi,t,x,s,y)\, d\xi\, dx\, dt\, dy\, ds
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{d} \int \partial^2_{x_i y_j}\phi \int_v^u \operatorname{sign}(\xi - v)\epsilon_{ij}(\xi,t,x,s,y)\,d\xi\,dx\,dt\,dy\,ds \\
&\qquad + \sum_{i,j=1}^{d} \int (\partial_{x_i} + \partial_{y_i})\phi \int_v^u \operatorname{sign}(\xi - v) \\
&\qquad\qquad \times \partial_{y_j}\epsilon_{ij}(\xi,t,x,s,y)\,d\xi\,dx\,dt\,dy\,ds \\
&\qquad + \sum_{i,j=1}^{d} \int \partial_{y_i}\phi \int_v^u \operatorname{sign}(\xi - v) \\
&\qquad\qquad \times \Big(\partial_{x_j}\epsilon_{ji} - \partial_{y_j}\epsilon_{ij}\Big)(\xi,t,x,s,y)\,d\xi\,dx\,dt\,dy\,ds \\
&\qquad + \sum_{i,j=1}^{d} \int \phi \int_v^u \operatorname{sign}(\xi - v)\partial^2_{x_i y_j}\epsilon_{ij}(\xi,t,x,s,y)\,d\xi\,dx\,dt\,dy\,ds \\
&=: \sum_{\ell=1}^{4} R_\ell(\rho_0,\rho).
\end{aligned}
\tag{4.11}
$$

Let us introduce the change of variables

$$
\tilde{x} = \frac{x+y}{2}, \quad \tilde{t} = \frac{t+s}{2}, \quad z = \frac{x-y}{2}, \quad \tau = \frac{t-s}{2},
$$
$$
x = \tilde{x} + z, \quad y = \tilde{x} - z, \quad t = \tilde{t} + \tau, \quad s = \tilde{t} - \tau,
$$

which maps $Q_T \times Q_T$ into

$$
\Omega = \mathbf{R}^d \times \mathbf{R}^d \times \left\{ (\tilde{t},\tau) \;:\; 0 \le \tilde{t} + \tau \le T, \quad 0 \le \tilde{t} - \tau \le T \right\}.
$$

The Jacobian is 4, that is,

$$
dx\,dt\,dy\,ds = 4\,d\tilde{x}\,d\tilde{t}\,dz\,d\tau.
$$

Estimate of $\lim_{\rho,\rho_0 \downarrow 0} R_1$. Let us start with $R_1(\rho_0,\rho)$. As

$$
\begin{aligned}
\partial^2_{x_i y_j}\phi &= \partial^2_{x_i y_j}\varphi\,\delta_{\rho_0}\omega_\rho + \partial_{x_i}\varphi\,\delta_{\rho_0}\partial_{y_j}\omega_\rho + \partial_{y_j}\varphi\,\delta_{\rho_0}\partial_{x_i}\omega_\rho + \varphi\,\delta_{\rho_0}\partial^2_{x_i y_j}\omega_\rho \\
&= \frac{1}{4}\partial^2_{\tilde{x}_i\tilde{x}_j}\varphi\left(\tilde{t},\tilde{x}\right)\,\delta_{\rho_0}(\tau)\omega_\rho(z) - \frac{1}{4}\partial_{\tilde{x}_i}\varphi\left(\tilde{t},\tilde{x}\right)\,\delta_{\rho_0}(\tau)\partial_{z_j}\omega_\rho(z) \\
&\qquad + \frac{1}{4}\partial_{\tilde{x}_j}\varphi\left(\tilde{t},\tilde{x}\right)\,\delta_{\rho_0}(\tau)\partial_{z_i}\omega_\rho(z) + \frac{1}{4}\varphi\left(\tilde{t},\tilde{x}\right)\,\delta_{\rho_0}(\tau)\partial^2_{z_i z_j}\omega_\rho(z),
\end{aligned}
$$

we may naturally write

$$
R_1(\rho_0,\rho) = \sum_{\ell=1}^{4} R_{1,\ell}(\rho_0,\rho),
$$

where the most difficult term is $R_{1,4}(\rho_0,\rho)$:

$$
\begin{aligned}
&R_{1,4}(\rho_0,\rho) \\
&= \int \sum_{i,j=1}^{d} \varphi\left(\tilde{t},\tilde{x}\right)\,\delta_{\rho_0}(\tau)\partial^2_{z_i z_j}\omega_\rho(z) \\
&\qquad \times \int_{v(\tilde{t}-\tau,\tilde{x}-z)}^{u(\tilde{t}+\tau,\tilde{x}+z)} \operatorname{sign}\left(\xi - v\left(\tilde{t}+\tau,\tilde{x}+z\right)\right) \\
&\qquad\qquad \times \epsilon_{ij}\left(\xi,\tilde{t}+\tau,\tilde{x}+z,\tilde{t}-\tau,\tilde{x}-z\right)\,d\xi\,d\tilde{x}\,d\tilde{t}\,dz\,d\tau.
\end{aligned}
$$

It is now crucial to exploit that "$\varphi\left(\tilde{t},\tilde{x}\right)\,\delta_{\rho_0}(\tau)\partial^2_{z_i z_j}\omega_\rho(z)$" is symmetric in i and j to obtain a favorable quadratic form:

$$
R_{1,4}(\rho_0,\rho)
$$

$$= \sum_{k=1}^{K} \sum_{i,j=1}^{d} \int \varphi\left(\tilde{t},\tilde{x}\right)\, \delta_{\rho_0}(\tau)\partial^2_{z_i z_j}\omega_\rho(z)$$
$$\times \int_{v\left(\tilde{t}-\tau,\tilde{x}-z\right)}^{u\left(\tilde{t}+\tau,\tilde{x}+z\right)} \operatorname{sign}\left(\xi - v\left(\tilde{t}+\tau,\tilde{x}+z\right)\right)$$
$$\times \left(\sigma^a_{ik}\left(\xi,\tilde{t}+\tau,\tilde{x}+z\right) - \sigma^a_{ik}\left(\tilde{t}-\tau,\tilde{x}-z\right)\right)$$
$$\times \left(\sigma^a_{jk}\left(\xi,\tilde{t}+\tau,\tilde{x}+z\right) - \sigma^a_{jk}\left(\tilde{t}-\tau,\tilde{x}-z\right)\right)\, d\xi\, d\tilde{x}\, d\tilde{t}\, dz\, d\tau.$$

By first sending $\rho_0 \downarrow 0$ and then using the Lipschitz continuity of the mapping $\tilde{x} \mapsto \sigma^a\left(\xi,\tilde{t},\tilde{x}\right)$, we get

$$|R_{1,4}(\rho)| := \lim_{\rho_0\downarrow 0} |R_{1,4}(\rho_0,\rho)|$$
$$\le C \max_{\tilde{t},\tilde{x}} |\varphi| \int_{\operatorname{supp}(\varphi)} \sum_{i,j=1}^{d} |z|^2 \left|\partial^2_{z_i z_j}\omega_\rho(z)\right| \left|u\left(\tilde{t},\tilde{x}+z\right) - v\left(\tilde{t},\tilde{x}-z\right)\right|\, d\tilde{x}\, d\tilde{t}\, dz.$$

Thus, since

$$\int |z|^2 \left|\partial^2_{z_i z_j}\omega_\rho(z)\right|\, dz \le C_\delta\, \mathbf{1}_{|z|<\varepsilon}, \qquad 1 \le i,j \le d,$$

we obtain

$$\left|\lim_{\rho\downarrow 0} R_{1,4}(\rho)\right| \le C_\delta \max_{\tilde{t},\tilde{x}} |\varphi| \int_{\operatorname{supp}(\varphi)} \left|u\left(\tilde{t},\tilde{x}\right) - v\left(\tilde{t},\tilde{x}\right)\right|\, d\tilde{x}\, d\tilde{t}.$$

Similarly, using

$$\left|\epsilon_{ij}\left(\xi,\tilde{t},\tilde{x}+z,\tilde{t},\tilde{x}-z\right)\right| \le C\,|z|$$

and

$$\int |z|\,|\partial_{z_i}\omega_\rho(z)|\, dz \le C_\delta\, \mathbf{1}_{|z|<\varepsilon}, \qquad 1 \le i \le d,$$

we deduce

$$\left|\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0}\Big(R_{1,2}(\rho_0,\rho) + R_{1,3}(\rho_0,\rho)\Big)\right|$$
$$\le C_\delta \max_{\tilde{t},\tilde{x},i} |\partial_{\tilde{x}_i}\varphi| \int_{\operatorname{supp}(\varphi)} \left|u\left(\tilde{t},\tilde{x}\right) - v\left(\tilde{t},\tilde{x}\right)\right|\, d\tilde{x}\, d\tilde{t}.$$

Finally,

$$\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0} R_{1,1}(\rho_0,\rho) = 0.$$

Estimates of $\lim_{\rho,\rho_0\downarrow 0} R_2$ and $\lim_{\rho,\rho_0\downarrow 0} R_4$. Using that

$$\left|\partial_{y_j}\epsilon_{ij}(\xi,t,x,s,y)\right| \le C, \qquad 1 \le i,j \le d,$$

we estimate $R_2(\rho_0,\rho)$ as follows:

$$|R_2(\rho_0,\rho)| = \left|\sum_{i,j=1}^{d} \int (\partial_{x_i} + \partial_{y_i})\varphi\, \delta_{\rho_0}\omega_\rho \times \int_v^u \operatorname{sign}(\xi - u)\partial_{y_j}\epsilon_{ij}(\xi,t,x,s,y)\, d\xi\, dx\, dt\, dy\, ds\right|$$
$$\le C \sum_{i,j=1}^{d} \int \left|\partial_{\tilde{x}_i}\varphi\left(\tilde{t},\tilde{x}\right)\right|\, \delta_{\rho_0}(\tau)\omega_\rho(z)$$
$$\times \left|u\left(\tilde{t},\tilde{x}+z\right) - v\left(\tilde{t},\tilde{x}-z\right)\right|\, d\tilde{x}\, d\tilde{t}\, dz\, d\tau,$$

so that

$$\left|\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0} R_2(\rho_0,\rho)\right| \le C \max_{\tilde{t},\tilde{x},i} |\partial_{\tilde{x}_i}\varphi| \int_{\operatorname{supp}(\varphi)} \left|u\left(\tilde{t},\tilde{x}\right) - v\left(\tilde{t},\tilde{x}\right)\right|\, d\tilde{x}\, d\tilde{t}.$$

Similarly, as

$$\left|\partial^2_{x_i y_j}\epsilon_{ij}(\xi,t,x,s,y)\right| \le C, \qquad 1 \le i,j \le d,$$

it follows that

$$\left|\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0} R_4(\rho_0,\rho)\right| \le C \max_{\tilde t,\tilde x,i} |\varphi| \int_{\mathrm{supp}(\varphi)} \left|u\left(\tilde t,\tilde x\right) - v\left(\tilde t,\tilde x\right)\right| \, d\tilde x\, d\tilde t.$$

Estimate of $\lim_{\rho,\rho_0\downarrow 0} R_3$. Notice that

$$\begin{aligned}
&\Big(\partial_{x_j}\epsilon_{ji} - \partial_{y_j}\epsilon_{ij}\Big)(\xi,t,x,s,y)\\
&\quad= a_{ij,x_j}(\xi,t,x) - a_{ij,y_j}(\xi,s,y)\\
&\qquad - 2\sum_{k=1}^K \Big(\sigma^a_{ik}(\xi,s,y)\sigma^a_{jk,x_j}(\xi,t,x) - \sigma^a_{ik}(\xi,t,x)\sigma^a_{jk,y_j}(\xi,s,y)\Big)\\
&\quad= a_{ij,x_j}(\xi,t,x) - a_{ij,y_j}(\xi,s,y)\\
&\qquad - 2\sum_{k=1}^K \sigma^a_{ik}(\xi,s,y)\Big(\sigma^a_{jk,x_j}(\xi,t,x) - \sigma^a_{jk,y_j}(\xi,s,y)\Big)\\
&\qquad + 2\sum_{k=1}^K \sigma^a_{jk,y_j}(\xi,s,y)\left(\sigma^a_{ik}(\xi,t,x) - \sigma^a_{ik}(\xi,s,y)\right),
\end{aligned}$$

and hence

$$\begin{aligned}
&R_3(\rho_0,\rho)\\
&= 2\sum_{i,j=1}^d \int \Big(\partial_{\tilde x_i}\varphi\left(\tilde t,\tilde x\right)\delta_{\rho_0}(\tau)\omega_\rho(z) - \varphi\left(\tilde t,\tilde x\right)\delta_{\rho_0}(\tau)\partial_{z_i}\omega_\rho(z)\Big)\\
&\quad\times \int_{v\left(\tilde t-\tau,\tilde x-z\right)}^{u\left(\tilde t+\tau,\tilde x+z\right)} \mathrm{sign}\left(\xi - v\left(\tilde t+\tau,\tilde x+z\right)\right)\\
&\quad\times \Bigg\{ a_{ij,\tilde x_j}\left(\xi,\tilde t+\tau,\tilde x+z\right) - a_{ij,\tilde x_j}\left(\xi,\tilde t-\tau,\tilde x-z\right)\\
&\qquad - 2\sum_{k=1}^K \sigma^a_{ik}\left(\xi,\tilde t-\tau,\tilde x-z\right)\\
&\qquad\qquad \times \Big(\sigma^a_{jk,\tilde x_j}\left(\xi,\tilde t+\tau,\tilde x+z\right) - \sigma^a_{jk,\tilde x_j}\left(\xi,\tilde t-\tau,\tilde x-z\right)\Big)\\
&\qquad + 2\sum_{k=1}^K \sigma^a_{jk,\tilde x_j}\left(\xi,\tilde t-\tau,\tilde x-z\right)\\
&\qquad\qquad \times \left(\sigma^a_{ik}\left(\xi,\tilde t+\tau,\tilde x+z\right) - \sigma^a_{ik}\left(\xi,\tilde t-\tau,\tilde x-z\right)\right)\Bigg\}\, d\xi\, d\tilde x\, d\tilde t\, dz\, d\tau.
\end{aligned}$$

Recalling

$$\tilde x \mapsto \sigma^a\left(\xi,\tilde t,\tilde x\right) \in W^{2,\infty}(\mathbf{R}^d;\mathbf{R}^{d\times K})$$

and taking into account

$$\lim_{\rho\downarrow 0}\int |z|\,|\omega_\rho(z)|\, dz = 0, \qquad \int |z|\,|\partial_{z_i}\omega_\rho(z)|\, dz \le C_\delta, \qquad 1 \le i \le d,$$

we conclude

$$\left|\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0} R_3(\rho_0,\rho)\right| \le C \max_{\tilde t,\tilde x,i} |\varphi| \int_{\mathrm{supp}(\varphi)} \left|u\left(\tilde t,\tilde x\right) - v\left(\tilde t,\tilde x\right)\right| \, d\tilde x\, d\tilde t.$$

Estimate of $\lim_{\rho,\rho_0\downarrow 0} R_0$. According to (4.8), we have

$$\begin{aligned} &R_0(\rho_0,\rho)\\ &= 4\sum_{i,j=1}^d \int \Big(r_{ij}\left(u(\tilde t+\tau,\tilde x+z), v(\tilde t-\tau,\tilde x-z),\tilde t+\tau,\tilde x+z\right)\\ &\qquad - r_{ij}\left(u(\tilde t+\tau,\tilde x+z), v(\tilde t-\tau,\tilde x-z),\tilde t-\tau,\tilde x-z\right)\Big)\\ &\qquad \times \partial_{\tilde x_j}\varphi\left(\tilde t,\tilde x\right)\, \delta_{\rho_0}(\tau)\partial_{z_i}\omega_\rho(z)\, d\tilde x\, d\tilde t\, dz\, d\tau. \end{aligned}$$

Using the $W^{1,\infty}$–regularity of $\tilde x\mapsto \sigma^a\left(\xi,\tilde t,\tilde x\right)$, it follows as before that

$$\left|\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0} R_0(\rho_0,\rho)\right| \le C\max_{\tilde t,\tilde x,j}\left|\partial_{\tilde x_j}\varphi\right| \int_{\mathrm{supp}(\varphi)} \left|u\left(\tilde t,\tilde x\right)-v\left(\tilde t,\tilde x\right)\right|\, d\tilde x\, d\tilde t.$$

Concluding the proof. For $1\le i,j\le d$,

$$\begin{aligned} &(\partial_t+\partial_s)\,\varphi\left(\tfrac{t+s}{2},\tfrac{x+y}{2}\right) = \varphi_{\tilde t}(\tilde t,\tilde x),\\ &(\nabla_x+\nabla_y)\,\varphi(t,x,s,y) = \nabla_{\tilde x}\varphi\left(\tilde t,\tilde x\right),\\ &(\partial^2_{x_ix_j}+\partial^2_{y_jx_i})\varphi = \frac12\partial^2_{\tilde x_i\tilde x_j}\varphi\left(\tilde t,\tilde x\right),\\ &(\partial^2_{x_jy_i}+\partial^2_{y_jy_i})\varphi = \frac12\partial^2_{\tilde x_i\tilde x_j}\varphi\left(\tilde t,\tilde x\right), \end{aligned}$$

and

$$\begin{aligned} &r_{ij,x_i}(u,v,t,x)+r_{ij,y_i}(u,v,s,y)\\ &\quad=\frac12\Big(r_{ij,\tilde x_i}\left(u\left(\tilde t+\tau,\tilde x+z\right), v\left(\tilde t-\tau,\tilde x-z\right),\tilde t+\tau,\tilde x+z\right)\\ &\qquad\quad + r_{ij,\tilde x_i}\left(u\left(\tilde t+\tau,\tilde x+z\right), v\left(\tilde t-\tau,\tilde x-z\right),\tilde t-\tau,\tilde x-z\right)\Big). \end{aligned}$$

Using these facts, along with the estimates derived above for $\left|\lim_{\rho\downarrow 0}\lim_{\rho_0\downarrow 0} R_\ell(\rho_0,\rho)\right|$, $\ell=0,\dots,4$, we send first $\rho_0\downarrow 0$ and second $\rho\downarrow 0$ in (4.9). The final result reads

$$\begin{aligned} &\int_{Q_T}\Bigg(|u-v|\,\partial_{\tilde t}\varphi + \sum_{i,j=1}^d \mathrm{sign}(u-v)\left(A_{ij}\left(u,\tilde t,\tilde x\right)-A_{ij}\left(v,\tilde t,\tilde x\right)\right)\partial^2_{\tilde x_i\tilde x_j}\varphi\\ &\qquad + \sum_{i,j=1}^d \mathrm{sign}(u-v)\left(A_{ij,\tilde x_i}\left(u,\tilde t,\tilde x\right)-A_{ij,\tilde x_i}\left(v,\tilde t,\tilde x\right)\right)\partial_{\tilde x_j}\varphi\left(\tilde t,\tilde x\right)\Bigg)\, d\tilde x\, d\tilde t\\ &\qquad\qquad \ge -C\Big(\max_{\tilde t,\tilde x}|\varphi| + \max_{\tilde t,\tilde x,i}\left|\partial_{\tilde x_i}\varphi\right|\Big)\int_{Q_T\cap\mathrm{supp}(\varphi)}|u-v|\, d\tilde x\, d\tilde t, \end{aligned} \tag{4.12}$$

where $u=u\left(\tilde t,\tilde x\right)$ and $v=v\left(\tilde t,\tilde x\right)$.

Condition (**D**.1) for u,v and the assumption $u_0-v_0\in L^1(\mathbf{R}^d)$ yield that $u-v$ belongs to $L^\infty(0,T;L^1(\mathbf{R}^d))$ and thus $L^1(Q_T)$. Therefore, by our assumptions, the integrand on the left-hand side of (4.12) belongs to $L^1(Q_T)$ for test functions φ that do not have compact support in x. Hence, from (4.12), we can conclude the proof of Theorem 3.1 by standard arguments (i.e., choosing a sequence of smooth functions $0\le\varphi\le 1$ that converges to $\mathbf{1}_{(0,t)\times\mathbf{R}^d}$ and using the initial conditions for u,v) and Gronwall's inequality.

Remark 4.1. For the quasi-isotropic case (1.4), that is,

$$a_{ij}\equiv 0\quad (i\ne j),\qquad a_{ii}=\sigma^a_{ii}\sigma^a_{ii},\qquad 1\le i,j\le d,$$

we can carry out the above proof under the mere assumption that σ^a is $W^{1,\infty}$ regular in the space variable $x\in\mathbf{R}^d$.

The key point is that the diagonal of the matrix ϵ in (1.7) automatically takes on a favorable quadratic form: $\epsilon_{ii}(\xi,t,x,y) = (\sigma^a_{ii}(\xi,t,x) - \sigma^a_{ii}(\xi,s,y))^2$, and hence

$$\partial_{x_i}\epsilon_{ii}(u,t,x,y) = 2\sigma^a_{ii,x_i}(\xi,t,x)\left(\sigma^a_{ii}(\xi,t,x) - \sigma^a_{ii}(\xi,s,y)\right),$$
$$\partial_{y_i}\epsilon_{ii}(u,t,x,y) = -2\sigma^a_{ii,y_i}(\xi,s,y)\left(\sigma^a_{ii}(\xi,t,x) - \sigma^a_{ii}(\xi,s,y)\right).$$

In view of this and the $W^{1,\infty}$ regularity of σ^a in the space variable, we can in particular estimate $\lim_{\rho,\rho_0\downarrow 0} R_3$ as before. Moreover, $\lim_{\rho,\rho_0\downarrow 0} R_2 = 0$. The rest of the proof remains the same. This proves the second part of Theorem 3.1.

5. Proof of Theorem 3.2

Throughout the proof, we set $f = g \equiv 0$, $k = l \equiv 0$ in (3.6) and (3.8). These terms can be included into the analysis by copying the arguments from [26] (we leave the details to the reader).

We follow Section 4 closely, but with a slightly different test function $\phi(t,x,s,y)$. Pick two (arbitrary but fixed) points $\nu,\tau \in (0,T)$, $\nu < \tau$. For any $\alpha_0 > 0$, we define

$$\varphi_{\alpha_0}(t) = H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau), \qquad H_{\alpha_0}(t) = \int_{-\infty}^{t} \delta_{\alpha_0}(\sigma)\, d\sigma.$$

With $0 < \alpha_0 < \min(\nu, T-\tau)$, we define $\phi \in \mathcal{D}(Q_T \times Q_T)$ by

$$\phi(x,t,y,s) = \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\omega_\rho(x-y) \geq 0.$$

One should observe that

$$(\partial_t + \partial_s)\phi = \Big(\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)$$

and $(\nabla_x + \nabla_y)\phi = 0$.

Thanks to the $BV(\mathbf{R}^d)$ regularity of $u(t,\cdot)$ and $v(s,\cdot)$, it makes sense to use this simplified test function (without cut-off in space).

Now proceeding as in Section 4, cf. equations (4.9) and (4.10), we derive the following inequality:

$$\begin{aligned}
&\int |u(t,x) - v(s,y)| \Big(\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\, dx\, dt\, dy\, ds \\
&\geq \sum_{i,j=1}^{d} \int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial^2_{x_i y_j}\omega_\rho(x-y) \\
&\qquad\qquad \times \int_v^u \operatorname{sign}(\xi - v)\epsilon^{a-b}_{ij}(\xi,x,y)\, d\xi\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^{d} \int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial_{x_i}\omega_\rho(x-y) \\
&\qquad\qquad \times \int_v^u \operatorname{sign}(\xi - v)\partial_{y_j}\epsilon^{a-b}_{ij}(\xi,x,y)\, d\xi\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^{d} \int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial_{y_j}\omega_\rho(x-y) \\
&\qquad\qquad \times \int_v^u \operatorname{sign}(\xi - v)\partial_{x_i}\epsilon^{a-b}_{ij}(\xi,x,y)\, d\xi\, dx\, dt\, dy\, ds \\
&\quad + \sum_{i,j=1}^{d} \int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\omega_\rho(x-y) \\
&\qquad\qquad \times \int_v^u \operatorname{sign}(\xi - v)\partial^2_{x_i y_j}\epsilon^{a-b}_{ij}(\xi,x,y)\, d\xi\, dx\, dt\, dy\, ds,
\end{aligned} \tag{5.1}$$

where the "continuous dependence term" $\epsilon_{ij}^{a-b}(\xi,x,y)$ is defined as

$$\epsilon_{ij}^{a-b}(\xi,x,y) = \sum_{k=1}^{K}\Big\{\sigma_{ik}^{a}(\xi,x)\sigma_{jk}^{a}(\xi,x) - 2\sigma_{ik}^{a}(\xi,x)\sigma_{jk}^{b}(\xi,y) + \sigma_{ik}^{b}(\xi,y)\sigma_{jk}^{b}(\xi,y)\Big\}. \tag{5.2}$$

Up to now, everything has been anisotropic. In the quasi-isotropic case (3.6) and (3.8), inequality (5.1) remains the same, except that the index j is replaced by i, all the sums over i,j are replaced by the sums over i, the sums over k disappear, and the index k is replaced by i. With this in mind, we do integration by parts in x in the first and second terms on the right-hand side of (5.1), exploiting that $u(t,\cdot)$ has the BV regularity. The final result is

$$\begin{aligned}\int |u(t,x)-v(s,y)|\,\Big(\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,dx\,dt\,dy\,ds\\ \ge -\sum_{i=1}^{d}\int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial_{y_i}\omega_\rho(x-y)\,\mathrm{sign}(u-v)\epsilon_{ii}^{a-b}(u,x,y)\partial_{x_i}u\,dx\,dt\,dy\,ds\\ -\sum_{i=1}^{d}\int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,\mathrm{sign}(u-v)\partial_{y_i}\epsilon_{ii}^{a-b}(u,x,y)\partial_{x_i}u\,dx\,dt\,dy\,ds.\end{aligned} \tag{5.3}$$

By the triangle inequality, we get

$$\begin{aligned}-\int |u(t,x)-v(s,y)|\,\Big(\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,dx\,dt\,dy\,ds\\ \le L+R^t+R^x,\end{aligned}$$

where

$$\begin{aligned}L &= -\int |u(t,y)-v(t,y)|\,\Big(\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,dx\,dt\,dy\,ds,\\ R^t &= -\int |v(t,y)-v(s,y)|\,\Big(\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,dx\,dt\,dy\,ds,\\ R^x &= -\int |u(t,x)-u(t,y)|\,\Big(\delta_{\alpha_0}(t-\nu)-\delta_{\alpha_0}(t-\tau)\Big)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,dx\,dt\,dy\,ds.\end{aligned}$$

It is standard to show that $\lim_{\rho_0\downarrow 0} R^t = 0$, and

$$\begin{aligned}\lim_{\alpha_0\downarrow 0} R^x &= \int\Big(|u(\tau,x)-u(\tau,y)| - |u(\nu,x)-u(\nu,y)|\Big)\omega_\rho(x-y)\,dx\,dy\\ &\le \rho \sup_{t\in(\nu,\tau)} |u(t,\cdot)|_{BV(\mathbf{R}^d)} \le \rho\,\|u\|_{L^\infty(0,T;BV(\mathbf{R}^d))},\\ \lim_{\alpha_0\downarrow 0} L &= \|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^1(\mathbf{R}^d)} - \|u(\nu,\cdot)-v(\nu,\cdot)\|_{L^1(\mathbf{R}^d)}.\end{aligned}$$

Hence, after sending $\alpha_0,\rho_0\downarrow 0$, we get the following approximation inequality:

$$\begin{aligned}&\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\\ &\quad\le \|u(\nu,\cdot)-v(\nu,\cdot)\|_{L^1(\mathbf{R}^d)} + \rho\,\|u\|_{L^\infty(0,T;BV(\mathbf{R}^d))} + R_1 + R_2,\end{aligned} \tag{5.4}$$

where

$$R_1 = \sum_{i=1}^{d}\int_\nu^\tau\int\int |\partial_{y_i}\omega_\rho(x-y)|\,\left|\epsilon_{ii}^{a-b}(u,x,y)\right|\,|\partial_{x_i}u(t,x)|\,dx\,dt\,dy,$$

and

$$R_2 = \sum_{i=1}^{d}\int_\nu^\tau\int\int \omega_\rho(x-y)\,\left|\partial_{y_i}\epsilon_{ii}^{a-b}(u,x,y)\right|\,|\partial_{x_i}u(t,x)|\,dx\,dt\,dy.$$

Since

$$\epsilon_{ii}^{a-b}(\xi,x,y) = \left(\sigma_{ii}^a(\xi,x) - \sigma_{ii}^b(\xi,y)\right)^2,$$

it follows that

$$|R_1| \le 2\sum_{i=1}^{d}\int_\nu^\tau \int\int \left(\left(\sigma_{ii}^a(\xi,x) - \sigma_{ii}^b(\xi,x)\right)^2 + \left(\sigma_{ii}^b(\xi,x) - \sigma_{ii}^b(\xi,y)\right)^2 \right) \times |\partial_{y_i}\omega_\rho(x-y)|\,|\partial_{x_i}u(t,x)|\,dx\,dt\,dy.$$

Hence, by the Lipschitz continuity of $x \mapsto \sigma^b(\xi,x)$ and $\int |\partial_{x_i}\omega_\rho(x-y)|\,dy \le C/\rho$,

$$|R_1| \le C(\tau-\nu)\,\|u\|_{L^\infty(0,T;BV(\mathbf{R}^d))} \times \left(\frac{\left\|\sigma^a(\cdot,\cdot) - \sigma^b(\cdot,\cdot)\right\|^2_{L^\infty(I\times\mathbf{R}^d;\mathbf{R}^{d\times d})}}{\rho} + \rho \right)$$

for some finite constant $C > 0$ independent of ρ.

Similarly, we can estimate

$$|R_2| \le C(\tau-\nu)\,\|u\|_{L^\infty(0,T;BV(\mathbf{R}^d))} \left(\left\|\sigma^a(\cdot,\cdot) - \sigma^b(\cdot,\cdot)\right\|_{L^\infty(I\times\mathbf{R}^d;\mathbf{R}^{d\times d})} + \rho \right)$$

for some finite constant $C > 0$ independent of ρ.

Equipped with these estimates of R_1 and R_2, we choose

$$\rho = \sqrt{\tau-\nu}\,\left\|\sigma^a(\cdot,\cdot) - \sigma^b(\cdot,\cdot)\right\|_{L^\infty(I\times\mathbf{R}^d;\mathbf{R}^{d\times d})}$$

in (5.4) and then we send $\nu \downarrow 0$. Since $\tau \in (0,T)$ is arbitrary, we obtain (3.12).

Remark 5.1. With no x–dependency in the diffusion coefficients, we can also derive a continuous dependence estimate in the anisotropic case (the second part of Theorem 3.2). In this case, (5.1) reads

$$\begin{aligned}&\int |u(t,x) - v(s,y)| \left(\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\right)\delta_{\rho_0}(t-s)\omega_\rho(x-y)\,dx\,dt\,dy\,ds \\ &\ge \sum_{i,j=1}^{d} \int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial^2_{x_iy_j}\omega_\rho(x-y)\int_v^u \operatorname{sign}(\xi - v)\epsilon_{ij}^{a-b}(\xi)\,d\xi\,dx\,dt\,dy\,ds \\ &:= R,\end{aligned} \tag{5.5}$$

where now

$$\epsilon_{ij}^{a-b}(\xi) = \sum_{k=1}^{K}\left\{ \sigma_{ik}^a(\xi)\sigma_{jk}^a(\xi) - 2\sigma_{ik}^a(\xi)\sigma_{jk}^b(\xi) + \sigma_{ik}^b(\xi)\sigma_{jk}^b(\xi) \right\}. \tag{5.6}$$

The main difference from the x–dependent diffusion case is that we are now able to exploit the symmetry property

$$\partial^2_{x_iy_j}\omega_\rho(x-y) = \partial^2_{x_jy_i}\omega_\rho(x-y)$$

to obtain a favorable quadratic form for R:

$$R = \sum_{k=1}^{K}\sum_{i,j=1}^{d}\int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial^2_{x_iy_j}\omega_\rho(x-y) \times \int_v^u \operatorname{sign}(\xi - v)\left(\sigma_{ik}^a(\xi) - \sigma_{ik}^b(\xi)\right)\left(\sigma_{jk}^a(\xi) - \sigma_{jk}^b(\xi)\right)\,d\xi\,dx\,dt\,dy\,ds.$$

Next, we do an integration by parts in x, exploiting the BV regularity of $u(t,\cdot)$, to obtain

$$R = -\sum_{k=1}^{K}\sum_{i,j=1}^{d}\int \varphi_{\alpha_0}(t)\delta_{\rho_0}(t-s)\partial_{y_j}\omega_\rho(x-y) \times \operatorname{sign}(u-v)\left(\sigma_{ik}^a(u) - \sigma_{ik}^b(u)\right)\left(\sigma_{jk}^a(u) - \sigma_{jk}^b(u)\right)\partial_{x_i}u\,dx\,dt\,dy\,ds.$$

From this, the following estimate can be easily obtained:

$$\left|\lim_{\alpha_0,\rho_0\downarrow 0} R\right| \leq C(t-\nu)\,\|u\|_{L^\infty(0,T;BV(\mathbf{R}^d))}\,\frac{\left\|\left(\sigma^a-\sigma^b\right)\left(\sigma^a-\sigma^b\right)^\top\right\|_{L^\infty(I;\mathbf{R}^{d\times d})}}{\rho}.$$

The remaining part of the proof goes as before, and thus the second part of Theorem 3.2 is proved.

6. Existence of entropy solutions

In this section we state a theorem about the existence of entropy solutions for the Cauchy problem (1.1) and (1.2). The proof is omitted, as it can be proved in a standard way by combining the arguments of [36] and [16].

Theorem 6.1. *Suppose the conditions in* (3.1) *and* (3.3) *hold, and* $u_0 \in L^1 \cap L^\infty(\mathbf{R}^d)$. *Suppose* $s(u,t,x)$ *is bounded in* (t,x) *and locally uniformly in* u*; is locally Lipschitz in* u *and uniformly in* (t,x)*; and either grows in* u *at most linearly when* $|u|$ *is sufficiently large or satisfies the maximal principle property. Then there exists an entropy solution* $u \in C\left([0,T);L^1(\mathbf{R}^d)\right)$ *of the Cauchy problem* (1.1) *and* (1.2).

Remark 6.1. The conditions on $s(u,t,x)$ in the theorem especially include various types of reaction terms $g(u)(1-u)$ for $0 \leq u \leq 1$ in reaction-convection-diffusion processes, such as the Kolmogorov-Petrovskii-Piskunov (KPP)-type $g(u) = \kappa u$, the Arrhenius-type $g(u) = \kappa e^{-A/u}$, and the ignition-type $g(u) = \kappa(u-\theta_0)_+$ for some $\theta_0 \in (0,1)$.

Remark 6.2. For simplicity, we state Theorem 6.1 under the assumption that $u_0 \in L^1 \cap L^\infty(\mathbf{R}^d)$. For more general initial data $u_0 \in L^\infty(\mathbf{R}^d)$ for which we can find a C^2 function $w : \mathbf{R}^d \to \mathbf{R}^d$ such that $u_0 - w \in L^1(\mathbf{R}^d)$, we can work with $u - w$ to conclude the existence of an entropy solution.

7. Initial condition

In Definition 2.1, we demand that the initial condition at $t = 0$ is satisfied in the strong L^1 sense. In the course of proving the convergence of certain approximate solution sequences, it can be difficult to verify condition (**D**.4) for a limit function. To have a more flexible framework, we can include the initial condition into the entropy formulation, that is, delete condition (**D**.4) and require instead the entropy inequality (2.1) to hold in $\mathcal{D}'([0,T)\times\mathbf{R}^d)$:

$$\begin{aligned}
&\int_{Q_T}\Bigg(\eta(u)\partial_t\phi + \sum_{i=1}^d q_i(u,t,x)\partial_{x_i}\phi + \sum_{i,j=1}^d r_{ij}(u,t,x)\partial^2_{x_ix_j}\phi \\
&\qquad + \eta'(u)s(u,t,x)\phi - \sum_{i=1}^d\Big(\eta'(u)f_{i,x_i}(u,t,x) - q_{i,x_i}(u,t,x)\Big)\phi \\
&\qquad + \sum_{i,j=1}^d r_{ij,x_i}(u,t,x)\partial_{x_j}\phi\Bigg)\,dx\,dt + \int_{\mathbf{R}^d}\eta(u_0)\phi(0,x)\,dx \\
&\geq \int_{Q_T}\eta''(u)\sum_{k=1}^K\left(\sum_{i=1}^d\Big(\partial_{x_i}\zeta^a_{ik}(u,t,x) - \zeta^a_{ik,x_i}(u,t,x)\Big)\right)^2\phi\,dx\,dt.
\end{aligned} \tag{7.1}$$

This weak formulation of the initial condition is easier to verify for limits of certain approximate solutions. In the context of conservation laws, this point goes back to [22]; see also [23] for isotropic degenerate parabolic equations. To prove Theorem 3.1 with a weak formulation of the initial condition, we simply have to combine the proof in Section 4 with a straightforward adaption of the arguments in [22, 23]. We leave the details to the reader.

Acknowledgments

The research of G.-Q. Chen was supported in part by the National Science Foundation and an Alexandre Humboldt Foundation fellowship. The research of K. H. Karlsen was supported in part by the BeMatA program of the Research Council of Norway and the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282. The writing of this paper started in June 2003 while K. H. Karlsen enjoyed the warm hospitality of the Department of Mathematics at Northwestern University.

References

[1] D. Aregba-Driollet, R. Natalini, and S. Tang. Explicit diffusive kinetic schemes for nonlinear degenerate parabolic systems. *Math. Comp.*, 73(245):63–94 (electronic), 2004.

[2] M. Bendahmane and K. H. Karlsen. Renormalized entropy solutions for quasilinear anisotropic degenerate parabolic equations. *SIAM J. Math. Anal.* To appear.

[3] P. Bénilan and M. G. Crandall. The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$. *Indiana Univ. Math. J.*, 30(2):161–177, 1981.

[4] P. Bénilan and H. Touré. Sur l'équation générale $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$ dans L^1. II. Le problème d'évolution. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 12(6):727–761, 1995.

[5] F. Bouchut, F. R. Guarguaglini, and R. Natalini. Diffusive BGK approximations for nonlinear multidimensional parabolic equations. *Indiana Univ. Math. J.*, 49(2):723–749, 2000.

[6] F. Bouchut and F. James. One-dimensional transport equations with discontinuous coefficients. *Nonlinear Anal.*, 32(7):891–933, 1998.

[7] F. Bouchut and B. Perthame. Kružkov's estimates for scalar conservation laws revisited. *Trans. Amer. Math. Soc.*, 350(7):2847–2870, 1998.

[8] R. Bürger, S. Evje, and K. H. Karlsen. On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes. *J. Math. Anal. Appl.*, 247(2):517–556, 2000.

[9] M. C. Bustos, F. Concha, R. Bürger, and E. M. Tory. *Sedimentation and Thickening: Phenomenological Foundation and Mathematical Theory.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[10] I. Capuzzo Dolcetta and B. Perthame. On some analogy between different approaches to first order PDE's with nonsmooth coefficients. *Adv. Math. Sci. Appl.*, 6(2):689–703, 1996.

[11] J. Carrillo. On the uniqueness of the solution of the evolution dam problem. *Nonlinear Anal.*, 22(5):573–607, 1994.

[12] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Rational Mech. Anal.*, 147(4):269–361, 1999.

[13] J. Carrillo and P. Wittbold. Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems. *J. Differential Equations*, 156(1):93–121, 1999.

[14] G.-Q. Chen and E. DiBenedetto. Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33(4):751–762 (electronic), 2001.

[15] G.-Q. Chen and K. H. Karlsen. L^1 framework for continuous dependence and error estimates for quasilinear degenerate parabolic equations. *Trans. Amer. Math. Soc.* To appear.

[16] G.-Q. Chen and B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):645–668, 2003.

[17] B. Cockburn and G. Gripenberg. Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Differential Equations*, 151(2):231–251, 1999.

[18] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[19] M. S. Espedal and K. H. Karlsen. Numerical solution of reservoir flow models based on large time step operator splitting algorithms. In *Filtration in Porous Media and Industrial Applications (Cetraro, Italy, 1998)*, volume 1734 of *Lecture Notes in Mathematics*, pages 9–77. Springer, Berlin, 2000.

[20] S. Evje and K. H. Karlsen. Monotone difference approximations of BV solutions to degenerate convection-diffusion equations. *SIAM J. Numer. Anal.*, 37(6):1838–1860 (electronic), 2000.

[21] S. Evje, K. H. Karlsen, and N. H. Risebro. A continuous dependence result for nonlinear degenerate parabolic equations with spatially dependent flux function. In *Hyperbolic problems: theory, numerics, applications, Vol. I (Magdeburg, 2000)*, pages 337–346. Birkhäuser, Basel, 2001.

[22] R. Eymard, T. Gallouët, and R. Herbin. Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation. *Chinese Ann. Math. Ser. B*, 16(1):1–14, 1995. A Chinese summary appears in Chinese Ann. Math. Ser. A **16** (1995), no. 1, 119.

[23] R. Eymard, T. Gallouët, R. Herbin, and A. Michel. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numer. Math.*, 92(1):41–82, 2002.

[24] K. H. Karlsen and M. Ohlberger. A note on the uniqueness of entropy solutions of nonlinear degenerate parabolic equations. *J. Math. Anal. Appl.*, 275(1):439–458, 2002.

[25] K. H. Karlsen and N. H. Risebro. Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients. *M2AN Math. Model. Numer. Anal.*, 35(2):239–269, 2001.

[26] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Contin. Dyn. Syst.*, 9(5):1081–1104, 2003.
[27] S. N. Kružkov. First order quasilinear equations in several independent variables. *Math. USSR Sbornik*, 10(2):217–243, 1970.
[28] B. J. Lucier. A moving mesh numerical method for hyperbolic conservation laws. *Math. Comp.*, 46(173):59–69, 1986.
[29] M. Maliki and H. Touré. Uniqueness of entropy solutions for nonlinear degenerate parabolic problems. *J. Evol. Equ.*, 3(4):603–622, 2003.
[30] C. Mascia, A. Porretta, and A. Terracina. Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations. *Arch. Ration. Mech. Anal.*, 163:87–124, 2002.
[31] A. Michel and J. Vovelle. Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods. Preprint, 2002.
[32] M. Ohlberger. A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations. *M2AN Math. Model. Numer. Anal.*, 35(2):355–387, 2001.
[33] B. Perthame and P. E. Souganidis. Dissipative and entropy solutions to non-isotropic degenerate parabolic balance laws. *Arch. Ration. Mech. Anal.*, 170(4):359–370, 2003.
[34] G. Petrova and B. Popov. Linear transport equations with discontinuous coefficients. *Comm. Partial Differential Equations*, 24(9-10):1849–1873, 1999.
[35] É. Rouvre and G. Gagneux. Solution forte entropique de lois scalaires hyperboliques-paraboliques dégénérées. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7):599–602, 1999.
[36] A. I. Vol'pert and S. I. Hudjaev. Cauchy's problem for degenerate second order quasilinear parabolic equations. *Math. USSR Sbornik*, 7(3):365–387, 1969.
[37] Z. Wu and J. Yin. Some properties of functions in BV_x and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations. *Northeastern Math. J.*, 5(4):395–422, 1989.
[38] Z. Wu, J. Zhao, J. Yin, and H. Li. *Nonlinear diffusion equations.* World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

(Gui-Qiang Chen)
Department of Mathematics
Northwestern University
Evanston, IL 60208-2730, USA
E-mail address: gqchen@math.northwestern.edu
URL: http://www.math.northwestern.edu/~gqchen/

(Kenneth Hvistendahl Karlsen)
Centre of Mathematics for Applications (CMA)
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: kennethk@math.uio.no
URL: http://www.math.uio.no/~kennethk/