Research Article

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Quasilinear Dirichlet problems with competing operators and convection

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Abstract: The paper deals with a quasilinear Dirichlet problem involving a competing (p,q)-Laplacian and a convection term. Due to the lack of ellipticity, monotonicity and variational structure, the known methods to find a weak solution are not applicable. We develop an approximation procedure permitting to establish the existence of solutions in a generalized sense. If in place of competing (p,q)-Laplacian we consider the usual (p,q)-Laplacian, our results ensure the existence of weak solutions.

Keywords: quasilinear Dirichlet problems, competing (p,q)-Laplacian, convection term, generalized solution, approximation

MSC 2020: 35H30, 35J92, 35D30

1 Introduction

The object of the paper is to study the following quasilinear problem with homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta_p u + \Delta_q u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with the boundary $\partial\Omega$. In the left-hand side of (1), we have the sum $-\Delta_p + \Delta_q$ of the negative p-Laplacian Δ_p and of the q-Laplacian Δ_q with $1 < q < p < +\infty$. The operator $-\Delta_p + \Delta_q$ has a completely different behavior in comparison to the operator $-\Delta_p - \Delta_q$, which is the (negative) (p, q)-Laplacian. Note that in $-\Delta_p + \Delta_q$ there is competition between $-\Delta_p$ and $-\Delta_q$ taking their difference and thus destroying the ellipticity in contrast to what happens in the case of $-\Delta_p - \Delta_q$. The right-hand side $f(x, u, \nabla u)$ of equation (1) is a so-called convection term meaning that it depends on the point $x \in \Omega$ in the domain, on the solution u and on its gradient ∇u. The convection term is expressed through the Nemytskii operator associated with a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, i.e., $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and is continuous in $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$. The function f will be subject to appropriate growth conditions (H1)–(H2) in Section 2.

Such a problem without any available ellipticity but with a variational structure that prevents to have convection was studied for the first time in [1]. Specifically, in [1] the following particular variational version of (1) was investigated:

$$\begin{cases} -\Delta_p u + \Delta_q u = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

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with a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$. In order to highlight the core of the problem we have skipped the nonsmooth formulation in [1]. The difference between problems (1) and (2) consists in the fact that the reaction term $f(x, u, \nabla u)$ of (1) depends on the gradient ∇u which is excluded in (2). This is an essential feature because the (somewhat) variational approach in [1] for (2) cannot be implemented for (1).

We briefly discuss some major characteristics of problem (1). Here we continue the study in [1] setting forth a nonvariational counterpart. The main aspect for the relevance of this work in comparison with the existing literature is the lack of ellipticity for the operator $-\Delta_p + \Delta_q$ in the principal part of (1). As pointed out in [1], taking a nonzero $u_0 \in W_0^{1,p}(\Omega)$ and a number $\lambda > 0$ it turns out that

$$\langle -\Delta_p(\lambda u_0) + \Delta_q(\lambda u_0), \lambda u_0 \rangle = \lambda^p \|\nabla u_0\|_p^p - \lambda^q \|\nabla u_0\|_q^q$$

is not of constant sign, being positive for λ sufficiently large and negative otherwise, thus the ellipticity is lost. For this reason we call the operator $-\Delta_p + \Delta_q$ the competing (p, q)-Laplacian. We also note that the left-hand side of the equation in (1) is in the divergence form $-\Delta_p u + \Delta_q u = -\mathrm{div}(a(|\nabla u|))$ with $a(t) = t^{p-2} - t^{q-2}$ for all $t > 0$. A minimal condition of ellipticity for an operator in divergence form $-\mathrm{div}(a(|\nabla u|))$ (see, e.g., [2]) is to have, among other things, $a(t) > 0$ for all $t > 0$, which is not satisfied in the case of $a(t) = t^{p-2} - t^{q-2}$.

Another important aspect of problem (1) is the presence of the convection term $f(x, u, \nabla u)$. It represents a real challenge with respect to [1] treating (2) because any variational method is inapplicable to (1). In [1] it was possible to build for (2) a variational approach through Ekeland's variational principle on finite dimensional spaces despite the lack of ellipticity. This is not anymore possible for (1), so here we proceed totally different using nonvariational arguments. A systematic study of nonvariational methods applied to elliptic problems can be found in [3]. However, such methods cannot be directly implemented in the case of problem (1) taking into account the lack of needed ellipticity. We overcome this difficulty by resolving finite dimensional approximated problems and then passing to the limit in an appropriate sense.

Moreover, in problem (1) there is a lack of any monotonicity property for the driving operator $-\Delta_p + \Delta_q$. This is the reason why the surjectivity theorem for pseudomonotone operators (see, e.g., [4, p. 40]) cannot be applied. A striking difference between the operators $-\Delta_p - \Delta_q$ and $-\Delta_p + \Delta_q$ is that the operator $-\Delta_p - \Delta_q$ is strictly monotone and continuous, so pseudomonotone, whereas this fails for $-\Delta_p + \Delta_q$. Let us note that even the linear continuous operator Δ on $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is not pseudomonotone. For any sequence $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ with $\limsup_{n\to\infty}\langle \Delta u_n, u_n - u\rangle \le 0$, the pseudomonotonicity of Δ would mean that $\liminf_{n\to\infty}\langle \Delta u_n, u_n - v\rangle \ge \langle \Delta u, u - v\rangle$ whenever $v \in H_0^1(\Omega)$, in particular (with $v = 0$), $\liminf_{n\to\infty}\langle \Delta u_n, u_n\rangle \ge \langle \Delta u, u\rangle$. Note that $\limsup_{n\to\infty}\langle \Delta u_n, u_n - u\rangle \le 0$ is always true being equivalent to $\liminf_{n\to\infty} \|\nabla u_n\|^2_{L^2(\Omega)} \ge \|\nabla u\|^2_{L^2(\Omega)}$ that holds thanks to the weak lower semicontinuity of the norm. Besides, $\liminf_{n\to\infty}\langle \Delta u_n, u_n\rangle \ge \langle \Delta u, u\rangle$ is equivalent to $\limsup_{n\to\infty} \|\nabla u_n\|^2_{L^2(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega)}$, which entails the strong convergence $u_n \to u$ since the space $H_0^1(\Omega)$ is uniformly convex. Thus, we reach the contradiction that every weakly convergent sequence is strongly convergent, which proves the claim.

Due to the deficit of ellipticity, monotonicity and variational structure, there are no available techniques to handle problem (1). A fundamental idea of the paper is to seek a solution to (1) as a limit of finite dimensional approximations. To this end, we develop a finite dimensional fixed point approach and then generate a passing to the limit process to get generalized solutions. Our assumptions on the convection term $f(x, u, \nabla u)$ are general and verifiable comprising solely conditions (H1)–(H2). Under a stronger assumption instead of (H1) and with (H2) as it is we are able to prove the existence of a generalized solution in a stronger sense. Finally, we observe that the same procedure applied to a problem driven by the ordinary (p, q)-Laplacian $-\Delta_p - \Delta_q$ and under the same hypotheses leads to the existence of a weak solution.

The rest of the paper consists of sections regarding mathematical background and hypotheses, approximate solutions and existence of generalized solutions to problem (1).

2 Mathematical background and hypotheses

In a Banach space, the strong convergence is denoted by $\to$ and the weak convergence by $\rightharpoonup$. The Euclidean norm on the Euclidean space $\mathbb{R}^m$ for any $m \ge 1$ is denoted by $|\cdot|$, while the standard scalar product is denoted by $\cdot$. For every real number $r > 1$, we set $r' = r/(r-1)$ (the Hölder conjugate of r). In particular, for $1 < q < p < +\infty$ we have $p' = p/(p-1) < q' = q/(q-1)$.

Given a bounded domain $\Omega \subset \mathbb{R}^N$, the Sobolev spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ are endowed with the norms $\|\nabla(\cdot)\|_{L^p(\Omega)}$ and $\|\nabla(\cdot)\|_{L^q(\Omega)}$, respectively, where $\|\cdot\|_{L^r(\Omega)}$ stands for the usual L^r-norm. The dual spaces of $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ are denoted $W^{-1,p'}(\Omega)$ and $W^{-1,q'}(\Omega)$, respectively. In order to avoid repetitive arguments, we assume that $N > p$. The case $N \le p$ is simpler and can be handled along the same lines. Under the assumption $N > p$, the critical exponent is $p^\star = Np/(N-p)$ with the conjugate $(p^\star)' = p^\star/(p^\star - 1)$.

We recall that the negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is expressed as

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

It is a strictly monotone and continuous operator, so pseudomonotone. The only linear case is when $p = 2$ giving rise to the ordinary Laplacian. Similarly, we have the negative q-Laplacian $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$. Due to the assumption $1 < q < p < +\infty$ there is a continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$. Consequently, the differential operator $-\Delta_p + \Delta_q$ in the left-hand side of (1) is well defined on $W_0^{1,p}(\Omega)$.

Next, we turn to the nonlinear term $f(x, u, \nabla u)$ in the right-hand side of equation (1). Such a term depending on the function u and on its gradient ∇u is often called convection. It prevents to settle a variational structure for problem (1). Our hypotheses on the convection term are as follows:

(H1) There exist a nonnegative function $\sigma \in L^{(p^\star)'}(\Omega)$ and constants $b \ge 0$ and $c \ge 0$ such that

$$|f(x,s,\xi)| \le \sigma(x) + b|s|^{p^\star-1} + c|\xi|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \xi \in \mathbb{R}^N.$$

(H2) There exist constants $c_0 < 1$, $c_1 > 0$ and $\alpha \in [1, p)$ such that

$$f(x,s,\xi)s \le c_0|\xi|^p + c_1(|s|^\alpha + 1) \quad \text{for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \quad \xi \in \mathbb{R}^N.$$

We provide a simple example of function verifying (H1)–(H2).

Example 2.1. The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ given by

$$f(x,s,\xi) = |s|^{\alpha-2}s + \frac{s}{1+s^2}(|\xi|^{p-1} + h(x)) \quad \text{for all } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

with a constant $\alpha \in [1, p)$ and some $h \in L^\infty(\Omega)$ satisfies conditions (H1)–(H2).

The next lemma will be useful in the sequel.

Lemma 2.2. *Under assumption* (H1) *one has the estimate*

$$\left|\int_\Omega f(x,u,\nabla u)v\,\mathrm{d}x\right| \le C\,(\|\sigma\|_{L^{(p^\star)'}(\Omega)} + \|u\|_{L^{p^\star}(\Omega)}^{p^\star-1} + \|\nabla u\|_{L^p(\Omega)}^{p-1})\|\nabla v\|_{L^p(\Omega)}$$

for all $u, v \in W_0^{1,p}(\Omega)$, *with a constant* $C > 0$.

Proof. Assumption (H1) and Hölder's inequality lead to

$$\begin{aligned}\left|\int_\Omega f(x,u,\nabla u)v\,\mathrm{d}x\right| &\le \int_\Omega |\sigma||v|\,\mathrm{d}x + b\int_\Omega |u|^{p^\star-1}|v|\,\mathrm{d}x + c\int_\Omega |\nabla u|^{p-1}|v|\,\mathrm{d}x \\ &\le \|\sigma\|_{L^{(p^\star)'}(\Omega)}\|v\|_{L^{p^\star}(\Omega)} + b\|u\|_{L^{p^\star}(\Omega)}^{p^\star-1}\|v\|_{L^{p^\star}(\Omega)} + c\|\nabla u\|_{L^p(\Omega)}^{p-1}\|v\|_{L^p(\Omega)}, \quad \forall u, v \in W_0^{1,p}(\Omega).\end{aligned}$$

Now it suffices to invoke the Sobolev embedding theorem for obtaining the stated conclusion. □

We introduce the notion of solution to problem (1) whose existence can be established under hypotheses (H1)–(H2).

Definition 2.3. Assume that hypothesis (H1) is verified. A function $u \in W_0^{1,p}(\Omega)$ is said to be a generalized solution to problem (1) if there exists a sequence $\{u_n\}_{n\geq 1}$ in $W_0^{1,p}(\Omega)$ such that
(a) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$;
(b) $-\Delta_p u_n + \Delta_q u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \rightharpoonup 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$;
(c) $\lim\limits_{n\to\infty}\Big[\langle -\Delta_p u_n, u_n - u\rangle + \langle \Delta_q u_n, u_n - u\rangle - \int_\Omega f(x, u_n(x), \nabla u_n(x))(u_n(x) - u(x))\,\mathrm{d}x\Big] = 0.$

Lemma 2.2 ensures that Definition 2.3 is correctly formulated.

Lemma 2.2 shows that the Nemytskii operator $N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ corresponding to the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, namely

$$N_f(w) = f(\cdot, w(\cdot), \nabla w(\cdot)), \quad \forall w \in W_0^{1,p}(\Omega),$$

is well defined. Moreover, again by Lemma 2.2, there exists a constant $C > 0$ such that the following estimate holds

$$\|N_f(w)\|_{W^{-1,p'}(\Omega)} \leq C\,(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|^{p^*-1}_{L^{p^*}(\Omega)} + \|\nabla w\|^{p-1}_{L^p(\Omega)}), \quad \forall w \in W_0^{1,p}(\Omega). \tag{3}$$

We focus a bit more on the integral term $\int_\Omega f(x, u_n(x), \nabla u_n(x))(u_n(x) - u(x))\mathrm{d}x$ strengthening the growth condition in (H1).

(H1)$'$ There exist constants $c_1 \geq 0$, $c_2 \geq 0$, $r \in [1, p^*)$, $r_1 \in [1, p^*)$, $r_2 \in [1, p)$ and a nonnegative function $\sigma \in L^{r'}(\Omega)$ such that

$$|f(x, s, \xi)| \leq \sigma(x) + c_1|s|^{\frac{p^*}{r_1'}} + c_2|\xi|^{\frac{p}{r_2'}} \quad \text{for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \ \xi \in \mathbb{R}^N.$$

Remark 2.4. Condition (H1)$'$ implies condition (H1) because if $r_1 \in [1, p^*)$ and $r_2 \in [1, p)$, then $r_1' > (p^*)'$ and $r_2' > p'$, which yields

$$\frac{p^*}{r_1'} < \frac{p^*}{(p^*)'} = p^* - 1 \quad \text{and} \quad \frac{p}{r_2'} < \frac{p}{p'} = p - 1.$$

With (H1)$'$ in place of (H1) we consider the existence of a solution to problem (1) in a stronger sense.

Definition 2.5. A function $u \in W_0^{1,p}(\Omega)$ is said to be a strong generalized solution to problem (1) if there exists a sequence $\{u_n\}_{n\geq 1}$ in $W_0^{1,p}(\Omega)$ such that (a) and (b) in Definition 2.3 are satisfied together with $(c)'$

(c)$'$ $$\lim_{n\to\infty}\langle -\Delta_p u_n + \Delta_q u_n, u_n - u\rangle = 0.$$

Our existence results for generalized and strong generalized solutions are given in Theorems 4.1 and 4.2 of Section 4.

We end this section with the following consequence of Brouwer's fixed point theorem that will be an essential tool in our approach. For a proof we refer to [5, p. 37].

Lemma 2.6. *Let X be a finite dimensional space with the norm $\|\cdot\|_X$ and let $A : X \to X^*$ be a continuous mapping. Assume that there is a constant $R > 0$ such that*

$$\langle A(v), v\rangle \geq 0 \ \textit{for all} \ v \in X \ \textit{with} \ \|v\|_X = R.$$

Then there exists $u \in X$ with $\|u\|_X \leq R$ satisfying $A(u) = 0$.

3 Finite dimensional approximate solutions

Since the Banach space $W_0^{1,p}(\Omega)$ with $1 < p < +\infty$ is separable, there exists a Galerkin basis of $W_0^{1,p}(\Omega)$, which means a sequence $\{X_n\}_{n\geq 1}$ of vector subspaces of $W_0^{1,p}(\Omega)$ satisfying

(i) $\dim(X_n) < \infty, \quad \forall n;$
(ii) $X_n \subset X_{n+1}, \quad \forall n;$
(iii) $\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{1,p}(\Omega).$

Fix a Galerkin basis $\{X_n\}_{n\geq 1}$ of $W_0^{1,p}(\Omega)$. The notation $|\Omega|$ stands for the Lebesgue measure of Ω.

Proposition 3.1. *Assume that conditions* (H1)–(H2) *are fulfilled. Then for each $n \geq 1$ there exists $u_n \in X_n$ such that*

$$\langle -\Delta_p u_n + \Delta_q u_n, v\rangle = \int_\Omega f(x, u_n(x), \nabla u_n(x))v(x)\,dx \quad \textit{for all } v \in X_n. \tag{4}$$

Proof. For each $n \geq 1$ we introduce the mapping $A_n : X_n \to X_n^\star$ by

$$\langle A_n(u), v\rangle = \langle -\Delta_p u + \Delta_q u, v\rangle - \int_\Omega f(x, u(x), \nabla u(x))v(x)\,dx \quad \text{for all } u, v \in X_n.$$

On the basis of assumption (H2), Hölder's inequality and Sobolev embedding theorem we find the estimate

$$\begin{aligned}\langle A_n(v), v\rangle &= \int_\Omega (|\nabla v|^p - |\nabla v|^q - f(x, v, \nabla v)v)\,dx \\ &\geq \int_\Omega ((1 - c_0)|\nabla v|^p - |\nabla v|^q - c_1(|v|^\alpha + 1))\,dx \\ &\geq (1 - c_0)\|\nabla v\|^p_{L^p(\Omega)} - |\Omega|^{\frac{p-q}{p}}\|\nabla v\|^q_{L^p(\Omega)} - \tilde{c}_1(\|\nabla v\|^\alpha_{L^p(\Omega)} + 1), \quad \forall v \in X_n,\end{aligned}$$

with a constant $\tilde{c}_1 > 0$. Using that $p > q$, $p > \alpha$ and $c_0 < 1$ we conclude that

$$\langle A_n(v), v\rangle \geq 0 \text{ whenever } v \in X_n \text{ with } \|\nabla v\|_{L^p(\Omega)} = R$$

provided $R > 0$ is sufficiently large. Lemma 2.6 can thus be applied for $X = X_n$ and $A = A_n$. Therefore, there exists $u_n \in X_n$ solving the equation $A_n(u_n) = 0$, which is just (4). The proof is complete. □

Corollary 3.2. *Assume that conditions* (H1)–(H2) *are fulfilled. Then the sequence $\{u_n\}_{n\geq 1}$, with $u_n \in X_n$ constructed in Proposition 3.1, is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Insert $v = u_n$ in (4). Through hypothesis (H2), Hölder's inequality and Sobolev embedding theorem, it gives

$$\|\nabla u_n\|^p_{L^p(\Omega)} = \|\nabla u_n\|^q_{L^q(\Omega)} + \int_\Omega f(x, v, \nabla u_n)u_n\,dx \leq |\Omega|^{\frac{p-q}{p}}\|\nabla u_n\|^q_{L^p(\Omega)} + c_0\|\nabla u_n\|^p_{L^p(\Omega)} + \tilde{c}_1(\|\nabla u_n\|^\alpha_{L^p(\Omega)} + 1),$$

with a constant $\tilde{c}_1 > 0$, or

$$(1 - c_0)\|\nabla u_n\|^p_{L^p(\Omega)} \leq |\Omega|^{\frac{p-q}{p}}\|\nabla u_n\|^q_{L^p(\Omega)} + \tilde{c}_1(\|\nabla u_n\|^\alpha_{L^p(\Omega)} + 1).$$

Recalling that $p > q$, $p > \alpha$ and $c_0 < 1$ we achieve the desired conclusion. □

4 Existence of generalized solutions

The statement below constitutes our existence result for generalized solutions to problem (1).

Theorem 4.1. *Assume that conditions* (H1)–(H2) *hold. Then there exists a generalized solution to problem* (1) *in the sense of Definition 2.3.*

Proof. By Corollary 3.2, we know that the sequence $\{u_n\}_{n\geq 1}$ constructed in Proposition 3.1 is bounded in $W_0^{1,p}(\Omega)$. Since the space $W_0^{1,p}(\Omega)$ is reflexive, along a relabeled subsequence we have that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \tag{5}$$

for some $u \in W_0^{1,p}(\Omega)$.

From estimate (3) we infer that the Nemytskii operator $N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (in the sense that it maps bounded sets into bounded sets). Taking into account (5) it holds

$$\{N_f(u_n)\}_{n\geq 1} \text{ is a bounded sequence in } W^{-1,p'}(\Omega). \tag{6}$$

Since $-\Delta_p + \Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is also a bounded operator, by (6) it follows that the sequence $\{-\Delta_p u_n + \Delta_q u_n - N_f(u_n)\}_{n\geq 1}$ is bounded in $W^{-1,p'}(\Omega)$. Thanks to the reflexivity of $W^{-1,p'}(\Omega)$, we can pass to a relabeled subsequence finding that

$$-\Delta_p u_n + \Delta_q u_n - N_f(u_n) \rightharpoonup \eta \text{ in } W^{-1,p'}(\Omega) \tag{7}$$

for some $\eta \in W^{-1,p'}(\Omega)$.

Let $v \in \bigcup_{n\geq 1} X_n$. There is an integer $m \geq 1$ such that $v \in X_m$. Applying Proposition 3.1, we see that equality (4) holds true for all $n \geq m$. Letting $n \to \infty$ in (4) entails

$$\langle \eta, v\rangle = 0 \text{ for all } v \in \bigcup_{n\geq 1} X_n.$$

We deduce that $\eta = 0$ because $\bigcup_{n\geq 1} X_n$ is dense in $W_0^{1,p}(\Omega)$ (see requirement (iii) in the definition of Galerkin basis in Section 3). Consequently, (7) becomes

$$-\Delta_p u_n + \Delta_q u_n - N_f(u_n) \rightharpoonup 0 \text{ in } W^{-1,p'}(\Omega). \tag{8}$$

Set $v = u_n$ in (4), which provides

$$\|\nabla u_n\|_{L^p(\Omega)}^p = \|\nabla u_n\|_{L^q(\Omega)}^q + \int_\Omega f(x, u_n(x), \nabla u_n(x))u_n(x)\,\mathrm{d}x, \quad \forall n \geq 1. \tag{9}$$

On the other hand, by (8) we get

$$\langle -\Delta_p u_n + \Delta_q u_n - N_f(u_n), u\rangle \to 0 \text{ as } n \to \infty. \tag{10}$$

Combining (9) and (10) it turns out

$$\lim_{n\to\infty} \langle -\Delta_p u_n + \Delta_q u_n - N_f(u_n), u_n - u\rangle = 0. \tag{11}$$

If we gather (5), (8) and (11), we obtain that $u \in W_0^{1,p}(\Omega)$ is a generalized solution to problem (1) in the sense of Definition 2.3. The proof is thus complete. □

Next we state our second existence result.

Theorem 4.2. *Assume that conditions* (H1)$'$–(H2) *hold. Then there exists a strong generalized solution to problem* (1) *in the sense of Definition 2.5.*

Proof. In view of Remark 2.4 we are allowed to use all the formulas in the proof of Theorem 4.1. By (H1)$'$ we are able to write

$$\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,\mathrm{d}x\right| \le \int_\Omega |f(x,u_n,\nabla u_n)||u_n-u|\mathrm{d}x \le \int_\Omega \left(\sigma(x)+c_1|u_n|^{\frac{p^*}{r'_1}}+c_2\,|\nabla u_n|^{\frac{p}{r'_2}}\right)|u_n-u|\mathrm{d}x.$$

Through the preceding estimate, in conjunction with Hölder's inequality, we arrive at

$$\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,\mathrm{d}x\right| \le \|\sigma\|_{L^{r'}(\Omega)}\|u_n-u\|_{L^r(\Omega)}+c_1\|u_n\|^{\frac{p^*}{r'_1}}_{L^{p^*}(\Omega)}\|u_n-u\|_{L^{r_1}(\Omega)}+c_2\,\|\nabla u_n\|^{\frac{p}{r'_2}}_{L^p(\Omega)}\|u_n-u\|_{L^{r_2}(\Omega)},\quad \forall n\ge 1. \tag{12}$$

The boundedness of $\{u_n\}_{n\ge1}$ in $W_0^{1,p}(\Omega)$ (see (5)) implies that $\{u_n\}_{n\ge1}$ is bounded in $L^{p^*}(\Omega)$ and $\{\nabla u_n\}_{n\ge1}$ is bounded in $(L^p(\Omega))^N$. Then (12) renders

$$\left|\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,\mathrm{d}x\right| \le M(\|u_n-u\|_{L^r(\Omega)}+\|u_n-u\|_{L^{r_1}(\Omega)}+\|u_n-u\|_{L^{r_2}(\Omega)}),\quad \forall n\ge 1, \tag{13}$$

with a constant $M>0$.

By the Rellich-Kondrachov theorem we know that $r\in[1,p^*)$, $r_1\in[1,p^*)$, $r_2\in[1,p)$ in (H1)$'$ and (5) ensure the strong convergence $u_n\to u$ in $L^r(\Omega)$, $L^{r_1}(\Omega)$ and $L^{r_2}(\Omega)$ as $n\to\infty$. Then from (13) we derive

$$\lim_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,\mathrm{d}x = 0. \tag{14}$$

On the basis of (11) and (14) we find that

$$\lim_{n\to\infty}\langle -\Delta_p u_n+\Delta_q u_n, u_n-u\rangle = 0. \tag{15}$$

Altogether (5), (8) and (15) show that u is a strong generalized solution to equation (1), thus completing the proof. □

Finally, we note that contrary to what happens for problem (1), our notion of strong generalized solution in the case of the problem

$$\begin{cases} -\Delta_p u-\Delta_q u = f(x,u,\nabla u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega\end{cases} \tag{16}$$

coincides with the classical notion of weak solution. Specifically, under hypothesis (H1)$'$, a function $u\in W_0^{1,p}(\Omega)$ is by definition a strong generalized solution to problem (16) if there exists a sequence $\{u_n\}_{n\ge1}$ in $W_0^{1,p}(\Omega)$ such that $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $n\to\infty$,

$$-\Delta_p u_n-\Delta_q u_n - f(\cdot,u_n(\cdot),\nabla u_n(\cdot))\rightharpoonup 0 \;\text{ in }\; W^{-1,p'}(\Omega)\;\text{ as }\; n\to\infty \tag{17}$$

and

$$\lim_{n\to\infty}\langle -\Delta_p u_n-\Delta_q u_n, u_n-u\rangle = 0. \tag{18}$$

We claim that $u \in W_0^{1,p}(\Omega)$ is a strong generalized solution to problem (16) if and only if it is a weak solution to problem (16), that is,

$$\langle -\Delta_p u - \Delta_q u, v\rangle = \int_\Omega f(x, u(x), \nabla u(x))v(x)\,\mathrm{d}x \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{19}$$

Indeed, if $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem (16), then posing $u_n = u$ it is clear that u is a strong generalized solution to problem (16). Conversely, assume that $u \in W_0^{1,p}(\Omega)$ is a strong generalized solution to problem (16). Hence, there exists a sequence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$ for which (17) and (18) hold. From (18), the monotonicity of $-\Delta_q$ and the weak convergence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ it follows that

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle \le \lim_{n\to\infty} \langle -\Delta_p u_n - \Delta_q u_n, u_n - u\rangle = 0.$$

As the operator $-\Delta_p$ on $W_0^{1,p}(\Omega)$ fulfills the (S_+)-property (see, e.g., [4, p. 45]), we infer the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then the continuity of the operators $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ and $N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ enables us to deduce from (17) the validity of (19), thus u is a weak solution to problem (16).

Remark 4.3. Proceeding as in the proof of Theorem 4.2 one obtains under Assumptions (H1)′–(H2) that there exists a strong generalized solution to problem (16). Thereby, in view of what has been said in the preceding comments, under Assumptions (H1)′–(H2) there exists a weak solution to problem (16).

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