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Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces

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ABSTRACT

In this paper, we are interested in the existence of infinitely many weak solutions for a non-homogeneous eigenvalue Dirichlet problem. By using variational methods, in an appropriate Orlicz–Sobolev setting, we determine intervals of parameters such that our problem admits either a sequence of non-negative weak solutions strongly converging to zero provided that the non-linearity has a suitable behaviour at zero or an unbounded sequence of non-negative weak solutions if a similar behaviour occurs at infinity.

1. Introduction

Classical Sobolev and Orlicz–Sobolev spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasiconformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, and probability theory.

In this framework, we study in the present paper the non-homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) = \lambda h(x) f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$D^{f,h}_{\alpha,\lambda}$}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, while $f : \mathbb{R} \to \mathbb{R}$ and $h : \bar{\Omega} \to [0, +\infty[$ are continuous functions, λ is a positive parameter and $\alpha : (0, \infty) \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Precisely, the main goal is to establish the existence of a precise interval of positive parameters λ such that problem $(D^{f,h}_{\alpha,\lambda})$ admits either a sequence of non-negative weak solutions strongly converging to zero provided that the non-linearity has a suitable behaviour at zero or an unbounded sequence of non-negative weak solutions if a similar behaviour occurs at infinity. The interest in analysing this kind of problems is motivated by some recent advances in the study of eigenvalue problems involving non-homogeneous operators in the divergence form; see, for instance, the papers [1–12].

The study of nonlinear elliptic equations involving quasilinear homogeneous type operators is based on the theory of Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. In the case of nonhomogeneous differential operators, the natural setting for this approach is the use of Orlicz–Sobolev spaces. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. The basic idea is to replace the Lebesgue spaces $L^p(\Omega)$ by more general spaces $L_\Phi(\Omega)$, called Orlicz spaces. The spaces $L_\Phi(\Omega)$ were thoroughly studied in the monograph by Kranosel'skii and Rutickii [13] and also in the doctoral thesis of Luxemburg [14]. If the role played by $L^p(\Omega)$ in the definition of the Sobolev spaces $W^{m,p}(\Omega)$ is assigned instead to an Orlicz space $L_\Phi(\Omega)$ then the resulting space is denoted by $W^m L_\Phi(\Omega)$ and it is called an Orlicz–Sobolev space.

Many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces, mainly by Dankert [15], Donaldson and Trudinger [16], and O'Neill [17] (see also [18] for an excellent account of those works). Orlicz–Sobolev spaces have been used in the last decades to model various phenomena.

Chen et al. [19] proposed a framework for image restoration based on a variable exponent Laplacian. A second application which uses variable exponent type Laplace operators is the modelling of some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as 'smart fluids'), cf. [20–22,9,23]. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium polymethacrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics consult Halsey [24] and for some technical applications Pfeiffer et al. [25]. An overview of Orlicz–Sobolev spaces is given in the monographs by Rao and Ren [26].

The main tool in order to prove our multiplicity result is the following critical point theorem obtained in [27] that we recall here in a convenient form; see also the variational principle of Ricceri [28].

Theorem 1.1 ([27, Theorem 2.1]). *Let X be a reflexive real Banach space, let $J, I : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that J is strongly continuous, sequentially weakly lower semi-continuous and coercive and I is sequentially weakly upper semi-continuous. For every $r > \inf_X J$, put*

$$\widetilde{\varphi}(r) := \inf_{u \in J^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in J^{-1}(]-\infty, r[)} I(v)\right) - I(u)}{r - J(u)},$$

and

$$\gamma := \liminf_{r \to +\infty} \widetilde{\varphi}(r), \qquad \delta := \liminf_{r \to (\inf_X J)^+} \widetilde{\varphi}(r).$$

Therefore

(a) *If $\gamma < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds:*
either
(a_1) *$g_\lambda := J - \lambda I$ possesses a global minimum,*
or
(a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of g_λ such that $\lim_{n\to\infty} J(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds:*
either
(b_1) *there is a global minimum of J which is a local minimum of g_λ,*
or
(b_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of g_λ which weakly converges to a global minimum of J, with $\lim_{n\to\infty} J(u_n) = \inf_X J$.*

We point out that, by using the above theoretical result, in [29] it was proved the existence of a well determined open interval of positive parameters for which a non-homogeneous Neumann problem admits infinitely many weak solutions in $W^1L_\Phi(\Omega)$, that strongly converges to zero; see also the paper [30].

In this paper, treating a non-homogeneous Dirichlet problem, our framework is more complicated. For instance, we cannot use constant functions in some steps of our proof. We overcome this difficulty by using a suitable sequence of cut-off maps belonging to $W_0^1L_\Phi(\Omega)$. In addition, through a careful analysis of the Young function associated to the operator in divergence form, we prove that the solutions are non-negative.

As pointed out before, this paper is motivated by recent advances on existence results for elliptic problems involving non-homogeneous quasilinear operators. In this respect, the results contained in the present paper can be related with various contributions present in literature involving general operators in divergence form.

The nonlinear boundary value problem

$$\begin{cases} -\mathrm{div}(\log(1+|\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u(x)) = f(u(x)), & \text{for } x \in \Omega \\ u(x) = 0, & \text{for } x \in \partial\Omega \end{cases}$$

was analysed by Mihăilescu and Rădulescu in [10]. In their paper two cases has been considered, where either

$$f(u) = -\lambda|u|^{p-2}u + |u|^{r-2}u, \quad \text{or} \quad f(u) = \lambda|u|^{p-2}u - |u|^{r-2}u,$$

with $p, q > 1$, $p + q < \min\{N, r\}$, and $r < (Np - N + p)/(N - p)$. In the first case the existence of infinitely many weak solutions for any $\lambda > 0$ is proved, while in the second case, the existence of a nontrivial weak solution is established, provided that λ is sufficiently large.

Previously, under appropriate conditions on the Young function Φ associated to the non-homogeneous potential α and requiring a suitable growth of f, the existence of nontrivial solutions for the problem $(D_{\alpha,1}^{f,1})$, which are of mountain pass type, was obtained by Clément, García-Huidobro, Manásevich and Schmitt in [1].

Moreover, let $q \in \bar{\Omega} \to \mathbb{R}$ be a continuous function. In [7, Theorem 2], the following anisotropic problem

$$\begin{cases} -\mathrm{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = \lambda|u(x)|^{q(x)-2}u(x), & \text{for } x \in \Omega \\ u(x) = 0, & \text{for } x \in \partial\Omega \end{cases}$$

is investigated. More precisely, through variational arguments, sufficient conditions on α and q such that the above problem admits continuous families of eigenvalues have been established.

We just mention that Omari and Zanolin [31], by using lower and super solutions method and requiring an oscillating behaviour of the potential $F(\xi) := \int_0^\xi f(t)dt$ at infinity, proved the existence of infinitely many solutions for a perturbed Dirichlet problem involving a quasilinear elliptic second order differential operator defined by

$$\mathcal{A}u := \mathrm{div}(a(|\nabla u|^2)\nabla u),$$

where $a : (0, +\infty) \to (0, +\infty)$ is a map of class C^1 satisfies the following ellipticity and growth conditions of Leray–Lions type:

(LL) *There are constants* $\gamma, \Gamma > 0$, $\kappa \in [0, 1]$ *and* $p \in (1, +\infty)$ *such that, for every* $t > 0$,

$$\gamma(\kappa + t)^{p-2} \le a(t^2) \le \Gamma(\kappa + t)^{p-2},$$

and

$$\left(\gamma - \frac{1}{2}\right)a(t) \le a'(t)t \le \Gamma a(t).$$

More precisely, Omari and Zanolin require that

$$-\infty < \liminf_{|\xi|\to\infty} \frac{F(\xi)}{A(\xi^2)} \le 0, \quad \text{and} \quad \limsup_{|\xi|\to\infty} \frac{F(\xi)}{A(\xi^2)} = +\infty, \tag{h_A}$$

where $A(\xi) := \int_0^\xi a(t)dt$, for every $\xi \in \mathbb{R}^+$.

The class of quasilinear differential operators $\mathcal{A}$ considered in the cited work includes in particular the p–Laplacian operator Δ_p, with $1 < p < \infty$, corresponding to $a(t) := t^{p/2-1}$, for every $t \in]0, +\infty[$. Further, in [32], for p–Laplacian equations, the existence of infinitely many solutions, has been studied requiring the following hypothesis on F at zero:

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = 0, \quad \text{and} \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = +\infty. \tag{OZ}$$

A careful analysis between our approach, for equations involving the p–Laplacian with Dirichlet boundary condition, and the above cited result can be found in [33].

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It is easy to see that no results from [31] can be applied for a class of elliptic problems whose prototype is

$$\begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right) = \lambda f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{D_λ^f}$$

where $p > N+1$; see Remark 4.2.

The plan of the paper is as follows. In Section 2 we introduce our notation and the abstract Orlicz–Sobolev spaces setting. Section 3 is devoted to main theorem and finally, in Section 4, as application, we prove that, for every $\lambda > 0$, there exists a sequence of pairwise distinct non-negative solutions for the problem (D_λ^f) that strongly converges to zero in $W_0^1 L_\Phi(\Omega)$; see Theorem 4.1.

We cite the very recent monograph by Kristály et al. [34] as general reference on this subject.

2. Orlicz–Sobolev spaces setting

A complete description regarding the development of variable exponent function spaces, based on a rich bibliography, can be found in the recent monograph by Diening et al. [35].

This section summarizes those aspects of the theory of Orlicz–Sobolev spaces that will be used in the present paper. This function spaces provide an appropriate venue for the analysis of quasilinear elliptic partial differential equations with rapidly or slowly growing principal parts.

Variable exponent function spaces had already appeared in the literature for the first time in a article by Orlicz [36]. In the 1950s, this study was carried on by Nakano [37], who made the first systematic study of spaces with variable exponent (called *modular spaces*). Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered, see [37, p. 284]. Later, the Polish mathematicians investigated the modular function spaces (e.g. [38]).

Set

$$\Phi(t) = \int_0^t \varphi(s)ds, \qquad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)ds, \quad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty,$$

then Φ is called an N–function. The function $\Phi^\star$ is called the complementary function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s); s \geq 0\}, \quad \text{for all } t \geq 0.$$

We observe that $\Phi^\star$ is also an N–function and the following Young's inequality holds true:

$$st \leq \Phi(s) + \Phi^\star(t), \quad \text{for all } s, t \geq 0.$$

Assume that Φ satisfying the following structural hypotheses

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq p^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < +\infty; \tag{Φ_0}$$

$$N < p_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{Φ_1}$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N–function Φ (see for instance [18,1]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{\int_\Omega u(x)v(x)dx; \int_\Omega \Phi^\star(|v(x)|)dx \leq 1\right\} < +\infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0; \int_\Omega \Phi\left(\frac{u(x)}{k}\right)dx \leq 1\right\}.$$

For Orlicz spaces the Hölder's inequality reads as follows (see [26, Inequality 4, p. 79]):

$$\int_\Omega uv dx \leq 2\|u\|_{L_\Phi}\|v\|_{L_{\Phi^\star}} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^\star}(\Omega).$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space for problem $(D^{f,h}_{\alpha,\lambda})$, defined by

$$W^1L_\Phi(\Omega)=\left\{u\in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i}\in L_\Phi(\Omega),\ i=1,\dots,N\right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi}=\||\nabla u|\|_\Phi+\|u\|_\Phi.$$

We refer to [18,1] for proofs and related results.

Following [18] we will define $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$, which is also reflexive under assumption (Φ_0). Also, from Lemma 5.7 in [39] we obtain that $W_0^1L_\Phi(\Omega)$ may be (equivalently) renormed by using as norm

$$\|u\|:=\||\nabla u|\|_\Phi,$$

for every $u\in W_0^1L_\Phi(\Omega)$; see also [1].

These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t\mapsto|t|^p/p$ is assumed by a more general convex function $\Phi(t)$.

We recall the following result.

Lemma 2.1. *Let $u\in W_0^1L_\Phi(\Omega)$. Then*

$$\int_\Omega\Phi(|\nabla u(x)|)dx\le\|u\|^{p_0},\quad\text{if } \|u\|<1;\tag{1}$$

$$\int_\Omega\Phi(|\nabla u(x)|)dx\ge\|u\|^{p_0},\quad\text{if } \|u\|>1;\tag{2}$$

and

$$\int_\Omega\Phi(|\nabla u(x)|)dx\ge\|u\|^{p^0},\quad\text{if } \|u\|<1;\tag{3}$$

$$\int_\Omega\Phi(|\nabla u(x)|)dx\le\|u\|^{p^0},\quad\text{if } \|u\|>1.\tag{4}$$

For the proof of the previous result see, for instance, Lemma 1 of [40] and Lemma C.9 in [2].

Moreover, we say that $u\in W_0^1L_\Phi(\Omega)$ is a *weak solution* for problem $(D^{f,h}_{\alpha,\lambda})$ if

$$\int_\Omega\alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)dx=\lambda\int_\Omega h(x)f(u(x))v(x)dx,$$

for every $v\in W_0^1L_\Phi(\Omega)$.

Finally, the following remarks will be useful in the sequel.

Remark 2.1. Let $u\in W_0^1L_\Phi(\Omega)$ and assume that $\|u\|=1$. Then

$$\int_\Omega\Phi(|\nabla u(x)|)dx=1.\tag{5}$$

Indeed, in our hypothesis, there exists a sequence $\{u_n\}\subset W_0^1L_\Phi(\Omega)$ such that $u_n\to u$ in $W_0^1L_\Phi(\Omega)$ and $\|u_n\|>1$ for every $n\in\mathbb{N}$. For instance, one can take $u_n:=\frac{n+1}{n}u$. Now, from (2) and (4) in Lemma 2.1, the following inequality holds

$$\|u_n\|^{p_0}\le\int_\Omega\Phi(|\nabla u_n(x)|)dx\le\|u_n\|^{p^0}.$$

Then

$$\lim_{n\to\infty}\int_\Omega\Phi(|\nabla u_n(x)|)dx=1.$$

Finally, the thesis is achieved taking into account that

$$\lim_{n\to\infty}\int_\Omega\Phi(|\nabla u_n(x)|)dx=\int_\Omega\Phi(|\nabla u(x)|)dx,$$

from the continuity of the map $u\mapsto\int_\Omega\Phi(|\nabla u(x)|)dx$.

Remark 2.2. We just point out that Lemma 2.1 and Remark 2.1 ensure that for every $u \in W_0^1 L_\Phi(\Omega)$ such that, for some $0 < r < 1$,

$$\int_\Omega \Phi(|\nabla u(x)|)dx \le r, \tag{6}$$

it follows that $\|u\| < 1$; see, for instance, the recent paper [41] for a direct proof involving some properties of the Young function. Moreover

$$\|u\| \le \max\left\{ \left(\int_\Omega \Phi(|\nabla u(x)|)dx\right)^{1/p_0}, \left(\int_\Omega \Phi(|\nabla u(x)|)dx\right)^{1/p^0} \right\}, \tag{7}$$

for every $u \in W_0^1 L_\Phi(\Omega)$.

Finally, let $\{u_n\} \subset W_0^1 L_\Phi(\Omega)$. A straightforward computation shows that if

$$\int_\Omega \Phi(|\nabla u_n(x)|)dx \to +\infty, \quad \text{then } \|u_n\| \to +\infty. \tag{8}$$

3. Main results

From hypothesis (Φ_1), by Lemma D.2 in [1] it follows that $W^1 L_\Phi(\Omega)$ is continuously embedded in $W^{1,p_0}(\Omega)$. On the other hand, since we assume $p_0 > N$, we deduce that $W_0^{1,p_0}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

Thus, $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\,\|u\|, \quad \forall u \in W_0^1 L_\Phi(\Omega), \tag{9}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$.

The aim of the paper is to prove the following result concerning the existence of infinitely many weak solutions of the problem $(D_{\alpha,\lambda}^{f,h})$.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Assume that*

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p^0}} < +\infty \quad \text{and} \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p_0}} = +\infty. \tag{h_0}$$

Suppose $h : \bar{\Omega} \to [0, +\infty)$ is a continuous and non-identically zero continuous function. Then, for every

$$\lambda \in \left]0, \frac{1}{c^{p^0}\|h\|_{L^1(\Omega)} \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p^0}}}\right[,$$

there exists a sequence $\{v_n\}$ of pairwise distinct non-negative weak solutions of problem $(D_{\alpha,\lambda}^{f,h})$ such that $\lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|v_n\|_\infty = 0$.

Proof. Let us consider the truncated problem

$$\begin{cases} -\mathrm{div}(\alpha(|\nabla u|)\nabla u) = \lambda h(x) f^\star(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$D_{\alpha,\lambda}^{f^\star,h}$}$$

where

$$f^\star(t) := \begin{cases} f(t) & \text{if } t \in]0, +\infty[\\ 0 & \text{otherwise}. \end{cases}$$

Since, from our assumptions, $f(0) = 0$ it follows that $f^\star \in C^0(\mathbb{R}, \mathbb{R})$. We will prove that problem $(D_{\alpha,\lambda}^{f^\star,h})$ admits a sequence of weak solutions that strongly converges to zero. Set $X := W_0^1 L_\Phi(\Omega)$. Hypothesis (Φ_0) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2-condition (at infinity), see [18, p. 232] and [1]. In particular, both (Φ, Ω) and $(\Phi^\star, \Omega)$ are Δ-regular, see [18, p. 232]. Consequently, the spaces $L_\Phi(\Omega)$ and $W^1 L_\Phi(\Omega)$ are separable, reflexive Banach spaces, see [18, p. 241 and p. 247]. Now, define the functionals $J, I : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega \Phi(|\nabla u(x)|)dx \quad \text{and} \quad I(u) = \int_\Omega h(x) F(u(x))dx,$$

where $F(\xi) := \int_0^\xi f^\star(t)dt$ for every $\xi \in \mathbb{R}$ and put

$$g_\lambda(u) := J(u) - \lambda I(u), \quad u \in X.$$

The functionals J and I satisfy the regularity assumptions of Theorem 1.1. Indeed, similar arguments as those used in [4, Lemma 3.4] and [1, Lemma 2.1] imply that $J, I \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\langle J'(u), v\rangle = \int_\Omega \alpha(|\nabla u(x)|)\nabla u(x) \cdot \nabla v(x)dx,$$
$$\langle I'(u), v\rangle = \int_\Omega h(x)f^\star(u(x))v(x)dx,$$

for any $u, v \in X$.

Moreover, since Φ is convex, it follows that J is a convex functional, hence J is sequentially weakly lower semi-continuous. Finally we observe that J is a coercive functional. Indeed, by formula (2) of Lemma 2.1, we have $J(u) \geq \|u\|^{p_0}$ for every $u \in X$ with $\|u\| > 1$. On the other hand, since X is compactly embedded into $C^0(\overline{\Omega})$ then the operator $I' : X \to X^\star$ is compact. Consequently, the functional $I : X \to \mathbb{R}$ is sequentially weakly (upper) continuous, see [42, Corollary 41.9]. Let us observe that $u \in X$ is a weak solution of problem $(D^{f,h}_{\alpha,\lambda})$ if u is a critical point of the functional g_λ. Hence, we can seek for weak solutions of problem $(D^{f,h}_{\alpha,\lambda})$ by applying part (b) of Theorem 1.1. Now, let $\{c_n\} \subset]0, \infty[$ be a sequence such that $\lim_{n\to\infty} c_n = 0$ and

$$\lim_{n\to\infty} \frac{F(c_n)}{c_n^{p^0}} = \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p^0}}.$$

Put $r_n = \left(\frac{c_n}{c}\right)^{p^0}$ for every $n \in \mathbb{N}$. Then, by inequality (7), it follows that

$$\{v \in W_0^1 L_\Phi(\Omega) : J(v) < r_n\} \subseteq \left\{v \in W_0^1 L_\Phi(\Omega) : \|v\| < \max\{r_n^{1/p_0}, r_n^{1/p^0}\}\right\}.$$

On the other hand, since $r_n < 1$, we have

$$\max\{r_n^{1/p_0}, r_n^{1/p^0}\} = r_n^{1/p^0},$$

and

$$\left\{v \in W_0^1 L_\Phi(\Omega) : \|v\| < r_n^{1/p^0}\right\} = \left\{v \in W_0^1 L_\Phi(\Omega) : \|v\| < \frac{c_n}{c}\right\}.$$

Moreover, due to (9), we have

$$|v(x)| \leq \|v\|_\infty \leq c\|v\| \leq c_n, \quad \forall x \in \overline{\Omega}.$$

Hence

$$\left\{v \in W_0^1 L_\Phi(\Omega) : \|v\| < \frac{c_n}{c}\right\} \subseteq \left\{v \in W_0^1 L_\Phi(\Omega) : \|v\|_\infty \leq c_n\right\}.$$

Set $u_0(x) = 0$ for every $x \in \bar{\Omega}$. Taking into account that $J(u_0) = I(u_0) = 0$, we deduce that

$$\begin{aligned}\widetilde{\varphi}(r_n) &= \inf_{J(u)<r_n} \frac{\sup_{J(v)<r_n} \int_\Omega h(x)F(v(x))dx - \int_\Omega h(x)F(u(x))dx}{r_n - J(u)} \\ &\leq \frac{\sup_{J(v)<r_n} \int_\Omega h(x)F(v(x))dx}{r_n} \leq \left(\int_\Omega h(x)dx\right) \frac{\max_{|\xi|\leq c_n} F(\xi)}{r_n} \\ &= \|h\|_{L^1(\Omega)} \frac{F(c_n)}{r_n} = c^{p^0} \|h\|_{L^1(\Omega)} \frac{F(c_n)}{c_n^{p^0}}.\end{aligned}$$

Therefore, since

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p^0}} < +\infty,$$

we have

$$\delta \leq \liminf_{n\to\infty} \widetilde{\varphi}(r_n) \leq c^{p^0} \|h\|_{L^1(\Omega)} \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p^0}} < +\infty.$$

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Now, take

$$\lambda \in \left]0, \frac{1}{c^{p^0}\|h\|_{L^1(\Omega)} \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p^0}}}\right[.$$

We show in what follows that 0, which is the unique global minimum of J, is not a local minimum of g_λ. Let $D \subset \Omega$ be a compact set of positive Lebesgue measure such that $h(x) > 0$ for every $x \in D$. Consider a cut-off function $\theta \in C_0^\infty(\Omega)$ (so $\theta \in W_0^1 L_\Phi(\Omega)$) such that $D \subset \mathrm{Supp}(\theta) \subseteq \Omega$, $0 \le \theta(x) \le 1$ for every $x \in \Omega$ and $\theta|_D = 1$. Bearing in mind that

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p_0}} = +\infty,$$

there exists a sequence $\{\xi_n\}$ in $]0, \rho[$ such that $\lim_{n\to\infty} \xi_n = 0$ and

$$\lim_{n\to\infty} \frac{F(\xi_n)}{\xi_n^{p_0}} = +\infty. \tag{10}$$

Consider the sequence of functions $\{\xi_n\theta\} \subset W_0^1 L_\Phi(\Omega)$. Clearly $\|\xi_n\theta\| \to 0$ and

$$g_\lambda(\xi_n\theta) = J(\xi_n\theta) - \lambda F(\xi_n)\int_D h(x)dx - \lambda \int_{\Omega\setminus D} h(x)F(\xi_n\theta(x))dx.$$

Now, bearing in mind (1) of Lemma 2.1, it follows that

$$J(\xi_n\theta) = \int_\Omega \Phi(\xi_n|\nabla\theta(x)|)dx \le \|\xi_n\theta\|^{p_0} = \|\theta\|^{p_0}\xi_n^{p_0}. \tag{11}$$

Moreover, since $h(x)F(\xi_n\theta(x)) \ge 0$ for every $x \in \bar{\Omega}$, we obtain

$$g_\lambda(\xi_n\theta) \le \|\theta\|^{p_0}\xi_n^{p_0} - \lambda F(\xi_n)\int_D h(x)dx,$$

for every n big enough. Thus

$$\frac{g_\lambda(\xi_n\theta)}{\xi_n^{p_0}} \le \|\theta\|^{p_0} - \lambda\frac{F(\xi_n)}{\xi_n^{p_0}}\int_D h(x)dx, \quad \forall n \ge \nu,$$

for some $\nu \in \mathbb{N}$.

Owing to $\int_D h(x)dx > 0$, where $\mathrm{meas}(D) > 0$, from (10) and taking into account the above inequality, we obtain

$$\lim_{n\to\infty} \frac{g_\lambda(\xi_n\theta)}{\xi_n^{p_0}} = -\infty.$$

Thus, $g_\lambda(\xi_n\theta) < 0$ definitively. Since $g_\lambda(0) = J(0) - \lambda I(0) = 0$, this means that 0 is not a local minimum of g_λ. Then, owing to J has 0 as unique global minimum, Theorem 1.1 ensures the existence of a sequence $\{v_n\}$ of pairwise distinct critical points of the functional g_λ (so a sequence of weak solution of the truncated problem) such that

$$0 \le \lim_{n\to\infty} \|v_n\|^{p^0} \le \lim_{n\to\infty} \int_\Omega \Phi(|\nabla v_n(x)|)dx = 0.$$

We deduce that $\lim_{n\to\infty} \|v_n\| = 0$. Moreover, by (9), it follows that $\lim_{k\to\infty} \|v_n\|_\infty = 0$. Finally, we prove here that the attained solutions are non-negative, in particular they are (non-negative) weak solutions of our initial problem. Hence, let $v_0 \in W_0^1 L_\Phi(\Omega)$ be one (non-trivial) weak solution of the truncated problem $(D^{f^\star;h}_{\alpha,\lambda})$. Arguing by contradiction, if we assume that v_0 is negative at a point of Ω the set

$$\Omega^- := \{x \in \Omega : v_0(x) < 0\},$$

is non-empty and open. Moreover, let us consider $v_0^\star := \min\{v_0, 0\}$. From either [5] or [1], we deduce that $v_0^\star \in W_0^1 L_\Phi(\Omega)$ and

$$|\nabla v_0^\star(x)| = \begin{cases} |\nabla v_0(x)| & \text{for } x \text{ such that } v_0(x) < 0 \\ 0 & \text{for } x \text{ such that } v_0(x) \ge 0. \end{cases}$$

Then, since

$$\int_\Omega \alpha(|\nabla v_0(x)|)\nabla v_0(x)\cdot\nabla v(x)dx = \lambda\int_\Omega h(x)f^\star(v_0(x))v(x)dx,$$

for every $v \in W_0^1 L_\Phi(\Omega)$, by setting $v = v_0^\star$ it follows that

$$\int_{\Omega^-} \alpha(|\nabla v_0(x)|)\nabla v_0(x) \cdot \nabla v_0^\star(x)dx = \lambda \int_{\Omega^-} h(x)f^\star(v_0(x))v_0^\star(x)dx.$$

Therefore

$$\int_{\Omega^-} \alpha(|\nabla v_0^\star(x)|)|\nabla v_0^\star(x)|^2 dx = 0,$$

which means

$$\int_{\Omega^-} \varphi(|\nabla v_0^\star(x)|)|\nabla v_0^\star(x)|dx = 0.$$

Now, from the previous relation and bearing in mind that $t\varphi(t) \geq \Phi(t)$ for every $t \in \mathbb{R}$, we find that

$$J(v_0^\star) = \int_\Omega \Phi(|\nabla v_0^\star(x)|)dx = \int_{\Omega^-} \Phi(|\nabla v_0^\star(x)|)dx = 0. \tag{12}$$

Moreover, it follows from [20] that, since Φ satisfies a global Δ_2-condition, there exists a best positive constant λ_1 such that

$$\lambda_1 \int_\Omega \Phi(|u(x)|)dx \leq \int_\Omega \Phi(|\nabla u(x)|)dx, \tag{13}$$

for every $u \in W_0^1 L_\Phi(\Omega)$. Finally, from (12) and (13) we obtain that

$$\int_{\Omega^-} \Phi(|v_0^\star(x)|)dx = 0,$$

that is $v_0^\star(x) = 0$ in Ω^-, an absurd. Hence v_0 is non-negative in Ω, which proves the claim. Since in this case $f^\star(v_0(x)) = f(v_0(x))$ for every $x \in \Omega$, it follows that v_0 is also a nontrivial weak solution to $(D^{f,h}_{\alpha,\lambda})$. The proof is complete. □

Remark 3.1. A concrete construction of a cut-off function $\theta \in C_0^\infty(\Omega)$ that appears in the proof of the above result can be given in the following standard way. Now, since h is a continuous and non-identically zero function in $\bar{\Omega}$, there exist $\tau > 0$ and $x_0 \in \Omega$ such that $B(x_0, \tau) \subset \Omega$ and $h|_{\bar{B}(x_0,\tau/2)} > 0$. Consider the function $\omega : \mathbb{R} \to \mathbb{R}$ given by

$$\omega(s) := \begin{cases} e^{-1/s^2} & \text{if } s > 0 \\ 0 & \text{if } s \leq 0. \end{cases}$$

Further, set $\psi : \mathbb{R} \to \mathbb{R}$ with

$$\psi(s) := \omega\left(s - \frac{\tau}{2}\right)\omega(\tau - s),$$

and define the map $\varsigma : \mathbb{R} \to \mathbb{R}$ such that

$$\varsigma(s) := \frac{\int_{-\infty}^{s} \psi(t)dt}{\int_{-\infty}^{+\infty} \psi(t)dt}.$$

We observe that $\omega, \psi, \varsigma \in C^\infty(\mathbb{R})$ and

$$\varsigma(s) = 0 \quad \text{if } s \leq \tau/2, \qquad \varsigma(s) = 1 \quad \text{if } s \geq \tau,$$
$$\varsigma(s) \in]0, 1[\quad \text{if } s \in]\tau/2, \tau[.$$

Finally, let

$$\theta(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \tau) \\ \varsigma\left(\frac{3}{2}\tau - |x - x_0|\right) & \text{if } x \in B(x_0, \tau) \setminus B(x_0, \tau/2) \\ 1 & \text{if } x \in B(x_0, \tau/2), \end{cases}$$

where $|\cdot|$ denotes the Euclidean distance.

Bearing in mind the properties of the function ς, we deduce that $\theta \in C_0^\infty(\Omega)$ (so $\theta \in W_0^1 L_\Phi(\Omega)$), $\bar{B}(x_0, \tau/2) \subset \mathrm{Supp}(\theta)$, $0 \leq \theta(x) \leq 1$ for every $x \in \Omega$ and $\theta|_{\bar{B}(x_0,\tau/2)} = 1$.

By the same method, applying part (a) instead of part (b) of Theorem 1.1, it is possible to prove the following result that involves a non-linear term $f : \mathbb{R} \to \mathbb{R}$ with an oscillating behaviour at infinity obtaining, in this case, a sequence $\{v_n\}$ of weak solutions of problem $(D^{f,h}_{\alpha,\lambda})$ such that $\lim_{n\to\infty} \|v_n\| = \infty$.

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Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function with $f(0) = 0$. Assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^{p_0}} < +\infty \quad \textit{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^{p^0}} = +\infty. \tag{h_∞}$$

Suppose $h : \bar{\Omega} \to [0, +\infty)$ is a continuous and non-identically zero continuous function. Then, for every

$$\lambda \in \left]0, \frac{1}{c^{p_0} \|h\|_{L^1(\Omega)} \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^{p_0}}}\right[,$$

there exists a sequence $\{v_n\}$ of non-negative weak solutions to problem ($D^{f,h}_{\alpha,\lambda}$) such that $\lim_{n\to\infty} \|v_n\| = +\infty$.

Proof. The strategy of the proof is similar to the previous one. Hence, in the sequel, we omit the details and we use the notations adopted in the proof Theorem 3.1. Then, from hypothesis

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^{p_0}} < +\infty,$$

by direct computations, it follows that $\gamma := \liminf_{r\to+\infty} \widetilde{\varphi}(r) < +\infty$. On the other hand, by

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^{p^0}} = +\infty,$$

there exists a sequence $\{\eta_n\}$ of positive constants such that $\lim_{n\to\infty} \eta_n = +\infty$ and

$$\lim_{n\to\infty} \frac{F(\eta_n)}{\eta_n^{p^0}} = +\infty. \tag{14}$$

Now, consider the sequence of functions $\{\eta_n \theta\} \subset W^1_0 L_\Phi(\Omega)$. Arguing as in Theorem 3.1, we obtain

$$\lim_{n\to\infty} \frac{g_\lambda(\eta_n v)}{\eta_n^{p^0}} = -\infty. \tag{15}$$

Indeed $\|\eta_n \theta\| \to +\infty$ and

$$g_\lambda(\eta_n\theta) = J(\eta_n\theta) - \lambda F(\eta_n) \int_D h(x)dx - \lambda \int_{\Omega\setminus D} h(x)F(\eta_n\theta(x))dx.$$

Again, thanks to Lemma 2.1, it follows that

$$J(\eta_n\theta) = \int_\Omega \Phi(\eta_n|\nabla\theta(x)|)dx \le \|\eta_n\theta\|^{p^0} = \|\theta\|^{0} \eta_n^{p^0}.$$

Moreover, since $h(x)F(\eta_n\theta(x)) \ge 0$ for every $x \in \bar{\Omega}$, we deduce that

$$g_\lambda(\eta_n\theta) \le \|\theta\|^{p^0} \eta_n^{p^0} - \lambda F(\eta_n) \int_D h(x)dx,$$

for every n big enough. Hence

$$\frac{g_\lambda(\eta_n\theta)}{\eta_n^{p^0}} \le \|\theta\|^{p^0} - \lambda \frac{F(\eta_n)}{\eta_n^{p^0}} \int_D h(x)dx, \quad \forall n \ge \nu,$$

for some $\nu \in \mathbb{N}$. So, condition (15) is fulfilled.

Hence, the functional g_λ is unbounded from below. Then, part (a) of Theorem 1.1 ensures the existence of a sequence $\{v_n\}$ of distinct critical points of g_λ such that

$$\lim_{n\to\infty} J(v_n) = \lim_{n\to\infty} \int_\Omega \Phi(|\nabla v_n(x)|)dx = +\infty.$$

In conclusion, by (8) in Remark 2.2, it follows that $\lim_{n\to\infty} \|v_n\| = +\infty$. Moreover, arguing as in the proof of Theorem 3.1, the attained solutions are non-negative. □

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Remark 3.2. We explicitly observe that, exploiting the proof of Theorem 3.1, our result also holds for sign-changing functions $f : \mathbb{R} \to \mathbb{R}$ just requiring that

$$-\infty < \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p_0}}, \qquad \liminf_{\xi\to 0^+} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^{p^0}} < +\infty \quad \text{and} \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p_0}} = +\infty$$

instead of condition (h_0). In this setting, for every

$$\lambda \in \left]0, \frac{1}{c^{p^0}\|h\|_{L^1(\Omega)} \liminf_{\xi\to 0^+} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^{p^0}}}\right[,$$

there exist a sequence $\{v_n\}$ of pairwise distinct non-negative weak solutions of problem $(D^{f,h}_{\alpha,\lambda})$ such that $\lim_{n\to\infty}\|v_n\| = \lim_{n\to\infty}\|v_n\|_\infty = 0$. Indeed, by using the same notations of Theorem 3.1, if

$$\liminf_{\xi\to 0^+} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^{p^0}} < +\infty,$$

direct computations ensure that $\delta < +\infty$. On the other hand, consider the sequence of functions $\{\eta_n\theta\} \subset W_0^1 L_\Phi(\Omega)$. Thanks to

$$-\infty < \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^{p_0}},$$

there exists $\rho > 0$ and a real constant ϱ such that $\frac{F(\xi)}{\xi^{p_0}} > \varrho$ for every $\xi \in]0, \rho[$. It follows that

$$F(\xi) \geq \varrho\xi^{p_0}, \quad \text{for every } \xi \in [0, \rho[. \tag{16}$$

From condition (11) and (16), bearing in mind that $h \geq 0$ in $\bar{\Omega}$, there exists $\nu \in \mathbb{N}$ such that

$$g_\lambda(\xi_n\theta) \leq \|\theta\|^{p_0}\xi_n^{p_0} - \lambda F(\xi_n)\int_D h(x)dx - \lambda\varrho\xi_n^{p_0}\int_{\Omega\setminus D} h(x)\theta^{p_0}(x)dx, \quad \forall n \geq \nu.$$

Thus

$$\frac{g_\lambda(\xi_n\theta)}{\xi_n^{p_0}} \leq \|\theta\|^{p_0} - \lambda\frac{F(\xi_n)}{\xi_n^{p_0}}\int_D h(x)dx - \lambda\varrho\int_{\Omega\setminus D} h(x)\theta^{p_0}(x)dx,$$

for every $n \geq \nu$. Again, since $\int_D h(x)dx > 0$, where $\operatorname{meas}(D) > 0$, from (16) and taking into account the above inequality, we deduce that

$$\lim_{n\to\infty} \frac{g_\lambda(\xi_n\theta)}{\xi_n^{p_0}} = -\infty.$$

Thus, $g_\lambda(\xi_n\theta) < 0$ definitively. The thesis is achieved from our theoretical result.

An analogous conclusion can be achieved if the potential F has the same behaviour at infinity instead of the origin obtaining, in this case, the existence of a sequence of weak solutions which is unbounded in $W_0^1 L_\Phi(\Omega)$. Indeed, with the notations of Theorem 3.2, if

$$\liminf_{\xi\to +\infty} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^{p_0}} < +\infty,$$

direct computations ensure that $\gamma < +\infty$. On the other hand, consider the sequence of functions $\{\eta_n\theta\} \subset W_0^1 L_\Phi(\Omega)$. Bearing in mind that $\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^{p^0}} > -\infty$, there exist $\varrho > 0$ and a real constant k such that

$$F(\xi) \geq k\xi^{p^0}, \quad \text{for every } \xi \in]\varrho, +\infty[. \tag{17}$$

Moreover, we have

$$g_\lambda(\eta_n\theta) \leq \eta_n^{p^0}\|\theta\|^{p^0} - \lambda F(\eta_n)\int_D h(x)dx - \lambda\int_{\Omega\setminus D} h(x)F(\eta_n\theta(x))dx,$$

for every n large enough.

Hence

$$g_\lambda(\eta_n\theta) = \eta_n^{p^0}\|\theta\|^{p^0} - \lambda F(\eta_n)\int_D h(x)dx - \lambda\int_{G_\varrho\cap(\Omega\setminus D)} h(x)F(\eta_n\theta(x))dx - \lambda\int_{G^\varrho\cap(\Omega\setminus D)} h(x)F(\eta_n\theta(x))dx,$$

where

$$G_\varrho := \{x\in\Omega : 0\le \eta_n\theta(x)\le\varrho\} \quad\text{and}\quad G^\varrho := \{x\in\Omega : \eta_n\theta(x) > \varrho\}.$$

Now, by using the mean value theorem, we obtain

$$\left|\int_{G_\varrho\cap(\Omega\setminus D)} h(x)F(\eta_n\theta(x))dx\right| \le \|h\|_\infty \text{meas}(\Omega)\max_{t\in[0,\varrho]}|f(t)|\varrho. \tag{18}$$

Then, inequalities (17) and (18) yield

$$g_\lambda(\eta_n\theta)\le \eta_n^{p^0}\|\theta\|^{p^0} - \lambda F(\eta_n)\int_D h(x)dx + \lambda\|h\|_\infty\text{meas}(\Omega)\max_{t\in[0,\varrho]}|f(t)|\varrho - \lambda k\eta_n^{p^0}\int_{\Omega\setminus D} h(x)\theta^{p^0}(x)dx,$$

for every n sufficiently large.

From (14) and the above inequality, we have

$$\lim_{n\to\infty}\frac{g_\lambda(\eta_n\theta)}{\eta_n^{p^0}} = -\infty.$$

Thus g_λ is unbounded from below. The proof is attained from part (a) of our theoretical result. In conclusion, for every

$$\lambda\in\left]0, \frac{1}{c^{p_0}\|h\|_{L^1(\Omega)}\liminf\limits_{\xi\to+\infty}\frac{\max_{t\in[-\xi,\xi]}F(t)}{\xi^{p_0}}}\right[,$$

there exists a sequence $\{v_n\}$ of weak solutions of problem $(D^{f,h}_{\alpha,\lambda})$ which is unbounded in $W^1_0L_\Phi(\Omega)$. If, in addition $f(0)=0$, the attained solutions are non-negative arguing as in the proof of Theorem 3.1.

4. Consequences and examples

Define

$$\varphi(s) = \frac{|s|^{p-2}}{\log(1+|s|)}s \quad \text{for } s\neq 0, \text{ and } \varphi(0)=0,$$

where $p > N+1$. Let $\Phi(t) := \int_0^t \varphi(s)\,ds$ and consider the space $W^1_0L_\Phi(\Omega)$. By [2, Example 3, p. 243] we have

$$p_0 = p-1 < p^0 = p = \liminf_{t\to\infty}\frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions (Φ_0) and (Φ_1) are verified. From the above observations, by using Theorem 3.1, the following result holds.

Theorem 4.1. *Let $p > N+1$ and $f:\mathbb{R}\to\mathbb{R}$ be a continuous non-negative function with potential $F(\xi) := \int_0^\xi f(t)\,dt$. Assume that*

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^{p-1}} = +\infty. \tag{h_0''}$$

Then, for each $\lambda > 0$, the problem

$$\begin{cases} -\text{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right) = \lambda f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{D^f_λ}$$

admits a sequence $\{v_n\}$ of pairwise distinct non-negative weak solutions of problem (D^f_λ) such that $\lim_{n\to\infty}\|v_n\| = \lim_{n\to\infty}\|v_n\|_\infty = 0$.

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Remark 4.1. From Theorem 3.2, by using the following condition

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^{p-1}}=0 \quad\text{and}\quad \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^{p}}=+\infty, \tag{h''_∞}$$

instead of (h''_0), in Theorem 4.1, our approach ensures a sequence $\{v_n\}$ of weak solutions to problem (D^f_λ) such that $\lim_{n\to\infty}\|v_n\|=\infty$. If, in addition, we have $f(0)=0$, the attained solutions are non-negative.

A concrete example of application of Theorem 4.1 is given as follows.

Example 4.1. Let $\{s_n\}$, $\{t_n\}$ and $\{\delta_n\}$ be sequences defined by

$$s_n:=2^{-\frac{n!}{2}},\qquad t_n:=2^{-2n!},\qquad \delta_n:=2^{-(n!)^2},$$

and consider $\nu\in\mathbb{N}$ such that

$$s_{n+1}<t_n<s_n-\delta_n,\quad \forall n\ge\nu.$$

Moreover, fix $p>N+1$ and let $f:\mathbb{R}\to\mathbb{R}$ be the non-decreasing continuous function given by

$$f(t):=\begin{cases}2^{-(p-1)\nu!} & \text{if } t\in\,]s_\nu-\delta_\nu,+\infty[\\ y(t) & \text{if } t\in\bigcup\limits_{n\ge\nu}]s_{n+1}-\delta_{n+1},s_{n+1}[\\ 2^{-(p-1)n!} & \text{if } t\in\bigcup\limits_{n\ge\nu}[s_{n+1},s_n-\delta_n]\\ 0 & \text{if } t\le 0,\end{cases}$$

where

$$y(t):=\left(2^{-(p-1)n!}-2^{-(p-1)(n+1)!}\right)\left(\frac{t-s_{n+1}+\delta_{n+1}}{\delta_{n+1}}\right)+2^{-(p-1)(n+1)!}.$$

Set $F(\xi):=\int_0^\xi f(t)\,dt$ for every $\xi\in\mathbb{R}$. Therefore

$$\frac{F(s_n)}{s_n^p}\le\frac{f(s_{n+1})s_n+f(s_n)\delta_n}{s_n^p},$$

and

$$\frac{F(t_n)}{t_n^{p-1}}\ge\frac{f(s_{n+1})(t_n-s_{n+1})}{t_n^{p-1}},$$

for every n large enough.

Owing to

$$\lim_{n\to\infty}\frac{f(s_{n+1})s_n+f(s_n)\delta_n}{s_n^p}=0,\qquad \lim_{n\to\infty}\frac{f(s_{n+1})(t_n-s_{n+1})}{t_n^{p-1}}=+\infty,$$

it follows that

$$\lim_{n\to\infty}\frac{F(s_n)}{s_n^p}=0,\qquad \lim_{n\to\infty}\frac{F(t_n)}{t_n^{p-1}}=+\infty.$$

Hence, condition (h''_0) holds and, thanks to Theorem 4.1, for each $\lambda>0$, the following Dirichlet problem

$$\begin{cases}-\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right)=\lambda f(u) & \text{in } \Omega\\ u|_{\partial\Omega}=0,\end{cases} \tag{D^f_λ}$$

admits a sequence of pairwise distinct non-negative weak solutions which strongly converges to zero in $W^1_0L_\Phi(\Omega)$.

Remark 4.2. The differential operators studied in this work do not necessarily satisfy Leray–Lions type conditions. For instance, we observe that the map $a:(0,+\infty)\to(0,+\infty)$ defined by

$$a(t):=\frac{t^{3/2}}{\log(1+\sqrt{t})}$$

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does not satisfy condition (LL). Thus, if Ω is a bounded domain of $\mathbb{R}^3$, the result contained in [31] cannot be applied in order to study the existence of infinitely many solutions for elliptic Dirichlet problems involving the operator

$$\mathcal{A}u := \operatorname{div}\left(\frac{|\nabla u|^3}{\log(1+|\nabla u|)}\nabla u\right).$$

In the next example we argue the existence of infinitely many non-negative solutions for a non-homogeneous Dirichlet problem involving a sign-changing function $f : \mathbb{R} \to \mathbb{R}$ whose potential F has a suitable growth at infinity.

Example 4.2. Let Ω be a nonempty bounded subset of the Euclidean plane $\mathbb{R}^3$ and let Φ be a Young function such that conditions (Φ_0) and (Φ_1) hold for $p_0 = 4$ and $p^0 = 5$. Moreover, put

$$a_1 := 2, \qquad a_{n+1} := n!(a_n)^{\frac{3}{2}} + 2,$$

for every $n \geq 1$. Further, set $S := \bigcup_{n\in\mathbb{N}}]a_{n+1}-1, a_{n+1}+1[$ and $S' := \bigcup_{n\geq 1}[a_n+1, a_{n+1}-1]$. Define the continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(t) := \begin{cases} (a_{n+1})^6 e^{\frac{1}{(t-(a_{n+1}-1))(t-(a_{n+1}+1))}+1} \dfrac{2(a_{n+1}-t)}{(t-(a_{n+1}-1))^2(t-(a_{n+1}+1))^2} & \text{if } t \in S \\ -\dfrac{\left(\frac{a_{n+1}+a_n}{2}\right)^5 e^{\frac{1}{(t-(a_{n+1}-1))(t-(a_{n+1}+1))}+\frac{4}{(a_n-a_{n+1}+2)^2}} (a_n+a_{n+1}-2t)}{(a_n+1-t)^2(a_{n+1}-1-t)^2} & \text{if } t \in S' \\ 0 & \text{otherwise.} \end{cases}$$

By straightforward computation we obtain

$$F(\xi) = \int_0^\xi f(t)dt = \begin{cases} (a_{n+1})^6 e^{\frac{1}{(\xi-(a_{n+1}-1))(\xi-(a_{n+1}+1))}+1} & \text{if } \xi \in S \\ -\left(\dfrac{a_{n+1}+a_n}{2}\right)^5 e^{\frac{1}{(\xi-a_n-1)(\xi-(a_{n+1}-1))}+\frac{4}{(a_n-a_{n+1}+2)^2}} & \text{if } \xi \in S' \\ 0 & \text{otherwise .} \end{cases}$$

Since $F(a_{n+1}) = (a_{n+1})^6$ for every $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} \frac{F(a_{n+1})}{a_{n+1}^5} = +\infty.$$

Hence

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^5} = +\infty.$$

Putting $y_n = a_{n+1} - 1$ for every $n \in \mathbb{N}$, we obtain $\max_{|\xi|\leq y_n} F(\xi) = (a_n)^6$ for every $n \geq 1$. Now, since

$$\lim_{n\to\infty} \frac{\max\limits_{|\xi|\leq y_n} F(\xi)}{{y_n}^4} = 0,$$

we deduce that

$$\liminf_{\xi\to+\infty} \frac{\max\limits_{|t|\leq\xi} F(t)}{\xi^4} = 0.$$

Finally, put $\xi_n := \frac{a_{n+1}+a_n}{2}$, for every $n \geq 1$. Owing to $F(\xi_n) = -\left(\frac{a_{n+1}+a_n}{2}\right)^5$, clearly

$$\lim_{n\to\infty} \frac{F(\xi_n)}{\xi_n^5} = -1.$$

Consequently

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^5} = -1.$$

Thanks to Remark 3.2, our result guarantee that for each $\lambda > 0$ and for every continuous and non-identically zero function $h : \bar{\Omega} \to [0, +\infty)$, the following problem

$$\begin{cases} -\mathrm{div}(\alpha(|\nabla u|)\nabla u) = \lambda h(x) f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$D^{f,h}_{\alpha,\lambda}$}$$

possesses a sequence of non-negative weak solutions which is unbounded in $W^1_0 L_\Phi(\Omega)$.

Remark 4.3. We just observe that the technical approach adopted here improves the existence results for elliptic Dirichlet problems involving the p–Laplacian contained in [33]. Indeed, in the cited paper, the potential F is assumed to be non-negative in $[0, +\infty)$.

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