

## QUASILINEAR EVOLUTION EQUATIONS AND PARABOLIC SYSTEMS

BY

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ABSTRACT. It is shown that general quasilinear parabolic systems possess unique maximal classical solutions for sufficiently smooth initial values, provided the boundary conditions are "time-independent". Moreover it is shown that, in the autonomous case, these equations generate local semiflows on appropriate Sobolev spaces. Our results apply, in particular, to the case of prescribed boundary values (Dirichlet boundary conditions).

**Introduction.** In this paper we study quasilinear evolution equations of the form

$$(1) \quad \dot{u} + A(t, u)u = f(t, u)$$

in a general Banach space  $X$ . We assume that  $A(t, y)$  is, for each fixed argument, the infinitesimal generator of a strongly continuous analytic semigroup, such that the domain  $X_1$  of  $A(t, y)$  is independent of  $(t, y)$ . Equation (1) is studied in general interpolation spaces  $X_\theta$  between  $X$  and  $X_1$ , and it is shown that (1) possesses a unique maximal solution, which depends (Lipschitz-)continuously upon its initial value. In the important autonomous case (that is, when  $A$  and  $f$  are independent of  $t$ ) this implies that (1) generates a local semiflow in  $X_\theta$ . In addition we give simple sufficient conditions, which guarantee that a given maximal solution exists globally, that is, for all time. Moreover, by imposing some additional hypotheses, we show that the solutions of (1) possess better regularity properties.

The general abstract results are then applied to quasilinear parabolic initial boundary value problems of the form

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t, x, u, Du, \dots, D^{2m-1}u)u \\ = f(t, x, u, Du, \dots, D^{2m-1}u) \quad \text{in } (s, T] \times \Omega, \\ \mathcal{B}u = 0 \quad \text{on } (s, T] \times \partial\Omega, \\ u(s, \cdot) = u_0 \quad \text{on } \Omega, \end{aligned}$$

where  $\Omega$  is a bounded smooth domain in  $\mathbf{R}^n$ ,  $\mathcal{B}$  denotes the Dirichlet boundary operator and  $\mathcal{A}(t, x, u, \dots, D^{2m-1}u)$  is a strongly parabolic differential operator of order  $2m$ , acting on vector-valued functions  $u: \Omega \rightarrow \mathbf{C}^N$  and depending smoothly upon the indicated quantities. We shall show that this problem possesses a unique

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maximal classical solution  $u(\cdot, s, u_0)$  for every  $s \in [0, T)$  and  $u_0 \in W_p^\sigma$  satisfying  $\mathcal{B}u_0 = 0$  on  $\partial\Omega$ , provided  $n < p < \infty$  and  $2m - 1 + n/p < \sigma < 2m$ , and that  $u(\cdot, s, u_0)$  depends continuously upon  $u_0$  in the topology of  $W_p^\sigma$ .

To be more specific we now state a special case of our general results of §10 in the important case of a strongly coupled real quasilinear second order autonomous system. For this we suppose that  $\Omega$  is a bounded domain in  $\mathbf{R}^n$  of class  $C^3$  and that

$$(2) \quad a_{jk} \in C^2(\bar{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{nN}, \mathcal{L}(\mathbf{R}^N)), \quad j, k = 1, \dots, n,$$

where  $\mathcal{L}(\mathbf{R}^N)$  is the space of all linear mappings ( $\equiv (N \times N)$ -matrices) on  $\mathbf{R}^N$ . Moreover we assume that

$$(3) \quad \operatorname{Re} \left\{ \sum_{r,s=1}^N \sum_{j,k=1}^n a_{jk}^{rs}(x, \eta, \zeta) \xi^j \xi^k \lambda_r \bar{\lambda}_s \right\} > 0$$

for all  $(x, \eta, \zeta) \in \bar{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{nN}$ , all  $\xi := (\xi^1, \dots, \xi^n) \in \mathbf{R}^n$  and all  $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$  with  $\xi \neq 0$  and  $\lambda \neq 0$ . Thus

$$\mathcal{A}(x, u, Du)u := - \sum_{j,k=1}^n a_{jk}(x, u, Du) D_j D_k u$$

is a quasilinear strongly coupled elliptic second order differential operator acting on  $N$ -vector-valued functions  $u: \Omega \rightarrow \mathbf{R}^N$ . Finally we assume that

$$(4) \quad f \in C^2(\bar{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{nN}, \mathbf{R}^N)$$

and consider the parabolic initial boundary value problem

$$(5) \quad \begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(x, u, Du)u &= f(x, u, Du) && \text{in } (0, \infty) \times \Omega, \\ u &= 0 && \text{on } (0, \infty) \times \partial\Omega, \\ u(0, \cdot) &= u_0 && \text{on } \Omega. \end{aligned}$$

**THEOREM.** *Suppose that  $n < p < \infty$  and  $1 + n/p < \sigma < 2$ . Then (5) possesses a unique maximal classical solution  $u(\cdot, u_0)$  for each  $u_0 \in W_{p,\mathcal{B}}^\sigma := \{u \in W_p^\sigma := W_p^\sigma(\Omega, \mathbf{R}^N) \mid u|_{\partial\Omega} = 0\}$ . The function  $(t, u_0) \mapsto u(t, u_0)$  is a local semiflow on  $W_{p,\mathcal{B}}^\sigma$  such that bounded orbits exist for all time. If an orbit is bounded in  $W_p^\tau$  for some  $\tau \in (\sigma, 2]$ , then it is relatively compact in  $W_p^\sigma$ .*

It should be noted that—besides the regularity assumptions (2)–(4)—we do not require any growth or compatibility conditions (except for the initial value  $u_0$ , of course).

It is well known that the fact that nonlinear parabolic equations generate semiflows is very important for the study of the qualitative behaviour of the solutions of these equations (cf. [12, 18 and 27] for recent surveys of some results of this type). Up to now this has only been known for semilinear parabolic equations and systems (e.g. [17 and 22]).

Our principal abstract results are contained in §§6–8. In the last section we give applications of the abstract theory to quasilinear parabolic systems of arbitrary order. We are particularly interested in classical solutions. For simplicity we do not

give the most general results but choose a relatively simple setting. Due to the length of this paper we do not give specific conditions guaranteeing the existence of global solutions.

The existence of solutions to abstract quasilinear evolution equations has already been studied by Sobolevskii [28] (cf. also Friedman [16]) and, more recently, by Potier-Ferry [24] and Lunardi [20, 21]. But our results about the continuous dependence of the solution upon the initial values are new, as are the simple criteria for global existence. Moreover the results of these authors do not give classical solutions to general parabolic systems as do our general theorems. A more detailed discussion of the relations between our results and those of other authors is given in the main body of this paper.

*Notations.* Throughout this paper we use standard notations. All vector spaces are over  $\mathbf{K} := \mathbf{R}$  or  $\mathbf{C}$ . If  $\mathbf{K} = \mathbf{R}$  and we use complex quantities (for example in connection with spectral theory), it is always understood that we work with the natural complexifications (of spaces and operators). Thus, by  $\rho(A)$ , the resolvent set, and by  $\sigma(A)$ , the spectrum of a linear operator with domain  $D(A)$  and range  $R(A)$ , we mean always the resolvent set and the spectrum, respectively, of its complexification.

$X, Y, Z$ , with or without indices, always denote Banach spaces, and  $\mathcal{L}(X, Y)$  is the Banach space of all continuous linear operators from  $X$  into  $Y$ . By  $\mathcal{L}_s(X, Y)$  we mean the same vector space, but endowed with the topology of pointwise convergence. Moreover  $\mathcal{L}(X) := \mathcal{L}(X, X)$ .

If  $S$  is a metric space, we denote by  $C(S, X)$  the space of all continuous functions from  $S$  into  $X$ , endowed with the topology of uniform convergence on compact subsets (the compact-open topology). If  $0 < \alpha < 1$ , then  $C^\alpha(S, X)$  is the subspace of all  $\alpha$ -Hölder continuous functions, where  $f: S \rightarrow X$  is said to be  $\alpha$ -Hölder continuous, if each point  $s_0 \in S$  has a neighbourhood  $U$  such that  $f$  is uniformly  $\alpha$ -Hölder continuous on  $U$ , that is,

$$[f]_\alpha^U := \sup_{s, t \in U} \frac{\|f(s) - f(t)\|}{[d(s, t)]^\alpha} < \infty,$$

where  $d$  denotes the metric in  $S$ . The space  $C^\alpha(S, X)$  is given the locally convex topology induced by the seminorms

$$(6) \quad \sup_{s \in U} \|f(s)\| + [f]_\alpha^U, \quad U \subset S, U \text{ compact.}$$

If  $\alpha = 1$ , we denote this space by  $C^1(S, X)$ , the space of Lipschitz continuous functions. More generally, if  $S$  is a subspace of some product space  $A \times B$ , then we write  $f \in C^{\alpha, \beta}(S, X)$  if each point  $(a, b) \in S$  has a product neighbourhood  $U \times V$  in  $S$  such that there exists a constant  $c$  with

$$(7) \quad \|f(s, t) - f(s', t')\| \leq c \left\{ [d_A(s, s')]^\alpha + [d_B(t, t')]^\beta \right\}$$

for all  $(s, t), (s', t') \in U \times V$ . Moreover,  $f \in C^{0, \beta}(S, X)$  means that  $f(\cdot, t)$  is continuous on  $U$  for each  $t \in V$  and  $f(s, \cdot)$  is uniformly  $\beta$ -Hölder continuous on  $V$ , uniformly with respect to  $s \in U$ . We shall repeatedly use the following important

fact: If  $0 \leq \alpha, \beta \leq 1$  and if  $f \in C^{\alpha,\beta}(S, X)$ , then each compact set  $K$  in  $S$  has a neighbourhood  $W$  in  $S$  such that (7) holds for all  $(s, t), (s', t') \in W$  (with the obvious modifications if  $\alpha = 0$  or  $\beta = 0$ , and where  $\alpha = 1$  is interpreted as  $\alpha = 1 -$  etc.). For a proof of this fact we refer to [6, Satz 6.4].

If  $S$  is compact, then  $C(S, X)$  is a Banach space with the maximum norm and  $C^\alpha(S, X)$  is a Banach space with the norm (6), where  $U$  is replaced by  $S$ . We use these norms throughout. In general we denote by  $B(S, X)$  the Banach space of all bounded functions from  $S$  into  $X$ , endowed with the supremum norm. Moreover  $BC(S, X) := B(S, X) \cap C(S, X)$ , and  $BUC(S, X)$  is the closed linear subspace of  $BC(S, X)$  consisting of all bounded and uniformly continuous functions. Similarly,  $BUC^\alpha(S, X)$  denotes the Banach space of all bounded and uniformly  $\alpha$ -Hölder continuous functions on  $S$ , endowed with the norm (6), where  $U$  is replaced by  $S$ .

If  $U$  is an open subset of some Banach space  $Y$ , then  $C^k(U, X)$ ,  $k \in \mathbf{N}^* := \mathbf{N} \setminus \{0\}$ , is the space of all  $k$ -times continuously differentiable functions from  $U$  into  $X$ , endowed with the usual locally convex topology. If  $Y = \mathbf{K}$ , then  $U$  can also be a perfect subset of  $\mathbf{K}$ , that is, a set such that each point of  $U$  is a limit point of  $U$ . Moreover  $C^{k-}(U, X)$  is the subspace of all  $f \in C^{k-1}(U, X)$  such that  $D^{k-1}f \in C^{1-}(U, X)$ , where  $D$  denotes the derivative. If  $U$  is a subset of some product space, we denote by  $D_1, D_2, \dots$  the partial derivatives.

Throughout this paper  $T$  is a fixed positive number. We let

$$T_\Delta := \{ (t, s) \in \mathbf{R}^2 \mid 0 \leq s \leq t \leq T \} \quad \text{and} \quad \dot{T}_\Delta := \{ (t, s) \in T_\Delta \mid s \neq t \}.$$

Moreover we denote by  $c$  positive constants, which may be different from formula to formula, but are always independent of the specific independent variables occurring at a given place. Thus we treat  $c$  in much the same way as the Landau symbol  $O$ . If the equations under consideration depend on additional parameters, say  $\alpha, \beta, \dots$ , then we sometimes write  $c(\alpha, \beta, \dots)$  to indicate that  $c$  depends on these parameters. Finally, (3.5) means formula 5 in §3, if we refer to it outside of §3.

**1. Convolution-type equations.** For each  $\alpha \in \mathbf{R}$  we denote by  $\mathfrak{R}(X, Y, \alpha)$  the Banach space of all functions  $k \in C(\dot{T}_\Delta, \mathcal{L}(X, Y))$  satisfying

$$\|k\|_{(\alpha)} := \sup_{(t,s) \in \dot{T}_\Delta} (t-s)^\alpha \|k(t,s)\| < \infty,$$

endowed with the norm  $\|\cdot\|_{(\alpha)}$ . It is obvious that

$$(1) \quad \mathfrak{R}(X, Y, \beta) \hookrightarrow \mathfrak{R}(X, Y, \alpha) \quad \text{for } \beta < \alpha,$$

where  $\hookrightarrow$  denotes continuous imbedding. If  $\alpha < 0$ , then it is easily seen that  $k \in \mathfrak{R}(X, Y, \alpha)$  can be continuously extended over  $T_\Delta$  by letting  $k(t, t) = 0$  for  $0 \leq t \leq T$ , so that

$$(2) \quad \mathfrak{R}(X, Y, \alpha) \hookrightarrow C(T_\Delta, \mathcal{L}(X, Y)) \quad \text{for } \alpha < 0.$$

Observe that

$$(3) \quad \mathfrak{R}(X, Y, 0) = BC(\dot{T}_\Delta, \mathcal{L}(X, Y)).$$

If  $X = \mathbf{K}$ , we identify  $\mathcal{L}(\mathbf{K}, Y)$  naturally with  $Y$  via  $\mathcal{L}(\mathbf{K}, Y) \ni B \leftrightarrow B \cdot 1 \in Y$ . Hence  $k \in \mathfrak{R}(\mathbf{K}, Y, \alpha)$  iff  $k \in C(\dot{T}_\Delta, Y)$  and  $\|k(t, s)\| \leq c(t - s)^{-\alpha}$  for  $(t, s) \in \dot{T}_\Delta$ . In particular,

$$(4) \quad C([0, T], Y) \hookrightarrow \mathfrak{R}(\mathbf{K}, Y, 0) = BC(\dot{T}_\Delta, Y)$$

by the obvious identification

$$(5) \quad C([0, T], Y) \ni u \leftrightarrow [(t, s) \mapsto u(t)] \in BC(\dot{T}_\Delta, Y).$$

Finally,  $\mathfrak{R}(X, \alpha) := \mathfrak{R}(X, X, \alpha)$ .

We define now a ‘‘convolution-type’’ operation  $*$  by

$$h * k(t, s) := \int_s^t h(t, \tau)k(\tau, s) d\tau, \quad (t, s) \in T_\Delta,$$

whenever  $k \in \mathfrak{R}(X, Y, \alpha)$  and  $h \in \mathfrak{R}(Y, Z, \beta)$  with  $\alpha, \beta < 1$ , so that the integral exists.

LEMMA 1.1. (i) Let  $\alpha, \beta \in (-\infty, 1)$ ,  $k \in \mathfrak{R}(X, Y, \alpha)$  and  $h \in \mathfrak{R}(Y, Z, \beta)$ . Then

$$h * k \in \mathfrak{R}(X, Z, \alpha + \beta - 1)$$

and

$$(6) \quad \|h * k\|_{(\alpha+\beta-1)} \leq B(1 - \alpha, 1 - \beta) \|h\|_{(\beta)} \|k\|_{(\alpha)},$$

where  $B(\cdot, \cdot)$  denotes the beta function.

(ii) The operation  $*$  is associative.

PROOF. (i) The estimate (6) is simple. The continuity of  $h * k$  on  $\dot{T}_\Delta$  follows essentially from Lebesgue’s theorem (for example, by modifying in an obvious way the proof of [7, Lemma 1.1]).

(ii) is a consequence of Fubini’s theorem.  $\square$

Let  $k \in \mathfrak{R}(X, Y, \alpha)$  with  $\alpha < 1$ . Then it is an immediate consequence of Lemma 1.1 and (1) that

$$(7) \quad k * \in \mathcal{L}(\mathfrak{R}(Z, X, \beta), \mathfrak{R}(Z, Y, \beta))$$

and

$$(8) \quad * k \in \mathcal{L}(\mathfrak{R}(Y, Z, \beta), \mathfrak{R}(X, Z, \beta)),$$

and that

$$(9) \quad \|k * \cdot\|, \|\cdot * k\| \leq T^{1-\alpha} B(1 - \alpha, 1 - \beta) \|k\|_{(\alpha)},$$

provided  $\beta < 1$ . In particular,

$$(10) \quad k * \in \mathcal{L}(\mathfrak{R}(Y, X, \beta)), \quad * k \in \mathcal{L}(\mathfrak{R}(X, Y, \beta))$$

for  $\beta < 1$ , if  $k \in \mathfrak{R}(X, \alpha)$  for some  $\alpha < 1$ .

An easy induction argument shows that

$$(11) \quad \underbrace{\|k * k * \cdots * k(t, s)\|}_{n+1} \leq (t - s)^{-\alpha} \Gamma(1 - \alpha) \|k\|_{(\alpha)} \frac{\left[ (t - s)^{1-\alpha} \Gamma(1 - \alpha) \|k\|_{(\alpha)} \right]^n}{\Gamma((n + 1)(1 - \alpha))}$$

for  $n \geq 1$ . This well-known estimate (e.g. [31, formula (5.16)]) implies, together with (9), that the linear operators (10) have spectral radii zero. Consequently the first part of the following theorem is obvious.

**THEOREM 1.2.** *Let  $\alpha, \beta \in (-\infty, 1)$  and  $k \in \mathfrak{K}(X, \alpha)$ . Then the “convolution-type equations”*

$$(12) \quad u = a + u * k$$

and

$$(13) \quad v = b + k * v$$

have for each  $a \in \mathfrak{K}(X, Y, \beta)$  and  $b \in \mathfrak{K}(Y, X, \beta)$  unique solutions  $u \in \mathfrak{K}(X, Y, \beta)$  and  $v \in \mathfrak{K}(Y, X, \beta)$ , respectively. The solution  $u$  of (12) can be represented by

$$(14) \quad u = a + a * w,$$

where  $w$ , the **resolvent kernel** of (12), is the unique solution in  $\mathfrak{K}(X, \alpha)$  of  $w = k + k * w$ . Moreover,  $w = k + w * k$ .

An analogous representation formula holds for (13).

**PROOF.** Since the solution  $u$  of (12) is given by the Neumann series,

$$u = (1 - *k)^{-1}a = \sum_{j=0}^{\infty} (*k)^j a = a + r,$$

where

$$r := \sum_{j=1}^{\infty} (*k)^j a = a * k + \sum_{j=2}^{\infty} (*k)^j a.$$

Let

$$l := \sum_{j=2}^{\infty} \underbrace{k * \dots * k}_j = \sum_{j=2}^{\infty} (*k)^{j-1} k.$$

It follows from (11) that  $l \in \mathfrak{K}(X, \alpha)$ . Moreover  $a * l = \sum_{j=2}^{\infty} (*k)^j a$ , so that  $r = a * w$ , where

$$\begin{aligned} w &:= k + l = k + \sum_{j=2}^{\infty} \underbrace{k * \dots * k}_j \\ &= k + \sum_{j=1}^{\infty} k * \left( \underbrace{k * \dots * k}_j \right) = k + \sum_{j=1}^{\infty} \left( \underbrace{k * \dots * k}_j \right) * k. \end{aligned}$$

Hence  $w = k + k * w = k + w * k$ .  $\square$

**REMARKS 1.3.** (a) By replacing the space  $\mathcal{L}(X, Y)$  in the definition of  $\mathfrak{K}(X, Y, \alpha)$  by  $\mathcal{L}_s(X, Y)$  we can introduce the spaces  $\mathfrak{K}_s(X, Y, \alpha)$ . Thus  $k \in \mathfrak{K}_s(X, Y, \alpha)$  iff  $k: T_\Delta \rightarrow \mathcal{L}(X, Y)$  is strongly continuous and  $\|k\|_{(\alpha)} < \infty$ . Then it is easily verified that everything said above remains true for the spaces  $\mathfrak{K}_s(X, Y, \alpha)$ , provided we substitute everywhere  $\mathcal{L}_s(X, Y)$  for  $\mathcal{L}(X, Y)$ .

(b) It is an obvious consequence of the estimate (11) that the norms of the solutions of the convolution-type equations (12) and (13) are majorized by constants depending only on bounds for the norms of  $a$  and  $k$ , or  $b$  and  $k$ , respectively, and on  $T$ ,  $\alpha$  and  $\beta$ , but not on the individual operators.  $\square$

Of course, the results of this section are known, in principle. For example, they are implicitly contained in [28] and in [31, §5.2]. Since they are fundamental for our work, and since we do not know of an explicit reference, we have shortly indicated their proofs.

**2. Families of analytic semigroups.** In the following we let  $\Sigma_\vartheta := \{z \in \mathbb{C} \mid |\arg z| \leq \vartheta + \pi/2\}$  for  $0 \leq \vartheta < \pi/2$ . Then we impose the following **Assumption (A)**:

*$\{A(t) \mid 0 \leq t \leq T\}$  is a family of closed densely defined linear operators in  $X$  with constant domain, that is,  $D(A(t)) = D(A(0))$  for  $0 \leq t \leq T$ . Moreover,  $\Sigma_0 \subset \rho(-A(t))$  and*

$$(1) \quad \|(\lambda + A(t))^{-1}\| \leq M/(1 + |\lambda|)$$

for  $\lambda \in \Sigma_0$  and  $0 \leq t \leq T$ .

It is well known that this implies the existence of a constant  $\vartheta \in (0, \pi/2)$ —which we fix now—such that  $\Sigma_\vartheta \subset \rho(-A(t))$  and that (1) holds (with a new constant, which we denote again by  $M$ ) for all  $\lambda \in \Sigma_\vartheta$  and all  $t \in [0, T]$ .

We let  $X_1 := D(A(0))$ , endowed with the norm  $x \mapsto \|x\|_1 := \|A(0)x\|$ , which is equivalent to the graph norm. Hence  $X_1$  is a Banach space such that  $X_1 \hookrightarrow X$ , and  $A(t) \in \mathcal{L}(X_1, X)$ .

We impose now the additional **Assumption (A $_\rho$ )**:

$$A(\cdot) \in C^\rho([0, T], \mathcal{L}(X_1, X)) \quad \text{for some } \rho \in (0, 1).$$

Observe that (A $_\rho$ ) implies the existence of a constant  $L$  such that

$$(2) \quad \|A(t) - A(s)\|_{\mathcal{L}(X_1, X)} \leq L|t - s|^\rho \quad \forall t, s \in [0, T].$$

Moreover (A $_\rho$ ) and the smoothness of the inversion  $B \rightarrow B^{-1}$  from  $\text{Isom}(X, Y)$  onto  $\text{Isom}(Y, X)$ , where  $\text{Isom}(X, Y)$  is the open subset in  $\mathcal{L}(X, Y)$  of all isomorphisms, imply

$$(3) \quad [(t, s) \mapsto A(t)(\lambda + A(s))^{-1}] \in C([0, T]^2, \mathcal{L}(X))$$

and

$$(4) \quad [(t, s) \mapsto (\lambda + A(s))^{-1}A(t)] \in C([0, T]^2, \mathcal{L}(X_1))$$

for every  $\lambda \in \Sigma_\vartheta$ . In particular there exists a constant  $N$  such that

$$(5) \quad \|A(t)A^{-1}(s)\| \leq N \quad \forall t, s \in [0, T].$$

We assume now that for each  $\theta \in (0, 1)$  there is given an interpolation functor  $\mathfrak{F}_\theta$  from the category  $\mathfrak{C}$  of compatible pairs of Banach spaces to the category of all Banach spaces, possessing the following property:

(IPF1):  $\mathfrak{F}_\theta$  is an interpolation functor of exponent  $\theta$ , that is,

$$\|T\|_{\mathcal{L}(\mathfrak{F}_\theta(A_0, A_1), \mathfrak{F}_\theta(B_0, B_1))} \leq c \|T\|_{\mathcal{L}(A_0, B_0)}^{1-\theta} \|T\|_{\mathcal{L}(A_1, B_1)}^\theta,$$

whenever  $(A_0, A_1)$  and  $(B_0, B_1)$  are objects and  $T$  is a morphism in the category  $\mathfrak{C}$ .

Observe that the complex interpolation functor  $[\cdot, \cdot]_\theta$  and the real interpolation functors  $(\cdot, \cdot)_{\theta, p}$ ,  $1 \leq p \leq \infty$ , possess the above property. (We refer to [10 and 32] for the basic facts about interpolation spaces.)

It is a consequences of (IPF1) that  $\mathfrak{F}_\theta$  possesses the following additional property:

(IPF2):  $\|x\|_{\mathfrak{F}_\theta(A_0, A_1)} \leq c \|x\|_{A_0}^{1-\theta} \|x\|_{A_1}^\theta \quad \forall x \in A_0 \cap A_1$ , whenever  $(A_0, A_1)$  is an object in  $\mathfrak{C}$  (cf. the proof of Theorem 1.3.3(g) in [32]).

In the following we let

$$X_\theta := \mathfrak{F}_\theta(X_0, X_1),$$

where  $X_0 := X$ , and  $\|\cdot\|_\theta$  denotes the norm in  $X_\theta$ .

It is well known that (A) implies that each  $-A(t)$  is the infinitesimal generator of a strongly continuous analytic semigroup  $\{e^{-sA(t)} | s \geq 0\}$  on  $X$ , which is explicitly given by

$$(6) \quad e^{-sA(t)} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda s} (\lambda + A(t))^{-1} d\lambda \quad \text{in } \mathcal{L}(X),$$

where  $\Gamma$  is any piecewise smooth curve in  $\Sigma_\vartheta$  running from  $\infty e^{-i(\vartheta + \pi/2)}$  to  $\infty e^{i(\vartheta + \pi/2)}$ .

LEMMA 2.1. *Let  $0 \leq \eta, \theta \leq 1$ . Then*

$$(7) \quad [(t, s) \mapsto A(t)e^{-sA(t)}] \in C([0, T] \times (0, \infty), \mathcal{L}(X_\eta, X_\theta))$$

and

$$(8) \quad \|A(t)e^{-sA(t)}\|_{\mathcal{L}(X_\eta, X_\theta)} \leq cs^{\eta-\theta-1} \quad \forall (t, s) \in [0, T] \times (0, \infty).$$

Moreover,

$$(9) \quad [(t, s) \mapsto e^{-sA(t)}] \in C([0, T] \times \mathbf{R}_+, \mathcal{L}_s(X_\theta)).$$

PROOF. Assumption (A) implies

$$(10) \quad \|A^j(t)e^{-sA(t)}\|_{\mathcal{L}(X)} \leq cs^j \quad \forall (t, s) \in [0, T] \times (0, \infty)$$

for  $j = 0, 1, 2$ . Now (8) follows easily by interpolation by observing that, due to (5),  $\{x \mapsto \|A(t)x\| | 0 \leq t \leq T\}$  is a family of uniformly equivalent norms on  $X_1$ . If  $\eta = \theta = 0$  or  $\eta = \theta = 1$  we obtain (7) from (6), (3), (4), (10) and Lebesgue's theorem. If  $0 \leq \eta, \theta \leq 1$  are arbitrary, (7) follows by interpolation. The strong continuity (9) at  $(t, 0)$ ,  $0 \leq t \leq T$ , in the case  $\theta = 0$  follows by an obvious modification of the standard proof of the strong continuity of  $s \mapsto e^{-sA(t)}$  at  $s = 0$  for fixed  $t$



(e.g. [31, pp. 66–67]). The same proof gives the result for  $\theta = 1$ , due to  $A(t)e^{-sA(t)} \supset e^{-sA(t)}A(t)$  and the already observed uniform equivalence of the norms  $x \mapsto \|A(t)x\|$  on  $X_1$ . The general case is now obtained by interpolation on the basis of (IPF2).  $\square$

**3. Estimates for parabolic evolution operators.** For convenience we introduce the following simplifying notation: Whenever  $U$  is a function of two real variables and  $V$  is a function of one real variable we write

$$VU(t, s) := V(t)U(t, s) \quad \text{and} \quad UV(t, s) := U(t, s)V(s),$$

provided the right-hand sides are meaningful.

By well-known results of Sobolevskii [28] and Tanabe [30] (cf. also [16, 23 and 31] for expositions) Assumptions (A) and  $(A_\rho)$  imply the existence of a unique function  $U: T_\Delta \rightarrow \mathcal{L}(X)$ , a **parabolic fundamental solution** for  $\{A(t) | 0 \leq t \leq T\}$ , possessing the following properties:

- (U1):  $U \in C(T_\Delta, \mathcal{L}_s(X))$  and  $R(U(t, s)) \subset X_1$  for  $(t, s) \in \dot{T}_\Delta$ .
- (U2):  $U(t, t) = \text{id}$  and  $U(t, s) = U(t, \tau)U(\tau, s)$  for  $0 \leq s \leq \tau \leq t \leq T$ .
- (U3):  $U(\cdot, s) \in C^1((s, T], \mathcal{L}_s(X))$  and  $D_1U = -AU$ . Moreover,  $U(t, \cdot) \in C^1([0, t], \mathcal{L}_s(X_1, X))$  and  $D_2Ux = UAx$  for all  $x \in X_1$ .
- (U4):  $AU \in C(\dot{T}_\Delta, \mathcal{L}(E))$  and

$$\|AU(t, s)\| \leq c(L, M, N, T, \rho)(t - s)^{-1}.$$

It should be observed that (U3) and (U4) imply that  $U(\cdot, s) \in C^1((s, T], \mathcal{L}(X))$ . It follows from (U1) and (U3) that

$$\frac{\partial}{\partial \tau} [U(t, \tau)e^{-(\tau-s)A(s)}] = U(t, \tau)[A(\tau) - A(s)]e^{-(\tau-s)A(s)}$$

in  $\mathcal{L}_s(X)$  for  $0 \leq s < \tau < t \leq T$ . Thus, by integration, which is possible due to  $(A_\rho)$ ,

$$(1) \quad U(t, s) = e^{-(t-s)A(s)} - \int_s^t U(t, \tau)[A(\tau) - A(s)]e^{-(\tau-s)A(s)} d\tau, \quad (t, s) \in T_\Delta,$$

in  $\mathcal{L}_s(X)$ . Similarly,

$$\frac{\partial}{\partial \tau} [e^{-(t-\tau)A(t)}U(\tau, s)] = e^{-(t-\tau)A(t)}[A(t) - A(\tau)]U(\tau, s)$$

in  $\mathcal{L}_s(X)$  for  $0 \leq s < \tau < t \leq T$ , and consequently,

$$(2) \quad U(t, s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(t)}[A(t) - A(\tau)]U(\tau, s) d\tau, \quad (t, s) \in T_\Delta,$$

in  $\mathcal{L}_s(X)$ . (It should be remarked that precisely the integral equations (1) and (2) were used by Sobolevskii and Tanabe to derive the existence of the fundamental solution  $U$  with the properties (U1)–(U4).)

We introduce now the following abbreviations:

$$a(t, s) := e^{-(t-s)A(s)}, \quad b(t, s) := e^{-(t-s)A(t)},$$

$$k(t, s) := -[A(t) - A(s)]a(t, s), \quad h(t, s) := b(t, s)[A(t) - A(s)]$$

for  $(t, s) \in \dot{T}_\Delta$ .

LEMMA 3.1. *Let  $0 \leq \eta \leq \theta \leq 1$ . Then*

- (i)  $a, b \in \mathfrak{R}(X_\theta, 0) \cap \mathfrak{R}(X, X_\theta, \theta) \cap \mathfrak{R}(X_\eta, X_1, 1 - \eta)$ ,
- (ii)  $Aa, Ab \in \mathfrak{R}(X_\eta, X_\theta, 1 + \theta - \eta)$ ,
- (iii)  $k \in \mathfrak{R}(X_\eta, X, 1 - \eta - \rho)$ ,
- (iv)  $h \in \mathfrak{R}(X_1, X_\theta, \theta - \rho)$ ,
- (v)  $Ah \in \mathfrak{R}(X_1, X_\theta, 1 + \theta - \rho)$ .

Moreover the norms of the above elements are bounded by constants depending only on  $L, M, N, \rho, \eta$ , and  $\theta$ , but not on the individual elements.

PROOF. (i) follows from Lemma 2.1 and (2.5) and by interpolation.

(ii) is also a consequence of Lemma 2.1 and (2.5).

(iii)–(v) are easily obtained from (A), the estimate (2.2), and from (i) and (ii).  $\square$

It follows from Lemma 3.1 and Theorem 1.2 that the convolution-type equations

$$(3) \quad u = a + u * k \quad \text{in } \mathfrak{R}(X, 0)$$

and

$$(4) \quad v = b + h * v \quad \text{in } \mathfrak{R}(X_1, 0)$$

possess unique solutions  $u$  and  $v$ , respectively. Since  $\mathfrak{R}(X, Y, \alpha) \subset \mathfrak{R}_s(X, Y, \alpha)$ , we can consider (3) and (4) also in the spaces  $\mathfrak{R}_s(X, 0)$  and  $\mathfrak{R}_s(X_1, 0)$ , respectively. However in these spaces they coincide with the integral equations (1) and (2), respectively. Thus  $u = v = U$ , by the unique solvability. Consequently,

$$(5) \quad U = a + U * k \quad \text{in } \mathfrak{R}(X, 0)$$

and

$$(6) \quad U = b + h * U \quad \text{in } \mathfrak{R}(X_1, 0).$$

It is now easy to derive further regularity properties of  $U$ , which we collect in the following

THEOREM 3.2. (i)  $U \in \mathfrak{R}(X, X_\theta, \theta) \cap \mathfrak{R}(X_\theta, X_1, 1 - \theta)$  for  $0 \leq \theta \leq 1$ .

(ii)  $U \in C(T_\Delta, \mathcal{L}_s(X_\theta))$  for  $0 \leq \theta \leq 1$ .

(iii) If  $0 \leq \eta, \theta \leq 1$  and  $\theta < \rho$ , then

$$AU \in \mathfrak{R}(X_\eta, X_\theta, 1 + \theta - \eta).$$

Moreover the norms of  $U$  and  $AU$  in the respective Banach spaces  $\mathfrak{R}$  are bounded by constants depending only on  $L, M, N, \rho, \eta, \theta$  and  $T$ .

PROOF. (i) Since  $U \in \mathfrak{R}(X, 0) \cap \mathfrak{R}(X_1, 0) \cap \mathfrak{R}(X, X_1, 1)$  by (5), (6) and (U4), the assertion follows by interpolation.

(ii) Since  $k \in \mathfrak{R}(X_\theta, X, 1 - \theta - \rho)$  by Lemma 3.1 and  $U \in \mathfrak{R}(X, X_\theta, \theta)$  by (i),  $U * k \in \mathfrak{R}(X_\theta, -\rho)$  by Lemma 1.1. Thus  $U * k \in C(T_\Delta, \mathcal{L}(X, Y))$  by (1.2). Since  $a \in C(T_\Delta, \mathcal{L}_s(X_\theta))$  by Lemma 2.1, the assertion follows from (5).

(iii) From (6) we obtain  $AU = Ab + (Ah) * U$ . Since  $U \in \mathfrak{R}(X_\eta, X_1, 1 - \eta)$  by (i) and  $Ah \in \mathfrak{R}(X_1, X_\theta, 1 + \theta - \rho)$ , it follows from (1.1) and Lemma 1.1 that  $(Ah) * U \in \mathfrak{R}(X_\eta, X_\theta, 1 + \theta - \eta)$  provided  $\theta < \rho$ . Now the assertion is a consequence of Lemma 3.1(v).

The statement about the norms of  $U$  and  $AU$  is an obvious consequence of (U4), Lemma 3.1 and (1.6).  $\square$

It should be observed that the estimates contained in Theorem 3.2(ii), (iii) are counterparts to some estimates of Sobolevskii involving fractional powers of  $A(t)$  (cf. [28, Theorem 2 and p. 26]).

Results which are related to Theorem 3.2(i), (ii) have also been obtained in the autonomous case and for particular interpolation functors by Sinestrari and Vernole [26, Proposition 1.1] and others.

**4. Linear evolution equations.** Let  $f \in C([0, T], X)$  be given. By a **solution** of the **linear Cauchy problem**

$$(LCP)_{(s,x)} \quad u + A(t)u = f(t), \quad s < t \leq T, \quad u(s) = x,$$

we mean a function  $u \in C([s, T], X) \cap C^1((s, T], X)$ , such that  $u(t) \in X_1$  for  $s < t \leq T$  and such that (LCP)<sub>(s,x)</sub> is pointwise satisfied. Let

$$(1) \quad u(\cdot, \cdot, x) := Ux + U * f \quad \forall x \in X$$

(cf. (1.4) and (1.5)). Then  $u(\cdot, s, x)$  is said to be a **mild solution** of (LCP)<sub>(s,x)</sub>. It is well known that every solution of (LCP)<sub>(s,x)</sub> is a mild solution. As for the converse we have the following

**THEOREM 4.1.** *Let  $0 < \theta \leq 1$  and suppose that either*

$$(2) \quad f \in C^\theta([0, T], X)$$

or

$$(3) \quad f \in C([0, T], X_\theta).$$

*Then every mild solution is a solution. Moreover, if  $f \in C([0, T], X)$  and  $0 \leq \eta < 1$ , then*

$$(4) \quad [(t, s, x) \mapsto u(t, s, x)] \in C^{0,1}(T_\Delta \times X_\eta, X_\eta).$$

*If (3) is satisfied, then (4) holds also for  $\eta = 1$ .*

**PROOF.** If (2) is true, the first assertion has been proven by Sobolevskii [28] and Tanabe [30, 31]. Thus let (3) be satisfied. Since  $U(\cdot, s)x$  is a solution of (LCP)<sub>(s,x)</sub> with  $f = 0$ , by (U1) and (U3), it suffices to show that  $U * f(\cdot, s)$  is a solution of (LCP)<sub>(s,0)</sub>. Since  $f \in \mathfrak{R}(K, X_\theta, 0)$  by (3) and (1.4), and since  $U \in \mathfrak{R}(X_\theta, X_1, 1 - \theta)$  by Theorem 3.2(i), it follows from Lemma 1.1 and (1.2) that

$$(5) \quad U * f \in C(T_\Delta, X_1) \hookrightarrow C(T_\Delta, X_\theta),$$

since  $X_1 \hookrightarrow X$  implies  $X_\theta \hookrightarrow X$ .

If  $(t, s) \in \dot{T}_\Delta$  and  $0 < h < T - t$ , then

$$(6) \quad \begin{aligned} h^{-1}[U * f(t + h, s) - U * f(t, s)] &= h^{-1} \int_t^{t+h} U(t + h, \tau) f(\tau) d\tau \\ &+ \int_s^t h^{-1}[U(t + h, \tau) - U(t, \tau)] f(\tau) d\tau. \end{aligned}$$

Due to Theorem 3.2(ii) the first term on the right converges to  $f(t)$  in  $X_\theta$ , hence in  $X$ , if  $h \rightarrow 0$ . By Theorem 3.2(iii) and (U3) we see that

$$\begin{aligned} \|U(t+h, \tau) - U(t, \tau)\|_{\mathcal{L}(X_\theta, X)} &= \left\| \int_t^{t+h} -AU(\sigma, \tau) d\sigma \right\|_{\mathcal{L}(X_\theta, X)} \\ &\leq c \left[ (t+h-\tau)^\theta - (t-\tau)^\theta \right]. \end{aligned}$$

By means of the substitution  $\xi := h/(t-\tau)$  it is easily verified that

$$h^{-1} \left[ (t+h-\tau)^\theta - (t-\tau)^\theta \right] \leq c(t-\tau)^{\theta-1} \quad \forall h > 0, (t, \tau) \in \dot{T}_\Delta.$$

Thus

$$\|h^{-1}[U(t+h, \tau) - U(t, \tau)]f(\tau)\| \leq c(t-\tau)^{\theta-1}.$$

Since the integrand of the second integral in (6) converges towards  $-AUf(t, \tau)$  as  $h \rightarrow 0$ , by (U3), Lebesgue's theorem implies that  $U * f(\cdot, s)$  is right differentiable and that  $D_1^+(U * f)(t, s) = f(t) - A(U * f)(t, s)$  for  $(t, s) \in \dot{T}_\Delta$ . Since  $D_1^+(U * f)$  is continuous, by (5) and  $(A_\rho)$ , it follows that  $U * f(\cdot, s)$  is continuously differentiable on  $(s, T]$  and that

$$D_1(U * f)(t, s) + A(U * f)(t, s) = f(t), \quad s < t \leq T.$$

Thus  $U * f(\cdot, s)$  is a solution of  $(LCP)_{(s,0)}$ .

Theorem 3.2(ii) implies

$$[(t, s, x) \mapsto U(t, s)x] \in C^{0,1}(T_\Delta \times X_\theta, X_\theta)$$

for  $0 \leq \theta \leq 1$ . Since  $U \in \mathfrak{R}(X, X_\theta, \theta)$  by Theorem 3.2(i), and since

$$f \in C([0, T], X) \subset \mathfrak{R}(\mathbf{K}, X, 0),$$

Lemma 1.1 and (1.2) give  $U * f \in C(T_\Delta, X_\theta)$ . This implies (4). If (3) is satisfied, then (5) implies the validity of (4) for  $\eta = 1$ .  $\square$

It should be remarked that Theorem 4.1 is essentially known. Indeed, the fact that  $(LCP)_{(s,x)}$  has a unique solution if  $f$  takes its values in certain interpolation spaces has apparently first been observed by Da Prato and Grisvard [14] and has been exploited in a series of papers by Da Prato and his students (cf. [13] for a recent survey). It should also be noted that so-called ‘‘sharp regularity’’ results have been used by Da Prato and Sinestrari [15] to study the linear Cauchy problem without using fundamental solutions (cf. also [1, 2]).

The above simple proof for the differentiability of  $U * f(\cdot, s)$ , if (3) is satisfied, follows an argument of Potier-Ferry [24].

Using the estimates of Theorem 3.2(iii) we can establish better regularity properties of the solution as a function of  $t$ . For this purpose the space  $X_\beta$  is said to be  $(X_\alpha, X_\gamma)$ -compatible if  $0 \leq \alpha \leq \beta \leq \gamma \leq 1$  and  $X_\gamma \hookrightarrow X_\beta \hookrightarrow X_\alpha$  and if

$$(7) \quad \|x\|_\beta \leq c \|x\|_\alpha^{(\gamma-\beta)/(\gamma-\alpha)} \|x\|_\gamma^{(\beta-\alpha)/(\gamma-\alpha)} \quad \forall x \in X_\gamma.$$

With this definition we have the following regularity

**THEOREM 4.2.** *Suppose that  $f \in C([0, T], X_\zeta)$  for some  $\zeta \in (0, 1)$ , that  $0 \leq \eta \leq \theta \leq 1$ , that  $X_\eta$  is  $(X, X_\theta)$ -compatible, and that  $x \in X_\theta$ . Then*

$$(8) \quad u(\cdot, s, x) \in C^{\theta-\eta}([0, T], X_\eta).$$

Moreover, if  $x \in X$  and  $0 \leq \sigma < \min(\rho, \zeta)$ , then

$$(9) \quad u(\cdot, s, x) \in C^1((s, T], X_\sigma).$$

PROOF. Due to Theorem 4.1 we can assume that  $\theta > 0$ . By Theorem 3.2(iii)

$$\|(t - s)^{1-\theta} AUx(t, s)\| \leq c \|x\|_\theta$$

and, due to Lemma 1.1,

$$\|(AU) * f\| \leq c \max_{s \leq t \leq T} \|f(t)\|_\zeta$$

for all  $(t, s) \in \dot{T}_\Delta$ . Then, since  $A(U * f) = (AU) * f$ ,

$$(10) \quad (t - s)^{1-\theta} \|A(Ux + U * f)(t, s)\| \leq c \quad \forall (t, s) \in \dot{T}_\Delta.$$

Let now  $(s, x) \in [0, T] \times X_\theta$  be fixed and let  $v := u(\cdot, s, x)$ . Then  $\dot{v}(t) + A(t)v(t) = f(t)$  for  $s < t \leq T$  by Theorem 4.1, that is,

$$(11) \quad \dot{v}(t) = -A(Ux + U * f)(t, s) + f(t) \quad \text{for } s < t \leq T.$$

Consequently (10) implies the estimate

$$(t - s)^{1-\theta} \|\dot{v}(t)\| \leq c, \quad s < t \leq T.$$

Hence we obtain from

$$\|v(t) - v(r)\| = \int_r^t \|\dot{v}(\tau)\| d\tau \leq \int_r^t (\tau - s)^{1-\theta} \|\dot{v}(\tau)\| (\tau - r)^{\theta-1} d\tau$$

the estimate

$$(12) \quad \|v(t) - v(r)\| \leq c(t - r)^\theta, \quad s < r < t \leq T.$$

Since  $v \in C([s, T], X_\theta)$  by Theorem 4.1, the first assertion follows from (7), (12), and the  $(X, X_\theta)$ -compatibility of  $X_\eta$ , provided  $\theta - \eta < 1$ .

If  $\theta = 1$ , then  $v \in C([s, T], X_1)$  by Theorem 4.1. Since

$$A(\cdot) \in C([s, T], \mathcal{L}(X_1, X)),$$

it follows that  $\dot{v} = -Av + f \in C([s, T], X)$ , which implies  $v \in C^1([s, T], X)$ .

Since  $AU \in \mathfrak{R}(X, X_\sigma, 1 + \sigma)$  and  $(AU) * f \in \mathfrak{R}(\mathbf{K}, X_\sigma, \sigma - \zeta)$  by Theorem 3.2 and Lemma 1.1, it follows from (11) that  $\dot{v} \in C((s, T], X_\sigma)$ , which implies (9).  $\square$

REMARK 4.3. It follows from the above proof that, for  $\theta < 1$ , the norm of  $u(\cdot, s, x)$  in  $C^{\theta-\eta}([0, T], X_\eta)$  is bounded by a constant depending only on  $L, M, N, T, \rho, \eta, \theta, \zeta$  and on bounds for  $\|x\|_\theta$  and  $\max_{s \leq t \leq T} \|f(t)\|_\zeta$ .  $\square$

**5. Perturbation results.** In order to study the dependence of the mild solutions on  $A(\cdot)$  and  $f$  we assume that there is given a second family  $\{A_1(t) | 0 \leq t \leq T\}$  of operators satisfying Assumptions (A) and  $(A_\rho)$ . Moreover we assume (without loss of generality) that  $\{A_1(t) | 0 \leq t \leq T\}$  satisfies the estimates (2.1), (2.2) and (2.5) with the same constants  $M, L$  and  $N$ , respectively, as the family  $\{A(t) | 0 \leq t \leq T\}$ .

We now occasionally write  $A_0$  for  $A$  and  $a_0 := a, b_0 := b, k_0 := k$  and  $h_0 := h$ . Then we define the functions  $a_1, b_1, k_1, h_1$  by replacing  $A_0$  in the definition of  $a_0, b_0, k_0, h_0$  by  $A_1$ . Finally we denote by  $\|\cdot\|_{C^\mu}$  the norm in  $C^\mu([0, T], \mathcal{L}(X_1, X))$ , where  $C := C^0$ .

It should be observed that the constants  $c$  in the following lemmas depend only on  $L, M, N, T, \rho, \eta$  and  $\theta$ .

LEMMA 5.1. *Let  $0 \leq \eta, \theta \leq 1$ . Then  $a_0 - a_1$  and  $b_0 - b_1$  belong to the space  $\mathfrak{R}(X_\theta, X_\eta, \eta - \theta)$  and the norms of*

(i)  $a_0 - a_1$  and  $b_0 - b_1$  in  $\mathfrak{R}(X_\theta, X_\eta, \eta - \theta)$ ,

(ii)  $k_0 - k_1$  in  $\mathfrak{R}(X_\theta, X, 1 - \theta)$ , and

(iii)  $h_0 - h_1$  in  $\mathfrak{R}(X_1, X_\eta, \eta)$

are bounded by  $c\|A_0(\cdot) - A_1(\cdot)\|_C$ .

The norms of

(iv)  $k_0 - k_1$  in  $\mathfrak{R}(X_\theta, X, 1 - \theta - \rho)$  and

(v)  $h_0 - h_1$  in  $\mathfrak{R}(X_1, X_\eta, \eta - \rho)$

are bounded by  $c\|A_0(\cdot) - A_1(\cdot)\|_{C^\rho}$ .

PROOF. (i) Observe that  $A_j(t)(\lambda + A_j(t))^{-1} = 1 - \lambda(\lambda + A_j(t))^{-1}$ . Hence  $\|A_j(t)(\lambda + A_j(t))^{-1}\| \leq 1 + M$ , which implies, by (2.5), that

$$\|(\lambda + A_j(t))^{-1}\|_{\mathcal{L}(X, X_1)} \leq N(1 + M).$$

Moreover,

$$\begin{aligned} \|(\lambda + A_j(t))^{-1}\|_{\mathcal{L}(X_1)} &\leq \|A(0)(\lambda + A_j(t))^{-1}A^{-1}(0)\| \\ &\leq N^2\|A_j(t)(\lambda + A_j(t))^{-1}A_j^{-1}(t)\| \\ &= N^2\|(\lambda + A_j(t))^{-1}\| \leq N^2M/|\lambda|. \end{aligned}$$

Thus, by interpolation,

$$(1) \quad \|(\lambda + A_j(t))^{-1}\|_{\mathcal{L}(X_\theta, X_\eta)} \leq c(M, N, \eta, \theta)|\lambda|^{\eta(1-\theta)-1}$$

for  $0 \leq \eta, \theta \leq 1, \lambda \in \Sigma_\vartheta, t \in [0, T]$  and  $j = 0, 1$  (cf. [7, Lemma 8.1]).

Since

$$\begin{aligned} &(\lambda + A_0(t))^{-1} - (\lambda + A_1(t))^{-1} \\ &= (\lambda + A_0(t))^{-1}(A_1(t) - A_0(t))(\lambda + A_1(t))^{-1}, \end{aligned}$$

it follows that

$$e^{-sA_0(t)} - e^{-sA_1(t)} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda s} (\lambda + A_0(t))^{-1} (A_1(t) - A_0(t)) (\lambda + A_1(t))^{-1} d\lambda.$$

By (1) the integrand can be estimated in the norm of  $\mathcal{L}(X_\theta, X_\eta)$  by

$$c|e^{\lambda s}| |\lambda|^{\eta-\theta-1} \|A_0(\cdot) - A_1(\cdot)\|_C,$$

which implies, by standard arguments (e.g. [31, p. 66]), that

$$(2) \quad \|e^{-sA_0(t)} - e^{-sA_1(t)}\|_{\mathcal{L}(X_\theta, X_\eta)} \leq cs^{\theta-\eta} \|A_0(\cdot) - A_1(\cdot)\|_C,$$

whence the assertion.

(ii) Let  $B_j(t, s) := -[A_j(t) - A_j(s)]$  and observe that

$$(3) \quad \|B_0(t, s) - B_1(t, s)\|_{\mathcal{L}(X_1, X)} \leq 2\|A_0(\cdot) - A_1(\cdot)\|_C$$

and

$$(4) \quad \|B_1(t, s)\|_{\mathcal{L}(X_1, X)} \leq LT^\rho.$$

Moreover,

$$(5) \quad k_0 - k_1 = [B_0(\cdot, \cdot) - B_1(\cdot, \cdot)]a_0 + B_1(\cdot, \cdot)[a_0 - a_1].$$

Hence (i) and Lemma 3.1(i) imply the estimate

$$\|k_0(t, s) - k_1(t, s)\|_{\mathcal{L}(X_\theta, X)} \leq c\|A_0(\cdot) - A_1(\cdot)\|_C(t - s)^{\theta - 1},$$

whence the assertion.

(iii) follows by analogous arguments.

(iv) and (v) are obtained from (5) by replacing (3) and (4) by the estimates

$$\|B_0(t, s) - B_1(t, s)\|_{\mathcal{L}(X_1, X)} \leq |t - s|^\rho \|A_0(\cdot) - A_1(\cdot)\|_{C^\rho}$$

and

$$\|B_1(t, s)\|_{\mathcal{L}(X_1, X)} \leq L|t - s|^\rho,$$

respectively.  $\square$

We now denote by  $w_j$  and  $r_j$  the resolvent kernels of  $u = k_j + u * k_j$  and  $v = h_j + h_j * v$ , respectively.

LEMMA 5.2. *Let  $0 \leq \eta, \theta \leq 1$ . Then  $w_j \in \mathfrak{R}(X_\theta, X, 1 - \theta - \rho)$  and*

$$(6) \quad \|w_0 - w_1\|_{\mathfrak{R}(X_\theta, X, 1 - \theta - \rho)} \leq c\|A_0(\cdot) - A_1(\cdot)\|_C \quad \text{if } \theta > 0,$$

and

$$(7) \quad \|w_0 - w_1\|_{\mathfrak{R}(X_\theta, X, 1 - \theta - \rho)} \leq c\|A_0(\cdot) - A_1(\cdot)\|_{C^\rho} \quad \text{for } 0 \leq \theta \leq 1.$$

Similarly,  $r_j \in \mathfrak{R}(X_1, X_\eta, \eta - \rho)$  and

$$\|r_0 - r_1\|_{\mathfrak{R}(X_1, X_\eta, \eta - \rho)} \leq c\|A_0(\cdot) - A_1(\cdot)\|_C \quad \text{if } \eta < 1,$$

and

$$\|r_0 - r_1\|_{\mathfrak{R}(X_1, X_\eta, \eta - \rho)} \leq c\|A_0(\cdot) - A_1(\cdot)\|_{C^\rho} \quad \text{for } 0 \leq \eta \leq 1.$$

PROOF. Since  $k_j \in \mathfrak{R}(X_\theta, X, 1 - \theta - \rho)$  by Lemma 3.1(iii), thus in particular,  $k_j \in \mathfrak{R}(X, 1 - \rho)$ , it follows from Theorem 1.2 that  $w_j \in \mathfrak{R}(X, 1 - \rho)$ . Hence  $w_j * k_j \in \mathfrak{R}(X_\theta, X, 1 - \theta - 2\rho) \subset \mathfrak{R}(X_\theta, X, 1 - \theta - \rho)$  by Lemma 1.1 and (1.1). Consequently, by Theorem 1.2,  $w_j = k_j + w_j * k_j \in \mathfrak{R}(X_\theta, X, 1 - \theta - \rho)$ . Moreover

$$w_0 - w_1 = k_0 - k_1 + w_0 * (k_0 - k_1) + (w_0 - w_1) * k_1,$$

and Lemma 5.1 and (1.1) imply

$$\|[k_0 - k_1 + w_0 * (k_0 - k_1)](t, s)\|_{\mathcal{L}(X_\theta, X)} \leq c(t - s)^{\theta - 1}\|A_0(\cdot) - A_1(\cdot)\|_C.$$

Thus

$$\begin{aligned} \|(w_0 - w_1)(t, s)\|_{\mathcal{L}(X_\theta, X)} &\leq c(t - s)^{\theta - 1} \|A_0(\cdot) - A_1(\cdot)\|_C \\ &\quad + c \int_s^t (\tau - s)^{\rho - 1} \|(w_0 - w_1)(t, \tau)\|_{\mathcal{L}(X_\theta, X)} d\tau \end{aligned}$$

for  $(t, s) \in \hat{T}_\Delta$ . From this we deduce, by a well-known generalization of Gronwall's lemma (e.g. [5, Corollary 2.4 or 17, §7.1]), the estimate

$$\|(w_0 - w_1)(t, s)\|_{\mathcal{L}(X_\theta, X)} \leq c(t - s)^{\theta - 1} \|A_0(\cdot) - A_1(\cdot)\|_C,$$

that is, the estimate (6). The proof of (7) is obtained by obvious modifications of the above arguments, as it is the case for the proof of the second part of the assertion.

□

Finally we put  $U_0 := U$  and denote by  $U_1$  the parabolic fundamental solution for  $\{A_1(t) | 0 \leq t \leq T\}$ .

LEMMA 5.3. *Let  $0 \leq \eta, \theta \leq 1$  satisfy  $|\eta - \theta| < 1$ . Then*

$$U_0 - U_1 \in \mathfrak{K}(X_\theta, X_\eta, \eta - \theta)$$

and

$$\|U_0 - U_1\|_{\mathfrak{K}(X_\theta, X_\eta, \eta - \theta)} \leq c \|A_0(\cdot) - A_1(\cdot)\|_{C^\mu},$$

where  $\mu := 0$  if  $\theta > 0$  and  $\eta < 1$ , and where  $\mu := \rho$  otherwise.

PROOF. Formula (3.5) and Theorem 1.2 imply

$$U_0 - U_1 = a_0 - a_1 + (a_0 - a_1) * w_0 + a_1 * (w_0 - w_1).$$

Thus, if  $\eta < 1$  the assertion follows from Lemmas 5.1, 5.2, 3.1(i) and 1.1. If  $\eta = 1$ , we use in a similar way

$$U_0 - U_1 = b_0 - b_1 + r_0 * (b_0 - b_1) + (r_0 - r_1) * b_1,$$

which follows from (3.6) and Theorem 1.2, to obtain the assertion. □

After these preparations we can easily prove the following important perturbation:

THEOREM 5.4. *Let  $0 < \eta, \theta, \zeta \leq 1$  and suppose that  $f_j \in C([0, T], X_\zeta)$ . Denote by  $u_j$  the solutions of*

$$\dot{u} + A_j(t)u = f_j(t), \quad s < t \leq T, u(s) = x_j,$$

respectively. Then

$$\begin{aligned} &\|u_0(t, s, x_0) - u_1(t, s, x_1)\|_\eta \\ &\leq c \left\{ \|A_0(\cdot) - A_1(\cdot)\|_{C^\mu([s, t], \mathcal{L}(X_1, X))} \right. \\ &\quad \times \left( \|x_0\|_\theta (t - s)^{\theta - \eta} + (t - s)^{1 + \zeta - \eta} \max_{s \leq \tau \leq t} \|f_0(\tau)\|_\zeta \right) \\ &\quad \left. + (t - s)^\rho \max_{s \leq \tau \leq t} \|f_0(\tau) - f_1(\tau)\|_\zeta + (t - s)^\lambda \|x_0 - x_1\|_\theta \right\}, \end{aligned}$$



where

$$\mu := \begin{cases} 0 & \text{if } \eta < 1, \\ \rho & \text{if } \eta = 1, \end{cases} \quad \nu := \max\{1 - \eta, \zeta\}, \quad \lambda := \begin{cases} 0 & \text{if } X_\theta \hookrightarrow X_\eta, \\ \theta - 1 & \text{otherwise.} \end{cases}$$

Moreover the constant  $c$  depends only on  $L, M, N, \rho, \eta, \theta, \zeta$  and  $T$ .

PROOF. Since

$$u_0 - u_1 = (U_0 - U_1)x_0 + (U_0 - U_1)*f_0 + U_1*(f_0 - f_1) + U_1(x_0 - x_1),$$

the assertion is an easy consequence of Lemma 5.3, Theorem 3.2(i), (ii), the uniform boundedness principle, Lemma 1.1, and  $X_1 \hookrightarrow X_\eta$ .  $\square$

**6. Quasilinear evolution equations.** We denote now by  $X_1$  and  $X_0 := X$  two fixed Banach spaces such that

(Q1)  $X_1$  is continuously and densely imbedded in  $X$ .

Then we suppose that

(Q2)  $0 < \eta < \theta \leq 1$  and that  $X_\eta$  is  $(X, X_\theta)$ -compatible,

where, of course,  $X_\xi := \mathfrak{F}_\xi(X, X_1)$  for  $0 < \xi < 1$ . Moreover we assume that

$V$  is an open subset of  $X_\eta$  and

(Q3)  $[(t, y) \mapsto A(t, y)] \in C^{\rho, 1^-}([0, T] \times V, \mathcal{L}(X_1, X))$

for some  $\rho \in (0, 1)$ .

We suppose also that

for each point  $y_0 \in V$  there exist a neighbourhood  $V_{y_0}$  and constants  $M > 0$  and  $\omega \in \mathbf{R}$  such that

(Q4)  $\omega + \Sigma_0 \subset \rho(-A(t, y))$

and

$$\|(\lambda + A(t, y))^{-1}\| \leq M/(1 + |\lambda - \omega|)$$

for all  $\lambda \in \omega + \Sigma_0$  and  $(t, y) \in [0, T] \times V_{y_0}$ , where  $A(t, y)$  is considered as a linear operator in  $X$  with domain  $X_1$ .

Finally we assume that

(Q5)  $f \in C^{0, 1^-}([0, T] \times V, X_\zeta)$  for some  $\zeta \in (0, 1)$ .

By a **solution of the quasilinear Cauchy problem**

(QCP)<sub>(s, x)</sub>  $\dot{u} + A(t, u)u = f(t, u), \quad s < t \leq T, u(s) = x,$

on  $J$  we mean a function  $u \in C(J, V) \cap C^1(J, X)$  such that  $J$  is a perfect subinterval of  $[s, T]$  containing  $s$  and  $\dot{J} := J \setminus \{s\}$ , and such that  $u(s) = x, u(t) \in X_1$  and  $\dot{u}(t) + A(t, u(t))u(t) = f(t, u(t))$  for  $t \in \dot{J}$ . A solution  $u$  is **maximal** if there does not exist a solution of (QCP)<sub>(s, x)</sub> which is a proper extension of  $u$ . In this case  $J$  is a **maximal interval of existence**.

For each  $v: [0, T] \rightarrow V$  put

$$A_v(t) := A(t, v(t)) \quad \text{and} \quad f_v(t) := f(t, v(t)), \quad 0 \leq t \leq T.$$

Moreover, fix  $\rho_0 \in (0, \rho \wedge (\theta - \eta))$  and let

$$W := \left\{ v \in C([0, T], V) \mid \|v(s) - v(t)\|_\eta \leq |s - t|^{\rho_0} \text{ for } 0 \leq s, t \leq T \right\}.$$

If  $v \in W$ , then the compactness of  $v([0, T])$  in  $V$  and (Q3) and (Q4) imply the existence of constants  $\omega$  and  $L$  such that

$$\|(\omega + A_v)(t) - (\omega + A_v)(s)\|_{\mathcal{L}(X_1, X)} \leq L|s - t|^{\rho_0}, \quad 0 \leq s, t \leq T.$$

Hence  $\{(\omega + A_v)(t) \mid 0 \leq t \leq T\}$  satisfies Assumption  $(A_{\rho_0})$ . Moreover it is an easy consequence of (Q1) and (Q4) that  $\{(\omega + A_v)(t) \mid 0 \leq t \leq T\}$  also satisfies (A). Hence there exists a unique parabolic fundamental solution  $U_{v,\omega}$  for  $\{(\omega + A_v)(t) \mid 0 \leq t \leq T\}$ . It is easily verified that

$$U_v(t, s) := e^{\omega t} U_{v,\omega}(t, s) e^{-\omega s}, \quad (t, s) \in T_\Delta,$$

is a parabolic fundamental solution for  $\{A_v(t) \mid 0 \leq t \leq T\}$ , and the only one.

Let  $w \in W$  and suppose that  $v[[\sigma, \tau]] = w[[\sigma, \tau]]$  for some  $(\tau, \sigma) \in \dot{T}_\Delta$ . Then it follows from (3.5), for example, that

$$(1) \quad U_v(t, s) = U_w(t, s) \quad \text{for } \sigma \leq s \leq t \leq \tau.$$

Finally it is clear that all results of §§3–5 are true for  $U_v$  and the corresponding linear Cauchy problem (provided the constant  $N$  is chosen so that

$$\|(\omega + A_v(t))(\omega + A_v(s))^{-1}\| \leq N$$

for all  $(t, s) \in T_\Delta$ , and where the various constants now depend also on  $\omega$ , of course).

For each  $v \in W$  and  $x \in X$  we put

$$\psi(v, x) := U_v x + U_v * f_v.$$

If  $J$  is a perfect compact subinterval of  $[0, T]$  and  $w: J \rightarrow X$ , then we denote by  $\bar{w}$  the function on  $[0, T]$ , defined by

$$\bar{w}(t) := \begin{cases} w(\alpha) & \text{for } 0 \leq t \leq \alpha := \min J, \\ w(t) & \text{for } \alpha \leq t \leq \beta := \max J, \\ w(\beta) & \text{for } \beta \leq t \leq T. \end{cases}$$

Moreover we let

$$\begin{aligned} \psi_s(v, x) &:= \psi(v, x)(\cdot, s), \\ J_{s,\delta} &:= [s, s + \delta] \cap [0, T] \end{aligned}$$

and

$$\psi(v, x, J_{s,\delta}) := \overline{\psi_s(v, x)|_{J_{s,\delta}}}$$

for  $0 \leq s < T$  and  $\delta > 0$ . Finally, if  $S$  is a nonempty subset of  $X_\xi$ ,  $0 \leq \xi \leq 1$ , then  $\bar{B}_\xi(S, \varepsilon)$  denotes the closed  $\varepsilon$ -neighbourhood of  $S$  in  $X_\xi$ , and  $S_\theta := S \cap X_\theta$ , endowed with the topology of  $X_\theta$ .

LEMMA 6.1. *Let  $S$  be a compact subset of  $V$  such that  $S_\theta$  is nonempty and bounded in  $X_\theta$ . Then there exist positive numbers  $\alpha, \epsilon, \delta$  and  $K_0$  such that  $\alpha \geq \epsilon$  and*

- (i)  $\overline{B}_\theta(S_\theta, \epsilon) \subset \overline{B}_\eta(S, \alpha) \subset \overline{B}_\eta(S, 3\alpha) \subset V$ ;
- (ii)  $\|\psi_s(v, x) - \psi_s(w, x)\|_{C(J_{s,\delta}, X_\eta)} \leq \frac{1}{2}\|v - w\|_{C(J_{s,\delta}, X_\eta)}$  for all  $v, w \in W$ , all  $s \in [0, T]$ , and all  $x \in \overline{B}_\theta(S_\theta, 3\epsilon)$ , provided  $v([0, T]), w([0, T]) \subset \overline{B}_\eta(S, 3\alpha)$ ;
- (iii)  $\|\psi_s(v, x) - \psi_s(v, y)\|_{C([s,t], X_\eta)} \leq K_0\|x - y\|_\theta$  for all  $(t, s) \in T_\Delta$ ,  $v \in W$ , and  $x, y \in \overline{B}_\theta(S_\theta, 3\epsilon)$  provided  $v([0, T]) \subset \overline{B}_\eta(S, 3\alpha)$ .
- (iv)  $(v, x, t, s) \mapsto \psi(v, x)(t, s)$  is continuous from  $\{v \in W | v([0, T]) \subset \overline{B}_\eta(S, 3\alpha)\} \times \overline{B}_\theta(S_\theta, 3\epsilon) \times T_\Delta$  into  $X_\eta$ .

PROOF. (i) is an obvious consequence of the fact that  $X_\theta \hookrightarrow X_\eta$  by (Q2) and the compactness of  $S$  in the open set  $V$ .

(ii), (iii) By decreasing  $\alpha$ , if necessary, it follows from the compactness of  $S$  and the assumptions (Q3)–(Q5) that we can assume that

- (2) the constant  $M$  in (Q4) is independent of  $y \in \overline{B}_\eta(S, 3\alpha)$ ,
- (3)  $\|A(t, y) - A(s, z)\|_{\mathcal{L}(X_1, X)} \leq c(|t - s|^{\rho_0} + \|y - z\|_\eta)$ ,
- (4)  $\|(\omega + A(t, y))(\omega + A(s, z))^{-1}\| \leq N$  for some  $\omega \in \mathbf{R}$ ,
- (5)  $\|f(t, y)\|_f \leq c$

and

$$(6) \quad \|f(t, y) - f(t, z)\|_f \leq c\|y - z\|_\eta$$

for all  $s, t \in [0, T]$  and  $y, z \in \overline{B}_\eta(S, 3\alpha)$ . Hence

$$(7) \quad \|A_v(t) - A_v(s)\|_{\mathcal{L}(X_1, X)} \leq (K_0 + 1)|t - s|^{\rho_0}$$

and

$$(8) \quad \|(\omega + A_v(t))(\omega + A_v(s))^{-1}\| \leq N$$

for  $s, t \in [0, T]$  and all  $v \in W$  with  $v([0, T]) \subset \overline{B}_\eta(S, 3\alpha)$ . Now (3), (5), (6), (Q2) and Theorem 5.4 imply the estimate

$$(9) \quad \|\psi(v, x)(t, s) - \psi(w, y)(t, s)\|_\eta \leq c\{(t - s)^{\theta - \eta}\|v - w\|_{C([s,t], X_\eta)} + \|x - y\|_\theta\}$$

for all  $(t, s) \in T_\Delta$ ,  $x, y \in \overline{B}_\theta(S_\theta, 3\epsilon)$  and  $v, w \in W$  with  $v([0, T]), w([0, T]) \subset \overline{B}_\eta(S, 3\alpha)$ . The assertions are now obvious.

(iv) It follows from Theorem 4.1 that  $\psi(v, x) \in C(T_\Delta, X_\theta)$  for each  $(v, x) \in W \times X_\theta$ . Hence the assertion is a consequence of  $X_\theta \hookrightarrow X_\eta$  and (9).  $\square$

LEMMA 6.2. *Let  $S$  be a compact subset of  $V$  such that  $S_\theta$  is nonempty and compact in  $X_\theta$ . Then there exist positive numbers  $\alpha, \epsilon, \delta, K_1$  and  $K_2$  such that  $\alpha \geq \epsilon$  and*

- (i)  $\overline{B}_\theta(S_\theta, \epsilon) \subset \overline{B}_\eta(S, \alpha) \subset \overline{B}_\eta(S, 3\alpha) \subset V$ ;
- (ii) (QCP) $_{(s,x)}$  has for each  $(s, x) \in [0, T] \times \overline{B}_\theta(S_\theta, \epsilon)$  a unique solution  $v(\cdot, s, x)$  on  $J_{s,\delta}$  such that  $v(J_{s,\delta}) \subset \overline{B}_\eta(S, 3\alpha)$ .
- (iii)  $\|v(t, s, x) - v(t, s, y)\|_\eta \leq K_1\|x - y\|_\theta$  for all  $s \in [0, T]$ ,  $t \in J_{s,\delta}$ , and  $x, y \in \overline{B}_\theta(S_\theta, \epsilon)$ .

(iv) If  $\theta < 1$ , then

$$\|v(t, s, x) - v(t, s, y)\|_\theta \leq K_2 \|x - y\|_\theta$$

for all  $s \in [0, T]$ ,  $t \in J_{s,\delta}$ , and  $x, y \in \bar{\mathbf{B}}_\theta(S_\theta, \epsilon)$ .

PROOF. (i) follows from Lemma 6.1(i).

(ii) We deduce from (2), (7), (8), (Q2), Theorem 4.2 and Remark 4.3 that

$$\|\psi(v, x)(\sigma, x) - \psi(v, x)(\tau, s)\|_\eta \leq c\delta^{\theta - \eta - \rho_0} |\sigma - \tau|^{\rho_0}$$

for all  $s \in [0, T]$ ,  $\sigma, \tau \in J_{s,\delta}$ ,  $x \in \bar{\mathbf{B}}_\theta(S_\theta, 3\epsilon)$  and  $v \in W$  with  $v([0, T]) \subset \bar{\mathbf{B}}_\eta(S, 3\alpha)$ . Hence, by decreasing  $\delta$  further, if necessary, we see that

$$(10) \quad \psi(v, x, J_{s,\delta}) \in W$$

for all  $s \in [0, T]$ ,  $x \in \bar{\mathbf{B}}_\theta(S_\theta, 3\epsilon)$  and  $v \in W$  with  $v([0, T]) \subset \bar{\mathbf{B}}_\eta(S, 3\alpha)$ . In particular

$$(11) \quad \psi(x, x, J_{s,\delta}) \in W \quad \forall x \in \bar{\mathbf{B}}_\theta(S_\theta, \epsilon), s \in [0, T],$$

since the constant function  $t \mapsto x$  belongs to  $W$ . Since  $\psi(x, x)(s, s) = x$ , we deduce from Lemma 6.1(iv) and the compactness of  $S_\theta$  in  $X_\theta$  that we can decrease  $\epsilon$  and  $\delta$  further so that

$$(12) \quad \|\psi(x, x, J_{s,\delta})(t, s) - x\|_\eta \leq \alpha \quad \forall x \in \bar{\mathbf{B}}_\theta(S_\theta, \epsilon), (t, s) \in T_\Delta.$$

Now let  $(s, x) \in [0, T] \times \bar{\mathbf{B}}_\theta(S_\theta, \epsilon)$  be fixed and let

$$W_{s,x} := \left\{ v \in C(J_{s,\delta}, X_\eta) \mid \bar{v} \in W, v(s) = x, \right. \\ \left. v(J_{s,\delta}) \subset \mathbf{B}_\eta(S, 3\alpha) \text{ and } \|v - \psi(x, x, J_{s,\delta})\|_{C(J_{s,\delta}, X_\eta)} \leq \alpha \right\}.$$

Then  $W_{s,x}$  is a closed nonempty subset of the Banach space  $C(J_{s,\delta}, X_\eta)$ , hence a complete metric space. It follows from (i), (10)–(12) and Lemma 6.1(ii) that  $\psi_s(\cdot, x)|_{J_{s,\delta}}$  maps  $W_{s,x}$  into itself and is a strict contraction. Hence, by Banach’s fixed point theorem there exists a unique function  $v(\cdot, s, x) \in C(J_{s,\delta}, X_\eta)$  such that  $\bar{v}(\cdot, s, x) \in W$ ,  $v(s, s, x) = x$ ,  $v(J_{s,\delta}, s, x) \in \bar{\mathbf{B}}_\eta(S, 3\alpha)$ , and

$$(13) \quad v(\cdot, s, x) = (U_{\bar{v}}x + U_{\bar{v}} * f_{\bar{v}})(\cdot, s)|_{J_{s,\delta}}.$$

Now it follows from Theorem 4.1 that  $v(\cdot, s, x)$  is a solution of (QCP) $_{(s,x)}$  on  $J_{s,\delta}$ , and the only one.

(iii) Let  $s \in [0, T]$  and  $x, y \in \bar{\mathbf{B}}_\theta(S_\theta, \epsilon)$ . Then

$$v(\cdot, s, x) - v(\cdot, s, y) = \psi_s(\bar{v}(\cdot, s, x), x) - \psi_s(\bar{v}(\cdot, s, y), x) \\ + \psi_s(\bar{v}(\cdot, s, y), x) - \psi_s(\bar{v}(\cdot, s, y), y)$$

on  $J_{s,\delta}$ , and Lemma 6.1(ii), (iii) imply the estimate

$$\|v(t, s, x) - v(t, s, y)\|_\eta \leq 2K_0 \|x - y\|_\theta, \quad t \in J_{s,\delta},$$

which proves the assertion.

(iv) is now an easy consequence of (3)–(6), (iii), Theorem 5.4 and  $X_\theta \hookrightarrow X_\eta$ .  $\square$

After these preparations we can now prove the main result of this section:

**THEOREM 6.3.** *The quasilinear Cauchy problem (QCP)<sub>(s,x)</sub> possesses for each  $(s, x) \in [0, T] \times V_\theta$  a unique maximal solution  $u(\cdot, s, x)$ , and*

$$(14) \quad u(\cdot, s, x) \in C^{\theta-\xi}(J, X_\xi) \quad \text{for } 0 \leq \xi \leq \theta,$$

*provided  $X_\xi$  is  $(X, X_\theta)$ -compatible. Moreover, if  $0 \leq \sigma < \min\{\rho, \theta - \eta, \zeta\}$ , then*

$$(15) \quad u(\cdot, s, x) \in C^1(J, X_\sigma).$$

*The maximal interval of existence  $J(s, x) := J$  is open in  $[s, T]$ .*

**PROOF.** Let  $(s, x) \in [0, T] \times V_\theta$  be fixed. Then (QCP)<sub>(s,x)</sub> possesses, according to Lemma 6.2, a unique solution  $u_0$  on some nontrivial interval  $[s, t_0]$ . Suppose that  $t_0 < T$ . Then, by applying Lemma 6.2 to (QCP)<sub>(t\_0, u\_0(t\_0))</sub>, we find that the equation

$$\dot{u} + A(t, u)u = f(t, u)$$

has a unique solution  $u_1$  on some nontrivial interval  $[t_0, t_1]$  satisfying  $u(t_0) = u_0(t_0)$ . By Theorem 4.2,  $u_0 \in C^{\theta-\eta}([s, t_0], X_\eta)$  and  $u_1 \in C^{\theta-\eta}([t_0, t_1], X_\eta)$ . Thus, defining  $u: [s, t_1] \rightarrow X_\eta$  by  $u|_{[s, t_0]} := u_0$  and  $u|_{[t_0, t_1]} := u_1$ , it follows that

$$u \in C^{\theta-\eta}([s, t_1], X_\eta).$$

Then it is clear that  $\{A_{\bar{u}}(t) | 0 \leq t \leq T\}$  satisfies Assumptions (A) and  $(A_{\rho_1})$ , where  $\rho_1 := \min\{\rho, \theta - \eta\}$ . Hence there exists a unique parabolic fundamental solution  $U_{\bar{u}}$  for  $\{A_{\bar{u}}(t) | 0 \leq t \leq T\}$ , and, by (1), we see that  $U_{\bar{u}}(t, \sigma) = U_{u_j}(t, \sigma)$  for  $t_{j-1} \leq \sigma \leq t \leq t_j$ ,  $j = 0, 1$ , where  $t_{-1} := s$ . Using these facts and property (U2) it is easily verified that

$$u(t) = (U_{\bar{u}}x + U_{\bar{u}} * f_{\bar{u}})(t, s) \quad \text{for } s \leq t \leq t_1.$$

Now we deduce from Theorem 4.1 that  $u$  is a solution of (QCP)<sub>(s,x)</sub> on  $[s, t_1]$ .

Let

$$J(s, x) := \bigcup \{ [s, t] \subset [s, T] | \text{(QCP)}_{(s,x)} \text{ has a solution on } [s, t] \}.$$

Then  $J(s, x)$  is a perfect subinterval of  $[s, T]$  containing  $s$ . Moreover  $J(s, x)$  is open in  $[s, T]$ , since otherwise an application of Lemma 6.2 to its right endpoint would give a contradiction. Clearly  $J(s, x)$  is a maximal interval of existence of a solution  $u(\cdot, s, x)$  of (QCP)<sub>(s,x)</sub>, and there is only one maximal solution.

Since  $u := u(\cdot, s, x) \in C^{\theta-\eta}(J(s, x), [s, t_0])$ , where  $t_0 \in J(s, u_0)$  is arbitrary, it follows from (3) that  $A_{\bar{u}}(\cdot) \in C^{\rho_1}([0, T], \mathcal{L}(X_1, X))$ . Hence (14) and (15) are consequences of Theorem 4.2.  $\square$

Similar but less precise results have been obtained by Sobolevskii [28] (cf. also Friedman [16]), Potier-Ferry [24] and Lunardi [20]. Sobolevskii and Lunardi prove the existence of a local solution, that is, a solution on some interval  $[s, t_0]$ , provided the initial value  $x$  has “better regularity properties than the solution itself”. (This would correspond to an assumption of the form  $x \in V \cap X_\alpha$  for some  $\alpha > \theta$ .) Thus, in particular, these authors cannot admit the value  $\theta = 1$ . Sobolevskii uses fractional powers and Lunardi works in specific interpolation spaces (namely in  $(X, X_1)_{\theta, \infty}$  and in “continuous interpolation spaces” introduced by Da Prato and Grisvard [14]). Hence, also as far as the possible choice of the spaces is concerned, our results are more general.

Lunardi does not use fundamental solutions, but works with the (essentially equivalent) theory developed by Sinestrari [25] and Acquistapace and Terreni [1, 2]. Moreover she does not assume that  $X_1$  is dense in  $X$ . This is mainly done in order to be able to work in spaces of continuous functions, which will yield almost classical solutions in the case of parabolic partial differential equations (cf. however the remarks at the end of §10).

Potier-Ferry works in  $X_1$  (that is, he considers the case  $\theta = 1$  only) and uses Sobolevskii’s theory of fractional powers. However he can only prove the existence of solutions for initial values close to 0, and this restriction is essential for his proof. Moreover he has to assume more regularity, namely that

$$[(t, y) \mapsto A(t, y)] \in C^{\rho, 1+\rho}([0, T] \times V, \mathcal{L}(X_1, X))$$

for some  $\rho \in (0, 1)$ .

**7. Global existence.** A solution  $u$  of  $(QCP)_{(s,x)}$  on  $J$  is said to be **global** if  $J := [s, T]$ . Clearly every global solution is maximal. In this section we give simple and useful sufficient criteria for a maximal solution to be global. For this we let

$$t^+(s, x) := \sup J(s, x) \quad \forall (s, x) \in [0, T] \times V_\theta.$$

We now fix  $(s, x) \in [0, T] \times V_\theta$  and put  $u := u(\cdot, s, x)$ ,  $J := J(s, x)$  and  $t^+ := t^+(s, x)$ .

**THEOREM 7.1.** *Suppose that  $u \in BUC^\varepsilon(J, X_\eta)$  for some  $\varepsilon \in (0, 1)$ . Then **either**  $u(t) \rightarrow y \in \partial V$  in  $X_\eta$  as  $t \rightarrow t^+$ , **or**  $u$  is a global solution.*

**PROOF.** The assumption implies that  $u(t) \rightarrow y$  in  $X_\eta$  for some  $y \in X_\eta$  as  $t \rightarrow t^+$ . Assume that  $y \notin \partial V$  and let  $v(t) := u(t)$  for  $t \in J$  and  $v(t^+) := y$ . Then  $v \in C^\varepsilon([s, t^+], V)$ . Hence  $A_{\bar{v}}(\cdot) \in C^{\varepsilon_1}([0, T], \mathcal{L}(X_1, X))$ , where  $\varepsilon_1 := \min\{\varepsilon, \rho\}$ , and there exists a unique parabolic fundamental solution  $U_{\bar{v}}$  for  $\{A_{\bar{v}}(\cdot) | 0 \leq t \leq T\}$ . It follows from (6.1) and the construction of  $u$  (cf. (6.13)), that

$$u = (U_{\bar{v}}x + U_{\bar{v}} * f_{\bar{v}})(\cdot, s) \quad \text{on } J.$$

Since both sides are continuous in  $t$  on  $[s, t^+]$ , it follows that

$$\bar{v} = (U_{\bar{v}}x + U_{\bar{v}} * f_{\bar{v}})(\cdot, s) \quad \text{on } [s, t^+].$$

Hence  $\bar{v}$  is a solution of  $(QCP)_{(s,x)}$  on  $[s, t^+]$ . If  $t^+ < T$ , this contradicts the maximality of  $J$ . Thus  $J = [s, T]$ .  $\square$

**THEOREM 7.2.** *Suppose that  $u(J)$  is relatively compact in  $X_\theta$  and has a positive distance in  $X_\eta$  from  $\partial V$ . Then  $u$  is a global solution.*

**PROOF.** Let  $S$  denote the closure of  $u(J)$  in  $X_\theta$ . Since  $X_\theta \hookrightarrow X_\eta$ , by (Q2), and since  $S$  is compact in  $X_\theta$ , it follows that  $S$  is compact in  $X_\eta$  and contains  $u(J)$ . Hence  $S$  contains the closure of  $u(J)$  in  $X_\eta$  and, using  $X_\theta \hookrightarrow X_\eta$  again, we see that  $S$  equals the closure of  $u(J)$  in  $X_\eta$ . Thus  $S$  is a compact subset of  $V$  and  $S_\theta$  is nonempty and compact.

Suppose now that  $t^+ < T$ . Lemma 6.2 guarantees that for each  $(\sigma, y) \in J \times u(J)$  there exists a unique solution of  $(QCP)_{(\sigma,y)}$  on  $J_{\sigma,\delta}$ , where  $\delta > 0$  is independent of  $(\sigma, y)$ . In particular, there exists a unique solution of  $(QCP)_{(\sigma,u(\sigma))}$  for each  $\sigma \in J$  with  $t^+ - \sigma < \delta/2$ , which exists on  $J_{\sigma,\delta}$ . Similar to the proof of Theorem 6.3, we see that this implies the existence of a solution of  $(QCP)_{(s,x)}$ , which is a proper extension of  $u$ . This contradiction proves the theorem.  $\square$

The following corollary will be particularly useful in applications. Here and in the following  $\hookrightarrow$  denotes compact imbedding.

**COROLLARY 7.3.** *Suppose that  $E$  is a Banach space such that  $E \hookrightarrow X_\theta$  and such that  $u([s_1, t^+))$  is contained and bounded in  $E$  for some  $s_1 \in J$ . Then  $u$  is a global solution, provided  $u(J)$  has a positive distance in  $X_\eta$  from  $\partial V$ .*

**PROOF.** Since  $u \in C(J, X_\theta)$  it follows that  $u([s, s_1])$  is compact in  $X_\theta$ . Since  $E \hookrightarrow X_\theta$ , it follows that  $u([s_1, t^+))$  is relatively compact in  $X_\theta$ . Hence  $u(J)$  is relatively compact in  $X_\theta$ .  $\square$

Of course, the above criteria become particularly simple if  $V = X_\eta$ , since in this case the conditions involving  $\partial V$  are vacuously satisfied.

Lunardi [21] has shown that there exists a global solution, provided one can find an a priori bound in  $X_1$  (if  $f$  is Hölder continuous from an appropriate interpolation space, namely  $(X, X_1)_{\theta,\infty}$  into  $X$ ), and if the problem is autonomous. However for practical applications it is essential to be able to work in spaces with a rather weak norm, that is, in an interpolation space  $X_\theta$  with  $\theta$  close to zero, to derive a priori bounds for the solutions. Deriving a priori bounds in  $X_1$  directly is almost impossible in most concrete situations.

**8. Continuity properties.** In this section we study the continuity of the function  $(t, x) \rightarrow u(t, s, x)$  for a fixed “initial time”  $s$ .

For each  $s \in [0, T)$  we put

$$\mathcal{D}(s) := \{ (t, x) \in [s, T] \times V_\theta \mid t \in J(s, x) \}.$$

Thus  $\mathcal{D}(s)$  is the domain of  $u(\cdot, s, \cdot)$ .

**THEOREM 8.1.** *Suppose that  $\theta < 1$ . Then  $\mathcal{D}(s)$  is open in  $[s, T] \times X_\theta$  and  $u(\cdot, s, \cdot) \in C^{0,1^-}(\mathcal{D}(s), X_\theta)$ .*

**PROOF.** Let  $(t_0, x_0) \in \mathcal{D}(s)$  be given and put  $S := u([s, t_0], s, x_0)$ . Then  $S$  is a nonempty compact subset of  $V$  and of  $X_\theta$ . Hence, by Lemma 6.2, there exist positive numbers  $\varepsilon, \delta$  and  $K$  such that the maximal solution  $u(\cdot, \sigma, x)$  of  $(QCP)_{(\sigma,x)}$  exists on  $J_{\sigma,\delta}$  for each  $(\sigma, x) \in [s, t_0] \times \bar{B}_\theta(S, \varepsilon)$  and satisfies

$$(1) \quad \|u(t, \sigma, x) - u(t, \sigma, y)\|_\theta \leq K \|x - y\|_\theta, \quad t \in J_{\sigma,\delta},$$

provided  $y \in \bar{B}_\theta(S, \varepsilon)$ .

We now fix points  $s := \sigma_0 < \sigma_1 < \dots < \sigma_m := t_0$  such that  $\sigma_{j+1} - \sigma_j \leq \delta$  for  $j = 0, 1, \dots, m - 1$ . Moreover we let  $\varepsilon_j := K^{j-m-1} \varepsilon$  for  $j = 0, 1, \dots, m - 1$ , where we can assume that  $K \geq 1$ . Hence  $\varepsilon_j \leq \varepsilon$  and  $\varepsilon_{j+1} = K \varepsilon_j$  for  $j = 0, 1, \dots, m - 1$ .

It follows from (1) that  $u(\cdot, \sigma_j, x)$  exists on  $J_{\sigma_j, \delta}$  and satisfies

$$u(t, \sigma_j, x) \in \bar{\mathbf{B}}_\theta(u(t, s, x_0), \varepsilon_{j+1}), \quad t \in J_{\sigma_j, \delta},$$

and

$$(2) \quad \|u(t, \sigma_j, x) - u(t, \sigma_j, y)\|_\theta \leq K\|x - y\|_\theta, \quad t \in J_{\sigma_j, \delta},$$

for all  $x, y \in \bar{\mathbf{B}}_\theta(u(\sigma_j, s, x_0), \varepsilon_j)$  and  $j = 0, 1, \dots, m$ . From this we obtain, by piecing together the local solutions similarly as in the proof of Theorem 6.3, that  $J(s, x) \supset [s, (t_0 + \delta) \wedge T]$  for each  $x \in \bar{\mathbf{B}}_\theta(x_0, \varepsilon_0)$ . Hence  $\mathcal{D}(s)$  is open in  $[s, T] \times (V \cap X_\theta)$ .

Since, due to the unique solvability,  $u(t, \sigma_j, u(\sigma_j, \sigma_{j-1}, x)) = u(t, \sigma_{j-1}, x)$  for  $\sigma_j \leq t \leq \sigma_{j+1}$ ,  $x \in \bar{\mathbf{B}}_\theta(u(\sigma_{j-1}, s, x_0), \varepsilon_{j-1})$  and  $j = 1, 2, \dots, m$ , we obtain from (2) that

$$(3) \quad \|u(t, s, x) - u(t, s, y)\|_\theta \leq c\|x - y\|_\theta$$

for all  $x, y \in \bar{\mathbf{B}}_\theta(x_0, \varepsilon_0)$  and  $t \in [s, (t_0 + \delta) \wedge T]$ . Now  $u(\cdot, s, \cdot) \in C^{0,1-}(\mathcal{D}(s), X_\theta)$  follows from (3) and the fact that  $u(\cdot, s, x) \in C(J(s, x), X_\theta)$  by Theorem 6.3.  $\square$

Let  $S$  be a metric space and let  $t^+ : S \rightarrow (0, \infty]$ . Put

$$\mathcal{D} := \bigcup_{x \in S} [0, t^+(x)) \times \{x\}$$

and suppose that  $\varphi : \mathcal{D} \rightarrow S$  is a map with the following properties:

- (i)  $\mathcal{D}$  is open in  $\mathbf{R}_+ \times S$ ;
- (ii)  $\varphi \in C(\mathcal{D}, S)$ ;
- (iii)  $\varphi(0, \cdot) = \text{id}_S$ ;
- (iv) if  $0 \leq \tau < t^+(x)$  and  $0 \leq t < t^+(\varphi(\tau, x))$ , then  $\tau + t < t^+(x)$  and  $\varphi(t, \varphi(\tau, x)) = \varphi(t + \tau, x)$ .

Then  $\varphi$  is a **(local) semiflow on  $S$** . If  $t^+(x) = \infty$  for all  $x \in S$ , then  $\varphi$  is a **global semiflow**. For each  $x \in S$  the set  $\gamma^+(x) := \{\varphi(t, x) \mid 0 \leq t < t^+(x)\}$  is the **(positive) orbit through  $x$**  and  $t^+(x)$  is the **(positive) exit time of  $x$**  (e.g. [6, 11]).

Consider now the **autonomous quasilinear evolution equation**

$$\dot{u} + A(u)u = f(u),$$

that is, suppose that  $A$  and  $f$  are independent of  $t$ . Then the proceeding results are valid for every  $T > 0$ . In particular the quasilinear Cauchy problem

$$\dot{u} + A(u)u = f(u), \quad s < t < \infty, \quad u(s) = x,$$

possesses a unique maximal solution  $u(\cdot, s, x)$ , which is defined on some open interval  $J(s, x)$  of  $[s, \infty)$  containing  $s$ , such that the regularity properties of Theorem 6.3 are satisfied.

Let

$$\varphi(t, x) := u(t, 0, x), \quad t^+(x) := \sup J(0, x)$$

and

$$\mathcal{D} := \left\{ (t, x) \in \mathbf{R}_+ \times V_\theta \mid 0 \leq t < t^+(x) \right\}.$$



Then the unique solvability of  $(QCP)_{(s,x)}$  implies easily that  $\varphi$  satisfies the properties (iii) and (iv). In fact, we have the following important

**THEOREM 8.2.** *Let  $A$  and  $f$  be independent of  $t$  and suppose that  $\theta < 1$ . Then  $\varphi$  is a semiflow on  $V_\theta$  and  $\varphi \in C^{0,1-}(\mathcal{D}, X_\theta)$ .*

**PROOF.** This is an easy consequence of the above remarks and Theorems 6.3 and 8.1.  $\square$

It is important to notice that the semiflow possesses a **smoothing property**, namely

$$(4) \quad \gamma^+(x) \setminus \{x\} \subset X_1 \quad \forall x \in V_\theta.$$

Hence suppose that  $E$  is a Banach space such that

$$X_1 \hookrightarrow E \hookrightarrow X_\theta$$

and suppose that, for some  $x \in V_\theta$  and some  $t_1 \in (0, t^+(x))$ , the set  $\{\varphi(t, x) | t_1 \leq t < t^+(x)\}$  is bounded in  $E$ . Then  $\gamma^+(x)$  is relatively compact in  $X_\theta$ . Hence, if  $\gamma^+(x)$  has a positive distance from  $\partial V_\theta$ , it follows that  $\gamma^+(x)$  is relatively compact in  $V_\theta$ . Consequently  $t^+(x) = \infty$  in this case by well-known abstract results (e.g. [6, Satz (10.12)]). Observe that this is also a special case of Corollary 7.3.

**9. Special regularity results.** Theorem 6.3 contains important regularity assertions for the solution  $u(\cdot, s, x)$ . In particular we obtain “smooth” solutions if we can choose  $\theta$  and  $\sigma$  “large”. However in some applications to parabolic differential equations—in particular in the case of Dirichlet boundary conditions considered in the next section—it turns out that a reasonable large choice of  $\sigma$  imposes compatibility conditions for the nonlinearity  $f$ . In order to avoid such restrictions we shall now prove a “higher regularity” result by imposing additional assumptions, which are motivated by the applications of the next section.

Let  $Y$  and  $Z$  be Banach spaces such that  $Y \hookrightarrow Z$ , and let  $B: D(B) \subset Z \rightarrow Z$  be a linear operator in  $Z$ . Then  $B_Y$ , the  **$Y$ -realization of  $B$**  (or “the maximal restriction” of  $B$  to  $Y$ ), is defined by

$$D(B_Y) := \{ y \in D(B) \cap Y | By \in Y \} \quad \text{and} \quad B_Y y := By.$$

It is easily verified that  $B_Y$  is closed in  $Y$  if  $B$  is closed.

We consider first linear evolution equations, that is, we assume that  $\{A(t) | 0 \leq t \leq T\}$  satisfies Assumptions (A) and  $(A_\rho)$ . Moreover we suppose that

$$Y \text{ and } Y_1 \text{ are Banach spaces such that } Y_1 \hookrightarrow X_1 \hookrightarrow Y \hookrightarrow X.$$

In this situation it is clear that

$$D(A_Y(t)) = \{ y \in D(A(t)) | A(t)y \in Y \}.$$

From this we deduce that

$$\rho(-A_Y(t)) \supset \Sigma_\vartheta \quad \forall t \in [0, T],$$

and that

$$(1) \quad [\lambda + A_Y(t)]^{-1} = (\lambda + A(t))^{-1} | Y \quad \forall (t, \lambda) \in [0, T] \times \Sigma_\vartheta.$$

We suppose now that

there exists a number  $\kappa \in (0, 1)$  such that

$$(2) \quad \left\| (\lambda + A_Y(t))^{-1} \right\|_{\mathcal{L}(Y)} \leq c |\lambda|^{-1+\kappa}$$

for all  $(t, \lambda) \in [0, T] \times \Sigma_\beta$ .

Thus it follows that

$$e^{-sA_Y(t)} := \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda s} (\lambda + A_Y(t))^{-1} d\lambda$$

is well defined for  $s > 0$  and, letting  $e^{0A_Y(t)} := \text{id}_Y$ , standard arguments show that  $\{e^{-sA_Y(t)} | s \geq 0\}$  is a semigroup on  $Y$ , which is differentiable for  $s > 0$  (but not strongly continuous at  $s = 0$ ) and satisfies

$$(3) \quad \left\| A_Y^j(t) e^{-sA_Y(t)} \right\|_{\mathcal{L}(Y)} \leq cs^{-j-\kappa}$$

for  $s > 0$ ,  $j = 0, 1$  and  $t \in [0, T]$ . Moreover (1) implies

$$(4) \quad e^{-sA_Y(t)} = e^{-sA(t)}|Y, \quad s > 0, t \in [0, T].$$

Finally we suppose that

$$(5) \quad \begin{aligned} D(A_Y(t)) &= Y_1 \text{ for } 0 \leq t \leq T \text{ and} \\ A_Y(\cdot) &\in C^\beta([0, T], \mathcal{L}(Y_1, Y)) \\ &\text{for some } \beta \in (2\kappa, 1). \end{aligned}$$

Similar to §2, it follows that the family of norms  $\{y \mapsto \|A_Y(t)\|_Y | 0 \leq t \leq T\}$  is uniformly equivalent to the original norm of  $Y_1$ . Hence we deduce from (1), (3) and (4) that

$$(6) \quad a_Y(t, s) := e^{-(t-s)A_Y(s)} = a(t, s)|Y$$

and

$$(7) \quad k_Y(t, s) := -[A_Y(t) - A_Y(s)]a_Y(t, s) = k(t, s)|Y$$

for  $(t, s) \in T_\Delta$ , and that

$$(8) \quad a_Y \in \mathfrak{R}(Y, \kappa) \cap \mathfrak{R}(Y, Y_1, 1 + \kappa)$$

and

$$(9) \quad k_Y \in \mathfrak{R}(Y, 1 - (\beta - \kappa)).$$

Now Theorem 1.2 shows that the convolution-type equation

$$(10) \quad u = k_Y + k_Y * u$$

has a unique solution

$$(11) \quad w_Y \in \mathfrak{R}(Y, 1 - (\beta - \kappa)),$$

and (7) implies that

$$w_Y = w|Y,$$

where  $w$  is the resolvent kernel of the equation  $u = a + u * k$  in  $X$ .

LEMMA 9.1. *Suppose that  $\kappa < \alpha < \beta - \kappa$ . Then*

$$\|w_Y(t, s) - w_Y(\tau, s)\|_{\mathcal{L}(Y)} \leq c(t - \tau)^\alpha (\tau - s)^{-\gamma}, \quad 0 \leq s < \tau < t \leq T,$$

where  $\gamma := 1 + \alpha + \kappa - \beta$ .

PROOF. This follows from (9)–(11) by obvious modifications of the proof of [31, Lemma 5.2.1] (where  $R_1$  and  $R$  correspond to  $k_Y$  and  $w_Y$ , respectively).  $\square$

LEMMA 9.2.  $U \in C(\dot{T}_\Delta, \mathcal{L}(Y, Y_1))$ .

PROOF. Recall that  $U = a + a * w$  by (3.5) and Theorem 1.2. Hence

$$(12) \quad U|Y = a_Y + (a * w)|Y$$

and, due to (8), it remains to show that  $(a * w)|Y \in C(\dot{T}_\Delta, \mathcal{L}(Y, Y_1))$ . Observe that

$$(13) \quad \begin{aligned} [(a * w)|Y](t, s) &= \int_s^t a_Y(t, \tau) [w_Y(t, s) - w_Y(\tau, s)] d\tau \\ &\quad + \int_s^t e^{-(t-\tau)A_Y(s)} d\tau w_Y(t, s), \end{aligned}$$

and that the first integrand can be estimated in  $\mathcal{L}(Y, Y_1)$  by  $c(t - \tau)^{\alpha - \kappa - 1} (\tau - s)^{-\gamma}$ , due to (8) and Lemma 9.1. Hence an application of Lebesgue’s theorem shows that the first term on the right-hand side of (13) has the desired continuity property. Since

$$A_Y(s) \int_s^t e^{-(t-\tau)A_Y(s)} d\tau = \text{id}_Y - e^{-(t-s)A_Y(s)}, \quad (t, s) \in \dot{T}_\Delta$$

(e.g. [7, Lemma 9.1]), it is easily verified that the second term on the right-hand side of (13) belongs also to  $C(\dot{T}_\Delta, \mathcal{L}(Y, Y_1))$ .  $\square$

LEMMA 9.3. *Let  $g \in C^\nu([s, T], Y)$  for some  $\nu \in (\kappa, 1)$ . Then*

$$[t \mapsto U * g(t, s)] \in C((s, T], Y_1).$$

PROOF. By (12),

$$U * g(t, s) = a_Y * g(t, s) + [(a * w)|Y] * g(t, s).$$

The above proof shows that

$$a_Y * g(\cdot, s) \in C((s, T], Y_1).$$

Since

$$\|a_Y(t, \tau) w_Y(\tau, \sigma) g(\sigma)\|_Y \leq c(t - \tau)^{-\kappa} (\tau - \sigma)^{1 - (\beta - \kappa)}$$

for  $s \leq \sigma < \tau < t \leq T$ , we can apply Fubini’s theorem (in  $Y$ ) to deduce that

$$[(a * w)|Y] * g(t, s) = a_Y * [w_Y * g](t, s), \quad s < t \leq T.$$

From Lemma 9.1 and [7, Proposition 1.4] we obtain

$$w_Y * g(\cdot, s) \in C^\alpha([s, T], Y).$$

Hence

$$a_Y * [w_Y * g](\cdot, s) \in C((s, T], Y_1),$$

as above, and the assertion follows.  $\square$

It should be noted that the above proofs are modifications of corresponding results in [7, §9].

After these preparations we can prove the desired *regularity*

**THEOREM 9.4.** *Suppose, in addition to the assumptions (Q1)–(Q5), that there are Banach spaces  $Y$  and  $Y_1$  such that*

$$(HR1) \quad Y_1 \hookrightarrow X_1 \hookrightarrow Y \hookrightarrow X,$$

such that

$$D(A_Y(t, y)) = Y_1 \quad \text{for } (t, y) \in [0, T] \times V \text{ and}$$

$$(HR2) \quad [(t, y) \mapsto A(t, y)] \in C^{\beta, 1-}([0, T] \times V, \mathcal{L}(Y_1, Y))$$

for some  $\beta \in (0, 1) \cup \{1 - \}$ ,

such that

there exists a number  $\kappa \in (0, \beta/2)$  such that

$$(HR3) \quad \|(\lambda + A_Y(t, y))^{-1}\|_{\mathcal{L}(Y)} \leq c/|\lambda - \omega|^{1-\kappa}$$

for each  $y_0 \in V$ , all  $(t, y) \in [0, T] \times V_{y_0}$ , and all  $\lambda \in \omega + \Sigma_0$ ,

and such that

$$(HR4) \quad f \in C^{\nu, 1-}([0, T] \times V, Y) \quad \text{for some } \nu \in (\kappa, 1) \cup \{1 - \}.$$

Finally suppose that

$$(14) \quad \theta - \eta > 2\kappa \quad \text{and} \quad X_\theta \hookrightarrow Y.$$

Then

$$u(\cdot, s, x) \in C(J(s, x), Y_1) \cap C^1(J(s, x), Y)$$

for each  $(s, x) \in [0, T] \times V_\theta$ .

**PROOF.** Let  $(s, x) \in [0, T] \times V_\theta$  and  $t_1 \in J(s, x)$  be arbitrary, and let  $u := u(\cdot, s, x)|_{[s, t_1]}$ . Clearly it suffices to show that

$$u \in C((s, t_1], Y_1) \cap C^1((s, t_1], Y).$$

Since  $u \in C^{\theta-\eta}([s, t_1], X)$  by Theorem 6.3, it follows from (HR2), (HR3) and (14), that

$$A_Y(t) := A_Y(t, \bar{u}(t)), \quad 0 \leq t \leq T,$$

satisfies (2) and (5) with  $\bar{\beta} := \min(\beta, \theta - \eta)$ , since we can assume without loss of generality that  $\omega = 0$ . Moreover,

$$(15) \quad g := f(\cdot, \bar{u}(\cdot)) \in C^{\bar{\nu}}([s, T], Y)$$

for  $\bar{\nu} := \min\{\nu, \theta - \eta\}$ . Thus, since  $x \in X_\theta \subset Y$ , we deduce from Lemmas 9.2 and 9.3 that

$$U(\cdot, s)x + U * g(\cdot, s) \in C([s, T], Y_1).$$

Of course,  $U(\cdot, s)x + U * g(\cdot, s)$  coincides on  $[s, t_1]$  with  $u$ . Hence

$$(16) \quad u \in C((s, t_1], Y_1).$$

Since

$$\dot{u}(t) = -A(t, u(t))u(t) + f(t, u(t)) = -A(t)u(t) + g(t), \quad s < t \leq t_1,$$

in  $X$ , we obtain from (15), (16) and (HR2) that  $\dot{u} \in C((s, t_1], Y)$ . This implies  $u \in C^1((s, t_1], Y)$ .  $\square$

**10. Quasilinear parabolic systems.** Throughout this section  $m, n, N \in \mathbf{N}^*$ , and  $k \in \mathbf{N}$  satisfies  $k \leq 2m - 1$ .

We denote by  $\Omega$  a bounded domain in  $\mathbf{R}^n$  of class  $C^{2m}$ , that is,  $\bar{\Omega}$  is a compact  $n$ -dimensional  $C^{2m}$ -submanifold of  $\mathbf{R}^n$  with boundary  $\partial\Omega$ . We let  $N(k) := N \sum_{|\alpha| \leq k} 1$ , where  $\alpha \in \mathbf{N}^n$  and  $|\alpha| := \alpha_1 + \dots + \alpha_n$  is the length of the multi-index  $\alpha$ , and we denote by  $G$  an open subset of  $\mathbf{K}^{N(k)}$ . Then we suppose that

there exists a number  $\rho \in (0, 1)$  such that

$$(1) \quad a_\alpha \in C^{\rho, 0, 1-}([0, T] \times \bar{\Omega} \times G, \mathcal{L}(\mathbf{K}^N))$$

for all  $\alpha \in \mathbf{N}^n$  with  $|\alpha| \leq 2m$ .

We consider a family of linear differential operators of order  $2m$ :

$$\mathcal{A}(t, \eta)u := (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(t, \cdot, \eta) D^\alpha u, \quad (t, \eta) \in [0, T] \times G,$$

acting on  $N$ -tuples of  $\mathbf{K}$ -valued functions  $u: \Omega \rightarrow \mathbf{K}^N$ . We let

$$a(t, x, \eta; \xi) := \sum_{|\alpha| = 2m} a_\alpha(t, x, \eta) \xi^\alpha \in \mathcal{L}(\mathbf{K}^N)$$

for  $\xi \in \mathbf{R}^n$  and denote by  $(\cdot | \cdot)$  the Euclidean inner product in  $\mathbf{C}^N$ . Then we suppose that  $\mathcal{A}$  is **strongly parabolic** in the sense that

$$(2) \quad \operatorname{Re}(a(t, x, \eta; \xi) \zeta | \zeta) > 0 \quad \forall (t, x, \eta, \xi, \zeta) \in [0, T] \times \bar{\Omega} \times G \times \mathbf{R}^n \times \mathbf{C}^N,$$

$$\xi \neq 0, \zeta \neq 0.$$

We denote by

$$\mathcal{B}u := (u, \partial u / \partial \nu, \dots, \partial^{m-1} u / \partial \nu^{m-1})$$

the **Dirichlet boundary operator** on  $\partial\Omega$ , where  $\nu$  is the outer normal on  $\partial\Omega$ . Moreover we suppose that

$$(3) \quad f \in C^{2-}([0, T] \times \bar{\Omega} \times G, \mathbf{K}^N).$$

Finally we denote by  $F$  the **substitution operator induced by  $f$** , that is,

$$F(t, u)(x) := f(t, x, u(x), Du(x), \dots, D^k u(x))$$

for  $u: \bar{\Omega} \rightarrow G$  and  $x \in \bar{\Omega}$ .

For  $1 < p < \infty$  and  $0 \leq s \leq 2m$  we let  $W_p^s := W_p^s(\Omega, \mathbf{K}^N)$ , so that  $W_p^0 = L_p := L_p(\Omega, \mathbf{K}^N)$ . Moreover

$$W_{p, \mathcal{A}}^s := \left\{ u \in W_p^s \mid \partial^j u / \partial \nu^j = 0 \text{ on } \partial\Omega \text{ for } j = 0, 1, \dots, m-1 \text{ and } j < s - 1/p \right\}.$$

Observe that

$$(4) \quad W_{p, \mathcal{A}}^s = W_p^s \quad \text{if } s < 1/p.$$

We let

$$\mathcal{E}_p := \{ j + 1/p \mid j = 0, 1, \dots, m - 1 \}.$$

If  $0 < \theta < 1$  and  $2m\theta \notin \mathcal{E}_p$ , we know by [7, Theorem 13.3] that

$$W_{p,\mathcal{A}}^{2m\theta} = (L_p, W_{p,\mathcal{A}}^{2m})_{\theta,p} \quad \text{if } 2m\theta \notin \mathbf{N},$$

and

$$W_{p,\mathcal{A}}^{2m\theta} = [L_p, W_{p,\mathcal{A}}^{2m}]_{\theta} \quad \text{if } 2m\theta \in \mathbf{N}.$$

Thus, due to reiteration and commutativity properties of the real and complex interpolation functors (cf. [10, Theorems 3.5.3, 4.6.1, and 4.7.2]), it follows that

$$(5) \quad \begin{aligned} W_{p,\mathcal{A}}^s \text{ is } (W_{p,\mathcal{A}}^\sigma, W_{p,\mathcal{A}}^\tau)\text{-compatible for} \\ 0 \leq \sigma \leq s \leq \tau \leq 2m \text{ and } s, \sigma, \tau \notin \mathcal{E}_p. \end{aligned}$$

Recall the well-known Sobolev-type imbedding theorems:

$$(6) \quad W_p^s \hookrightarrow W_q^t \quad \text{if } 1/p \geq 1/q \text{ and } s - n/p \geq t - n/q,$$

$$(7) \quad W_p^s \hookrightarrow W_q^t \quad \text{if } 1/p \geq 1/q \text{ and } s - n/p > t - n/q,$$

$$(8) \quad W_p^s \hookrightarrow C^t \quad \text{if } s - n/p > t$$

and

$$(9) \quad W_p^s \hookrightarrow C^t \quad \text{if } s - n/p = t \notin \mathbf{N},$$

where  $C^t := C^t(\bar{\Omega}, \mathbf{K}^N)$  and  $0 \leq s, t \leq 2m, 1 < p, q < \infty$  (cf. [32], for example).

It is an easy consequence of (8) and the fact that  $W_{p,\mathcal{A}}^\sigma$  is a closed linear subspace of  $W_p^\sigma$ , that

$$(10) \quad \begin{aligned} V_p^\sigma := \{ u \in W_{p,\mathcal{A}}^\sigma \mid (u, Du, \dots, D^k u)(\bar{\Omega}) \subset G \} \text{ is open in} \\ W_{p,\mathcal{A}}^\sigma, \text{ if } k + n/p < \sigma \leq 2m \text{ and } \sigma \notin \mathcal{E}_p \end{aligned}$$

(cf. [7, Proposition 15.4]).

For each  $(t, v) \in [0, T] \times V_p^\sigma$  with  $k + n/p < \sigma \leq 2m$  and  $\sigma \notin \mathcal{E}_p$  we define a linear operator  $A_p(t, v)$  in  $L_p$  with domain  $W_{p,\mathcal{A}}^{2m}$  by

$$(11) \quad A_p(t, v)u := \mathcal{A}(t, v, Dv, \dots, D^k v)u,$$

where

$$\begin{aligned} \mathcal{A}(t, v, Dv, \dots, D^k v)u(x) \\ := (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(t, x, v(x), Dv(x), \dots, D^k v(x)) D^\alpha u(x) \end{aligned}$$

for  $x \in \Omega$ .

LEMMA 10.1. *Let  $n < p < \infty$  and put  $X := L_p, X_1 := W_{p,\mathcal{A}}^{2m}$ , and  $X_\xi := W_{p,\mathcal{A}}^{2m\xi}$  for  $0 < \xi < 1$ . Then*

- (i) *conditions (Q1) and (Q2) are satisfied if  $2m\eta, 2m\theta \notin \mathcal{E}_p$ .*
- (ii)  *$A_p(\cdot, \cdot)$  satisfies conditions (Q3) and (Q4) provided  $k + n/p < 2m\eta \leq 2m$  and  $2m\eta \notin \mathcal{E}_p$ .*
- (iii) *If  $0 \leq \varepsilon < 2m\eta - k - n/p$  and  $\varepsilon < 1$ , then*

$$F \in C^{1-}([0, T] \times V_p^{2m\eta}, W_p^\varepsilon).$$

PROOF. (i) (Q1) is clear, (Q2) follows from (5).

(ii) (Q3) is an easy consequence of (10), (8) and (1). (Q4) follows from [8, Theorem 6.6] and (2), by using again (8) and the fact that (2) implies a uniform estimate on compact sets.

(iii) is a consequence of (3) and [7, Proposition 15.6].  $\square$

After these preparations we consider now the **quasilinear parabolic initial boundary value problem**

(QIBVP)<sub>(s,u<sub>0</sub>)</sub>

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t, u, D_2 u, \dots, D_2^k u)u &= f(t, x, u, D_2 u, \dots, D_2^k u) \quad \text{in } (s, T] \times \Omega, \\ \mathcal{B}u &= 0 \quad \text{on } (s, T] \times \partial\Omega, \\ u(s, \cdot) &= u_0 \quad \text{on } \Omega. \end{aligned}$$

By an  $L_p$ -solution,  $n < p < \infty$ , of (QIBVP)<sub>(s,u<sub>0</sub>)</sub> we mean a solution of the quasilinear Cauchy problem

$$\dot{u} + A_p(t, u)u = F(t, u), \quad s < t \leq T, \quad u(s) = u_0,$$

in  $L_p$ . A function  $u: \bar{\Omega} \rightarrow G$  with

$$u \in C(J \times \bar{\Omega}, G) \cap C^{0,2m}(J \times \bar{\Omega}, \mathbf{K}^N) \cap C^{1,0}(J \times \bar{\Omega}, \mathbf{K}^N),$$

defined on a perfect subinterval  $J$  of  $[s, T]$  containing  $s$ , is a **classical solution** of (QIBVP)<sub>(s,u<sub>0</sub>)</sub>, provided it satisfies (QIBVP)<sub>(s,u<sub>0</sub>)</sub> pointwise. Clearly every classical solution is a  $L_p$ -solution for  $p \in (n, \infty)$ .

**THEOREM 10.2.** *Suppose that  $n < p < \infty$  and that  $k + n/p < \sigma \leq 2m$  with  $\sigma \notin \mathcal{E}_p$ . Then (QIBVP)<sub>(s,u<sub>0</sub>)</sub> possesses for each  $(s, u_0) \in [0, T) \times V_p^\sigma$  a unique maximal  $L_p$ -solution  $u(\cdot, s, u_0)$ , and*

$$(12) \quad u(\cdot, s, u_0) \in C^{\sigma-\tau}(J, W_p^\tau) \quad \text{for } 0 \leq \tau \leq \sigma, \tau \notin \mathcal{E}_p.$$

Moreover

$$(13) \quad u(\cdot, s, u_0) \in C^1(J, W_p^\varepsilon) \quad \text{for } 0 \leq \varepsilon < \min\{\rho, \sigma - k - n/p, 1/p\}.$$

The maximal interval of existence  $J(s, u_0) := J$  is open in  $[s, T]$ . If  $\sigma < 2m$ , then

$$(14) \quad \mathcal{D}_p^\sigma(s) := \left\{ (t, u_0) \in [s, T] \times V_p^\sigma \mid t \in J(s, u_0) \right\}$$

is open in  $[s, T] \times W_{p,\emptyset}^\sigma$ , and

$$(15) \quad u(\cdot, s, \cdot) \in C^{0,1-}( \mathcal{D}_p^\sigma(s), W_p^\sigma )$$

for every  $s \in [0, T)$ .

If

$$\gamma^+(s, u_0) := \left\{ u(t, s, u_0) \mid t \in J(s, u_0) \right\}$$

is bounded in  $W_p^\sigma$  and has a positive distance from  $\partial V_p^\sigma$  for some  $(s, u_0) \in [0, T) \times V_p^\sigma$ , then  $u(\cdot, s, u_0)$  is a global solution, that is,  $J(s, u_0) = [s, T]$ .

PROOF. Let  $\theta := \sigma/2m$  and choose  $\eta \in (0, \theta)$  such that  $2m\eta \notin \mathcal{E}_p$  and  $k + n/p < 2m\eta$ . Moreover let  $\zeta \in (0, 1)$  satisfy  $\varepsilon := 2m\zeta < 1/p$  and  $2m\zeta + k + n/p < 2m\eta$ . Then everything, except the last assertion, follows from (4), Lemma 10.1(i), (ii) and (iv), Theorem 6.3 and Theorem 8.1.

Let now  $\eta < \xi < \theta$  such that  $\tau := 2m\xi \notin \mathcal{E}_p$ . Since  $u_0 \in X_\theta \hookrightarrow X$  we can apply Theorem 6.3 also in the space  $X_\xi$ . Denoting the solution in this case by  $u_\xi(\cdot, s, u_0)$ , it is clear that  $u_\xi(\cdot, s, u_0) \supset u(\cdot, s, u_0)$ . On the other hand,

$$u_\xi(t, s, u_0) \in W_{p, \mathcal{B}}^{2m} \hookrightarrow X_\theta = W_{p, \mathcal{B}}^\sigma \quad \text{for } t > s,$$

since  $u_\xi(\cdot, s, u_0)$  is an  $L_p$ -solution. This shows that  $u_\xi(\cdot, s, u_0) = u(\cdot, s, u_0)$ . Since  $\gamma^+(s, u_0)$  is bounded in  $X_\theta$ , by assumption, and since  $X_\theta \hookrightarrow X_\xi$ , by (7), the last assertion follows from Corollary 7.3.  $\square$

COROLLARY 10.3. *Suppose that  $n < p < \infty$  and  $k + n/p < \sigma < 2m$  with  $\sigma \notin \mathcal{E}_p$ . Moreover suppose that  $\mathcal{A}$  and  $f$  are independent of  $t$ , and let  $\varphi(t, u_0) := u(t, 0, u_0)$  and  $\mathcal{D}_p^\sigma := \mathcal{D}_p^\sigma(0)$ . Then  $\varphi$  is a semiflow on  $V_p^\sigma$  such that  $\varphi \in C^{0,1-}(\mathcal{D}_p^\sigma, W_p^\sigma)$ . Moreover, bounded orbits, which are bounded away from  $\partial V_p^\sigma$ , exist globally. If they are also bounded in  $W_p^\tau$  for some  $\tau > \sigma$ , then they are relatively compact in  $W_p^\sigma$ .*

Of course, an orbit is bounded in  $W_p^\tau$  for  $\tau > \sigma$  if it is bounded “after” some positive initial time (if  $u_0 \notin W_p^\tau$ ).

Our next theorem shows that  $u(\cdot, s, u_0)$  has much better regularity properties for  $t > s$ .

THEOREM 10.4. *Suppose that  $n < p < \infty$  and  $k + n/p < \sigma \leq 2m$  with  $\sigma \notin \mathcal{E}_p$ , and that  $(s, u_0) \in [0, T) \times V_p^\sigma$ . Then*

$$u(\cdot, s, u_0) \in C(J, W_q^{2m}) \quad \forall q \in [p, \infty).$$

PROOF. Let  $p < q < \infty$  and choose a  $\delta \in (k + n/q, 2m - n(p^{-1} - q^{-1})) \setminus \mathcal{E}_q$  which is possible by  $2m - n/p > k$ . Hence  $u_1 := u(s_1, s, u_0) \in W_{p, \mathcal{B}}^{2m} \hookrightarrow W_{q, \mathcal{B}}^\delta$  by (6), where  $s_1 \in J$  is arbitrary. Since  $u_1 \in V_q^\delta$ , we obtain from Theorem 10.2 that  $(\text{QIBVP})_{(s_1, u_1)}$  has a unique maximal  $L_q$ -solution  $u_1(\cdot, s_1, u_1)$ . Similar to the proof of Theorem 10.2, we find that  $u_1(\cdot, s_1, u_1) = u(\cdot, s, u_0)|_{(J \cap [s_1, T])}$ . Since  $s_1 \in J$  is arbitrary, the assertion follows.  $\square$

It follows from Theorem 10.4 and (8) that

$$u(\cdot, s, u_0) \in C(J, C^{2m-1+\mu}(\bar{\Omega}, \mathbf{K}^N))$$

for every  $\mu \in (0, 1)$ . In the following we shall show that we obtain classical solutions if we impose further mild regularity assumptions. For simplicity we do not give the most general assumptions (for  $\Omega$  and the coefficients  $a_\alpha$ ), but choose a simple setting. Namely we assume that  $\Omega$  belongs to class  $C^{(2m+1)-}$  and

$$a_\alpha \in C^{2-}([0, T] \times \bar{\Omega} \times G, \mathcal{L}(\mathbf{K}^N)), \quad |\alpha| \leq 2m.$$



Then we prove the following crucial

LEMMA 10.5. *Suppose that  $2n < q < \infty$  and that  $\eta \in (0, 1)$  satisfies  $2m\eta > k + n/q$ . Moreover let  $0 < \mu < (2m\eta - k - 2n/q) \wedge 1$  and put  $X := L_q$ ,  $X_1 := W_q^{2m}$ ,  $Y := C^\mu$  and  $Y_1 := \{u \in C^{2m+\mu} | \mathcal{B}u = 0 \text{ on } \partial\Omega\}$ . Then the conditions (HR1)–(HR4) are satisfied with  $\beta = \nu = 1 -$  and  $\kappa = \mu/2m$ .*

PROOF. It follows from (8) that (HR1) is true. Moreover, by using (8) and the mean-value theorem (in integral form), it is not difficult to verify that

$$(16) \quad \left[ (t, y) \mapsto A(t, y) | Y_1 \right] \in C^{1-}([0, T] \times V, \mathcal{L}(Y_1, Y)).$$

Consider now the operators

$$\mathcal{A}_\psi(t, v)w := \mathcal{A}(t, v, Dv, \dots, D^k v)w + (-1)^m e^{i\psi} \frac{\partial^{2m} w}{\partial \tau^{2m}}$$

on  $\Omega \times \mathbf{R}$ , where  $(t, v) \in [0, T] \times V$  and  $\psi \in \mathbf{R}$ . Then it follows from [8, Lemmas 6.1 and 6.3, and 7, Theorem 9.3] (cf. also [29, Theorem 2.1]) that we can assume that, for each fixed  $y_0 \in V$ , the neighbourhood  $V_{y_0}$  and the constant  $\vartheta \in (0, \pi/2)$  are chosen such that

$$(17) \quad \|w\|_{2m+\mu}^{\bar{\Omega} \times \mathbf{R}} \leq c \left\{ \|\mathcal{A}_\psi(t, y)w\|_\mu^{\bar{\Omega} \times \mathbf{R}} + \|w\|_0^{\bar{\Omega} \times \mathbf{R}} \right\}$$

for all  $(t, y) \in [0, T] \times V_{y_0}$ ,  $\psi \in [-\vartheta - \pi/2, \vartheta + \pi/2]$  and all

$$w \in C^{2m+\mu}(\bar{\Omega} \times \mathbf{R}, \mathbf{C}^N)$$

whose supports are contained in a fixed compact subset of  $\bar{\Omega} \times \mathbf{R}$ , and which satisfy  $\mathcal{B}w = 0$  on  $\partial\Omega \times \mathbf{R}$ . Here  $\|\cdot\|_\lambda^M$  denotes the norm in  $C^\lambda(M, \mathbf{K}^L)$ , where  $M$  is a compact subset of some euclidean space and  $L \in \mathbf{N}^*$ . Following an idea of Agmon [3], we fix a function  $\varphi \in C^\infty(\mathbf{R}, \mathbf{R})$  with  $\text{supp}(\varphi) \subset (-2, 2)$  and  $\varphi[-\frac{3}{2}, \frac{3}{2}] = 1$ . Then we put  $w(x, \tau) := \varphi(\tau) e^{ir\tau} u(x)$ , where  $r \geq 1$  is fixed and  $u \in C^{2m+\mu}$  satisfies  $\mathcal{B}u = 0$  on  $\partial\Omega$ . Then an easy calculation shows that

$$(18) \quad \begin{aligned} \|\mathcal{A}_\psi(t, y)w\|_\mu^{\bar{\Omega} \times \mathbf{R}} &\leq c \left\{ \|\varphi e^{ir\tau}\|_\mu^{[-2,2]} \|(A(t, y) + r^{2m} e^{i\psi})u\|_Y \right. \\ &\quad \left. + r^{2m-1} \|e^{ir\tau}\|_\mu^{[-2,2]} \|u\|_Y \right\} \end{aligned}$$

(cf. [7, formula (12.7)]). On the other hand,

$$(19) \quad \begin{aligned} \|w\|_{2m+\mu}^{\bar{\Omega} \times \mathbf{R}} &\geq \sum_{j+|\alpha| \leq 2m} \|D^j(e^{ir\tau})D^\alpha u\|_0^{\bar{\Omega} \times [-1,1]} \\ &\quad + \sum_{j+|\alpha|=2m} [D^j(e^{ir\tau})D^\alpha u]_\mu^{\bar{\Omega} \times [-1,1]} \\ &\geq \sum_{k=0}^{2m} r^{2m-k} \left\{ \|u\|_k^{\bar{\Omega}} + \sum_{|\alpha|=k} [D^\alpha u]_\mu^{\bar{\Omega}} \right\} = \sum_{k=0}^{2m} r^{2m-k} \|u\|_{k+\mu}^{\bar{\Omega}}. \end{aligned}$$

Since

$$\|\varphi e^{ir\tau}\|_\mu^{[-2,2]}, \|e^{ir\tau}\|_\mu^{[-2,2]} \leq cr^\mu,$$

we deduce from (17)–(19) the estimate

$$\sum_{k=0}^{2m} r^{2m-k-\mu} \|u\|_{k+\mu}^{\bar{\Omega}} \leq c \| (A(t, y) + r^{2m} e^{i\psi}) u \|_Y,$$

provided  $r$  is sufficiently large. Hence, letting  $\lambda := r^{2m} e^{i\psi}$ , we see that

$$(20) \quad \sum_{k=0}^{2m} |\lambda|^{1-(k+\mu)/2m} \|u\|_{k+\mu}^{\bar{\Omega}} \leq c \| (A(t, y) + \lambda) u \|_{\mu}^{\bar{\Omega}}$$

for all  $u \in C^{2m+\mu}$  satisfying  $\mathcal{B}u = 0$  on  $\partial\Omega$ , all  $(t, y) \in [0, T] \times V_{y_0}$ , and  $\lambda \in \Sigma_{\vartheta}$  with  $|\lambda| \geq \lambda_0$  for some  $\lambda_0 \geq 1$ . This implies, in particular, that for each point  $y_0 \in V$  there exists a neighbourhood  $V_{y_0}$  and positive constants  $\omega$  and  $\vartheta$  such that

$$(21) \quad |\lambda - \omega|^{1-\mu/2m} \|x\|_Y \leq c \| (\lambda + A(t, y)) x \|_Y$$

for all  $(t, y, x) \in [0, T] \times V_{y_0} \times Y_1$  and  $\lambda \in \omega + \Sigma_{\vartheta}$ .

The arguments of the proof of Theorem 6.6 in [8] now show that  $\lambda + A(t, y)$  is an isomorphism from  $Y_1$  onto  $Y$  for  $\lambda > \omega$  and  $(t, y) \in [0, T] \times V_{y_0}$ . This implies, in particular, that  $A_Y(t, y) = A(t, y)|_Y$  for all  $(t, y) \in [0, T] \times V$ . Now (HR2) and (HR3) follow from (16) and (21).

Finally (HR4) is an obvious consequence of Lemma 10.1(iii) and (8), since we can choose  $\zeta \in (0, 1)$  such that  $\mu + n/q < 2m\zeta =: \varepsilon < 2m\eta - k - n/q$  and  $\varepsilon < 1$ .  $\square$

After these preparations we can prove the following *regularity*

**THEOREM 10.6.** *Suppose that  $\Omega$  belongs to class  $C^{(2m+1)-}$  and that*

$$a_{\alpha} \in C^{2-}([0, T] \times \bar{\Omega} \times G, \mathcal{L}(\mathbf{K}^N)), \quad |\alpha| \leq 2m.$$

*Moreover suppose that  $n < p < \infty$  and  $k + n/p < \sigma \leq 2m$  with  $\sigma \notin \mathcal{E}_p$ , and that  $(s, u_0) \in [0, T] \times V_p^{\sigma}$ . Then*

$$(22) \quad u(\cdot, s, u_0) \in C(J, C^{2m+\mu}) \cap C^1(J, C^{\mu})$$

*for every  $\mu \in [0, 1 \wedge (2m - k)m/(1 + m)]$ . In particular,  $u(\cdot, s, u_0)$  is a classical solution.*

**PROOF.** Choose any  $q \in (p, \infty)$  with  $q > 2n$ . By replacing  $u_0$  by  $u(s_1, s, u_0)$  and  $s$  by  $s_1$ , where  $s_1 \in J$  is arbitrary, we can assume, due to Theorem 10.4, that  $u_0 \in V_q^{2m}$ . Thus we can choose  $X := L_q$ ,  $X_1 := W_{q, \mathcal{B}}^{2m}$ ,  $\theta := 1$ ,  $\eta \in (0, 1)$  with  $2m\eta > k + n/q$  and  $2m\eta \notin \mathcal{E}_q$ ,  $\mu \in (0, 1)$  with  $2m(1 - \eta) > \mu/m$  and  $\mu < 2m\eta - k - 2n/q$ ,  $Y := C^{\mu}$  and  $Y_1 := C^{2m+\mu} \cap X_1$ . Then it follows from Lemma 10.5 that the hypotheses of Theorem 9.4 are satisfied with  $\beta := \nu := 1 -$  and  $\kappa := \mu/2m$ . Hence Theorem 9.4 implies (22) for this choice of  $\mu$ . Since we can choose  $q$  arbitrarily large, it follows that  $\mu$  has to satisfy the restrictions  $2m(1 - \eta) > \mu/m$  and  $\mu < 2m\eta - k$ , where  $1 > \eta > k/2m$ , and, of course,  $\mu < 1$ . This leads to the asserted range for  $\mu$ . The last part of the assertion is now obvious, since  $u \in C(J, W_p^{\sigma})$  by Theorem 10.2 and  $W_p^{\sigma} \hookrightarrow C$  by (8).  $\square$

REMARKS 10.7. For the sake of an easy presentation we have chosen a relatively simple setting. However it should be observed that our abstract results apply to much more general situations. Namely:

(a) The differential operator does not need to be strongly parabolic. It suffices that it satisfies the  $\alpha$ -root condition for some  $\alpha \in (0, \pi/2)$ , uniformly in  $t \in [0, T]$ , in the sense of [7].

(b) The Dirichlet boundary operator  $\mathcal{B}$  can be replaced by much more general systems, provided it remains independent of  $t$  and  $(\mathcal{A}, \mathcal{B})$  is a strongly  $\alpha$ -regularly elliptic boundary value problem, uniformly in  $t \in [0, T]$ , in the sense of [7]. For concrete instances of such systems we refer to [8, §6].

(c)  $F$  does not need to be a substitution operator. It can be a nonlocal operator.

(d)  $\Omega$  can be unbounded (cf. [7] for details). Of course, in this case the above assertions guaranteeing global existence and compactness of orbits are not valid since their proofs depend upon the compact imbeddings (7) and (8).  $\square$

The existence of solutions for quasilinear parabolic equations has been studied extensively by Ladyženskaja, Solonnikov and Ural'ceva [19] for a single second order equation. Their method is completely different. Namely it is based upon a priori estimates and fixed point arguments. This requires restrictive structure conditions for the nonlinearities (since it means essentially that all solutions have to be bounded in the maximum norm), but gives global solutions. In addition these authors have to impose compatibility conditions. None of these restrictions is necessary in our approach, which is “dynamical” in contrast to the “static” approach in [19]. Moreover we can handle—essentially with the same ease—systems of arbitrary even order. Due to the above results the question of global existence has been reduced to the problem of finding a priori bounds in an appropriate norm. The problem of existence and regularity has been completely settled.

As mentioned earlier, the existence of a local  $L_p$ -solution for quasilinear parabolic equations has been shown by Sobolevskii [28] (cf. also [16, §II.17]), and our abstract approach follows essentially his method. However our regularity results (e.g. Theorem 10.6) are new, as are, in particular, the results concerning the continuous dependence on the initial values.

The existence of a classical solution to certain quasilinear parabolic equations has also been shown by Lunardi [20, 21]. She works in the space  $C(\bar{\Omega})$  and does not assume that the operators are densely defined in order to avoid compatibility conditions. However she can only treat equations with a principal part of the very particular form  $a(t, x, u, Du)\Delta u$ , where  $\Delta$  is the Laplace operator.

A related but different approach to quasilinear parabolic equations has been given by Da Prato and Grisvard [14] (cf. also [13] for a survey). These authors use a linearization method based upon “maximal regularity” results. However it is not clear whether their method can be applied to general quasilinear parabolic systems to give the same precise results as the ones of this paper.

We should also like to mention the results for quasilinear parabolic equations obtained by the theory of nonlinear semigroups, that is, the theory of monotone and accretive operators (e.g. [9]). These results apply essentially to different classes of

problems and are closer to the “static” approach of [19] than to our “dynamic” method.

Finally we want to point out that von Wahl [33, 34] was the first to apply Agmon’s technique to Hölder spaces. He derived an estimate similar to our estimate (20) for a single linear operator (that is,  $N = 1$  and  $\mathcal{A}$  is independent of  $v \in V$ ). He then used these estimates to define fractional powers in  $C^\mu$ -spaces and to derive regularity results for semilinear parabolic equations.

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