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QUASILINEAR PARABOLIC BOUNDARY VALUE PROBLEMS. APPROXIMATE SOLUTIONS AND ERROR BOUNDS BY LINEAR PROGRAMMING

by

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ABSTRACT

Constrained minimization problems are formulated from a quasilinear parabolic boundary value problem (probably with non-linear boundary conditions), making use of the latter's (conditional) inverse-positive property. Approximate solutions and three error bounds can be obtained by solving these minimization problems by linear programming and discretization techniques. Numerical results are obtained using splines as basis functions.

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1. Introduction

Let S (Figure 1) be a simply-connected open domain in the xt-plane bounded by a closed interval B_0 on $t = 0$, an open interval B_T on $t = T$, and two curves T_i defined by the continuous functions $x = x_i(t)$, $0 < t \leq T$, $i = 1,2$.

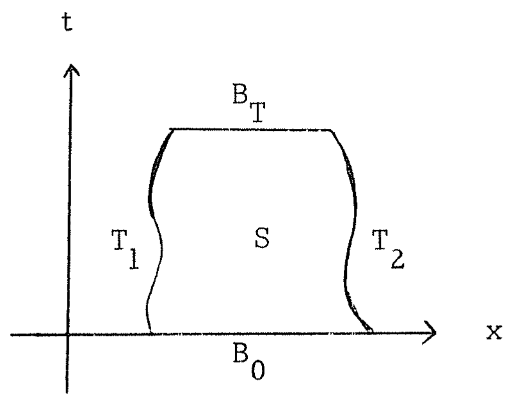


Figure 1

Let $R_2 = S \cup B_T$ and $\bar{R} = R_2 \cup T_1 \cup T_2 \cup B_0$.

Consider two linear operators

$$L_1 \equiv \frac{\partial}{\partial \nu} \quad \text{and} \quad L_2 \equiv \frac{\partial}{\partial t} - a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t) \frac{\partial}{\partial x} ,$$

where a and b are continuous on R_2, $\frac{\partial}{\partial \nu}$ is an outward non-tangential directional derivative and L_2 is a uniform parabolic operator.

In this paper, we apply the linear programming method to determine approximate solutions and error bounds for the first and third boundary value problems (BVP) of the quasilinear parabolic equations

$$Q(g_j) \begin{cases} L_2[u] + g_2(x,t,u) = r_2(x,t) \text{ in } R_2 & (1.1) \\ L_1[u] + g_1(x,t,u) = r_1(x,t) \text{ in } R_1 & (1.2) \\ u = r_0(x,t) \text{ in } R_0 & (1.3) \end{cases}$$

where g_j, $j = 1,2$, may be nonlinear in x,t and u; r_j is bounded in R_j, $j = 0,1,2$; and R_0 and R_1 are defined as follows:

i) For first BVP : $R_0 = T_1 \cup B_0 \cup T_2$, R_1 is empty.

ii) For third BVP : $R_0 = B_0$, $R_1 = T_1 \cup T_2$ (other combinations are also possible).

In Section 2, we present some definitions and preliminary results. In Section 3, three error bounds are derived for any given approximate solution to the problem $Q(g_j)$, making use of the conditionally inverse-positive property of L_j and the local Lipschitz condition of g_j, $j = 1,2$. The first error bound depends on the maximum of the absolute defects of the operator equations. The second error bound is a positively weighted sum of these defects. The third error bound is a conditional improvement of the second one. In Section 4, by means of these error bound formulas, theoretical constrained minimization problems (CMP) are formulated, by means of which we can determine approximate solutions and error bounds for $Q(g_j)$. Section 5 contains some discussions on our approaches. In Section 6, computational schemes making use of linear programming (LP) are suggested to solve these CMP. Section 7 presents some numerical results.

Rosen [1966], at a different approach, seems to be the first one who applied LP to solve the first BVP of nonlinear parabolic equations. By similar approach as this paper, Rosen [1970a] and Cheung [1971] solved BVP of nonlinear elliptic equations.

2. Preliminary

Def. 1 Let f be a function defined on a set $X \subset E^2$. Set

$$\|f\|_X \equiv \inf_{(x,t)\in X} |f(x,t)|$$

If there is no ambiguity, the subscript X may be omitted.

Def. 2 $V(\bar{R}) \equiv C^0(\bar{R}) \cap C^1(\bar{R}_1) \cap C^{2,1}(R_2)$, where $\bar{R}_1$ is the closure of R_1 and $C^{2,1}(R_2)$ denotes the class of functions $u(x,t)$ whose second order derivative with respect to x and first order derivative with respect to t are continuous in R_2.

Def. 3 $g_j(u) \equiv g_j(x,t,u)$; $g_j'(u) \equiv \dfrac{\partial g_j(x,t,u)}{\partial u}$.

Def. 4 For $j = 1,2$, suppose g_j' exists and is bounded. For a fixed $v \in V(\bar{R})$ and a constant $\xi > 0$, define

$$p_j \equiv p_j(\xi, v, x, t) = \min_{\eta} \{g_j'(\eta) \,\big|\, |\eta - v| \le \xi\} \text{ for each}$$

$(x,t) \in R_j$,

and

$$\hat{g}_j(u) \equiv \hat{g}_j(x,t,u) = \begin{cases} g_j(u) & v+\xi \ge u \ge v-\xi \quad \text{in } R_j \\ g_j(v+\xi)+(u-v-\xi)g_j'(v+\xi) & u > v+\xi \quad \text{in } R_j \\ g_j(v-\xi)-(v-\xi-u)g_j'(v-\xi) & v-\xi > u \quad \text{in } R_j \end{cases} \tag{2.1}$$

The following lemma is obvious:

Lemma 1

Let p_j and $\hat{g}_j$ be defined in Def. 4. Then $\hat{g}_j$ is differentiable with respect to u, and, for a fixed $(x,t) \in R_j$ and arbitrary $u \in V(\bar{R})$, we have

$$\hat{g}_j'(u) \equiv \frac{\partial \hat{g}_j(x,t,u)}{\partial u} \geq p_j(\xi, v, x, t) \qquad (x,t) \in R_j,\ j = 1,2 \tag{2.2}$$

or equivalently

$$\hat{g}_j(x,t,v_1) - \hat{g}_j(x,t,v_2) \geq p_j(\xi,v,x,t)(v_1 - v_2) \qquad (x,t) \in R_j,\ j=1,2 \tag{2.3}$$

where $v_1 \geq v_2$, $v_1, v_2 \in V(\bar{R})$.

In particular,

$$g_j(x,t,v_1) - g_j(x,t,v_2) \geq p_j(\xi,v,x,t)(v_1 - v_2) \qquad (x,t) \in R_j,\ j=1,2 \tag{2.4}$$

where $v_1 \geq v_2$, $|v_i - v| \leq \xi$, $v_i \in V(\bar{R})$, $i = 1,2$.

Def. 5 (2.4) is called a one-sided local Lipschitz condition.

Def. 6 The problem $Q(g_j)$ is said to be inverse-positive if, for every $v_1, v_2 \in V(\bar{R})$, we have

$$\left.\begin{array}{ll} L_j[v_1] + g_j(x,t,v_1) \geq L_j[v_2] + g_j(x,t,v_2) & \text{in } R_j,\ j=1,2 \\ v_1 \geq v_2 & \text{in } R_0 \end{array}\right\} \Rightarrow v_1 \geq v_2 \text{ on } \bar{R}.$$

Notation We say: $[(L_j + g_j, R_j),\ j = 1,2;\ (I, R_0)]$ is inverse-positive, where I is an identity operator.

An important feature of an inverse-positive problem is that it can have at most one solution. However, the nonlinear problem $Q(g_j)$ is not always inverse-positive. Let us first consider the linear case with $g_j(x,t,u) \equiv uk_j(x,t)$. The following lemma is a direct consequence of Theorem 4 of Protter and Weinberger [1967, p. 172].

Lemma 2

The problem $Q(uk_j)$ is inverse-positive if $k_1 > 0$ in R_1 and $k_2 \geq 0$ in R_2.

From this lemma we shall derive some other results. First we relax the restriction on k_2.

Corollary 3

Suppose that $\frac{\partial}{\partial \nu} \equiv + \frac{\partial}{\partial x}$ (or $- \frac{\partial}{\partial x}$). The problem $Q(uk_j)$ is inverse-positive if $k_1 > 0$ in R_1 and k_2 is bounded below in R_2.

Proof. Suppose that $u \in V(\bar{R})$ satisfies

$$u \geq 0 \text{ in } R_0,\quad L_j[u] + k_j u \geq 0 \text{ in } R_j,\ j = 1,2 \tag{2.5}$$

Let $v = ue^{\lambda t}$, where λ is a constant. Then

$$\begin{cases} L_2[v] + (k_2+\lambda)v = e^{\lambda t}\,(L_2[u] + k_2 u) \geq 0 & \text{in } R_2 \\ L_1[v] + k_1 v = e^{\lambda t}\,(L_1[u] + k_1 u) \geq 0 & \text{in } R_1 \\ v = e^{\lambda t}\, u \geq 0 & \text{in } R_0 \end{cases}$$

Since k_2 is bounded below, λ may be chosen so large that $k_2 + \lambda \geq 0$ in R_2. By lemma 2, we have $v \geq 0$ on $\bar{R}$. Hence $u \geq 0$ on $\bar{R}$. ∎

In particular, the parabolic first BVP (R_1 is empty) is inverse-positive regardless of the sign of k_2. This is an important difference from the elliptic first BVP which requires k_2 to be nonnegative.

Next, we omit both the requirements $k_1 > 0$ in R_1 and $k_2 \geq 0$ in R_2 and consider the following conditionally inverse-positive property.

<u>Lemma 4</u>

Suppose there exists a positive $\mu \in V(\bar{R})$ such that

$$\begin{cases} (L_2 + k_2)[\mu] \geq 0 & \text{in } R_2 \\ (L_1 + k_1)[\mu] > 0 & \text{in } R_1 \end{cases} \tag{2.6}$$

Then, $Q(uk_j)$ is inverse-positive.

<u>Proof.</u> Let u satisfy (2.5). By the transformation $v = u/\mu$, we get

$$\begin{cases} \dfrac{\partial v}{\partial t} - a\dfrac{\partial^2 v}{\partial x^2} + b^* \dfrac{\partial v}{\partial x} + k_2^* v = \dfrac{1}{\mu}\,(L_2 + k_2)[u] \geq 0 & \text{in } R_2 \\ \dfrac{\partial v}{\partial \nu} + k_1^* v = \dfrac{1}{\mu}\,(L_1 + k_1)[u] \geq 0 & \text{in } R_1 \\ v = \dfrac{u}{\mu} \geq 0 & \text{in } R_0 \end{cases}$$

where $b^* = b - \dfrac{2a}{\mu}\dfrac{\partial \mu}{\partial x}$, $k_1^* = \dfrac{1}{\mu}\left(\dfrac{\partial \mu}{\partial \nu} + k_1\mu\right) > 0$ in R_1 and $k_2^* = \dfrac{1}{\mu}(L_2 + k_2)[\mu] \geq 0$ in R_2. By Lemma 2, we have $v \geq 0$ on $\bar{R}$. Hence $u \geq 0$ on $\bar{R}$. ∎

For $k_1 > 0$ and $k_2 \geq 0$, the function $\mu \equiv 1$ satisfies (2.6). This reduces to Lemma 2 again.

3. Derivation of Error Bounds

In this section, three error bounds ρ_1, ρ_2 and ρ_3 are derived for any given approximate solution making use of the local Lipschitz condition (2.4) and the conditionally inverse-positive property of Lemma 4.

ρ_1 depends on the maximum of the absolute defects of the operation equations.

Theorem 5 (Error bound ρ_1)

For the problem $Q(g_j)$, the following assumptions are made:

1) For any $u \in V(\bar{R})$, $g_j'(u)$ exists and is bounded on $\bar{R}_j$ (the closure of R_j), $j = 1, 2$.

2) Let $\hat{g}_j$ be defined in Def. 4 and $Q(\hat{g}_j)$ have a solution in $V(\bar{R})$.

3) $v \in V(\bar{R})$ is an approximate solution to $Q(g_j)$. λ_0 is a scalar such that

$$\begin{cases} \lambda_0 \geq |L_j[v] + g_j(v) - r_j| & \text{in } R_j,\ j = 1, 2 \\ \lambda_0 \geq |v - r_0| & \text{in } R_0 \end{cases} \tag{3.1}$$

4) For $p_j(\xi, v, x, t)$ as defined in Def. 4, there exists a solution $\mu \in V(\bar{R})$ to

$$\left\{\begin{array}{ll} \left.\begin{array}{ll}(L_j + p_j(\xi, v, x, t))[\mu] \geq 1 & \text{in } R_j,\ j = 1,2 \\ \mu \geq 0 & \text{in } R_1 \cup R_2 \end{array}\right\} & (3.2) \\ \mu \geq 1 \quad \text{on } R_0 & (3.3) \\ \xi \geq \lambda_0 \mu \quad \text{on } \bar{R} & (3.4) \end{array}\right.$$

Then, there exists exactly one solution $u \in V(\bar{R})$ of $Q(g_j)$ such that

$$|u(x,t) - v(x,t)| \leq \rho_1(x,t) \equiv \lambda_0 \mu(x,t) \quad \text{on } \bar{R} \tag{3.5}$$

Proof. We first prove that

$$[(L_j + \hat{g}_j', R_j),\ j = 1,2;\ (I, R_0)] \text{ is inverse-positive.} \tag{3.6}$$

In fact, by (2.2) and (3.2), there exists a positive $\mu \in V(\bar{R})$ such that

$$(L_j + \hat{g}_j')[\mu] \geq (L_j + p_j)[\mu] \geq 1 \quad \text{in } R_j,\ j = 1,2. \tag{3.7}$$

Hence (3.6) follows from Lemma 4 with k_j replaced by $\hat{g}_j'$.

Now, let u be a solution of $Q(\hat{g}_j)$. For $j = 1,2$, let $\hat{g}_j(v) - \hat{g}_j(u) = \hat{g}_j'(v_j)(v - u)$ where $\bar{v}_j = \theta_j v + (1 - \theta_j)u$, $0 < \theta_j < 1$. Since $\hat{g}_j(v) = g_j(v)$, it follows from (3.1) and (3.7) that

$$\begin{aligned}(L_j+\hat{g}_j'(\bar{v}_j))[v+\rho_1-u] &= (L_j+\hat{g}_j)[v]-(L_j+\hat{g}_j)[u] + (L_j+\hat{g}_j'(\bar{v}_j))[\rho_1] \\ &= (L_j+g_j)[v] - r_j + \lambda_0(L_j+\hat{g}_j'(\bar{v}_j))[\mu] \geq -\lambda_0 + \lambda_0 = 0 \quad \text{in } R_j\end{aligned} \tag{3.8}$$

On the boundary segment R_0, (3.1) and (3.3) imply

$$v+\rho_1-u = v+\lambda_0\mu-u \geq v+\lambda_0-r_0 \geq 0 \quad \text{in } R_0 \tag{3.9}$$

Hence, by (3.6), (3.8) and (3.9) we get

$$v+\rho_1 \geq u \quad \text{on } \bar{R}.$$

Similarly, we have

$$v-\rho_1 \leq u \quad \text{on } \bar{R}.$$

Next, by (3.4)

$$|v-u| \leq \rho_1 \leq \xi \tag{3.10}$$

But then (2.1) implies that u is also a solution of $Q(g_j)$.

Lastly, suppose that $Q(g_j)$ has two solutions u_1 and $u_2 \in V(\bar{R})$, both satisfying (3.10). For $j = 1,2$, set $g_j'(\bar{v}_j)(u_1-u_2) = g_j(u_1) - g_j(u_2)$, where $\bar{v}_j = \theta_j u_1 + (1-\theta_j)u_2$, $0 < \theta_j < 1$. Then, $|\bar{v}_j - v| \leq \xi$,

$$\hat{g}_j'(\bar{v}_j) = g_j'(\bar{v}_j) \quad \text{in } R_j,\ j = 1,2,$$

and

$$\begin{cases} (L_j+\hat{g}_j')[u_1-u_2] = (L_j+g_j)[u_1] - (L_j+g_j)[u_2] = 0 & \text{in } R_j,\ j = 1,2 \\ u_1-u_2 = 0 & \text{in } R_0 \end{cases} \tag{3.11}$$

Since $[(L_j+\hat{g}_j', R_j),\ j = 1,2;\ (I,R_0)]$ is inverse-positive, (3.11) has the only solution $u_1 - u_2 \equiv 0$ on $\bar{R}$. ∎

The following theorem, which gives the second error bound ρ_2, can be proved in a similar manner.

<u>Theorem 6</u> (Error bound ρ_2)

For the problem $Q(g_j)$, the following assumptions are made:

1) For any $u \in V(\bar{R})$, $g_j'(u)$ exists and is bounded on $\bar{R}_j$ (the closure of R_j), $j = 1,2$.

2) Let $\hat{g}_j$ be defined in Def. 4 and $Q(\hat{g}_j)$ have a solution in $V(\bar{R})$.

3) $v \in V(\bar{R})$ is an approximate solution to $Q(g_j)$. λ_j, $j = 0,1,2$, are scalars such that

$$\begin{cases} \lambda_j \geq |L_j[v] + g_j(v) - r_j| & \text{in } R_j,\ j = 1,2 \\ \lambda_0 \geq |v - r_0| & \text{in } R_0 \end{cases}$$

4) For $p_j(\xi,v,x,t)$ as defined in Def. 4, let $\bar{p}_j \equiv \bar{p}_j(\xi,v)$ satisfy $\bar{p}_j \leq p_j(\xi,v,x,t)$ $(x,t) \in R_j$ and $\lambda_j - \lambda_0\bar{p}_j \geq 0$.

5) Let there exist a solution $\mu \in V(\bar{R})$ to

$$\left\{\begin{array}{lll} (L_j + p_j(\xi,v,x,t))[\mu] \geq 1 & \text{in} & R_j,\ j = 1,2 \\ \mu \geq 0 & \text{on} & \bar{R} \\ \xi \geq \lambda_0 + \mu \sum_{j=1}^{2} (\lambda_j - \lambda_0\bar{p}_j) & \text{on} & \bar{R} \end{array}\right\} \tag{3.12}$$

then, there exists exactly one solution $u \in V(\bar{R})$ of $Q(g_j)$ such that

$$\begin{cases} |u(x,t) - v(x,t)| \leq \rho_2(x,t) \equiv \lambda_0 + \mu(x,t) \sum_{j=1}^{2} (\lambda_j - \lambda_0 \bar{p}_j) \\ \text{on } \bar{R}. \end{cases} \tag{3.13}$$

<u>Remark</u> (Error bound ρ_3)

In addition to ρ_2, it can be shown by similar proof as Theorem 6 that

$$\rho_3 \equiv \lambda_0 + (\hat{\lambda} - \lambda_0 \hat{p})\, \mu(x,t) \tag{3.14}$$

is also an error bound, where $\hat{\lambda} = \max(\lambda_1, \lambda_2)$, $\hat{p} = \min(\bar{p}_1, \bar{p}_2)$. If $\bar{p}_j \leq 0$, $j = 1,2$, then $(\hat{\lambda} - \lambda_0 \hat{p}) \leq \Sigma(\lambda_j - \lambda_0 \bar{p}_j)$ and hence $\rho_3 \leq \rho_2$. However, this is not necessarily true if some $\bar{p}_j$ are positive.

The basic idea of Theorem 5 and Theorem 6 is as follows: In case $[(L_j + g_j', R_j),\ j = 1,2;\ (I, R_0)]$ is not inverse-positive in the whole domain of u, we want to find a set $Z = \{u \mid v - \rho \leq u \leq v + \rho, (x,y) \in \bar{R}\}$ in which it has this property (i.e. locally inverse-positive) and hence $Q(g_j)$ has at most one solution in Z. This is done by first taking an approximate solution v as the 'center' of Z. If ρ is determined in Theorem 5 or Theorem 6, it can be a possible 'radius' of Z. The constrained minimization problem formulated in the next section is devised for finding the 'smallest possible' ρ such that Z may 'trap' a solution of $Q(g_j)$.

4. Constrained Minimization Problems

In this section, we formulate from the error bound formulas (3.5) and (3.13) of the last section two constrained minimization problems. In solving them by some numerical techniques and linear programming method, we can obtain approximate solutions and the corresponding error bounds for the problem $Q(g_j)$.

Notations

For a given function v^k, define

$$g_j(v^k) \equiv g_j(x,t,v^k);$$

$$q_j^k \equiv g_j'(v^k) \equiv \frac{\partial g_j(x,t,v^k)}{\partial u} \quad ; \text{ and}$$

$$G_j^k \equiv v^k q_j^k - g_j(v^k) + r_j \;.$$

Constrained minimization problem one (CMP1)

Suppose $Q(g_j)$ satisfy the assumptions 1) and 2) of Theorem 5.

Step 1 Starting with suitable $v^0 \in V(\bar{R})$, let $v^k \in V(\bar{R})$ and $\delta_0 \in E^1$ solve

$$\min_{v,\delta_0} \left\{ \delta_0 \;\middle|\; \begin{array}{ll} \delta_0 \geq (L_j + q_j^{k-1})[v] - G_j^{k-1} \geq -\delta_0 & \text{in } R_j,\ j=1,2 \\ \delta_0 \geq v - r_0 \geq -\delta_0 & \text{in } R_0 \end{array} \right\} \qquad (4.1)$$

$k = 1,2,\ldots$

Step 2 For any approximate solution v obtained in Step 1, let

$$\lambda_0 = \max\{\|v - r_0\|_{R_0},\ \|L_j[v]+g_j(v)-r_j\|_{R_j},\ j=1,2\} \quad (4.2)$$

Starting with a suitable ξ^0, compute the sequences $\{\xi^k\}_{k=1}^{\infty}$ and $\{\mu^k\}_{k=1}^{\infty}$ by the following iterative process:

Let $\mu^k \in V(\bar{R})$ and $\hat{\mu}^k \in E^1$ solve

$$\min_{\mu,\hat{\mu}} \left\{ \hat{\mu} \;\middle|\; \begin{array}{rl} (L_j + p_j(\xi^{k-1}, v, x, t))[\mu] \geq 1 & \text{in } R_j,\ j=1,2 \\ \mu \geq 1 & \text{in } R_0 \\ \hat{\mu} \geq \mu \geq 0 & \text{on } \bar{R} \end{array} \right\} \quad (4.3)$$

Set $\xi^k = \lambda_0 \hat{\mu}^k$ (4.4)

Theorem 7

In the above iterative process, if at the k^{th} cycle we have $\xi^k \leq \xi^{k-1}$, then the error bound (3.5) holds with

$$\rho_1(x,t) \equiv \lambda_0 \mu^k(x,t) \quad (4.5)$$

Proof: Clearly, we have only to show that (3.2) and (3.4) hold for $\xi = \xi^k$ and $\mu = \mu^k(x,t)$.

Since $\hat{\mu}^k \geq \mu^k \geq 0$ and, for fixed v, p_j is a monotonic non-decreasing function of ξ, it follows from (4.3) and (4.4) that

$$(L_j + p_j(\xi^k, v, x, t))[\mu^k] \geq (L_j + p_j(\xi^{k-1}, v, x, t))[\mu^k] \geq 1 \quad \text{in } R_j,\ j=1,2$$

$$\xi^k = \lambda_0 \hat{\mu}^k \geq \lambda_0 \mu^k(x,t) \quad \text{on } \bar{R} \qquad \blacksquare$$

Constrained minimization problem two (CMP2)

Suppose $Q(g_j)$ satisfy the Assumptions 1) and 2) of Theorem 6. Given suitable initial approximation v^0 and constants ξ^0, $\hat{\mu}^0$ and $\bar{p}_j^0$, $j = 1,2$, the k^{th} cycle of the following iterative process starts with known v^{k-1}, ξ^{k-1}, $\hat{\mu}^{k-1}$ and $\bar{p}_j^{k-1}$:

Step 1 Let $v^k(x,t) \in V(\bar{R})$ and $\delta_j^k \in E^1$, $j = 0,1,2$, solve

$$\min_{v,\delta_j} \left\{ (1 - \hat{\mu}^{k-1} \sum_{j=1}^{2} \bar{p}_j^{k-1})\delta_0 + \hat{\mu}^{k-1} \sum_{j=1}^{2} \delta_j \right|$$

$$\left. \begin{aligned} &\delta_j \geq (L_j + q_j^{k-1})[v] - G_j^{k-1} \geq -\delta_j, \quad \text{in } R_j,\ j = 1,2 \\ &\delta_0 \geq v - r_0 \geq -\delta_0 \quad \text{in } R_0 \end{aligned} \right\} \tag{4.6}$$

Step 2 For $j = 1,2$, evaluate

$$\lambda_0^k = \| v^k - r_0 \|_{R_0}, \quad \lambda_j^k = \| L_j[v^k] + g_j(v^k) - r_j \|_{R_j} \tag{4.7}$$

$$p_j^k(x,t) \equiv p_j(\xi^{k-1}, v^k, x, t) = \min_{\eta} \{ g_j'(\eta) \,|\, |\eta - v^k| \leq \xi^{k-1} \}$$

$$\text{for fixed } (x,t) \tag{4.8}$$

and

$$\bar{p}_j^k = \begin{cases} \min \{ \inf\limits_{(x,t)\in R_j} p_j^k(x,t),\ \lambda_j^k/\lambda_0^k \} & \text{if } \lambda_0^k \neq 0 \\ \inf\limits_{(x,t)\in R_j} p_j^k(x,t) & \text{if } \lambda_0^k = 0 \end{cases} \tag{4.9}$$

Step 3 Let the scalar $\hat{\mu}^k$ and the function $\mu^k \in V(\bar{R})$ solve

$$\min_{\mu,\hat{\mu}} \left\{ \hat{\mu} \;\middle|\; \begin{array}{ll} (L_j + p_j^k(x,t))[\mu] \geq 1 & \text{in } R_j,\ j = 1,2 \\ \hat{\mu} \geq \mu \geq 0 & \text{on } \bar{R} \end{array} \right\} \tag{4.10}$$

Set $\xi^k = \lambda_0^k + \hat{\mu}^k \sum_{j=1}^{2} (\lambda_j^k - \lambda_0^k \bar{p}_j^k)$.

Theorem 8

In the above iterative process, if at the k^{th} cycle we have $\xi^k \leq \xi^{k-1}$, then the error bound (3.13) holds with

$$\rho(x,t) \equiv \lambda_0^k + \mu^k(x,t) \sum (\lambda_j^k - \lambda_0^k \bar{p}_j(\xi^k, v^k)) \tag{4.11}$$

where $\bar{p}_j(\xi, v) = \inf_{(x,t)\in R_j} p_j(\xi, v, x, t)$.

Proof: Clearly we have only to show that (3.12) holds with $\xi = \xi^k$, $v = v^k$ and $\mu = \mu^k$.

Since $\hat{\mu}^k \geq \mu^k \geq 0$, and, for fixed v^k and (x,t), p_j and $\bar{p}_j$ are both monotone non-increasing functions of ξ, it follows from (4.10) that

$$(L_j + p_j(\xi^k, v^k, x, t))[\mu^k] \geq (L_j + p_j(\xi^{k-1}, v^k, x, t))[\mu^k] \geq 1 \text{ in } R_j,\ j = 1,2.$$

$$\xi^k = \lambda_0^k + \hat{\mu}^k \sum(\lambda_j^k - \lambda_0^k \bar{p}_j(\xi^{k-1}, v^k)) \geq \lambda_0^k + \mu^k(x,t) \sum(\lambda_j^k - \lambda_0^k \bar{p}_j(\xi^k, v^k)) \text{ on } \bar{R} \quad \blacksquare$$

5. Discussion

(1) Numerical results show that CMP2 gives more accurate approximate solutions than CMP1.

(2) The error bound ρ_1 (4.5) and ρ_2 (4.11) are monotonic decreasing functions of each of the quantities λ_j, $j = 0,1,2$, and $\mu(x,t)$. Hence we may say that each approximate solution is obtained by minimizing its error bound in certain sense. Again, it can be shown both theoretically (see Cheung [1970]) and numerically (see section 7) that ρ_2 is a better error bound than ρ_1.

(3) Suppose, instead of the local Lipschitz condition (2.4), g_j satisfies a global Lipschitz condition, i.e. there exists a bounded function k_j, independent of u, such that

$$g_j(v_1) - g_j(v_2) \geq (v_1 - v_2)k_j \quad \text{in } R_j, \text{ for all } v_1 \geq v_2, \quad v_i \in V(\bar{R}) \tag{5.1}$$

Theorem 5, Theorem 6 and the constrained minimization problems still hold, with $p_j = k_j$. However, a local Lipschitz condition has at least two advantages (for detail, see Cheung [1970]):

i) A global Lipschitz condition may not exist; whereas a local Lipschitz condition can always be constructed, provided g_j' exists and is bounded.

ii) A local Lipschitz condition gives a 'better' error bound than a global one.

6. Computational Schemes

This section presents some computational schemes of solving the constrained minimization problems of Section 4. Let $\{\phi_i(x,t)\}_{i=1}^{m} \subset V(\bar{R})$ by a set of suitably chosen functions. Assume the function μ and the approximate solution v to be of the form

$$\mu(\beta;x,t) = \sum_{i=1}^{m} \beta_i \, \phi_i(x,t) \quad \text{and} \quad v(\alpha;x,t) = \sum_{i=1}^{m} \alpha_i \, \phi_i(x,t).$$

Let D_j and D_j^* be two discretizations of the region R_j, where D_j^* has finer grid size than D_j (Figure 2)

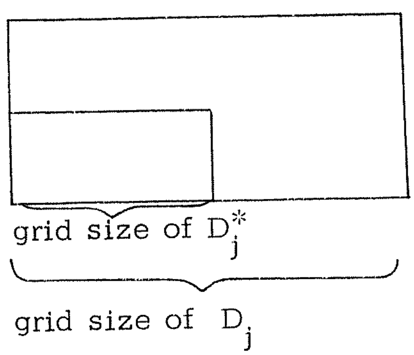


Figure 2

Computational Schemes One (CS1)

Step 1 Starting with suitable $v^0 \in V(\bar{R})$, let α_i^k, $i = 1,2,\ldots,m$ and δ_0^k solve

$$\min_{\alpha_i,\delta_0} \left\{ \delta_0 \;\middle|\; \begin{array}{ll} \delta_0 \geq \sum_{i=1}^{m} \alpha_i (L_j + q_j^{k-1})[\phi_i] - G_j^{k-1} \geq -\delta_0 & \\ \qquad\qquad \text{in } D_j,\ j = 1,2 & \\ \delta_0 \geq \sum_{i=1}^{m} \alpha_i \, \phi_i - r_0 \geq -\delta_0 & \text{in } D_0 \end{array} \right. \tag{6.1}$$

$k = 1,2,3,\ldots$

Step 2 For any approximate solution $v \equiv \Sigma \alpha_i^k \phi_i$ obtained in Step 1, let

$$\lambda_0 = \max \{ \| v - r_0 \|_{D_0^*}, \ \| L_j[v] + g_j(v) - r_j \|_{D_j^*}, \ j = 1,2 \}.$$

(The superscript k is omitted from v and λ_0).

Step 3 Let ε_j, $j = 0,1,2$, be suitably chosen small positive quantities. With a suitable initial ξ^0, let $\{\xi^k\}$ and $\{\beta_j^k\}_{j=0}^m$ be determined by the following iterative process:
Let β_i^k, $i = 0,1,\ldots,m$, solve

$$\min_{\beta_i} \left\{ \beta_0 \;\middle|\; \begin{array}{ll} \sum_{i=1}^{m} \beta_i (L_j + p_j(\xi^{k-1}, v, x, t))[\phi_i] \geq 1 + \varepsilon_j, & \text{in } D_j, \ j = 1,2 \\ \Sigma \beta_i \phi_i \geq \varepsilon_0 & \text{in } D_1 \cup D_2 \\ \Sigma \beta_i \phi_i \geq 1 + \varepsilon_0 & \text{in } D_0 \\ \beta_0 \geq \Sigma \beta_i \phi_i & \text{in } \bar{D} \end{array} \right\} \quad (6.2)$$

where $\bar{D} = D_0 \cup D_1 \cup D_2$.
Let $\hat{\mu}^k = \max_{(x,t)\in D^*} \Sigma \beta_i^k \phi_i(x,t)$ and $\xi^k = \lambda_0 \hat{\mu}^k$, where
$D^* = D_0^* \cup D_1^* \cup D_2^*$.

Computational Scheme Two (CS2)

Given suitable initial approximation v^0 and constants ξ^0, $\hat{\mu}^0$ and $\bar{p}_j^0$, the k^{th} cycle of the following iterative process starts with known ξ^{k-1}, $\hat{\mu}^{k-1}$, $\bar{p}_j^{k-1}$ and $v^{k-1}(x,t) \equiv \Sigma\, \alpha_i^{k-1}\, \phi_i(x,t)$:

<u>Step 1</u> Let α_i^k, $i = 1,\ldots m$, and δ_j^k, $j = 0,1,2$, solve

$$\min_{\alpha_i,\delta_j} \left\{ (1 - \hat{\mu}^{k-1} \sum_{j=1}^{2} \bar{p}_j^{k-1})\delta_0 + \hat{\mu}^{k-1} \sum_{j=1}^{2} \delta_j \;\middle|\; \right.$$

$$\left.\begin{aligned} &\delta_j \geq \Sigma\, \alpha_i (L_j + q_j^{k-1})[\phi_i] - G_j^{k-1} \geq -\delta_j \text{ in } D_j,\ j = 1,2 \\ &\delta_0 \geq \Sigma\, \alpha_i\, \phi_i - r_0 \geq -\delta_0 \quad \text{in } D_0 \end{aligned}\right\} \qquad (6.3)$$

<u>Step 2</u> For $j = 1,2$, evaluate

$$\lambda_0^k = \| v^k - r_0 \|_{D_0^*},\ \lambda_j^k = \| L_j[v^k] + g_j(v^k) - r_j \|_{D_j^*}$$

$$p_j^k(x,t) \equiv p_j(\xi^{k-1}, v^k, x, t) \approx \min_{\eta} \{ g_j'(\eta) \mid |\eta - v^k| \leq \xi^{k-1} \} \text{ for fixed } (x,t)$$

and

$$\bar{p}_j^k = \begin{cases} \min\{ \min_{(x,t)\in D_j^*} p_j^k(x,t),\ \lambda_j^k/\lambda_0^k \} & \text{if } \lambda_0^k \neq 0 \\ \min_{(x,t)\in D_j^*} p_j^k(x,t) & \text{if } \lambda_0^k = 0 \end{cases}$$

<u>Step 3</u> Choose suitable small positive constants ε_j, $j = 0,1,2$. Let

β_i^k, $i = 0,1,\ldots,m$, solve

$$\min_{\beta_i} \left\{ \beta_0 \;\middle|\; \begin{aligned} &\sum_{i=1}^{m} \beta_i (L_j + p_j^k(x,t))[\phi_i] \geq 1 + \varepsilon_j \text{ in } D_j,\ j = 1,2 \\ &\beta_0 \geq \sum_{i=1}^{m} \beta_i\, \phi_i \geq \varepsilon_0 \quad \text{on } \bar{D} \equiv D_0 \cup D_1 \cup D_2 \end{aligned} \right. \qquad (6.4)$$

Let $\hat{u}^k = \max_{D^*} \sum_{i=1}^{m} \beta_i^k \phi_k^k$ and $\xi^k = \lambda_0^k + \hat{u}^k \sum_{j=1}^{2} (\lambda_j^k - \lambda_0^k \bar{P}_j^k)$, where $D^* = D_0^* \cup D_1^* \cup D_2^*$.

Choice of initial values for the parameters ξ^0, $\hat{u}^0$, $\bar{p}_j^0$ and approximation v^0

Theorems 7 and 8 imply that the initial ξ^0 should overestimate the error of the initial approximation v^0. Usually if ξ^0 is large enough we should have $\xi^0 \geq \xi^1$ and hence an error bound is obtained at a single cycle. However difficulty may arise that if ξ^0 is too large, (6.2) or (6.4) may have no feasible solution.

Without better values, we may set $\hat{u}^0 = 1$, $\bar{p}_j^0 = 0$, $j = 1,2$ and $v^0(x,t) \equiv 0$ (or 1).

Choice of the parameters ε_j, $j = 0,1,2$

In (6.2) and (6.4), the positive quantities ε_j are added to the right sides. If the density of discretization is fine enough and the differential operators satisfy some Lipschitz conditions, it can be shown (for detail, see Cheung [1970]) that a solution $u = \Sigma\, \beta_i\, \phi_i$ of the discretized problem (6.2) (or (6.4)) also satisfies the inequalities (4.3) (or (4.10)) over the whole region. Therefore, the error bound is valid over $\bar{R}$.

Criterion for terminating the iterative process

The iterative process may stop whenever $\xi^k \leq \xi^{k-1}$. However, in practice, this is usually satisfied at the first cycle. To obtain better accuracy, we may use the following criterion:

Let $\delta_2^k = \| L_2[v^k] + q_2^{k-1} v^k - G_2^{k-1} \|_{D_2}$ and $\lambda_2^k = \| L_2[v^k] + g_2(v^k) - r_2 \|_{D_2^*}$.

For a preassigned quantity ε_N (convergence tolerance of Newton's method), the iterative process is stopped at the k^{th} cycle if

$$\frac{|\lambda_2^k - \delta_2^k|}{\lambda_2^k} \leq \varepsilon_N. \tag{6.5}$$

Since λ_2^k and δ_2^k are quantities obtained during the iterative process, only little additional computation is required.

Linear programming formulation

It is easy to show that (6.1), (6.2), (6.3) and (6.4) are linear programs of the form (for detail, see Rosen [1970]):

$$\min_{\pi} \; \{d'\pi \,|\, A'\pi \geq -c\} \tag{6.6}$$

where d, π and c are vectors and A is a matrix. $'$ denotes the transpose. With $w = -\pi$, (6.6) is equivalent to the dual problem of a standard linear program (see Dantzig [1963]):

$$\max_{w} \; \{d'w \,|\, A'w \leq c\} \tag{6.7}$$

Instead of solving (6.7) directly, most available computer linear programming code (e.g. SIMPLX [1969]) are devised so as to solve its primal problem

$$\min_{z} \; \{c'z \,|\, Az = d, \; z \geq 0\} \tag{6.8}$$

(6.6) and (6.8) have the following relations which are well known in the duality theory of linear programming:

(i) If (6.8) has an optimal solution z^* with optimal base B^*, then $\pi^* = -w^* = -(B^*)^{-1} z^*$ is an optimal solution to (6.6). In some computer linear programming codes, π^* is one of the output data. Hence, we can directly obtain an optimal solution to (6.6) by solving (6.8).

(ii) If (6.8) has an infinite (negative) solution, (6.6) has no feasible solution. This fact can be used to test the inverse-positive property of the given problem $Q(g_j)$.

Sizes of the linear programs

Let m = total number of basis functions $\{\phi_i\}_{i=1}^m$;

n = total number of grid points over $\bar{D} = D_0 \cup D_1 \cup D_2$; and

n_0 = number of grid points over D_0.

The following table shows the dimensions of the different linear programs:

Table 1

linear program	dimension of A
(6.1)	$(m + 1) \times 2n$
(6.3)	$(m + 3) \times 2n$
(6.2) or (6.4)	$(m + 1) \times (3n - n_0)$

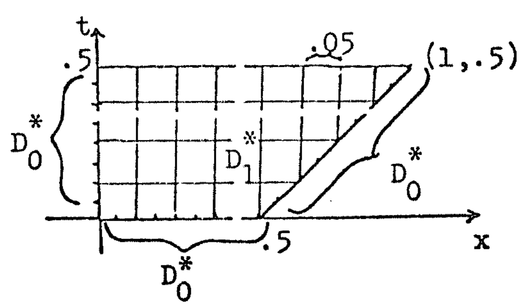
7. Numerical Results

Example 1 (first BVP on a Trapezoid)

$$\begin{cases} \dfrac{\partial u}{\partial t} - (2+t)\dfrac{\partial^2 u}{\partial x^2} + g(u) = r & 0 < x < t + .5,\ 0 < t \le .5 \text{ — } R_0 \\ u = x^2 + \cos x & 0 \le x \le .5,\quad t = 0 \\ u = \cos t^2 & x = 0,\quad 0 \le t \le .5 \\ u = x^2 + \cos(x+t^2) & x = t + .5,\ 0 \le t \le .5 \end{cases} \Bigg\} R_0$$

where $g(u) = \exp(.1(u - \cos(x+t^2)))$ and $r = -2t \sin(x+t^2) - (2+t)(2 - \cos(x+t^2)) + \exp(.1x^2)$.

Figure 3



Exact solution $u = x^2 + \cos(x+t^2)$

Algorithm Since $g'(u)$ is bounded below, Example 1 is inverse-positive. We determine approximate solutions and error bounds by CS1 and CS2. (See Section 6).

Both algorithms are started with $v^0 \equiv 0$, $\xi^0 = 1$, $\hat{u}^0 = 1$ and $\bar{p}^0 = 0$.

$\|\cdot\|_{R_j}$ is approximately by $\|\cdot\|_{D_j^*}$, where D_j^* are meshes described in

Figure 3. The termination criterion (6.5) is applied with $\varepsilon_N = .0002$.

Discretization method The linear programs included in CS1 and CS2 are solved with $\varepsilon_0 = \varepsilon_1 = .0001$. D_0^n and D_1^n are the same as D_0^* and D_1^* but with their grid sizes doubled.

Function space $S_2^{(3+4,3+2)}$ (bicubic B-spline with 5 knots in the x-direction and 3 knots in the t-direction. See Appendix.)

Computer and LP code CDC 3600, SIMPLX [1969].

Time 6 minutes 57 seconds.

Numerical results After 2 iterations the termination criterion is satisfied and the following results are obtained (omitting superscripts):

	CS 1	CS 2
λ_0	.025593	.000088
λ_1		.020331
$\bar{p}$		.079323
max. value of μ	1.040319	.044913
max. error	5.49 E-3 at (.1,.0)	-1.09 E-4 at (.25,.15)
max. error bound	2.66 E-2 at (.30,.0)	1.00 E-3 at (.35,.0)
min. error bound	2.56 E-2 on t = 0	8.98 E-5 on x = 0

Table 2

Table 2 shows that CS2 gives a much smaller error function μ than CS1. This is the main reason why the error bound ρ_2 obtained by CS2 is much better than the error bound ρ_1 obtained by CS1 (by a factor of $10^{-1.5}$ on the average). The error in the approximate solution obtained by CS2 is also much better than that obtained by CS1 (by a factor of 10^{-2} on the average (Table 3).)

Figure 4 and Figure 5 show respectively the errors of the approximate solutions v_1 and v_2 along 5 lines parallel to the x-axis, Table 4 contains the coefficients of v_1 and v_2. Within each lattice, the upper entry is for v_1 while the lower for v_2. Each entry is the coefficient of the product of functions on the top row and leftmost column.

Table 3

Approximate Solution v, Errors (u-v), and Error Bounds ρ

			CS 1			CS 2		
	x	t	v_1	$u-v_1$	ρ_1	v_2	$u-v_2$	ρ_2
*	.05	.00	.99577	5.48 E-3	2.59 E-2	1.00124	1.40 E-5	3.16 E-4
*	.10	.00	.99952	5.48 E-3	2.61 E-2	1.00499	1.05 E-5	5.12 E-4
*	.35	.00	1.05706	4.81 E-3	2.66 E-2	1.06188	-5.43 E-6	1.00 E-3
	.45	.00	1.09878	4.16 E-3	2.65 E-2	1.10295	-4.93 E-7	9.08 E-4
	.10	.05	1.00132	3.43 E-3	2.59 E-2	1.00472	2.77 E-5	3.83 E-4
	.30	.05	1.04201	2.59 E-3	2.62 E-2	1.04460	-6.15 E-7	6.41 E-4
	.45	.05	1.10048	1.38 E-3	2.61 E-2	1.10184	1.96 E-5	5.18 E-4
	.25	.15	1.02742	-1.82 E-3	2.61 E-2	1.02571	-1.09 E-4	5.55 E-4
*	.00	.20	1.00264	-3.44 E-3	2.56 E-2	.99922	-1.77 E-5	8.89 E-5
	.25	.20	1.02432	-3.57 E-3	2.62 E-2	1.02083	-8.95 E-5	6.54 E-4
	.55	.20	1.13810	-4.66 E-3	2.62 E-2	1.13350	-5.56 E-5	6.36 E-4
*	.70	.20	1.23391	-5.44 E-3	2.57 E-2	1.22854	-7.31 E-5	1.80 E-4
	.25	.25	1.01870	-4.63 E-3	2.63 E-2	1.01412	-5.02 E-5	7.54 E-4
	.50	.25	1.10058	-4.66 E-3	2.65 E-2	1.09594	-1.44 E-5	8.86 E-4
*	.75	.25	1.25481	-4.63 E-3	2.59 E-2	1.25020	-1.77 E-5	3.79 E-4
*	.00	.40	.99013	-2.91 E-3	2.56 E-2	.98725	-1.77 E-5	8.98 E-5
	.35	.40	.99735	-2.10 E-3	2.65 E-2	.99524	-3.69 E-6	9.26 E-4
*	.70	.40	1.14160	8.39 E-4	2.63 E-2	1.14242	1.98 E-5	7.04 E-4
	.90	.40	1.29543	3.44 E-3	2.56 E-2	1.29885	1.77 E-5	8.98 E-5
	.15	.45	.96105	-3.39 E-5	2.61 E-2	.96100	1.38 E-5	5.66 E-4
*	.00	.45	.96348	5.44 E-3	2.56 E-2	.96893	-1.77 E-5	8.98 E-5
*	.30	.45	.93932	3.20 E-3	2.65 E-2	.94257	-4.10 E-5	8.60 E-4
	.65	.45	1.04036	3.75 E-3	2.65 E-2	1.04416	-4.88 E-5	8.72 E-4
	.95	.45	1.30989	5.44 E-3	2.60 E-2	1.31534	-1.77 E-5	8.98 E-4

*grid points

Table 4

Coefficients of v_1 and v_2 (multiplied by 100)

		β(4x+1)	β(4x)	β(4x-1)	β(4x-2)	β(4x-3)	β(4x-4)	β(4x-5)
β(4t+1)		2.8455 2.8714	2.7370 2.7585	2.7889 2.8130	3.0084 3.048	3.4225 3.4806	4.0828 4.1875	0 0
β(4t)		2.7971 2.8173	2.7213 2.7456	2.8253 2.8472	3.1138 3.1257	3.5000 3.5970	4.3093 4.2870	4.9201 5.1306
β(4t-1)		2.8971 2.8689	2.7777 2.7532	2.8340 2.8107	3.0715 3.0470	3.5075 3.4814	4.1654 4.1400	5.1514 5.0900
β(4t-2)		2.9481 2.9484	2.7087 2.7070	2.670 2.6310	2.7493 2.7508	3.0719 3.0868	3.6377 3.6715	4.4269 4.5129
β(4t-3)		2.6774 2.8738	2.2746 2.4085	2.0692 2.1405	2.0343 2.0925	2.2755 2.3009	2.8223 2.8035	3.9031 3.7437

Fig. 4 Error Curves of $(u-v_1)$
(Example 1, using CS1)

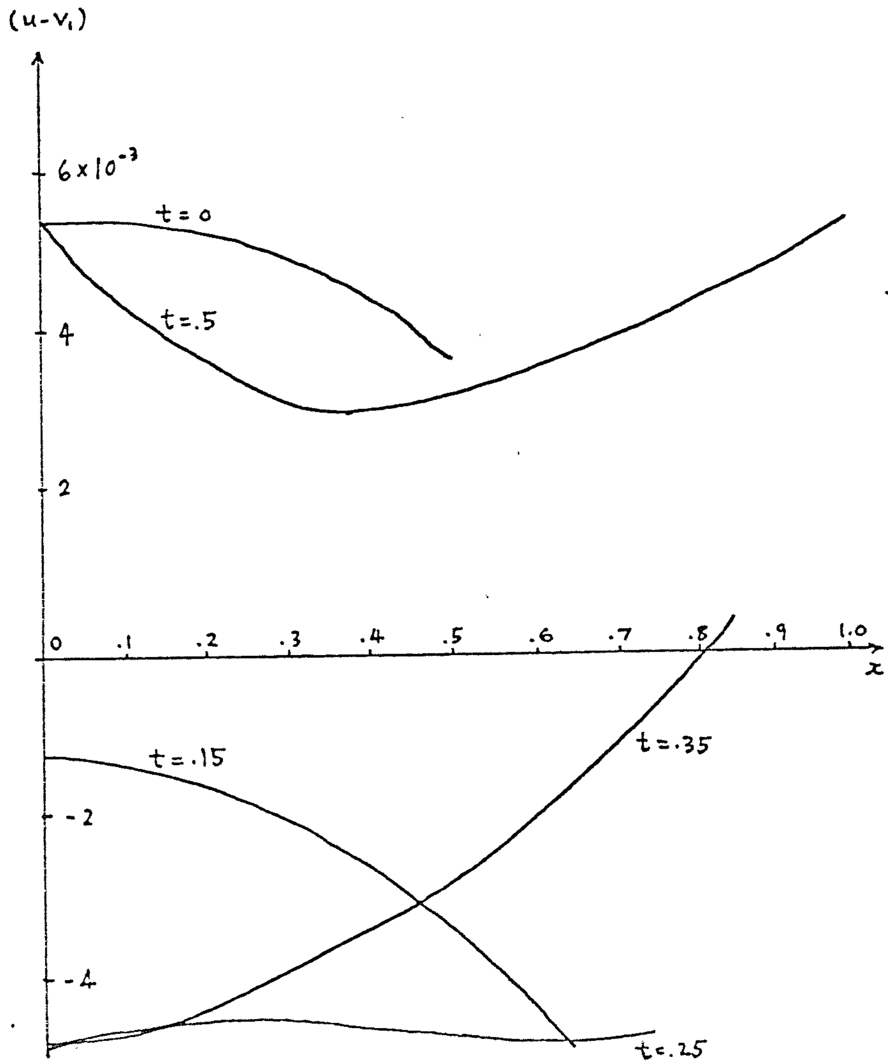
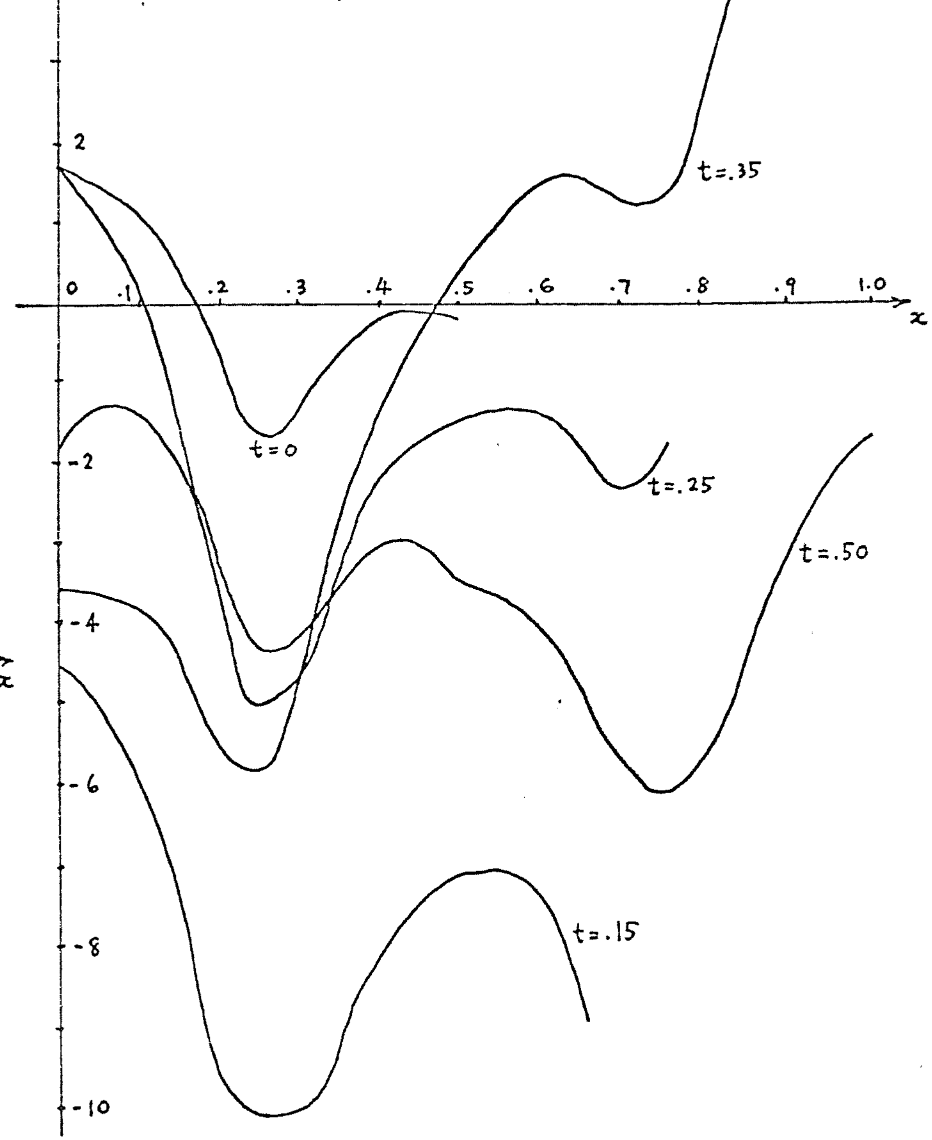


Fig. 5 Error Curves of $(u-v_2)$
(Example 1, using CS2)



Example 2 (third BVP on a rectangular domain)

$$\begin{cases} (x+1)\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + g_2(u) = r_2 & 0 < x < .45, \quad 0 < t \le .8 \text{ --- } R_2 \\ -\dfrac{\partial u}{\partial x} + g_1(u) = r_1 & x = 0, \quad 0 < t \le .8 \text{ --- } R_1 \\ \left.\begin{array}{ll} u = 2\cos x & 0 \le x \le .45, \quad t = 0 \\ u = 2\cos(.45+t^2) & x = .45, \quad 0 < t \le .8 \end{array}\right\} R_0 \end{cases}$$

where $g_1 = -.01tu^3$, $g_2 = -.1t \sin u$, $r_1 = 2 \sin t^2 - .08t^2 \cos^3 t^2$ and $r_2 = 2\cos(x+t^2) - 4(x+1)t \sin(x+t^2) - .1t \sin(2\cos(x+t^2))$.

Exact solution $u = 2\cos(x+t^2)$

Algorithm Since $g_1'(u) = -.03tu^2$ may vanish somewhere in R_1, inverse-positivity is not guaranteed. We apply CS2 to obtain an approximate solution and an error bound ρ_2.

CS2 is applied with $v^0 \equiv 1$, $\xi^0 = .5$, $\hat{u}^0 = 1$, and $\bar{p}_1^0 = \bar{p}_2^0 = 0$. $\|\cdot\|_{R_j}$ is approximated by $\|\cdot\|_{D_j^*}$, where D_j^* are meshes described in Figure 6. The termination criterion (6.5) is applied with $\varepsilon_N = .0002$. Besides the error bound ρ_2, we also calculate the error bound ρ_3 (3.14).

Figure 6

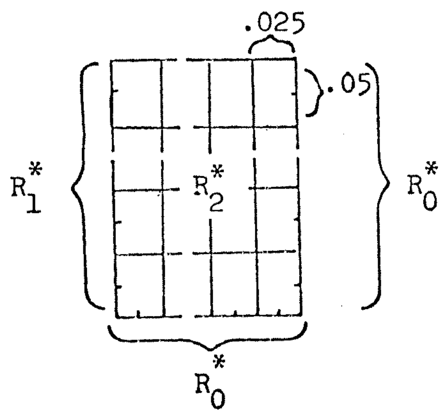
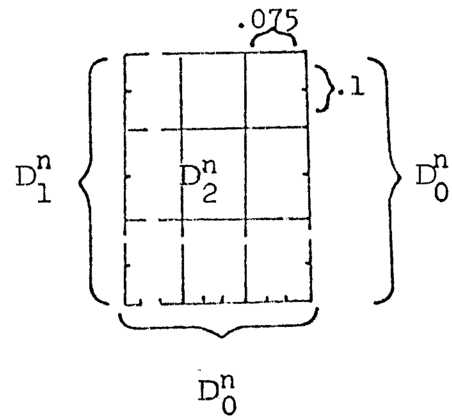


Figure 7



Discretization method   The linear programs included in CS2 are solved with $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = .0001$. The meshes D_j^n are described in Figure 7.

Function space $S_1^{(3+3,2+4)}$ (product of elementary cubic spline in x and quadratic spline in t, with 4 knots in the x-direction and 5 knots in the t-direction. See Appendix).

Computer and LP code   CDC 3600, RS MSUB (Clasen [1961]).

Time   7 minutes 49 seconds.

Numerical results   After 3 cycles, the termination criterion is satisfied and the following results are obtained (omitting superscripts):

$\lambda_0 = 2.11$ E-2, $\lambda_1 = 4.04$ E-2, $\lambda_2 = 3.16$ E-2; $\xi = 4.32$ E-2;
$\bar{p}_1 = -.1$, $\bar{p}_2 = -.08$; max. value of error function $\mu = 5.69$ E-1.

Each entry in Table 5 is the coefficient of a basis function which is the product of the corresponding functions on the top row and the left-most column.

Table 5

Coefficients of Approximate Solution

	1	x	x^2	$(x)_+^3$	$(x-.15)_+^3$	$(x-.3)_+^3$
1	1.99857	0.00074	-1.00116	0.01102	0.08195	-0.01331
t	-0.03019	0.00940	0.15975	-0.33221	0.61872	0.49820
$(t)_+^2$	0.09610	-2.05745	-0.42922	2.08791	-3.42521	-1.00641
$(t-.2)_+^2$	-0.66026	0.12379	0.58758	-3.33028	6.36580	0.17891
$(t-.4)_+^2$	-0.96210	0.23835	0.61809	2.70660	-4.69479	2.31576
$(t-.6)_+^2$	-1.04285	0.77683	1.23289	-2.38896	2.93268	-3.42271

Table 6

Approximate Solution v, Error (u-v) and

Error Bounds ρ_2 and ρ_3

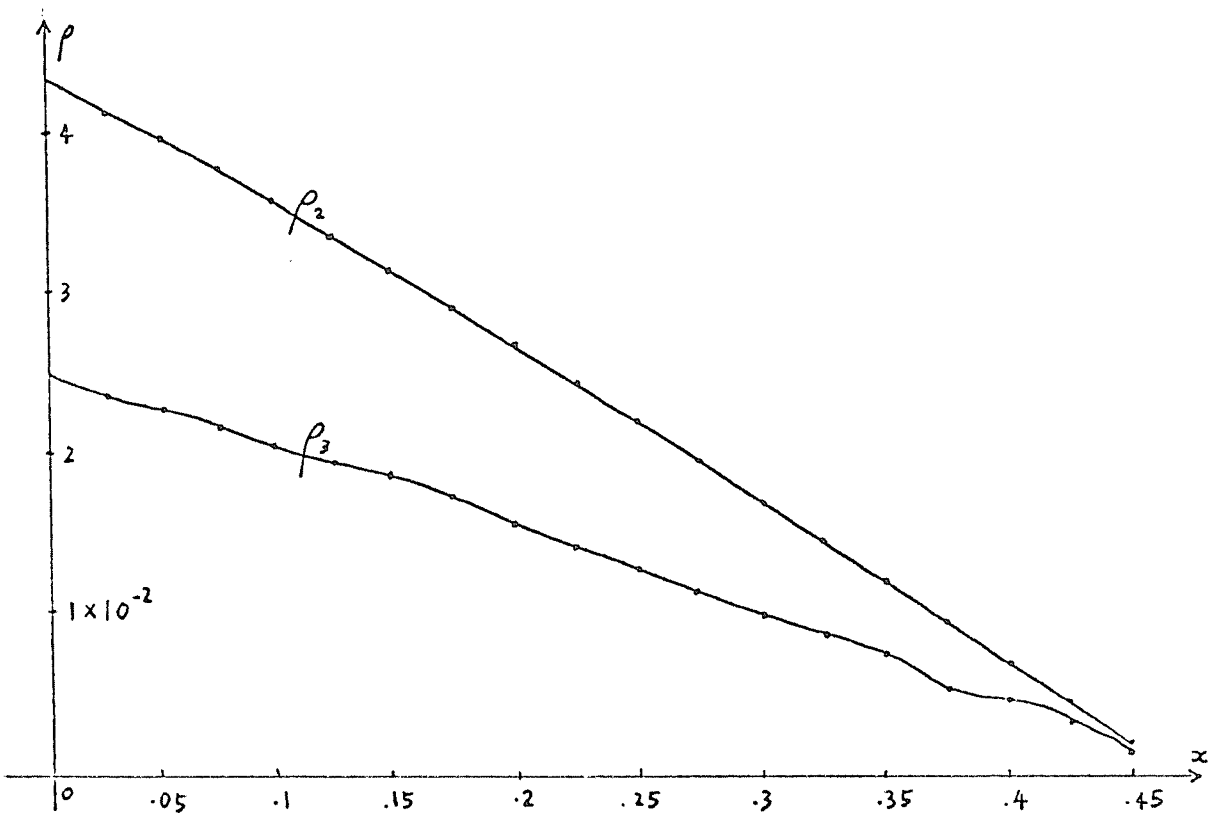
	x	t	v	u-v	ρ_2	ρ_3
*	.00	.00	1.998573	-1.43 E-3	2.96 E-2	1.76 E-2
*	.20	.00	1.958772	-1.36 E-3	1.65 E-2	1.02 E-2
*	.30	.00	1.909264	-1.41 E-3	1.07 E-2	6.95 E-3
*	.45	.00	1.799342	-1.55 E-3	2.12 E-2	2.11 E-3
o*	.00	.10	1.996514	-3.39 E-3	4.00 E-2	2.34 E-2
*	.30	.10	1.902116	-2.55 E-3	1.59 E-2	9.83 E-3
	.25	.20	1.915299	-1.19 E-3	2.22 E-2	1.34 E-2
	.40	.20	1.809574	7.03 E-5	7.46 E-3	5.11 E-3
	.00	.35	1.984922	-9.08 E-5	4.29 E-2	2.50 E-2
	.15	.35	1.926220	1.83 E-5	3.10 E-2	1.83 E-2
	.30	.35	1.824817	6.84 E-4	1.74 E-2	1.07 E-2
	.45	.35	1.682757	1.66 E-3	2.19 E-3	2.15 E-3
	.25	.45	1.800130	1.42 E-3	2.21 E-2	1.33 E-2
	.45	.45	1.590241	1.10 E-3	2.13 E-3	2.12 E-3
*	.38	.50	1.621948	2.18 E-5	9.95 E-3	6.51 E-3
*	.45	.50	1.529429	-2.56 E-4	2.12 E-3	2.11 E-3
	.20	.60	1.693365	-1.15 E-3	2.67 E-2	1.59 E-2
	.43	.60	1.413037	-1.74 E-3	4.77 E-3	3.60 E-3
	.08	.75	1.604364	-2.81 E-3	3.74 E-2	2.19 E-2
	.20	.75	1.443795	-2.43 E-3	2.68 E-2	1.59 E-2
	.40	.75	1.141855	-1.09 E-3	7.41 E-3	5.08 E-3
ǂ*	.00	.80	1.602582	-1.61 E-3	4.32 E-2	2.52 E-2
	.33	.80	1.138715	-1.19 E-4	1.50 E-2	9.35 E-3
*	.45	.80	0.926523	1.55 E-3	2.11 E-3	2.11 E-3

*grid points, o max. error, ǂ max. error bounds.

Figure 8

Comparison of the Error Bound Curves ρ_2 and ρ_3

Along the Line $t = .4$



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APPENDIX

Monovariate splines

In the following we present two convenient ways of forming a basis for the space B^{m+n} of spline functions of degree m with $n + 1$ knots in terms of the basic splines defined by

$$(x)_+^m = \begin{cases} x^m & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

1) Elementary splines

Let $x_0, x_1, \ldots, x_n$ be a set of knots over $[x_0, x_n]$. An arbitrary spline of degree m is given by

$$\alpha_i \phi_i(x) = \sum_{i=0}^{m-1} \alpha_i x^i + \sum_{i=0}^{n-1} \alpha_{m+i}(x-x_i)_+^m$$

This space is of dimension $m + n$.

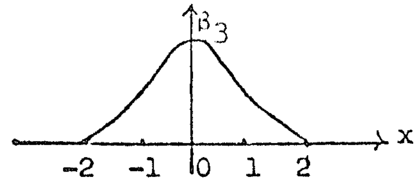
2) B-splines

Define the following function

$$\beta_m(x) = \sum_{j=-k}^{k} (-1)^{j+k} \binom{2k}{j+k} (j-x)_+^m \quad m = 2k - 1.$$

It looks like Figure A.1 and has the following properties:

Figure A.1 $k = 2$



$\beta_m(x) > 0$ for $-k < x < k$

$\beta_m(x) = 0$ for $|x| \geq k$