Nonlinear Differ. Equ. Appl. (2016) 23:24

1021-9722/16/030001-30
published online April 26, 2016
DOI 10.1007/s00030-016-0380-3

Nonlinear Differential Equations and Applications NoDEA

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Quasilinear parabolic problem with $p(x)$-laplacian: existence, uniqueness of weak solutions and stabilization

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Abstract. We discuss the existence and uniqueness of the weak solution of the following quasilinear parabolic equation

$$\begin{cases} u_t - \Delta_{p(x)} u = f(x,u) & \text{in} \quad Q_T \stackrel{\text{def}}{=} (0,T)\times\Omega, \\ u = 0 & \text{on} \quad \Sigma_T \stackrel{\text{def}}{=} (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega \end{cases} \qquad (P_T)$$

involving the $p(x)$-laplacian operator. Next, we discuss the global behaviour of solutions and in particular some stabilization properties.

Mathematics Subject Classification. Primary 35K55, 35J62; Secondary 35B65.

Keywords. $p(x)$-laplacian, Parabolic equation, Stabilization.

1. Introduction

Our main goal in this paper is to study the existence, uniqueness and global behaviour of the weak solutions to the following parabolic equation involving the $p(x)$-laplacian operator

$$\begin{cases} u_t - \Delta_{p(x)} u = f(x,u) & \text{in} \quad Q_T = (0,T)\times\Omega, \\ u = 0 & \text{on} \quad \Sigma_T = (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega \end{cases} \qquad (P_T)$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a smooth bounded domain, $p : \Omega \to [1, +\infty]$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.
Let $\mathcal{P}(\Omega)$ be the family of all measurable functions $q : \Omega \to [1, +\infty]$ and set

$$\mathcal{P}^{log}(\Omega) \stackrel{\text{def}}{=} \left\{ q \in \mathcal{P}(\Omega) \, : \, \frac{1}{q} \text{ globally log-Hölder continuous} \right\}.$$

In particular, for any $p \in \mathcal{P}^{\log}(\Omega)$, there exists a function w such that

$$\forall (x,y) \in \Omega^2,\ |p(x)-p(y)| \leq w(|x-y|) \text{ and } \limsup_{t\to 0^+} -w(t)\ln t < +\infty.$$

There is an abundant literature devoted to questions on existence and uniqueness of solutions to (P_T) for $p(x) \equiv p$ (see for instance [7] and references therein). More recently, parabolic and elliptic problems with variable exponents have been studied quite extensively, see for example [1,3–5,9,21,23]. The importance of investigating these problems lies in their occurrence in modeling various physical problems involving strong anisotropic phenomena related to electrorheological fluids (an important class of non-Newtonian fluids) [1,22,23], image processing [9], elasticity [29], the processes of filtration in complex media [6], stratigraphy problems [18] and also mathematical biology [16].

Regarding the current literature, we bring in this paper new results about the regularity of weak solutions and about the behaviour of global weak solutions. In particular, we investigate the question of asymptotic convergence to a steady state. To prove the existence of weak solutions, we follow a semi-group approach, involving a semi-discretization in time method, that provides the existence of mild solutions belonging to $C([0,T];W_0^{1,p(x)}(\Omega))$ and $C([0,T];L^\infty(\Omega))$. Then the existence of subsolutions and supersolutions and the weak comparison principle reveal the stabilization property for a suitable class of nonlinearities f.

In our knowledge, the existence of mild solutions and the convergence to a stationary solution for quasilinear parabolic equations with variable exponents were not investigated previously in the literature and all the corresponding results brought in the present paper are new. To establish these results, we use some former contributions about the validity of a strong comparison principle (see [28]), the regularity of solutions (see in particular [1,12,17]) and some extensions proved in Appendix C. We point out that other aspects of global behaviour of weak solutions (extinction in finite time, localization, blow-up in finite time) are discussed in [4,5]. In [3], the Galerkin method is used alternatively to prove existence of weak solutions. In the same way, in the case where $f \equiv 0$ or f depends only to $(t,x) \in Q_T$, the authors of [2,6] use a perturbation method to establish the existence of solutions of (P_T).

Concerning quasilinear elliptic equations with variable exponents, striking results about existence and nonexistence of eigenvalues contrasting with the constant exponent case are proved in [21,24] showing the complex nature of the $p(x)$-laplacian operator. Furthermore, in [24], the authors established also multiplicity results for combined concave-convex function f.

Before going further, we recall the definitions and useful results on the variable exponent Lebesgue and Sobolev spaces. For more details, we refer to the book [10] and the paper [19]. Let $p \in \mathcal{P}(\Omega)$. We define the semimodular

$$\rho_p(u) \stackrel{\text{def}}{=} \int_{\Omega\setminus\Omega_\infty} |u|^{p(x)}dx + ess\sup_{\Omega_\infty} |u(x)|$$

where $\Omega_\infty = \{x \in \Omega \,|\, p(x) = \infty\}$. Then the variable exponent Lebesgue space is defined as follows:

$$L^{p(x)}(\Omega) \stackrel{\text{def}}{=} \{u \,|\, u \text{ is measurable on } \Omega \text{ and } \rho_p(\lambda u) < \infty \text{ for some } \lambda > 0\}.$$

If $p \in L^\infty(\Omega)$, this definition is equivalent to (see Theorem 3.4.1 in [10])

$$L^{p(x)}(\Omega) = \{u \,|\, u \text{ is measurable on } \Omega \text{ and } \rho_p(u) < \infty\}.$$

This is a normed linear space equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \ \rho_p\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Define $p_- = \inf_{x\in\Omega} p(x)$ and $p_+ = \sup_{x\in\Omega} p(x)$. Then

$$L^{p_+}(\Omega) \subset L^{p(x)}(\Omega) \subset L^{p_-}(\Omega).$$

We denote by p_c the conjugate exponent of p defined as

$$p_c(x) = \frac{p(x)}{p(x)-1}.$$

We have the following well-known properties on $L^{p(x)}$ spaces (see [19]).

Proposition 1.1. *Let $p \in L^\infty(\Omega)$. Then for any $u \in L^{p(x)}(\Omega)$ we have:*

(i) $\rho_p(u/\|u\|_{L^{p(x)}(\Omega)}) = 1$.
(ii) $\|u\|_{L^{p(x)}(\Omega)} \to 0$ *if and only if* $\rho_p(u) \to 0$.
(iii) $L^{p_c(x)}(\Omega)$ *is the dual space of* $L^{p(x)}(\Omega)$.

Proposition 1.1 (i) implies that: if $\|u\|_{L^{p(x)}(\Omega)} \ge 1$,

$$\|u\|^{p_-}_{L^{p(x)}(\Omega)} \le \rho_p(u) \le \|u\|^{p_+}_{L^{p(x)}(\Omega)} \tag{1.1}$$

and if $\|u\|_{L^{p(x)}(\Omega)} \le 1$

$$\|u\|^{p_+}_{L^{p(x)}(\Omega)} \le \rho_p(u) \le \|u\|^{p_-}_{L^{p(x)}(\Omega)}. \tag{1.2}$$

Moreover, we have also the generalized Hölder's inequality: for $p \in \mathcal{P}(\Omega)$, there exists a constant $C = C(p_+, p_-) \ge 1$ such that for any $f \in L^{p(x)}(\Omega)$ and $g \in L^{p_c(x)}(\Omega)$

$$\int_\Omega |f(x)g(x)|dx \le C\|f\|_{L^{p(x)}(\Omega)}\|g\|_{L^{p_c(x)}(\Omega)}. \tag{1.3}$$

The corresponding Sobolev space is defined as follows:

$$W^{1,p(x)}(\Omega) \stackrel{\text{def}}{=} \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

A natural norm defined on $W^{1,p(x)}(\Omega)$ is

$$\|u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

For the sake of convenience, we define $\mathbb{W} = W_0^{1,p(x)}(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ for $p \in \mathcal{P}^{log}(\Omega) \cap L^\infty(\Omega)$. Since the domain Ω is a bounded domain, the Poincaré inequality holds and thus we define the norm on $\mathbb{W}$ as $\|u\|_{\mathbb{W}} = \|\nabla u\|_{L^{p(x)}(\Omega)}$.

$L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $\mathbb{W}$ are Banach spaces. Moreover they are separable if $p \in L^\infty(\Omega)$ and reflexive if $1 < p_- \le p_+ < \infty$ (see [19]). Furthermore we have the following Sobolev embedding Theorem (see [10]):

Theorem 1.2. *Let $p \in \mathcal{P}^{log}(\Omega)$ satisfies $1 \le p_- \le p_+ < d$. Then, $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ for any $\alpha \in L^\infty(\Omega)$ such that for all $x \in \Omega$, $\alpha(x) \le p^*(x) = \frac{d\,p(x)}{d-p(x)}$. Also the previous embedding is compact for $\alpha(x) < p^*(x) - \varepsilon$ a.e. in Ω for any $\varepsilon > 0$.*

The paper is organized as follows. The next section (Sect. 2) contains the statements of our main results on the existence, uniqueness, regularity of solutions to (P_T) (see Theorems 2.4, 2.5, 2.8) and on the global behaviour of solutions (see Theorems 2.5 and 2.10). In Sect. 3, we deal with the existence of weak solutions to the auxilary problem (S_T). Main results concerning the existence of weak solutions to (P_T) are established in Sect. 4. Finally the existence of mild solutions and stabilization properties are proved in Sect. 5. The appendices A, B contain some technical lemmata about monotonicity and compactness properties of $p(x)$-laplacian operator. In Appendix C, we establish some new regularity results (L^∞-bound) about quasilinear elliptic equations involving $p(x)$-laplacian used in Sects. 3–5.

2. Main results

In the rest of the paper, we assume that $p \in \mathcal{P}^{log}(\Omega)$ such that

$$\frac{2d}{d+2} < p_- \le p_+ < d.$$

First we consider the following problem:

$$\begin{cases} u_t - \Delta_{p(x)} u = h(t,x) & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad \Sigma_T, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \qquad (S_T)$$

where $T > 0$, $h \in L^2(Q_T) \cap L^q(Q_T)$, $q > \dfrac{d}{p_-}$. Considering the initial data in $u_0 \in \mathbb{W} \cap L^\infty(\Omega)$, we study the weak solution to (S_T) defined as follows:

Definition 2.1. *A weak solution to (S_T) is any function $u \in L^\infty(0,T;\mathbb{W})$ such that $u_t \in L^2(Q_T)$ and satisfying for any $\phi \in C_0^\infty(Q_T)$*

$$\int_0^T\int_\Omega u_t\phi\,dxdt + \int_0^T\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\phi\,dxdt = \int_0^T\int_\Omega h(t,x)\phi\,dxdt$$

and $u(0,.) = u_0$ a.e. in Ω.

Similarly we define a weak solution to the problem (P_T) as follows:

Definition 2.2. *A solution to (P_T) is a function $u \in L^\infty(0,T;\mathbb{W})$ such that $u_t \in L^2(Q_T)$, $f(.,u) \in L^\infty(0,T;L^2(\Omega))$ and for any $\phi \in C_0^\infty(Q_T)$*

$$\int_0^T\int_\Omega u_t\phi\,dxdt + \int_0^T\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\phi\,dxdt = \int_0^T\int_\Omega f(x,u)\phi\,dxdt$$

and $u(0,.) = u_0$ *a.e. in* Ω.

Hence with the above definitions, we establish the following local existence results:

Theorem 2.3. *Let* $T > 0$, $u_0 \in \mathbb{W}\cap L^\infty(\Omega)$ *and* $h \in L^2(Q_T)\cap L^q(Q_T)$, $q > \frac{d}{p_-}$. *Then,* (S_T) *admits a unique solution* u *in the sense of Definition 2.1. Moreover* $u \in C([0,T];\mathbb{W})$.

Theorem 2.4. *Let* $f : \Omega\times\mathbb{R} \to \mathbb{R}$ *be a Caratheodory function satisfying the following two conditions:*

($\mathbf{f_1}$) $t \to f(x,t)$ *is locally Lipschitz uniformly in* $x \in \Omega$*;*
($\mathbf{f_2}$) *there exists* $\alpha \in \mathbb{R}$ *such that* $x \to f(x,\alpha) \in L^2(\Omega)\cap L^q(\Omega)$, $q > \frac{d}{p_-}$.

Assume in addition that one of the following hypotheses holds:

(H1) *there exists a nondecreasing locally Lipschitz function* L_0 *such that*

$$|f(x,v)| \le L_0(v), \quad a.e.\ (x,v) \in \Omega\times\mathbb{R};$$

(H2) *there exist two nondecreasing locally Lipschitz functions* L_1 *and* L_2 *such that*

$$L_1(v) \le f(x,v) \le L_2(v), \quad a.e.\ (x,v) \in \Omega\times\mathbb{R}.$$

Then, for any $u_0 \in \mathbb{W}\cap L^\infty(\Omega)$, *there exists* $\tilde{T} \in (0,+\infty]$ *such that for any* $T \in [0,\tilde{T})$, (P_T) *admits a unique solution* u *in sense of Definition 2.2. Moreover for any* $r > 1$, $u \in C([0,T];L^r(\Omega))\cap C([0,T];\mathbb{W})$.

Under additional hypothesis about the growth of f and regularity of the initial data, we are able to prove the existence of global solutions. Precisely, we have the following result:

Theorem 2.5. *Let* f *be a Caratheodory function satisfying* ($\mathbf{f_1}$) *and the additional condition:*

($\mathbf{f_3}$) *there exists* $C > 0$ *such that* $\forall(x,s) \in \Omega\times\mathbb{R}$, $|f(x,s)| \le C(1+|s|^\beta)$ *where* $\beta < p_- - 1$.

Assume in addition that one of the following conditions is valid:

(C1) $u_0 \in \mathbb{W}$ *such that* $\Delta_{p(x)}u_0 \in L^q(\Omega)$ *where* $q > \frac{d}{p_-}$*;*
(C2) $u_0 \in C_0^1(\overline{\Omega})$ *and* $p \in C^1(\overline{\Omega})$.

Then, for any $T > 0$, (P_T) *admits a unique weak solution in the sense of Definition 2.2. Moreover* $u \in C([0,T];\mathbb{W})$.

Remark 2.6. 1. Theorem 2.5 is still valid, under the condition **(C2)**, replacing ($\mathbf{f_3}$) by the hypotheses on f:
($\mathbf{f_4}$) there exists $\zeta \in \mathbb{R}$ such that $x \to f(x,\zeta) \in L^\infty(\Omega)$;
($\mathbf{f_5}$) $\lim_{|s|\to+\infty} \frac{|f(x,s)|}{|s|^{p_- -1}} = 0$.

2. Under an additional asymptotic super homogeneous growth assumption on f and for initial data large enough, blow up in finite time of solutions can also occur. For instance, let $f(x,v) = v^q$ with $q > p^+$ and define the energy functional

$$E(u) \stackrel{\text{def}}{=} \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{u^{q+1}}{q+1}\,dx.$$

Then, using a well-known energy method and for any initial data u_0 satisfying $E(u_0) < 0$, the weak solution to (P_T) blows up in finite time. For further discussions of global behaviour of solutions (blow up, localization of solutions, extinction of solutions) to quasilinear anisotropic parabolic equations involving variable exponents, we refer to [4,5].

Next, we investigate the asymptotic behaviour of global solutions, in particular the convergence to a stationary solution. For that we appeal the theory of maximal accretive operators in Banach spaces (see Chapters 3 and 4 in [8]) that provides the existence of mild solutions. Precisely, observing that the operator $A \stackrel{\text{def}}{=} -\Delta_{p(x)}$, with Dirichlet boundary conditions, is m-accretive in $L^\infty(\Omega)$ with

$$\mathcal{D}(A) = \{u \in \mathbb{W} \cap L^\infty(\Omega) \,|\, Au \in L^\infty(\Omega)\}$$

as the domain of the operator A, we get the above results which essentially follow from Theorems 2.3 and 2.4 with Theorem 4.2 (page 130) and Theorem 4.4 (page 141) in [8]:

Theorem 2.7. *Let $T > 0$, $h \in L^\infty(Q_T)$ and let u_0 be in $\mathbb{W} \cap \overline{\mathcal{D}(A)}^{L^\infty}$. Then,*

(i) *the unique weak solution u to (S_T) belongs to $\mathcal{C}([0,T];\mathcal{C}_0(\overline{\Omega}))$.*

(ii) *If v is another mild solution to (S_T) with the initial datum $v_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A)}^{L^\infty}$ and the right-hand side $k \in L^\infty(Q_T)$, then the following estimate holds:*

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \|h(s) - k(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s, \quad 0 \le t \le T. \tag{2.1}$$

(iii) *If $u_0 \in \mathcal{D}(A)$ and $h \in W^{1,1}(0,T;L^\infty(\Omega))$ then $u \in W^{1,\infty}(0,T;L^\infty(\Omega))$ and $\Delta_{p(x)}u \in L^\infty(Q_T)$, and the following estimate holds:*

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^\infty(\Omega)} \le \|\Delta_{p(x)}u_0 + h(0)\|_{L^\infty(\Omega)} + \int_0^T \left\|\frac{\partial h}{\partial t}(t)\right\|_{L^\infty(\Omega)}\mathrm{d}\tau. \tag{2.2}$$

Concerning problem (P_T), we deduce the following similar result:

Theorem 2.8. *Assume that conditions and hypotheses on f in Theorem 2.4 are satisfied. Let $u_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A)}^{L^\infty}$. Then, the unique weak solution to (P_T) belongs to $C([0,T];C_0(\overline{\Omega}))$ and*

(i) *there exists $\omega > 0$ such that if v is another weak solution to (P_T) with the initial datum $v_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A)}^{L^\infty}$ then the following estimate holds for $T < \tilde{T}$:*

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le e^{\omega t}\|u_0 - v_0\|_{L^\infty(\Omega)}, \quad 0 \le t \le T.$$

(ii) *If* $u_0 \in \mathcal{D}(A)$ *then* $u \in W^{1,\infty}(0,T;L^\infty(\Omega))$ *and* $\Delta_{p(x)}u \in L^\infty(Q_T)$*, and the following estimate holds:*

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^\infty(\Omega)} \le e^{\omega t}\|\Delta_{p(x)}u_0 + f(x,u_0)\|_{L^\infty(\Omega)}.$$

Remark 2.9.

1. The constant ω in Theorem 2.8 is the Lipschitz constant of f in $[v_1(T), v_2(T)]$ (respectively $[-v_0(T), v_0(T)]$) given in (4.1). If f is non-increasing with respect to the second variable and $x \to f(x,0) \in L^\infty(\Omega)$, $\omega = 0$ can be taken in assertions (i) and (ii) above. In this case, note that $-\Delta_{p(x)} - f(x,\cdot)$ is m-accretive in $L^\infty(\Omega)$ (see Proposition 5.1).
2. If we assume hypotheses in Theorem 2.5, then the weak solution to (P_T) belongs to $C([0,+\infty), C_0(\overline{\Omega}))$.

Using the above results, we give some stabilization properties for (P_T) for global solutions. Precisely, we prove the following:

Theorem 2.10. *Assume that* f *satisfies* $(\mathbf{f_1})$, $(\mathbf{f_4})$ *and is nonincresing in respect to the second variable. Then, for any initial data* $u_0 \in C_0^1(\overline{\Omega})$*, the weak solution,* u*, to* (P_T) *is defined in* $(0,\infty)\times\Omega$*, belongs to* $C([0,+\infty); C_0(\overline{\Omega}))$ *and verifies*

$$u(t) \to u_\infty \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty$$

where u_∞ *is the unique stationary solution to* (P_T)*.*

Remark 2.11. [11] establish uniqueness results for quasilinear elliptic equations involving the $p(x)$-laplacian under different conditions on f (see Theorem 1.2 for instance). Theorem 2.10 is still valid in this case.

3. Existence of solutions of (S_T)

First, we consider the following quasilinear elliptic problem:

$$\begin{cases} u - \lambda\Delta_{p(x)}u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{(P)}$$

with $\lambda > 0$ and g a measurable function. Concerning (P), we have the following result.

Lemma 3.1. *Let* $g \in L^q(\Omega)$, $q > \frac{d}{p_-}$*. Then for any* $\lambda > 0$, (P) *admits a unique weak solution* $u \in \mathbb{W}$ *satisfying*

$$\int_\Omega u\varphi\,dx + \lambda\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi\,dx = \int_\Omega g\varphi\,dx, \quad \forall\varphi \in \mathbb{W}.$$

Furthermore, $u \in L^\infty(\Omega)$*.*

Remark 3.2. Lemma 3.1 still holds under the assumptions $p_- > d$ and $q > 1$.

Proof. Consider the energy functional J_λ associated to (P) given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega u^2\,dx + \lambda\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega gu\,dx.$$

Note that J_λ is well-defined and Gâteaux differentiable on $\mathbb{W}$. Indeed, $q > \frac{d}{p_-}$ and $1 < p_- \le p_+ < d$ imply that $L^q \subset (L^{p^*(x)})'$.

By Theorem 1.2 and (1.1), for $\|u\|_{\mathbb{W}} \ge 1$:

$$J_\lambda(u) = \frac{1}{2}\int_\Omega u^2\,dx + \lambda\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega gu\,dx \ge \frac{\lambda}{p_+}\|u\|_{\mathbb{W}}^{p_-} - C\|u\|_{\mathbb{W}}.$$

Thus J_λ is coercive. Furthermore J_λ is continuous and strictly convex on $\mathbb{W}$ and therefore admits a global minimizer $u \in \mathbb{W}$ which is a weak solution to (3.1). In addition, applying Corollary C.5 in Appendix C, $u \in L^\infty(\Omega)$. □

Proof of Theorem 2.3 Let $N \in \mathbb{N}^*$, $T > 0$ and set $\Delta_t = \frac{T}{N}$. For $0 \le n \le N$, we define $t_n = n\Delta_t$. We perform the proof along five steps.
Step 1. Approximation of h.
For $n \in \{1, \dots, N\}$, we define for $t \in [t_{n-1}, t_n)$ and $x \in \Omega$

$$h_{\Delta_t}(t,x) = h^n(x) \stackrel{\text{def}}{=} \frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(s,x)ds.$$

Then by Jensen's Inequality:

$$\begin{aligned}\|h_{\Delta_t}\|^q_{L^q(Q_T)} &= \Delta_t\sum_{n=1}^N \|h^n\|^q_{L^q} = \Delta_t\sum_{n=1}^N \|\frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(s,x)ds\|^q_{L^q}\\ &\le \sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|h(s,.)\|^q_{L^q}ds \le \|h\|^q_{L^q(Q_T)}.\end{aligned}$$

Thus $h_{\Delta_t} \in L^q(Q_T)$ and $h^n \in L^q(\Omega)$. Also note that $h_{\Delta_t} \to h$ in $L^q(Q_T)$. Indeed let $\varepsilon > 0$, there exists $h^\varepsilon \in C^1_0(Q_T)$ such that

$$\|h - h^\varepsilon\|_{L^q(Q_T)} < \frac{\varepsilon}{3}.$$

Hence

$$\|h_{\Delta_t} - (h^\varepsilon)_{\Delta_t}\|_{L^q(Q_T)} \le \|h - h^\varepsilon\|_{L^q(Q_T)} < \frac{\varepsilon}{3}.$$

Since $\|h^\varepsilon - (h^\varepsilon)_{\Delta_t}\|_{L^q(Q_T)} \to 0$ as $\Delta_t \to 0$, we have for Δ_t small enough

$$\begin{aligned}\|h_{\Delta_t} - h\|_{L^q(Q_T)} \le \|h_{\Delta_t} - h^\varepsilon_{\Delta_t}\|_{L^q(Q_T)} + \|h^\varepsilon - h^\varepsilon_{\Delta_t}\|_{L^q(Q_T)}\\ + \|h - h^\varepsilon\|_{L^q(Q_T)} < \varepsilon.\end{aligned}$$

Hence $h_{\Delta_t} \to h$ in $L^q(Q_T)$.
Step 2. Time-discretization of (S_T).
We define the following iterative scheme $u^0 = u_0$ and for $n \ge 1$,

$$u^n \text{ is solution of } \begin{cases} \dfrac{u^n - u^{n-1}}{\Delta_t} - \Delta_{p(x)}u^n = h^n & \text{in } \Omega,\\ u^n = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.1}$$

Note that the sequence $(u^n)_{n\in\{1,\dots,N\}}$ is well-defined. Indeed, existence and uniqueness of $u^1 \in \mathbb{W}\cap L^\infty(\Omega)$ follows from Lemma 3.1 with $g = \Delta_t h^1 + u^0 \in L^q(\Omega)$. Hence by induction we obtain in the same way the existence of (u^n), for any $n = 2,\dots,N$.

Defining the functions, for $n = 1,\dots,N$ and $t \in [t_{n-1}, t_n)$:

$$u_{\Delta_t}(t) = u^n \quad \text{and} \quad \tilde{u}_{\Delta_t}(t) = \frac{(t-t_{n-1})}{\Delta_t}(u^n - u^{n-1}) + u^{n-1}, \tag{3.2}$$

we get

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_{p(x)} u_{\Delta_t} = h_{\Delta_t} \quad \text{in } Q_T. \tag{3.3}$$

Step 3. A priori estimates for u_{Δ_t} and $\tilde{u}_{\Delta_t}$.

Multiplying the equation in (3.1) by $(u^n - u^{n-1})$ and summing from $n = 1$ to $N' \le N$, we get

$$\sum_{n=1}^{N'} \int_\Omega |\nabla u^n|^{p(x)-2}\nabla u^n.\nabla(u^n - u^{n-1})dx + \sum_{n=1}^{N'} \Delta_t \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta_t}\right)^2 dx = \sum_{n=1}^{N'} \int_\Omega h^n(u^n - u^{n-1})dx, \tag{3.4}$$

hence by Young's inequality and using the convexity of $u \to \displaystyle\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx$ we obtain:

$$\sum_{n=1}^{N'} \int_\Omega \frac{1}{p(x)}\left(|\nabla u^n|^{p(x)} - |\nabla u^{n-1}|^{p(x)}\right)dx + \frac{1}{2}\sum_{n=1}^{N'} \Delta_t \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta_t}\right)^2 dx \le \frac{1}{2}\|h\|^2_{L^2(Q_T)}.$$

Thus we obtain

$$\left(\frac{\partial \tilde{u}_{\Delta t}}{\partial t}\right)_{\Delta t} \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t, \tag{3.5}$$

$$(u_{\Delta_t}) \text{ and } (\tilde{u}_{\Delta_t}) \text{ are bounded in } L^\infty(0,T;\mathbb{W}) \text{ uniformly in } \Delta_t. \tag{3.6}$$

Furthermore, using (3.5) we have

$$\sup_{[0,T]} \|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{L^2(\Omega)} \le \max_{n=1,\dots,N} \|u^n - u^{n-1}\|_{L^2(\Omega)} \le C\Delta_t^{1/2}. \tag{3.7}$$

Therefore for $\Delta_t \to 0^+$, there exist $u, v \in L^\infty(0,T,\mathbb{W})$ such that (up to a subsequence)

$$\tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T,\mathbb{W}), \quad u_{\Delta_t} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T,\mathbb{W}), \tag{3.8}$$

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T). \tag{3.9}$$

It follows from (3.7) that $u \equiv v$. By (3.8), for any $r \ge 1$

$$\tilde{u}_{\Delta_t},\ u_{\Delta_t} \rightharpoonup u \text{ in } L^r(0,T;\mathbb{W}). \tag{3.10}$$

Step 4. u satisfies (S_T).

Plugging (3.5), (3.6) and since the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the Aubin-Simon's compactness result (see [25]) implies that (up to a subsequence),

$$\tilde{u}_{\Delta_t} \to u \in C([0,T]; L^2(\Omega)). \tag{3.11}$$

Now multiplying (3.3) by $(u_{\Delta_t} - u)$ we get

$$\int_0^T \int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t}(u_{\Delta_t} - u)dxdt - \int_0^T \langle \Delta_{p(x)} u_{\Delta_t}, u_{\Delta_t} - u\rangle dt$$
$$= \int_0^T \int_\Omega h_{\Delta_t}(u_{\Delta_t} - u)dxdt.$$

Rearranging the terms in the last equation and using (3.7)–(3.10) we have

$$\int_0^T \int_\Omega \left(\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t} \right)(\tilde{u}_{\Delta_t} - u)dxdt - \int_0^T \langle \Delta_{p(x)} u_{\Delta_t} - \Delta_{p(x)} u, u_{\Delta_t} - u\rangle dt$$
$$= o_{\Delta_t}(1)$$

where $o_{\Delta_t}(1) \to 0$ as $\Delta_t \to 0^+$. Thus we get

$$\frac{1}{2}\int_\Omega |\tilde{u}_{\Delta_t}(T) - u(T)|^2 dx - \int_0^T \langle \Delta_{p(x)} u_{\Delta_t} - \Delta_{p(x)} u, u_{\Delta_t} - u\rangle = o_{\Delta_t}(1).$$

Using (3.11), we obtain

$$\int_0^T \langle \Delta_{p(x)} u_{\Delta_t} - \Delta_{p(x)} u, u_{\Delta_t} - u\rangle dt = o_{\Delta_t}(1)$$

and by Lemma B.1 we conclude that

$$\int_0^T \int_\Omega |\nabla(u_{\Delta_t} - u)|^{p(x)} dxdt \to 0. \tag{3.12}$$

This implies ∇u_{Δ_t} converges to ∇u in $L^{p(x)}(Q_T)$ and u_{Δ_t} converges to u in $\mathbb{W}$. Furthermore

$$|\nabla u_{\Delta_t}|^{p(x)-2}\nabla u_{\Delta_t} \to |\nabla u|^{p(x)-2}\nabla u \text{ in } (L^{p_c(x)}(Q_T))^d. \tag{3.13}$$

Indeed, we write

$$\int_{Q_T} ||\nabla u_{\Delta_t}|^{p(x)-2}\nabla u_{\Delta_t} - |\nabla u|^{p(x)-2}\nabla u|^{\frac{p(x)}{p(x)-1}} dxdt$$
$$= \int_0^T \int_{\Omega \setminus \Omega_2} (\dots)dxdt + \int_0^T \int_{\Omega_2} (\dots)dxdt \tag{3.14}$$

where $\Omega_2 = \{x \in \Omega \,|\, p(x) > 2\}$. We apply inequality (A.1). Then the first term in the right-hand side converges to zero as $\Delta_t \to 0^+$. For the second term, we apply Hölder's inequality (1.3):

$$\int_0^T \int_{\Omega_2} (\dots)dxdt \le \int_0^T \int_{\Omega_2} |\nabla(u_{\Delta_t} - u)|^{\frac{p(x)}{p(x)-1}} (|\nabla u_{\Delta_t}| + |\nabla u|)^{\frac{p(x)(p(x)-2)}{p(x)-1}} dxdt$$
$$\le cXY$$

where

$$X = \| \, |\nabla(u_{\Delta_t} - u)|^{\frac{p(x)}{p(x)-1}} \|_{L^{p(x)-1}(Q_{T,2})},$$

$$Y = \| \, (|\nabla u_{\Delta_t}| + |\nabla u|)^{\frac{p(x)(p(x)-2)}{p(x)-1}} \|_{L^{\frac{p(x)-1}{p(x)-2}}(Q_{T,2})},$$

$Q_{T,2} = (0,T) \times \Omega_2$ and $c \geq 1$ is a constant independent of Δ_t. We define $r = p_{|\Omega_2}$ the restriction of p on Ω_2. Then $r_- = 2$ and $r_+ = p_+$. With the new notations and applying Lemma A.1, we have

$$X \leq \|\nabla(u_{\Delta_t} - u)\|_{L^{p(x)}(Q_T)}^{\frac{2}{p_+-1}} + \|\nabla(u_{\Delta_t} - u)\|_{L^{p(x)}(Q_T)}^{p_+} \tag{3.15}$$

and

$$\begin{aligned} Y &\leq 1 + \| \, |\nabla u_{\Delta_t}| + |\nabla u| \, \|_{L^{r(x)}(Q_{T,2})}^{p_+(p_+-2)} \\ &\leq 1 + \| \, |\nabla u_{\Delta_t}| + |\nabla u| \, \|_{L^{p(x)}(Q_T)}^{p_+(p_+-2)} \\ &\leq 1 + c(p)(\| \, \nabla u_{\Delta_t}\|_{L^{p(x)}(Q_T)}^{p_+(p_+-2)} + \|\nabla u\|_{L^{(x)}(Q_T)}^{p_+(p_+-2)}) \\ &\leq C(p) \end{aligned} \tag{3.16}$$

since (u_{Δ_t}) is bounded in $L^\infty(0,T;\mathbb{W})$ uniformly in Δ_t.
Plugging (3.12), (3.15) and (3.16), we deduce that the second term in the right-hand side of (3.14) converges to 0 as $\Delta_t \to 0^+$. Hence we have (3.13).
Finally, gathering Step 1., (3.9) and (3.13), we conclude passing to the limit, in the distribution sense, in Eq. (3.3) that u is a weak solution of (S_T). Furthermore u is the unique weak solution of (S_T). Indeed assume that there exists v another weak solution of (S_T). Then,

$$\int_0^T\int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)\, dxdt - \int_0^T \langle \Delta_{p(x)}u - \Delta_{p(x)}v, u-v\rangle \, dt = 0.$$

Since $u(0) = v(0)$, the above equality implies that $u \equiv v$.

Step 5. u belongs to $C([0,T];\mathbb{W})$.
Since $u \in C([0,T];L^2(\Omega)) \cap L^\infty([0,T];\mathbb{W})$ and $p \in \mathcal{P}^{\log}(\Omega)$, $u : t \in [0,T] \to \mathbb{W}$ is weakly continuous.
Fix $t_0 \in [0,T]$. Since ρ_p is weakly lower semicontinuous (see Theorem 3.2.9 in [10]) we have

$$\int_\Omega \frac{|\nabla u(t_0)|^{p(x)}}{p(x)} dx \leq \liminf_{t\to t_0} \int_\Omega \frac{|\nabla u(t)|^{p(x)}}{p(x)} dx.$$

From (3.4) with $\sum_{n=N''}^{N'}$ for $1 \leq N'' \leq N'$ and since $|\nabla u_{\Delta_t}|$ converges to $|\nabla u|$ in $L^{p(x)}$, it follows that u satisfies for any $t \in [t_0,T]$:

$$\begin{aligned} \int_{t_0}^t\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 \, dxds + \int_\Omega \frac{|\nabla u(t)|^{p(x)}}{p(x)}\, dx - \int_\Omega \frac{|\nabla u(t_0)|^{p(x)}}{p(x)}\, dx \\ \leq \int_{t_0}^t\int_\Omega h\frac{\partial u}{\partial t}\, dxds. \end{aligned} \tag{3.17}$$

Passing to the limit, we get

$$\limsup_{t\to t_0^+}\int_\Omega \frac{|\nabla u(t)|^{p(x)}}{p(x)}dx \le \int_\Omega \frac{|\nabla u(t_0)|^{p(x)}}{p(x)}dx.$$

Define $v(t)=\nabla u(t)/(p(x))^{1/p(x)}$. Thus we get $\lim_{t\to t_0^+}\rho_p(v(t))=\rho_p(v(t_0))$. Now we prove the left continuity. Let $0<k\le t-t_0$. Multiplying (S_T) by $\tau_k(u)(s)=\frac{u(s+k)-u(s)}{k}$ and integrating over $(t_0,t)\times\Omega$, the convexity gives

$$\begin{aligned}\int_{t_0}^{t}\int_\Omega \tau_k(u)\frac{\partial u}{\partial t}\,dxds &+ \int_{t}^{t+k}\int_\Omega \frac{|\nabla u(s)|^{p(x)}}{kp(x)}\,dxds\\ &- \int_{t_0}^{t_0+k}\int_\Omega \frac{|\nabla u(s)|^{p(x)}}{kp(x)}\,dxds\\ &\ge \int_{t_0}^{t}\int_\Omega \tau_k(u)h\,dxdt. \end{aligned}\tag{3.18}$$

By Dominated Convergence Theorem as $k\to 0^+$:

$$\int_{t}^{t+k}\int_\Omega \frac{|\nabla u(s)|^{p(x)}}{kp(x)}\,dxds \to \int_\Omega \frac{|\nabla u(t)|^{p(x)}}{p(x)}\,dx,$$
$$\int_{t_0}^{t_0+k}\int_\Omega \frac{|\nabla u(s)|^{p(x)}}{kp(x)}\,dxds \to \int_\Omega \frac{|\nabla u(t_0)|^{p(x)}}{p(x)}\,dx.$$

Hence (3.18) yields

$$\begin{aligned}\int_{t_0}^{t}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dxds + \int_\Omega \frac{|\nabla u(t)|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|\nabla u(t_0)|^{p(x)}}{p(x)}\,dx\\ \ge \int_{t_0}^{t}\int_\Omega h\frac{\partial u}{\partial t}\,dxds.\end{aligned}$$

From the above inequality, we deduce that we have the equality in (3.17). This implies, using the Dominated Convergence Theorem, that $\rho_p(v(t))\to\rho_p(v(t_0))$ as $t\to t_0$.
Since $v(t)\rightharpoonup v(t_0)$ in $L^{p(x)}(\Omega)$ and $\rho_p(v(t))\to\rho_p(v(t_0))$ as $t\to t_0$, Lemma B.2 implies the convergence of $v(t)$ to $v(t_0)$ in $L^{p(x)}(\Omega)$. Therefore we deduce that $u\in C([0,T];\mathbb{W})$. □

4. Existence of solution of (P_T)

Proof of Theorem 2.4 We proceed as in the proof of Theorem 2.3 splitting the proof in several steps.
Step 1. Existence of barrier functions.
Consider the equations, for $i\in\{0,1,2\}$

$$\begin{cases}\dfrac{dv_i}{dt}=L_i(v_i),\\ v_i(0)=(-1)^i\kappa,\end{cases}\tag{4.1}$$

where $\kappa=\|u_0\|_\infty$.

By Cauchy–Lipschitz Theorem, there exists $T_i^{max} \in (0, +\infty]$ and a unique maximal solution v_i to (4.1) on $[0, T_i^{max})$.
If **(H1)** holds, we take $T \in (0, T_0^{max})$ otherwise, if **(H2)** holds, we take $T \in (0, \min(T_1^{max}; T_2^{max}))$.
Let $N \in \mathbb{N}^*$. Set $\Delta_t = \frac{T}{N}$ and consider the family (v_i^n) defined by $v_i^n = v_i(t_n) = v_i(n\Delta_t)$ for $n \in \{1, \ldots, N\}$. Hence for any $i \in \{0, 1, 2\}$

$$v_i^{n+1} = v_i^n + \int_{t_n}^{t_{n+1}} L_i(v_i(s))ds, \quad \forall n \in \{0, \ldots, N-1\}.$$

Replacing L_1 (resp. L_2) by $\min(L_1, 0)$ (resp. $\max(L_2, 0)$) in **(H2)**, we can assume that $L_1 \leq 0$ and $L_2 \geq 0$. We get for $n \in \{0, \ldots, N\}$, $v_1(T) \leq v_1^n \leq -\kappa$ and for $i = 0$ or $i = 2$, $\kappa \leq v_i^n \leq v_i(T)$.
Step 2. Semi-discretization in time of (P_T).
Introduce the following iterative scheme (u^n) defined as

$$u^0 = u_0 \quad \text{and} \quad \begin{cases} u^n - \Delta_t \Delta_{p(x)} u^n = u^{n-1} + \Delta_t f(x, u^{n-1}) & \text{in} \quad \Omega, \\ u^n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We just prove the existence of u^1. The conditions $(\mathbf{f_1})$-$(\mathbf{f_2})$ insure that $f(., u^0) \in L^q(\Omega)$ with $q > \frac{d}{p_-}$. Thus Lemma 3.1 applying with $g = u^0 + \Delta_t f(x, u^0) \in L^q(\Omega)$ gives the existence of $u^1 \in \mathbb{W} \cap L^\infty(\Omega)$.
Let u_{Δ_t} and $\tilde{u}_{\Delta_t}$ be defined as in (3.2) and for $t < 0$, $u_{\Delta_t}(t) = u_0$. Thus (3.3) is satisfied with $h_{\Delta_t}(t) \stackrel{\text{def}}{=} f(x, u_{\Delta_t}(t - \Delta_t))$.
Step 3. (u^n) is bounded in $L^\infty(\Omega)$ uniformly in Δ_t. First we consider the case where **(H1)** is valid. We claim that for all n, $|u^n| \leq v_0^n$ in Ω. We just prove for $n = 1$. Since L_0 and v_0 are nondecreasing, we get

$$u^1 - v_0^1 - \Delta_t \Delta_{p(x)} u^1 = \int_0^{\Delta_t} f(x, u_0) - L_0(v_0(s))ds + u_0 - v_0^0 \leq 0.$$

Multiplying the previous inequality by $(u^1 - v_0^1)^+ = \max(u^1 - v_0^1, 0)$ and integrating on $\mathcal{O} = \{x \in \Omega \mid u^1(x) > v_0^1\}$, we get

$$\int_{\mathcal{O}} (u^1 - v_0^1)^2 dx + \Delta_t \int_{\mathcal{O}} |\nabla u^1|^{p(x)} dx \leq 0.$$

Hence, $u^1 \leq v_0^1$ and by the same method we have $-v_0^1 \leq u^1$.
For **(H2)** we claim that for all n, $v_1^n \leq u^n \leq v_2^n$ in Ω. Let $n = 1$. Since L_1, L_2, $-v_1$ and v_2 are nondecreasing:

$$u^1 - v_1^1 - \Delta_t \Delta_{p(x)} u^1 = \int_0^{\Delta_t} f(x, u_0) - L_1(v_1(s))ds + u_0 - v_1^0 \geq 0,$$
$$u^1 - v_2^1 - \Delta_t \Delta_{p(x)} u^1 = \int_0^{\Delta_t} f(x, u_0) - L_2(v_2(s))ds + u_0 - v_2^0 \leq 0.$$

Multiply the first inequality by $(v_1^1 - u^1)^+$ and the second inequality by $(u^1 - v_2^1)^+$. Integrating respectively on $\mathcal{O}_1 = \{x \in \Omega \mid v_1^1 > u^1(x)\}$ and $\mathcal{O}_2 = \{x \in \Omega \mid v_2^1 < u^1(x)\}$, we get

$$-\int_{\mathcal{O}_1}(u^1-v_1^1)^2dx-\Delta_t\int_{\mathcal{O}_1}|\nabla u^1|^{p(x)}dx\geq 0,$$
$$\int_{\mathcal{O}_2}(u^1-v_2^1)^2dx+\Delta_t\int_{\mathcal{O}_2}|\nabla u^1|^{p(x)}dx\leq 0.$$

Then $v_1^1\leq u^1\leq v_2^1$. By induction, we deduce that for $n\in\{0,\dots,N\}$, $v_1^n\leq u^n\leq v_2^n$ in Ω.
Thus we have

$$(u_{\Delta_t}),\ (\tilde{u}_{\Delta_t}) \text{ are bounded in } L^\infty(Q_T) \text{ uniformly in } \Delta_t. \tag{4.2}$$

and

$$(h_{\Delta_t}) \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t$$

Indeed, either **(H1)** holds which implies

$$|f(x,u^n)|\leq L_0(u^n)\leq L_0(v_0(T))$$

or **(H2)** holds, we have

$$|f(x,u^n)|\leq \max(-L_1(u^n),L_2(u^n))\leq \max(-L_1(v_1(T)),L_2(v_2(T))).$$

Hence

$$\|h_{\Delta_t}\|^2_{L^2(Q_T)}=\Delta_t\sum_{n=1}^N\|f(x,u^{n-1})\|^2_{L^2(\Omega)}\leq C.$$

Step 4. End of the proof.
By the same computations of Step 3. of the proof of Theorem 2.3, we obtain estimates and we prove there exists $u\in L^\infty(0,T,\mathbb{W})$ such that

$$\tilde{u}_{\Delta_t},\, u_{\Delta_t}\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T,\mathbb{W}) \ \text{ and } \ \frac{\partial\tilde{u}_{\Delta_t}}{\partial t}\rightharpoonup\frac{\partial u}{\partial t} \text{ in } L^2(Q_T).$$

(3.5) implies that $(\tilde{u}_{\Delta_t})$ is equicontinuous in $C([0,T];L^r(\Omega))$ for $1\leq r\leq 2$. By the interpolation inequality and (4.2) we obtain $(\tilde{u}_{\Delta_t})$ is equicontinuous in $C([0,T];L^r(\Omega))$ for any $r>1$.
By (3.6) and Theorem 1.2, we deduce applying the Ascoli-Arzela Theorem that (up to a subsequence) for any $r>1$

$$\tilde{u}_{\Delta_t}\to u \text{ in } C([0,T];L^r(\Omega)).$$

Since (u_{Δ_t}) is uniformly bounded in $L^\infty(Q_T)$, $(\mathbf{f_1})$ implies

$$\|h_{\Delta_t}(t)-f(.,u(t))\|_{L^2(\Omega)}\leq C\|u_{\Delta_t}(t-\Delta_t)-u(t)\|_{L^2(\Omega)}.$$

Hence we deduce that $h_{\Delta_t}\to f(.,u)$ in $L^\infty(0,T;L^2(\Omega))$. Next we follow Step 4 of Theorem **??** and obtain that u is a weak solution to (P_T).
Now, we prove the uniqueness of the solution to (P_T). Let w be another weak solution of (P_T). By $(\mathbf{f_1})$, for $t\in[0,T]$:

$$\frac{1}{2}\|u(t)-w(t)\|^2_{L^2(\Omega)}-\int_0^t\langle\Delta_{p(x)}u-\Delta_{p(x)}w,u-w\rangle ds$$
$$=\int_0^T\int_\Omega(f(x,u)-f(x,w))(u-w)dxds\leq C\int_0^t\|u(s)-w(s)\|^2_{L^2(\Omega)}ds.$$

Since $u \to \Delta_{p(x)} u$ is a monotone operator from $\mathbb{W}$ to $\mathbb{W}'$, the second term in the left-hand side is nonnegative. Then, by Gronwall's Lemma, we deduce that $u \equiv w$.

Step 5 of the proof of Theorem 2.3 again goes through and completes the proof. □

Now we give the proof of Theorem 2.5.

Proof of Theorem 2.5 First we introduce the stationary quasilinear elliptic problem associated to (P_T):

$$\begin{cases} -\Delta_{p(x)} u = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{E}$$

Thus we claim that if **(C1)** or **(C2)** holds there exist $\underline{u}, \overline{u} \in \mathbb{W} \cap L^\infty(\Omega)$, a sub- and a supersolution of (E) such that $\underline{u} \leq u_0 \leq \overline{u}$.

First, consider that **(C1)** holds. For $(x,s) \in \Omega \times \mathbb{R}$, define

$$G(x,s) = |\Delta_{p(x)} u_0(x)| + |f(x,s)|.$$

Consider the following problems:

$$\begin{cases} -\Delta_{p(x)} \underline{u} = -G(x, \underline{u}) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta_{p(x)} \overline{u} = G(x, \overline{u}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The existence of $\underline{u}$ and $\overline{u} \in \mathbb{W}$ follows from the sub-homogeneity of f given by $(\mathbf{f_3})$ (see Theorem 4.3 in [14]) and by Corollary C.5 we have $\underline{u},\ \overline{u} \in L^\infty(\Omega)$. Moreover

$$-\Delta_{p(x)} \underline{u} = -G(.,\underline{u}) \leq -\Delta_{p(x)} u_0 \qquad \text{a.e in} \ \ \Omega.$$

and

$$-\Delta_{p(x)} \overline{u} = G(.,\overline{u}) \geq -\Delta_{p(x)} u_0 \qquad \text{a.e in} \ \ \Omega.$$

Hence Lemma A.4 implies $\underline{u} \leq u_0$ and $\underline{u}$ is a subsolution of (E). Similarly we have that $\overline{u} \geq u_0$ and $\overline{u}$ is a supersolution of (E).

Now, if **(C2)** holds. We have the following lemma which follows from [13,27]:

Lemma 4.1. *Let $p \in C^1(\overline{\Omega})$ and $\lambda \in \mathbb{R}^+$. Let $w_\lambda \in \mathbb{W} \cap L^\infty(\Omega)$ be the unique solution of*

$$\begin{cases} -\Delta_{p(x)} w_\lambda = \lambda & in \quad \Omega, \\ w_\lambda = 0 & on \quad \partial\Omega. \end{cases} \tag{4.3}$$

Then, there exists two constants C_1 and C_2 which do not depend to λ such that

$$\|w_\lambda\|_{L^\infty} \leq C_1 \lambda^{\frac{1}{p_- - 1}} \quad \textit{and} \quad w_\lambda(x) \geq C_2 \lambda^{\frac{1}{p_+ - 1 + \mu}} \, dist(x, \partial\Omega)$$

where $\mu \in (0,1)$.

Fix $\lambda > 0$, let w_λ be the solution of (4.3). Since $\beta < p_- - 1$ and by Lemma 4.1: for λ large enough, w_λ verifies

$$- \Delta_{p(x)} w_\lambda = \lambda \geq C \left(1 + C_1^\beta \lambda^{\frac{\beta}{p_- - 1}}\right) \geq C(1 + w_\lambda^\beta) \geq |f(x, w_\lambda)|. \tag{4.4}$$

Moreover, since $u_0 \in C_0^1(\overline{\Omega})$, there exists $K > 0$ such that for any $x \in \Omega$, $|u_0(x)| \leq K\text{dist}(x, \partial\Omega)$. Hence choosing λ large enough, we have by Lemma 4.1 $w_\lambda \geq |u_0|$ in $\overline{\Omega}$.
Set $\overline{u} = w_\lambda$ and $\underline{u} = -w_\lambda$. We deduce for λ large enough, $\overline{u}$ and $\underline{u}$ are a super- and a subsolution of (E) such that $\underline{u} \leq u_0 \leq \overline{u}$.
Now we proceed as the proof of Theorem 2.4. We define the sequence (u^n) as follows.

$$\begin{cases} u^n - \Delta_t \Delta_{p(x)} u^n = u^{n-1} + \Delta_t f(x, u^{n-1}) & \text{in} \quad \Omega, \\ u^n = 0 & \text{on} \quad \partial\Omega \end{cases}$$

for $n = 1, 2, \ldots, N$ with $u^0 = u_0$. we prove for $n \geq 1$, $\underline{u} \leq u^n \leq \overline{u}$ in Ω. Indeed for $n = 1$, we have

$$\underline{u} - u^1 - \Delta_t(\Delta_{p(x)}\underline{u} - \Delta_{p(x)}u^1) \leq \underline{u} - u^0 + \Delta_t(f(x, u^0) - f(x, \underline{u})).$$

Let Λ be the Lipschitz constant of f on $[-M, M]$, where M is the maximum of $\|\underline{u}\|_{L^\infty}$ and $\|\overline{u}\|_{L^\infty}$. Then

$$\underline{u} - u^1 - \Delta_t(\Delta_{p(x)}\underline{u} - \Delta_{p(x)}u^1) \leq (Id - \Delta_t f)(\underline{u} - u^0).$$

For Δ_t small enough, the function $Id - \Delta_t f$ is nondecreasing. Then the right-hand side of the above inequality is nonpositive and thus by Lemma A.4 we have $\underline{u} \leq u^1$. Similarly we prove $u^1 \leq \overline{u}$.

By induction, for $n \geq 1$, $\underline{u} \leq u^n \leq \overline{u}$ in Ω. Thus (u^n) is uniformly bounded in $L^\infty(\Omega)$. The rest of the proof follows Step 3 and 4 of the proof of Theorem 2.4. □

5. Existence of mild solutions and stabilization

In this section we prove Theorems 2.7, 2.8 and 2.10. We first show the m-accretivity of $A = -\Delta_{p(x)}$:

Proposition 5.1. *Let f be locally Lipschitz and nonincreasing in respect to the second variable. Assume further that f sastifies* ($\mathbf{f_4}$). *Then, A_f defined by $A_f(u) \stackrel{\text{def}}{=} -\Delta_{p(x)}u - f(., u)$, is m-accretive in $L^\infty(\Omega)$.*

Proof. First, let $h \in L^\infty(\Omega)$ and $\lambda > 0$. Then,

$$\begin{cases} u + \lambda A_f(u) = h & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega \end{cases}$$

admits a unique solution, $u \in \mathbb{W} \cap L^\infty(\Omega)$. Indeed, for $\mu > 0$ large enough w_μ and $-w_\mu$ defined in (4.3) in Lemma 4.1 are respectively supersolution and subsolution to the above equation and then from the weak comparison principle, $u \in [-w_\mu, w_\mu]$ and u is obtained by a minimization argument and a truncation argument. The uniqueness of the solution follows from the strict convexity of the associated energy functional. Next we prove the accretivity of A_f. Let h and $g \in L^\infty(\Omega)$ and set u and v the unique solutions to

$$\begin{aligned} u + \lambda A_f u = h \quad \text{in } \Omega, \\ v + \lambda A_f v = g \quad \text{in } \Omega. \end{aligned}$$

Substracting the two above equations and using the test function $w = (u - v - \|h-g\|_{L^\infty(\Omega)})^+$, we get $u - v \leq \|h-g\|_{L^\infty(\Omega)}$ and reversing the roles of u and v, we get that $\|u-v\|_{L^\infty(\Omega)} \leq \|h-g\|_{L^\infty(\Omega)}$. This proves the proposition. □

Next, we prove Theorem 2.7.

Proof of Theorem 2.7 We follow the approach in the proof of Theorems 4.2 and 4.4 in [8]. Let u_0, v_0 be in $\overline{\mathcal{D}(A)}^{L^\infty(\Omega)}$. For $z \in \mathcal{D}(A)$ and r, k in $L^\infty(Q_T)$, set

$$\begin{aligned} \varphi(t,s) &= \|r(t) - k(s)\|_{L^\infty(\Omega)} \quad \forall\, (t,s) \in [0,T] \times [0,T]; \\ b(t,r,k) &= \|u_0 - z\|_{L^\infty(\Omega)} + \|v_0 - z\|_{L^\infty(\Omega)} + |t| \|Az\|_{L^\infty(\Omega)} \\ &\quad + \int_0^{t^+} \|r(\tau)\|_{L^\infty(\Omega)} \mathrm{d}\tau + \int_0^{t^-} \|k(\tau)\|_{L^\infty(\Omega)} \mathrm{d}\tau,\ t \in [-T,T], \end{aligned}$$

and

$$\Psi(t,s) = b(t-s,r,k) + \begin{cases} \displaystyle\int_0^s \varphi(t-s+\tau,\tau)\mathrm{d}\tau & \text{if} \quad 0 \leq s \leq t \leq T, \\ \displaystyle\int_0^t \varphi(\tau, s-t+\tau)\mathrm{d}\tau & \text{if} \quad 0 \leq t \leq s \leq T, \end{cases}$$

the solution of

$$\begin{cases} \dfrac{\partial \Psi}{\partial t}(t,s) + \dfrac{\partial \Psi}{\partial s}(t,s) = \varphi(t,s) & (t,s) \in [0,T] \times [0,T], \\ \Psi(t,0) = b(t,r,k) & t \in [0,T], \\ \Psi(0,s) = b(-s,r,k) & s \in [0,T]. \end{cases} \tag{5.1}$$

Moreover, let denote by (u_ϵ^n) the solution of (3.1) with $\Delta_t = \epsilon$, $h = r$, $r^n = \frac{1}{\epsilon}\int_{(n-1)\epsilon}^{n\epsilon} r(\tau,\cdot)d\tau$ and (u_η^n) the solution of (3.1) with $\Delta_t = \eta$, $h = k$, $k^n = \frac{1}{\eta}\int_{(n-1)\eta}^{n\eta} k(\tau,\cdot)d\tau$ respectively. For $(n,m) \in \mathbb{N}^*$ elementary calculations lead to

$$\begin{aligned} u_\epsilon^n - u_\eta^m + \frac{\epsilon\eta}{\epsilon+\eta}(Au_\epsilon^n - Au_\eta^m) &= \frac{\eta}{\epsilon+\eta}(u_\epsilon^{n-1} - u_\eta^m) \\ &\quad + \frac{\epsilon}{\epsilon+\eta}(u_\epsilon^n - u_\eta^{m-1}) + \frac{\epsilon\eta}{\epsilon+\eta}(r^n - k^m), \end{aligned}$$

and since A is m-accretive in $L^\infty(\Omega)$ we first verify that $\Phi_{n,m}^{\epsilon,\eta} = \|u_\epsilon^n - u_\eta^m\|_{L^\infty(\Omega)}$ obeys

$$\begin{aligned} &\Phi_{n,m}^{\epsilon,\eta} \leq \frac{\eta}{\epsilon+\eta}\Phi_{n-1,m}^{\epsilon,\eta} + \frac{\epsilon}{\epsilon+\eta}\Phi_{n,m-1}^{\epsilon,\eta} + \frac{\epsilon\eta}{\epsilon+\eta}\|r^n - k^m\|_\infty, \\ &\Phi_{n,0}^{\epsilon,\eta} \leq b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Phi_{0,m}^{\epsilon,\eta} \leq b(-s_m, r_\epsilon, k_\eta), \end{aligned}$$

and thus, with an easy inductive argument, that $\Phi^{\epsilon,\eta}_{n,m} \le \Psi^{\epsilon,\eta}_{n,m}$ where $\Psi^{\epsilon,\eta}_{n,m}$ satisfies

$$\Psi^{\epsilon,\eta}_{n,m} = \frac{\eta}{\epsilon+\eta}\Psi^{\epsilon,\eta}_{n-1,m} + \frac{\epsilon}{\epsilon+\eta}\Psi^{\epsilon,\eta}_{n,m-1} + \frac{\epsilon\eta}{\epsilon+\eta}\|h^n_\epsilon - h^m_\eta\|_\infty,$$
$$\Psi^{\epsilon,\eta}_{n,0} = b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Psi^{\epsilon,\eta}_{0,m} = b(-s_m, r_\epsilon, k_\eta).$$

For $(t,s) \in (t_{n-1}, t_n) \times (s_{m-1}, s_m)$, set

$$\varphi^{\epsilon,\eta}(t,s) = \|r_\epsilon(t) - k_\eta(s)\|_\infty,$$
$$\Psi^{\epsilon,\eta}(t,s) = \Psi^{\epsilon,\eta}_{n,m},$$
$$b_{\epsilon,\eta}(t,r,k) = b(t_n, r_\epsilon, k_\eta)$$

and

$$b_{\epsilon,\eta}(-s,r,k) = b(-s_m, r_\epsilon, k_\eta).$$

Then $\Psi^{\epsilon,\eta}$ satisfies the following discrete version of (5.1):

$$\frac{\Psi^{\epsilon,\eta}(t,s) - \Psi^{\epsilon,\eta}(t-\epsilon,s)}{\epsilon} + \frac{\Psi^{\epsilon,\eta}(t,s) - \Psi^{\epsilon,\eta}(t,s-\eta)}{\eta} = \varphi^{\epsilon,\eta}(t,s),$$
$$\Psi^{\epsilon,\eta}(t,0) = b_{\epsilon,\eta}(t,r,k) \quad \text{and} \quad \Psi^{\epsilon,\eta}(0,s) = b_{\epsilon,\eta}(s,r,k),$$

and from $b_{\epsilon,\eta}(\cdot,r,k) \to b(\cdot,r,k)$ in $L^\infty([0,T])$. Furthermore,

$$\sum_{n=1}^{N_n}\int_{t_{n-1}}^{t_n} \|r(s) - r_n\|_\infty \mathrm{d}s \to 0, \qquad \text{as } \epsilon \to 0^+,$$
$$\sum_{m=1}^{N_m}\int_{s_{m-1}}^{s_m} \|k(s) - k_m\|_\infty \mathrm{d}\tau \to 0 \qquad \text{as } \eta \to 0^+.$$

The above statements follow easily from the fact that $r, k \in L^1(0,T;L^\infty(\Omega))$ and a density argument.
Then, we deduce that $\rho_{\epsilon,\eta} = \|\Psi^{\epsilon,\eta} - \Psi\|_{L^\infty([0,T]\times[0,T])} \to 0$ as $(\epsilon,\eta) \to 0$ (see for instance [8, Chap.4,Lemma 4.3,p. 136] and [8, Chap.4, proof of Theorem 4.1, p. 138]). Then from

$$\|u_\epsilon(t) - u_\eta(s)\|_\infty = \Phi^{\epsilon,\eta}(t,s) \le \Psi^{\epsilon,\eta}(t,s) \le \Psi(t,s) + \rho_{\epsilon,\eta}, \tag{5.2}$$

we obtain with $t = s$, $r = k = h$, $v_0 = u_0$:

$$\|u_\epsilon(t) - u_\eta(t)\|_{L^\infty(\Omega)} \le 2\|u_0 - z\|_{L^\infty(\Omega)} + \rho_{\epsilon,\eta},$$

and since z can be chosen in $\mathcal{D}(A)$ arbitrary close to u_0, we deduce that u_ϵ is a Cauchy sequence in $L^\infty(Q_T)$ and then that $u_\epsilon \to u$ in $L^\infty(Q_T)$. Thus, passing to the limit in (5.2) with $r = k = h$, $v_0 = u_0$ we obtain

$$\begin{aligned}\|u(t) - u(s)\|_{L^\infty(\Omega)} \le &\int_0^{\max(t,s)} \|h(|t-s| + \tau) - h(\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau \\ &+ 2\|u_0 - z\|_{L^\infty(\Omega)} + \int_0^{|t-s|} \|h(\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau \\ &+ |t-s|\|Az\|_{L^\infty(\Omega)},\end{aligned}$$

which, together with the density $\mathcal{D}(A)$ in $L^\infty(\Omega)$ and $h \in L^1(0,T;L^\infty(\Omega))$, yields $u \in C([0,T];L^\infty(\Omega))$.
Analogously, from (5.2) with $\varepsilon = \eta = \Delta_t$, $r = k = h$, $v_0 = u_0$ and $t = s + \Delta_t$ we deduce that

$$\begin{aligned}\|u_{\Delta_t}(t) - \tilde{u}_{\Delta_t}(t)\|_{L^\infty(\Omega)} &\leq 2\|u_{\Delta_t}(t) - u_{\Delta_t}(t-\Delta_t)\|_{L^\infty(\Omega)} \\ &\leq 4\|u_0 - z\|_{L^\infty(\Omega)} + 2\int_0^t \|h(\Delta_t + \tau) - h(\tau)\|_\infty \mathrm{d}\tau \\ &\quad + 2\int_0^{\Delta_t} \|h(\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau + 2\Delta_t\|Az\|_{L^\infty(\Omega)}\end{aligned}$$

which gives the limit $\tilde{u}_{\Delta_t} \to u$ in $C([0,T];L^\infty(\Omega))$ as $\Delta_t \to 0^+$. Note that since $\tilde{u}_{\Delta_t} \in C([0,T];C_0(\overline{\Omega}))$, the uniform limit u belongs to $C([0,T];C_0(\overline{\Omega}))$. Moreover, passing to the limit in (5.2) with $t = s$ we obtain

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \leq \|u_0 - z\|_{L^\infty(\Omega)} + \|v_0 - z\|_{L^\infty(\Omega)} + \int_0^t \|r(\tau) - k(\tau)\|_\infty \mathrm{d}\tau,$$

and (2.1) follows since we can choose z arbitrary close to v_0. Finally, if $Au_0 \in L^\infty(\Omega)$ and $h \in W^{1,1}(0,T;L^\infty(\Omega))$ and if we assume (without loss of generality) that $t > s$ then with $z = v_0 = u(t-s)$ and $(r,k) = (h, h(\cdot + t - s))$ in the last above inequality we obtain

$$\begin{aligned}\|u(t) - u(s)\|_{L^\infty(\Omega)} &\leq \|u_0 - u(t-s)\|_{L^\infty(\Omega)} \\ &\quad + \int_0^s \|h(\tau) - h(\tau + t - s)\|_{L^\infty(\Omega)}\mathrm{d}\tau. \end{aligned} \tag{5.3}$$

From (2.1) with $v = u_0$, $k = Au_0$:

$$\|u_0 - u(t-s)\|_{L^\infty(\Omega)} \leq \int_0^{t-s} \|Au_0 - h(\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau. \tag{5.4}$$

Using (5.4) and gathering Fubini's Theorem and

$$h(\tau) - h(\tau + t - s) = \int_\tau^{\tau+t-s} \frac{\mathrm{d}h}{\mathrm{d}t}(\sigma)\mathrm{d}\sigma,$$

the right-hand side of (5.3) is smaller than

$$\begin{aligned}(t-s)\|Au_0 - h(0)\|_{L^\infty(\Omega)} &+ \int_0^{t-s} \|h(0) - h(\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau \\ &+ \int_0^s \|h(\tau) - h(\tau + t - s)\|_{L^\infty(\Omega)}\mathrm{d}\tau.\end{aligned}$$

Thus

$$\begin{aligned}\|u(t) - u(s)\|_{L^\infty(\Omega)} &\leq (t-s)\|Au_0 - h(0)\|_{L^\infty(\Omega)} \\ &\quad + (t-s)\int_0^T \left\|\frac{\mathrm{d}h}{\mathrm{d}t}(\tau)\right\|_{L^\infty(\Omega)} \mathrm{d}\tau. \end{aligned} \tag{5.5}$$

Dividing the expression (5.5) by $|t-s|$, we get that u is a Lipschitz function and since $\frac{\partial u}{\partial t} \in L^2(Q_T)$, passing to the limit $|t-s| \to 0$ we obtain that $\frac{u(t)-u(s)}{t-s} \to \frac{\partial u}{\partial t}$ as $s \to t$ weakly in $L^2(Q_T)$ and *-weakly in $L^\infty(Q_T)$. Furthermore,

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(\Omega)} \leq \liminf_{s\to t} \frac{\|u(t)-u(s)\|_{L^\infty(\Omega)}}{|t-s|}.$$

Therefore, we get $u \in W^{1,\infty}(0,T;L^\infty(\Omega))$ as well as inequality (2.2). □

The proof of Theorem 2.8 follows easily:

Proof of Theorem 2.8 The existence of mild solutions can be obtained similarly as in the proof of Theorem 2.7 taking into account the L^∞-bound given by the barrier functions v_0, v_1 and v_2. *(i)* is the consequence of (2.1) together with the fact that f is locally Lipschitz and the Gronwall's Lemma.

Regarding assertion *(ii)*, we follow the proof of Proposition 2.7: assume without loss of generality that $t > s$. Then,

$$\begin{aligned}\|u(t)-u(s)\|_{L^\infty(\Omega)} &\leq \|u_0-u(t-s)\|_{L^\infty(\Omega)} \\ &\quad + \int_0^s \|f(x,u(\tau))-f(x,u(\tau+t-s))\|_{L^\infty(\Omega)}\mathrm{d}\tau.\end{aligned}$$

From assertion (i) and the fact that f is Lipschitz on $[v_1(T),v_2(T)]$, it follows that

$$\begin{aligned}\|u(t)-u(s)\|_{L^\infty(\Omega)} &\leq \|u_0-u(t-s)\|_{L^\infty(\Omega)} + \omega\int_0^s e^{\omega\tau}\mathrm{d}\tau\|u_0-u(t-s)\|_{L^\infty(\Omega)} \\ &\leq e^{\omega s}\|u_0-u(t-s)\|_{L^\infty(\Omega)}.\end{aligned}$$

Now, we estimate the term $\|u_0-u(t-s)\|_{L^\infty(\Omega)}$ in the following way:

$$\begin{aligned}\|u_0-u(t-s)\|_{L^\infty(\Omega)} &\leq \int_0^{t-s} \|Au_0-f(x,u(\tau))\|_{L^\infty(\Omega)}\mathrm{d}\tau \\ &\leq (t-s)\|Au_0-f(x,u_0)\|_{L^\infty(\Omega)} \\ &\quad + \omega\int_0^{t-s}\|u_0-u(\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau.\end{aligned}$$

From Gronwall's lemma, we deduce that

$$\|u_0-u(t-s)\|_{L^\infty(\Omega)} \leq (t-s)e^{\omega(t-s)}\|Au_0-f(x,u_0)\|_{L^\infty(\Omega)}.$$

Gathering the above estimates, we get

$$\|u(t)-u(s)\|_{L^\infty(\Omega)} \leq (t-s)e^{\omega t}\|Au_0-f(x,u_0)\|_{L^\infty(\Omega)}.$$

Then, the rest of the proof follows with the same arguments as in the proof of Theorem 2.7. □

We are ready now to prove our stabilization result:

Proof of Theorem 2.10 From Proposition 5.1, $A_f(u) = -\Delta_{p(x)}u - f(x,u)$ is m-accretive in $L^\infty(\Omega)$ and according to the Remark 2.9, Theorem 2.8 holds with $u_0 \in C_0^1(\overline{\Omega})$ replacing A by A_f, the barrier functions v_0, v_1 and v_2 by the subsolution $-w_\mu$ and the supersolution w_μ respectively and for $\mu > 0$ large. Furthermore, $\omega = 0$ in assertion (i) of Theorem 2.8 and the solution is global. Now, from the comparison principle, the solution $u(t)$ emanating from u_0 belongs to the conical shell $[u_1(t), u_2(t)]$ where u_1 and u_2 are the mild solutions with initial data $-w_\mu$ and $w_\mu \in \mathcal{D}(A_f)$, respectively. Again from the weak comparision principle, $t \to u_1(t)$ and $t \to u_2(t)$ are nondecreasing and nonincreasing respectively. Furthermore, from the uniqueness of the mild solution, u_1 and u_2 converge in $L^\infty(\Omega)$ to the stationary solution, u_∞, to (P_T) which is unique from the monotonicity of A_f. Then, $u(t) \to u_\infty$ as $t \to \infty$. $\square$

Remark 5.2. If one assumes in addition that $f(x, 0) \geq 0$ for all $x \in \Omega$, then it is easy to prove that u_∞ is positive and if u_0 is nonegative, $u(t)$ is nonnegative for all $t \geq 0$.

Acknowledgements

We wish to thank Guy Vallet for his time and his advices. Our discussions about the problem always were very helpful and beneficial. The authors were funded by IFCAM (Indo-French Centre for Applied Mathematics) under the project "Singular phenomena in reaction diffusion equations and in conservation laws".

Appendix A. Algebraic tools

We recall suitable inequalities due to Simon [26]: for all $u,\ v \in \mathbb{R}^d$

$$\big||u|^{p-2}u - |v|^{p-2}v\big| \leq \begin{cases} c|u-v|(|u|+|v|)^{p-2} & \text{if } p \geq 2; \\ c|u-v|^{p-1} & \text{if } p \leq 2; \end{cases} \tag{A.1}$$

$$< |u|^{p-2}u - |v|^{p-2}v, u-v > \geq \begin{cases} \tilde{c}|u-v|^p & \text{if } p \geq 2; \\ \tilde{c}\dfrac{|u-v|^2}{(|u|+|v|)^{2-p}} & \text{if } p \leq 2 \end{cases} \tag{A.2}$$

where $c,\ \tilde{c}$ are positive constants and $< .,. >$ is the scalar product of $\mathbb{R}^d$.

Lemma A.1. *Let $p \in L^\infty(\Omega)$ such that $p \geq 0$, $p \not\equiv 0$. Let $q \in \mathcal{P}(\Omega)$ such that $p(x)q(x) \geq 1$ a.e. on Ω. Then for every $f \in L^{p(x)q(x)}(\Omega)$,*

$$\|f^{p(x)}\|_{L^{q(x)}(\Omega)} \leq \|f\|^{p_-}_{L^{p(x)q(x)}(\Omega)} + \|f\|^{p_+}_{L^{p(x)q(x)}(\Omega)}. \tag{A.3}$$

Proof. To simplify the notations we set $\alpha(.) = p(.)q(.)$. Let $f \in L^{\alpha(x)}(\Omega)$, we define $\beta = p_-$ if $\|f\|_{L^{\alpha(x)}(\Omega)} \leq 1$ and $\beta = p_+$ if $\|f\|_{L^{\alpha(x)}(\Omega)} > 1$. Then, for $\lambda = \|f\|_{L^{\alpha(x)}(\Omega)}$

$$\rho_q\left(\frac{f^{p(x)}}{\lambda^\beta}\right) = \int_\Omega \left|\frac{f}{\|f\|_{L^{\alpha(x)}(\Omega)}}\right|^{\alpha(x)} \cdot \frac{1}{\lambda^{\beta q(x) - \alpha(x)}} dx \leq 1.$$

Hence by the definition of the norm of $L^{\alpha(x)}(\Omega)$, we obtain the estimate. $\square$

Now we prove a technical inequality in the case $p_+ \leq 2$.

Lemma A.2. *Let $p \in C(\overline{\Omega})$, $1 < p_- \leq p_+ \leq 2$. Then, there exists $C > 0$ such that for any u, $v \in \mathbb{W}$*

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)dx \tag{A.4}$$

$$\geq C\left(\frac{\int_\Omega |\nabla(u-v)|^{p(x)}dx}{\|(|\nabla u| + |\nabla v|)^{\alpha(x)}\|_{L^{\frac{2}{2-p(x)}}(\Omega)}}\right)^\gamma, \tag{A.5}$$

where $\alpha(x) = \frac{p(x)(2-p(x))}{2}$ and $\gamma \in \{\frac{2}{p_+}; \frac{2}{p_-}\}$.

Proof. First, Hölder's inequality implies

$$\int_\Omega |\nabla(u-v)|^{p(x)}dx \left(\|(|\nabla u| + |\nabla v|)^{\alpha(x)}\|_{L^{\frac{2}{2-p(x)}}(\Omega)}\right)^{-1} \tag{A.6}$$

$$\leq C\left\|\frac{|\nabla(u-v)|^{p(x)}}{(|\nabla u| + |\nabla v|)^{\alpha(x)}}\right\|_{L^{\frac{2}{p(x)}}(\Omega)} \stackrel{\text{def}}{=} C\mathcal{I}. \tag{A.7}$$

On the other hand, since $p_+ \leq 2$ we have using (A.2):

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)dx \geq C\int_\Omega \frac{|\nabla(u-v)|^2}{(|\nabla u| + |\nabla v|)^{2-p(x)}}dx.$$

In the case where $\mathcal{I} < 1$, plugging the last inequality (A.6) and (1.2) we obtain (A.4) with $\gamma = \frac{2}{p_-}$. In the other case: $\mathcal{I} \geq 1$ then by (1.1), we get inequality (A.4) with $\gamma = \frac{2}{p_+}$. □

Remark A.3. In the case $p_- \geq 2$, using inequality (A.2) , on can be easily prove that there exists $\tilde{C} > 0$ such that:

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)dx \geq \tilde{C}\int_\Omega |\nabla(u-v)|^{p(x)}dx.$$

We have the following comparison principle:

Lemma A.4. *Let $u, v \in \mathbb{W}$ such that $-\Delta_{p(x)}u \geq -\Delta_{p(x)}v$ in Ω in the sense that*

$$\forall\varphi \in \mathbb{W},\ \varphi \geq 0, \quad \int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot\nabla\varphi\, dx \geq 0.$$

Then $u \geq v$ a.e. in Ω.

Proof. Set $\varphi = \max(v-u, 0) \in \mathbb{W}$. Then

$$\int_{\{v>u\}} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot\nabla\varphi\, dx \geq 0$$

By Lemma A.2 and Remark A.3 we deduce that

$$\int_{\{v>u\}} |\nabla\varphi|^{p(x)}dx \leq 0.$$

Hence $\varphi = 0$ a.e. in Ω. Therefore $u \geq v$ a.e. in Ω. □

Appendix B. Convergence tools

Lemma B.1. *Let (u_n) be a bounded sequence in $L^\infty(0,T,\mathbb{W})$ uniformly in n. Let $u \in L^\infty(0,T,\mathbb{W})$ such that*

$$\lim_{n\to+\infty}\int_0^T\int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u).\nabla(u_n-u)dxdt = 0.$$

Then, ∇u_n converges to ∇u in $L^{p(x)}(Q_T)$.

Proof. Define $\nabla X = |\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u$. Thus we have,

$$-\int_0^T \langle \Delta_{p(x)}u_n - \Delta_{p(x)}u, u_n-u\rangle dt = \int_0^T\int_{\Omega\setminus\Omega_2} \nabla X.\nabla(u_n-u)dxdt + \int_0^T\int_{\Omega_2} \nabla X.\nabla(u_n-u)dxdt$$

where $\Omega_2 = \{p(x) > 2\}$. By Lemma A.2, with $\alpha(x) = \frac{p(x)(2-p(x))}{2}$, and Remark A.3, we get as $n \to +\infty$

$$\int_0^T\int_{\Omega_2} |\nabla(u_n-u)|^{p(x)}dxdt \to 0$$

and

$$\int_0^T \frac{\int_{\Omega\setminus\Omega_2}|\nabla(u_n-u)|^{p(x)}dx}{\|(|\nabla u_n|+|\nabla u|)^{\alpha(x)}\|_{L^{\frac{2}{2-p(x)}}(\Omega\setminus\Omega_2)}}dt \to 0.$$

Applying Lemma A.1, we prove that the mapping

$$t \to \|(|\nabla u_n|+|\nabla u|)^{\alpha(x)}\|_{L^{\frac{2}{2-p(x)}}(\Omega\setminus\Omega_2)}$$

is bounded on $[0,T]$. Hence we obtain

$$\int_0^T\int_{\Omega\setminus\Omega_2} |\nabla(u_n-u)|^{p(x)}dxdt \to 0.$$

We conclude by Proposition 1.1 (ii) that ∇u_n converges to ∇u in $L^{p(x)}(Q_T)$. □

Finally we recall the following Corollary A.3 in [20].

Lemma B.2. *Let Ω be a smooth bounded domain. Consider (u_n), $u \in L^{p(x)}(\Omega)$ such that u_n converges to u weakly in $L^{p(x)}(\Omega)$. Then, $\rho_p(u_n)$ converges to $\rho_p(u)$ implies that u_n converges to u in $L^{p(x)}(\Omega)$.*

Appendix C. Regularity result

We begin by recalling the regularity result due to Fan and Zhao [15]:

Proposition C.1 (Theorem 4.1 in [15]). *Let* $p \in C(\overline{\Omega})$ *and* $u \in \mathbb{W}$ *satisfying*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\Psi dx = \int_\Omega f(x,u)\Psi dx, \quad \forall \Psi \in \mathbb{W},$$

where f *satisfies for all* $(x,t) \in \Omega\times\mathbb{R}$, $|f(x,t)| \leq c_1 + c_2|t|^{r(x)-1}$ *with* $r \in C(\overline{\Omega})$ *and* $\forall x \in \overline{\Omega}$, $1 < r(x) < p^*(x)$. *Then* $u \in L^\infty(\Omega)$.

For $f(x,.) = f(x)$, we have the following proposition.

Proposition C.2. *Let* $p \in C(\bar{\Omega})$ *with* $p^- < d$ *and* $u \in \mathbb{W}$ *satisfying*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\Psi dx = \int_\Omega f\Psi dx, \quad \forall \Psi \in \mathbb{W}, \tag{C.1}$$

where $f \in L^q(\Omega)$, $q > \frac{d}{p_-}$. *Then* $u \in L^\infty(\Omega)$.

To prove Proposition C.2, we need a regularity lemma.

Lemma C.3. *Let* $u \in W_0^{1,p}(\Omega)$, $1 < p < d$, *satisfying for any* B_R, $R < R_0$, *and for all* $\sigma \in (0,1)$, *and any* $k \geq k_0 > 0$

$$\int_{A_{k,\sigma R}} |\nabla u|^p dx \leq C\left[\int_{A_{k,R}} \left|\frac{u-k}{R(1-\sigma)}\right|^{p^*} dx + k^\alpha |A_{k,R}| + |A_{k,R}|^{\frac{p}{p^*}+\varepsilon} + \left(\int_{A_{k,R}} \left|\frac{u-k}{R(1-\sigma)}\right|^{p^*} dx\right)^{\frac{p}{p^*}} |A_{k,R}|^\delta\right]$$

where $A_{k,R} = \{x \in B_R \cap \Omega \mid u(x) > k\}$, $0 < \alpha < p^* = \frac{dp}{d-p}$ *and* ε, $\delta > 0$. *Then* $u \in L^\infty(\Omega)$.

Proof. Fusco and Sbordone have already proved in [17] the local boundedness of u in the case $u \in W^{1,p}(\Omega)$ and the inequality is satisfied for any $B_R \subset\subset \Omega$. We claim that the result is still valid in our situation. For that we will prove the boundednes of u in a neighborhood of the boundary $\partial\Omega$.
Let $x_0 \in \partial\Omega$, B_R be the ball centred in x_0. We define $K_R \overset{\text{def}}{=} B_R \cap \Omega$ and we set

$$r_h = \frac{R}{2} + \frac{R}{2^{h+1}}, \quad \tilde{r}_h = \frac{r_h + r_{h+1}}{2} \quad \text{and} \quad k_h = k\left(1 - \frac{1}{2^{h+1}}\right) \quad \text{for any } h \in \mathbb{N}.$$

Also define

$$I_h = \int_{A_{k_h,r_h}} |u(x) - k_h|^{p^*} dx \quad \text{and} \quad \varphi(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if} \quad t \geq \frac{3}{4} \end{cases}$$

satisfying $\varphi \in C^1([0,+\infty);[0,1])$. We set $\varphi_h(x) = \varphi\left(\frac{2^{h+1}}{R}(|x| - \frac{R}{2})\right)$. Hence $\varphi_h = 1$ on $B_{r_{h+1}}$ and $\varphi_h = 0$ on $\mathbb{R}^d \backslash B_{\tilde{r}_{h+1}}$.
We have

$$I_{h+1} = \int_{A_{k_{h+1},r_{h+1}}} |u(x) - k_{h+1}|^{p^*} dx = \int_{A_{k_{h+1},r_{h+1}}} (|u(x) - k_{h+1}|\varphi_h(x))^{p^*} dx$$
$$\leq \int_{K_R} ((u(x) - k_{h+1})^+ \varphi_h(x))^{p^*} dx.$$

Since $u \in W_0^{1,p}(\Omega)$, $(u-k_{h+1})^+\varphi_h \in W_0^{1,p}(K_R)$. Thus

$$I_{h+1} \lesssim \left(\int_{K_R} |\nabla((u-k_{h+1})^+\varphi_h)|^p dx\right)^{\frac{p^*}{p}}$$
$$\lesssim \left(\int_{A_{k_{h+1},\tilde{r}_h}} |\nabla u|^p dx + \int_{A_{k_{h+1},\tilde{r}_h}} (u-k_{h+1})^p dx\right)^{\frac{p^*}{p}}$$

where we use the notation $f \lesssim g$ in the sense there exists a constant $c > 0$ such that $f \le cg$. Since $\tilde{r}_h < r_h$, we have

$$I_{h+1}^{\frac{p}{p^*}} \lesssim 2^{hp^*} \int_{A_{k_{h+1},r_h}} |u-k_{h+1}|^{p^*} dx + k_{h+1}^{\alpha} |A_{k_{h+1},r_h}| + |A_{k_{h+1},r_h}|^{\frac{p}{p^*}+\varepsilon}$$
$$+ 2^{hp} \left(\int_{A_{k_{h+1},r_h}} |u-k_{h+1}|^{p^*} dx\right)^{\frac{p}{p^*}} |A_{k_{h+1},r_h}|^{\delta}$$
$$+ \int_{A_{k_{h+1},r_h}} |u-k_{h+1}|^{p^*} dx.$$

Moreover, for any h, $k_h \le k_{h+1}$, this implies

$$I_h = \int_{A_{k_h,r_h}} |u-k_h|^{p^*} dx \ge \int_{A_{k_{h+1},r_h}} |u-k_h|^{p^*} dx \tag{C.2}$$
$$\ge \int_{A_{k_{h+1},r_h}} |k_h-k_{h+1}|^{p^*} dx = |A_{k_{h+1},r_h}||k_{h+1}-k_h|^{p^*}. \tag{C.3}$$

Then, for any $k > k_0$ and $h \in \mathbb{N}$

$$|A_{k_{h+1},r_h}| + k_{h+1}^{\alpha} |A_{k_{h+1},r_h}| \lesssim 2^{hp^*} I_h$$

where the constant in the notation depends only on k_0, p and α. Replacing in (C.2), we obtain

$$I_{h+1}^{\frac{p}{p^*}} \lesssim 2^{hp^*} I_h + 2^{h(p+\varepsilon p^*)} I_h^{\frac{p}{p^*}+\varepsilon} + 2^{h(p+\delta p^*)} I_h^{\frac{p}{p^*}+\delta}. \tag{C.4}$$

Setting $M = \frac{p}{p^*}\max(p^*, p+\varepsilon p^*, p+\delta p^*)$ and $\theta = \min(1-\frac{p}{p^*}, \varepsilon, \delta)$ and noting that

$$I_h \le \int_{K_R} (|u-k_h|^+)^{p^*} dx \le \int_{K_R} |u|^{p*} \le \|u\|_{W_0^{1,p}}^{p*},$$

(C.4) becomes

$$I_{h+1} \lesssim 2^{hM} I_h^{1+\frac{\theta p^*}{p}}$$

where the constant depends on $||u||_{W_0^{1,p}}$, k_0, α and p. We need the following lemma to conclude.

Lemma C.4. (Lemma 4.7, Chapter 2, [20]) *Let (x_n) be a sequence such that $x_0 \le \lambda^{-\frac{1}{\eta}}\mu^{-\frac{1}{\eta^2}}$ and $x_{n+1} \le \lambda\mu^n x_n^{1+\eta}$, for any $n \in \mathbb{N}^*$ with λ, η and μ are positive constants and $\mu > 1$. Then (x_n) converges to 0 as $n \to +\infty$.*

It suffices to prove that I_0 is small enough. Indeed $u \in L^{p^*}(\Omega)$ implies

$$I_0 = \int_{A_{\frac{k}{2},R}} |u - \frac{k}{2}|^{p^*}\,dx \to 0 \quad \text{as} \quad k \to \infty.$$

Hence for k large enough, $I_0 \le C^{-\frac{1}{\eta}}(2^M)^{-\frac{1}{\eta^2}}$ with $\eta = \frac{\theta p^*}{p}$. Thus I_h converges to 0 as $h \to +\infty$ and

$$\int_{A_{k,\frac{R}{2}}} |u - k|^{p^*}\,dx = 0.$$

We deduce that $u \le k$ on $K_{\frac{R}{2}}$. In the same way, we prove that $-u \le k$ on $K_{\frac{R}{2}}$. Since $\overline{\Omega}$ is compact, we conclude that $u \in L^\infty(\Omega)$. □

Proof of Proposition C.2 We follow the idea of the proof of Theorem 4.1 in [15].
Let $x_0 \in \overline{\Omega}$, B_R be the ball of radius R centered in x_0 and $K_R = \Omega \cap B_R$. We define

$$p^+ \stackrel{\text{def}}{=} \max_{K_R} p(x) \quad \text{and} \quad p^- \stackrel{\text{def}}{=} \min_{K_R} p(x)$$

and we choose R small enough such that $p^+ < (p^-)^* = \frac{dp^-}{d-p^-}$.
Fix $(s,t) \in (\mathbb{R}_+^*)^2$, $t < s < R$ then $K_t \subset K_s \subset K_R$. Define $\varphi \in C^\infty(\Omega)$, $0 \le \varphi \le 1$ such that

$$\varphi = \begin{cases} 1 & \text{in} \quad B_t, \\ 0 & \text{in} \quad \mathbb{R}^d \backslash B_s \end{cases}$$

satisfying $|\nabla\varphi| \lesssim 1/(s-t)$. Let $k \ge 1$, using the same notations as previously $A_{k,\lambda} = \{y \in K_\lambda \mid u(y) > k\}$ and taking $\Psi = \varphi^{p^+}(u-k)^+ \in \mathbb{W}$ in (C.1), we obtain

$$\int_{A_{k,s}} |\nabla u|^{p(x)}\varphi^{p^+}\,dx + p^+ \int_{A_{k,s}} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\varphi^{p^+-1}(u-k)^+\,dx \tag{C.5}$$

$$= \int_{A_{k,s}} f\varphi^{p^+}(u-k)\,dx. \tag{C.6}$$

Hence by Young's inequality, for $\epsilon > 0$, we have

$$p^+ \int_{A_{k,s}} |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi\varphi^{p^+-1}(u-k)\,dx$$
$$\le \varepsilon \int_{A_{k,s}} |\nabla u|^{p(x)}\varphi^{(p^+-1)\frac{p(x)}{p(x)-1}}\,dx + c\varepsilon^{-1}\int_{A_{k,s}} (u-k)^{p(x)}|\nabla\varphi|^{p(x)}\,dx.$$

Since $|\nabla\varphi| \le c/(s-t)$ and for any $x \in K_R$, $p^+ \le (p^+-1)\frac{p(x)}{p(x)-1}$, we have $\varphi^{(p^+-1)\frac{p(x)}{p(x)-1}} \le \varphi^{p^+}$. This implies

$$p^+ \int_{A_{k,s}} |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi\varphi^{p^+-1}(u-k)\,dx \tag{C.7}$$

$$\le \varepsilon \int_{A_{k,s}} |\nabla u|^{p(x)}\varphi^{p^+}\,dx + c\varepsilon^{-1}\int_{A_{k,s}} \left(\frac{u-k}{s-t}\right)^{p(x)}\,dx. \tag{C.8}$$

Using Hölder's inequality we estimate the right-hand side of (C.5) as follows:

$$\int_{A_{k,s}} f\varphi^{p^+}(u-k)\,dx \le \|f\|_{L^q}\left(\int_{A_{k,s}} (u-k)^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}.$$

Since $q > \frac{d}{p^-}$, we have $\frac{(p^-)^*}{p^-}\frac{q-1}{q} > 1$. So, applying once again the Hölder's inequality, we obtain

$$\int_{A_{k,s}} f\varphi^{p^+}(u-k)\,dx \le C\left(\int_{A_{k,s}} (u-k)^{\frac{(p^-)^*}{p^-}}\,dx\right)^{\frac{p^-}{(p^-)^*}} |A_{k,s}|^{\delta}, \tag{C.9}$$

where $\delta = \frac{q-1}{q} - \frac{p^-}{(p^-)^*} > 0$. Set $\{u-k > s-t\} = \{x \in K_R \mid u(x)-k > s-t\}$ and its complement as $\{u-k \le s-t\}$. Now we split the integral in the right-hand side of (C.9) on $\Theta = A_{k,s} \cap \{u-k > s-t\}$ and $A_{k,s}\backslash\Theta$:

$$\mathcal{I} \overset{\text{def}}{=} \int_{A_{k,s}} \left(\frac{u-k}{s-t}\right)^{(p^-)^*} dx + |A_{k,s}| \tag{C.10}$$

$$\gtrsim \int_{\Theta} \left(\frac{u-k}{s-t}\right)^{\frac{(p^-)^*}{p^-}} (s-t)^{\frac{(p^-)^*}{p^-}}\,dx \tag{C.11}$$

$$+\int_{A_{k,s}\backslash\Theta} \left(\frac{u-k}{s-t}\right)^{\frac{(p^-)^*}{p^-}} (s-t)^{\frac{(p^-)^*}{p^-}}\,dx. \tag{C.12}$$

In the same way, the second term in the right-hand side of (C.7) can be estimated as follows.

$$\int_{\Theta} \left(\frac{u-k}{s-t}\right)^{p(x)} dx + \int_{A_{k,s}\backslash\Theta} \left(\frac{u-k}{s-t}\right)^{p(x)} dx \lesssim \mathcal{I}. \tag{C.13}$$

Finally, plugging (C.7)–(C.13) and we obtain for ε small enough

$$\int_{A_{k,s}} |\nabla u|^{p(x)}\varphi^{p^+}\,dx \lesssim \mathcal{I} + |A_{k,s}|^{\delta}\mathcal{I}^{\frac{p^-}{(p^-)^*}}$$

where the constant depends on p, R and ε. Moreover we have

$$\mathcal{I}^{\frac{p^-}{(p^-)^*}} \lesssim \left(\int_{A_{k,s}} \left(\frac{u-k}{s-t}\right)^{(p^-)^*} dx\right)^{\frac{p^-}{(p^-)^*}} + |A_{k,s}|^{\frac{p^-}{(p^-)^*}}.$$

Hence using the Young's inequality, we obtain the following estimate.

$$\begin{aligned}\int_{A_{k,t}} |\nabla u|^{p^-}\,dx &\le \int_{A_{k,s}} |\nabla u|^{p(x)}\varphi^{p^+}\,dx + |A_{k,s}| \\ &\lesssim \int_{A_{k,s}} \left(\frac{u-k}{s-t}\right)^{(p^-)^*} dx + |A_{k,s}|^{\frac{p^-}{(p^-)^*}+\delta} \\ &\quad + |A_{k,s}| + |A_{k,s}|^{\delta}\left(\int_{A_{k,s}} \left(\frac{u-k}{s-t}\right)^{(p^-)^*} dx\right)^{\frac{p^-}{(p^-)^*}}.\end{aligned}$$

By Lemma C.3, we deduce that u bounded in Ω. □

Combining Propositions C.1 and C.2, we have the following corollary:

Corollary C.5. *Let $p \in C(\bar{\Omega})$ such that $p^- < d$ and $u \in W_0^{1,p(x)}(\Omega)$ satisfying*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \Psi dx = \int_\Omega (f(x,u) + g)\Psi dx, \quad \forall \Psi \in \mathbb{W},$$

where f satisfies $|f(x,t)| \leq c_1 + c_2|t|^{r(x)-1}$ with $r \in C(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $1 < r(x) < p^*(x)$ and $g \in L^q$, $q > \frac{d}{p_-}$. Then $u \in L^\infty(\Omega)$.

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Received: 12 June 2015.
Accepted: 5 March 2016.