QUASIMODES AND BOHR-SOMMERFELD CONDITIONS FOR THE TOEPLITZ OPERATORS

L. CHARLES

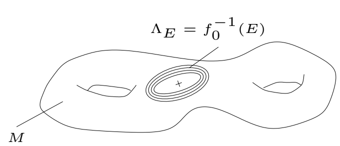
Abstract. This article is devoted to the quantization of the Lagrangian submanifold in the context of geometric quantization. The objects we define are similar to the Lagrangian distributions of the cotangent phase space theory. We apply this to construct quasimodes for the Toeplitz operators and we state the Bohr-Sommerfeld conditions under the usual regularity assumption. To compare with the Bohr-Sommerfeld conditions for a pseudodifferential operator with small parameter, the Maslov index, defined from the vertical polarization, is replaced with a curvature integral, defined from the complex polarization. We also consider the quantization of the symplectomorphisms, the realization of semi-classical equivalence between two different quantizations of a symplectic manifold and the microlocal equivalences.

Let (M, ω) be a symplectic compact manifold of dimension $2n$ endowed with a prequantization bundle, that is a complex line bundle $L \to M$ with a Hermitian structure h and a covariant derivation ∇ whose curvature is ω. To quantize these data, we assume that M is endowed with a complex structure J which is integrable and compatible with $-i\omega$. The quantum space $\mathcal{H}_k$ is defined as the space of the holomorphic sections of $L^k \to M$. k is any positive integer and the semi-classical limit is $k \to \infty$. The quantum semi-classical observables are the Berezin-Toeplitz operators (cf. [2], [3], [4], [5]). The purpose of this article is to quantize the Lagrangian manifolds of M, by generalising the ansatz for the Schwartz kernel of a Toeplitz operator that we proposed in [5]. We will apply this to produce quasimodes of Toeplitz operators and deduce the Bohr-Sommerfeld conditions.

Let us state this last result in the case M is 2-dimensional. Consider the Toeplitz operator

$$T_k := \Pi_k M_{f_0 + k^{-1} f_1} : \mathcal{H}_k \to \mathcal{H}_k$$

where Π_k is the orthogonal projector of $L^2(M, L^k)$ onto $\mathcal{H}_k$, f_0 and f_1 are some functions of $C^\infty(M)$ and $M_{f_0+k^{-1}f_1}$ is the multiplication operator by $f_0 + k^{-1} f_1$. Assume that E^0 is a regular value of the principal symbol f_0 of (T_k) and that $f_0^{-1}(E^0)$ is connected. Then if E belongs to some neighborhood U of E^0, the level set $f^{-1}(E) = \Lambda_E$ is a circle.



Theorem 0.1. *For all sequences (E_α, k_α) of $U \times \mathbb{N}$,*

$$(1) \qquad E_\alpha \in \operatorname{Spec}(T_{k_\alpha}) + O(k_\alpha^{-2}) \quad \Leftrightarrow \quad g_{-1}(E_\alpha) + k_\alpha^{-1} g_0(E_\alpha) \in k_\alpha^{-1}\mathbb{Z} + O(k_\alpha^{-2})$$

where

Key words: Geometric quantization, Semi-classical analysis, Toeplitz operators, Lagrangian submanifolds, Quasimodes, Bohr-Sommerfeld conditions, Quantum maps.

- *$g_{-1}(E)$ is the holonomy of Λ_E for the prequantization bundle L,*
- *$g_0(E)$ is the sum of the integral of the geodesic curvature of Λ_E and the integral over Λ_E of the Weyl subsymbol $f_1^w = f_1 + \frac{1}{2}\Delta f_0$.*

We refer the reader to section 3 for a more precise statement. Let us compare this with the known result for a pseudodifferential operator with a small parameter. In that case, the phase space is a cotangent bundle T^*C and the action of Λ_E replaces its holonomy. Actually, this action can be interpreted as a holonomy for the trivial bundle $T^*C \times \mathbb{C}$ endowed with the connection form $-ipdq$. The second order term is more unexpected. It involves the Riemannian metric of M defined by the symplectic and complex structures. Its analog for the pseudodifferential operator is the Maslov index, an invariant of the cotangent bundles. Hence in the language of geometric quantization, these second order invariants come from the polarizations: the vertical polarization in the cotangent case, the complex polarization in the Kählerian case.

For the proof we construct quasimodes, that is Lagrangian sections (u_α) associated to the circles Λ_{E_α} such that

$$u_\alpha \in \mathcal{H}_{k_\alpha} \quad \text{and} \quad T_{k_\alpha} u_\alpha = E_\alpha u_\alpha + O(k_\alpha^{-\infty})$$

The quantization condition to define these quasimodes is the Bohr-Sommerfeld condition and this will prove the converse of (1). To show the direct sense, we will prove by using microlocal equivalence that the eigenvectors of T_k are necessarily Lagrangian sections associated to the Λ_E.

Let us briefly explain how we will construct the Lagrangian sections. In the usual semi-classical theory, the semi-classical observables are the pseudodifferential operators with a small parameter $\hbar$. The Schwartz kernel of these operators is of the form

$$\left(\frac{1}{2\pi\hbar}\right)^n \int e^{i\hbar^{-1}(x-y).\xi} a(x,\xi,\hbar)|d\xi| \tag{2}$$

Our main result in [5] was to give a similar expression for the Schwartz kernel of a Toeplitz operator:

$$T_k(x_l, x_r) = \left(\frac{k}{2\pi}\right)^n E^k(x_l,x_r) a(x_l,x_r,k) + O(k^{-\infty}) \tag{3}$$

where E is a section of $L \boxtimes L^{-1} \to M \times M$ and $(a(.,k))$ a sequence of $C^\infty(M \times M)$ which correspond respectively to $e^{i(x-y).\xi}$ and $a(x,\xi,\hbar)$.

The oscillatory integrals, as (2), can also be used to define the Lagrangian functions or the Fourier integral operators (cf. [7]). In a similar way, we extend (3) to define sequence of holomorphic sections associated to a closed Lagrangian submanifold Λ of M. Assume that Λ satisfies the following quantization condition: the flat bundle $(L|_\Lambda, \nabla)$ is trivial. Then a Lagrangian section associated to Λ is a sequence (u_k) such that

$$u_k \in \mathcal{H}_k, \ \forall k \quad \text{and} \quad u_k(x) = \left(\frac{k}{2\pi}\right)^m F^k(x) a(x,k) + O(k^{-\infty}) \tag{4}$$

where m is a real constant and

- F is a section of $L \to M$ which restricts on Λ to a flat section with a constant norm equal to 1 and such that $\nabla_{X-iJX} F$ vanishes to order ∞ along Λ for every vector field X of M.
- $(a(.,k))$ is a sequence of $C^\infty(M)$ which admits an asymptotic expansion $\sum_l k^{-l} a_l(x)$ for the C^∞ topology.

The symbol of (u_k) is the formal series $\sum_l \hbar^l a_l|_\Lambda$ of $C^\infty(\Lambda)[[\hbar]]$. This is a full symbol, meaning that it vanishes if and only if (u_k) is $O(k^{-\infty})$. There is an associated symbolic calculus corresponding to the estimate of the norm of a Lagrangian section. If (T_k) is a Toeplitz operator we can also compute the symbol of the Lagrangian section $(T_k u_k)$ in terms of the symbols of (u_k) and (T_k).

We will also define quantum maps by quantizing the Lagrangian manifolds of M^2. We proved in [6] that the quantum propagator of a Toeplitz operator is an operator whose Schwartz kernel is a Lagrangian section associated to the graph of the Hamiltonian flow of its principal symbol. Another application is to prove that the quantization is independent of the complex structure in a semi-classical sense: we introduce unitary operators $(U_k : \mathcal{H}_k^a \rightarrow \mathcal{H}_k^b)$, where $\mathcal{H}_k^a$ and $\mathcal{H}_k^b$ are the quantum spaces associated to two complex structures J_a and J_b. These maps have good semi-classical properties: they send the Lagrangian sections into the Lagrangian sections, the Toeplitz operators of $\mathcal{H}_k^a$ into the Toeplitz operators of $\mathcal{H}_k^b$, etc... Using a local version of these maps, we can also realize microlocal equivalences, which leads to some normal forms and can be used to apply the usual techniques of microlocal analysis in this context.

To end this introduction let us mention some previous results. Lagrangian sections were already introduced by Borthwick, Paul and Uribe [2]. Their approach consists in using the homogeneous theory of the Toeplitz operator of Boutet de Monvel and Guillemin [4]. Let us identify the sections of L^k to functions defined on the circle principal bundle $P \rightarrow M$ associated to L. Then the Lagrangians sections are obtained by projecting the usual Lagrangian distributions defined on P onto $\bigoplus \mathcal{H}_k$. The quantum maps considered by Zelditch [11] are defined in a similar way from the Fourier integral operators $C^\infty(P) \rightarrow C^\infty(P)$. These objects are viewed as Hermite distributions, which leads to the definition of their principal symbol. The symbolic calculus is then a consequence of the symbolic calculus of the Hermite distributions.

To compare, our definition is more concrete and leads to the definition of a full symbol map, from which we control the Lagrangian sections modulo $O(k^{-\infty})$. Furthermore, the products and the symbolic calculus are obtained by a direct application of the stationary phase lemma. Also, we have an explicit description of the subsymbolic calculus, which involves Riemannian invariants, whereas the subsymbolic calculus in the homogeneous theory of the Toeplitz operators has never been achieved.

Finally, let us mention that the main part of the article [2] is devoted to the Lagrangian sections of the Riemann surfaces with genus $\geqslant 2$. In the article [11], the quantization of some symplectomorphisms of the torus illustrates the results about the quantum maps.

1. Preliminaries

First we present some general notations and conventions. We state some technical lemmas that we need to apply the complex stationary phase lemma. Finally we define the Weyl symbol of a Toeplitz operator, which will be useful for the Bohr-Sommerfeld conditions.

1.1. **Geometric notations.** Let (M,ω) be a symplectic manifold endowed with a complex structure which is integrable and compatible with ω, that is

$$\omega(JX,JY)=\omega(X,Y), \quad \omega(X,JX)\geqslant 0 \quad \text{and} \quad (\omega(X,JX)=0 \Rightarrow X=0).$$

In other words M is a Kähler manifold with fundamental 2-form ω. We denote by g the Riemannian metric induced by the symplectic and complex structures

$$g(X,Y)=\omega(X,JY)$$

and by μ_M the measure induced by g . μ_M is also the Liouville measure $\frac{1}{n!}|\omega^{\wedge n}|$.

Let $(L\to M,h,\nabla)$ be a prequantization bundle. We denote by $|u|$ the norm of $u\in L_x$ and by $h(u,v)$ the scalar product of $u,v\in L_x$. The scalar product (s,t) of two sections $s,t\in C^\infty(M,L)$ is defined in the usual way by

$$(s,t)=\int_M h(s,t)\mu_M.$$

We denote by $L\boxtimes L^{-1}\to M^2$ the bundle $\pi_l^{\#}L\otimes\pi_r^{\#}L^{-1}\to M\times M$, where π_l and π_r are the projections $M^2\to M$ onto the first and the second factor. Observe that $L\boxtimes L^{-1}$ endowed with the induced Hermitian structure and covariant derivation is a prequantization bundle, whose symplectic structure of the base M^2 is given by $\pi_l^*\omega-\pi_r^*\omega$. We identify the Schwartz kernel of an operator $T:C^\infty(M,L)\to C^\infty(M,L)$ with a section $(x_l,x_r)\to T(x_l,x_r)$ of $L\boxtimes L^{-1}\to M^2$ by the following formula

$$(Ts)(x_l)=\int_M T(x_l,x_r).s(x_r)\mu_M(x_r), \quad \forall s\in C^\infty(M,L).$$

We use the same notations and definitions for the induced data on the bundle $L^k\to M$, where k is any positive integer.

1.2. **Admissible and negligible sequences.** Let $(u_k)_k$ be a sequence such that $u_k\in C^\infty(M,L^k)$ for every k. We say that (u_k) is *admissible* if for every positive integer l, for every vector fields $X_1,...,X_l$ of M and for every compact set K of M, there exists C and an integer N such that

$$\left|\nabla_{X_1}...\nabla_{X_l}s_k(x)\right|\leqslant Ck^{-N} \text{ on } K. \tag{5}$$

We say that (u_k) is negligible if for every positive integers l,N, for every vector fields $X_1,...,X_l$ of M and for every compact K of M, there exists C such that (5) holds. We say that (u_k) is *negligible* over an open set U if the previous estimates are verified for every compact set of U. We denote by $O_\infty(k^{-\infty})$ any negligible sequence or the set of negligible sequences. The microsupport of (u_k) is the complementary set of

$$\big\{x\in M/(u_k) \text{ is negligible on a neighborhood of } x\big\}.$$

Recall that the Toeplitz operators reduce microsupport.

We will also consider some sequences $(u_\alpha,k_\alpha)_\alpha$ such that $u_\alpha\in C^\infty(M,L^{k_\alpha})$ for every α. We will always assume that $k_\alpha\to\infty$ even if we do not mention it. As previously we may say that (u_α) is admissible or negligible over an open set U when in the previous estimates k,u_k are replaced by k_α,u_α. The microsupport is defined in the same way.

1.3. **Asymptotic and Taylor expansions.** If X is any manifold, the space $S^0(X)$ consists of the sequences $(f(.,k))$ of $C^\infty(X)$ which admit an asymptotic expansion of the form

$$f(.,k) = \sum_{l=0}^{\infty} k^{-l} f_l + O(k^{-\infty})$$

for the C^∞ topology. By the Borel process, if $\sum \hbar^l f_l$ is a formal series of $C^\infty(X)[[\hbar]]$, there exists a symbol of $S^0(X)$ which admits the asymptotic expansion $\sum k^{-l} f_l$ and this symbol is unique modulo $O(k^{-\infty})$.

Let Y be a closed submanifold of X of codimension k. We denote by $\mathcal{I}^N(Y)$ the ideal of $C^\infty(X)$ which consists of the functions which vanish to order N along Y and by $\mathcal{I}^\infty(Y)$ the ideal $\cap_N \mathcal{I}^N(Y)$ whose functions vanish to any order along Y.

Let $\partial_1, ..., \partial_k$ be vector fields of X such that on an open set U of X

- $[\partial_i, \partial_j] = 0$ on U
- $\langle \partial_1|_x, ..., \partial_k|_x \rangle \oplus (T_x Y \otimes \mathbb{C}) = T_x X \otimes \mathbb{C}, \quad \forall\, x \in U \cap Y.$

To solve some equations, we will use the following lemma.

Lemma 1.1. *There exists functions $\underline{Z}^1, ..., \underline{Z}^k$ of $C^\infty(U)$ such that*

$$\underline{Z}^j|_{Y\cap U} = 0, \qquad \partial_l.\underline{Z}^j \equiv \delta_{jl} \mod \mathcal{I}^\infty(Y \cap U).$$

These functions are unique modulo $\mathcal{I}^\infty(Y \cap U)$. If $f \in C^\infty(U \cap Y)$, there exists $F \in C^\infty(U)$ such that

$$F|_{Y\cap U} = f, \qquad \partial_l.F \equiv 0 \mod \mathcal{I}^\infty(Y \cap U).$$

F is unique modulo $\mathcal{I}^\infty(Y \cap U)$.

To deal with the Taylor expansions along a submanifold, we will use the following result which can be proved by the Borel process.

Lemma 1.2. *The map which sends $f \in C^\infty(U)$ into the formal series*

$$\textstyle\sum_\alpha f_\alpha Z^\alpha, \qquad \text{with } f_\alpha = \partial^\alpha f|_{Y\cap U}$$

induces an algebra isomorphism from $C^\infty(U)/\mathcal{I}^\infty(Y \cap U)$ onto the space $C^\infty(Y \cap U)[[Z^1, ..., Z^k]]$. The inverse of this isomorphism sends the formal series $\sum_\alpha g_\alpha Z^\alpha$ into $[g]$ with $g \in C^\infty(U)$ such that

$$g \equiv \textstyle\sum_{|\alpha|\leqslant N} G_\alpha \underline{Z}^\alpha \mod \mathcal{I}^{N+1}(Y \cap U), \quad \forall\, N$$

where the functions $G_\alpha \in C^\infty(U)$ restrict on Y to the functions g_α and satisfy $\partial_l G_\alpha \equiv 0$ modulo $\mathcal{I}^\infty(Y \cap U)$ for every l.

With respect to the notation, ∂^α is the differential operator $\partial_1^{\alpha(1)} ... \partial_k^{\alpha(k)}$. Finally we recall the following result proved in [5].

Lemma 1.3. *Let $d \in C^\infty(X, \mathbb{R}^+)$ be a positive function outside Y which vanishes to order 2 along Y and whose kernel of its Hessian is $T_x Y$ for every x in Y. Let $\big(a(.,k)\big)$ be a sequence of $C^\infty(X)$ which admits the asymptotic expansion $\sum_{l=0}^\infty a_l(x) k^{-l}$ for the C^0 topology. Let N be a non negative integer. Then the following two assertions are equivalent.*

i. $\forall$ *compact subset K of X, $\exists\, C$ such that* $\big|\, e^{-kd(x)} a(x,k) \big| \leqslant C\, k^{-\frac{N}{2}}$ *on K.*

ii. $a_l \in \mathcal{I}^{N-2l}(Y), \quad$ *for every l such that $2l \leqslant N$.*

1.4. **Toeplitz operators.** A Toeplitz operator is a sequence (T_k) of the form

$$T_k := \Pi_k M_{f(.,k)} + O(k^{-\infty}) : \mathcal{H}_k \to \mathcal{H}_k$$

where $(f(.,k))$ is a symbol of $S^0(M)$, $M_{f(.,k)}$ is the multiplication by $f(.,k)$ and Π_k is the orthogonal projector of $L^2(M, L^k)$ onto $\mathcal{H}_k$. The big O is for the uniform norm of operators. Recall that the Schwartz kernel of a Toeplitz operator is of the form

$$T_k(x_l, x_r) = \Big(\frac{k}{2\pi}\Big)^n E^k(x_l, x_r) g(x_l, x_r, k) + O_\infty(k^{-\infty}) \tag{6}$$

where

- E is a section of $L \boxtimes L^{-1}$ such that $E(x,x) = 1$, $|E(x_l, x_r)| < 1$ if $x_l \neq x_r$ and $\nabla_{\bar{Z}} E \equiv 0 \mod \mathcal{I}^\infty(\operatorname{diag} M)$ for every holomorphic vector field Z of $(M^2, J \times -J)$,
- $(g(.,k))$ is a symbol of $S^0(M^2)$ with asymptotic expansion $\sum k^{-l} g_l$ such that $\bar{Z}.g_l \equiv 0 \mod \mathcal{I}^\infty(\operatorname{diag} M)$, for every holomorphic vector field Z of $(M^2, J \times -J)$.

The σ symbol of (T_k) is the formal series $\sum \hbar^l g_l(x,x)$ of $C^\infty(M)[[\hbar]]$. Here it is convenient to introduce the Weyl symbol:

$$\sigma_w(T_k) = g_0 + \hbar(g_1 - \tfrac{1}{2} r.g_0 - \Delta g_0) + O(\hbar^2) \tag{7}$$

where r is the scalar curvature of (M, g) and Δ the holomorphic Laplacian. The product of the Weyl symbols induced by the composition of the Toeplitz operators is

$$f *_w h = f.h + \frac{\hbar}{2i}\{f, h\} + O(\hbar^2).$$

The formulas describing the spectrum of (T_k) in the semi-classical limit are simpler when we write them in terms of this symbol. As instance, assume that T_k is self-adjoint for every k and denote by

$$E_k^1 \leqslant E_k^2 \leqslant ... \leqslant E_k^{d_k}$$

its eigenvalues. Then using the functional calculus of the Toeplitz operators (cf. [5]), we can prove that for every C^∞ function φ

$$\sum_{i=1}^{d_k} \varphi(E_k^i) = \Big(\frac{k}{2\pi}\Big)^n \int_M \big(\varphi(f_0 + k^{-1} f_1)\big)(1 + k^{-1} \tfrac{r}{2}) \mu_M + O(k^{n-2})$$

where $f_0 + \hbar f_1$ is the Weyl symbol of (T_k).

2. Lagrangian sections

In this part, we consider a symplectic manifold (M, ω) endowed with a prequantization bundle and an integrable positive complex structure J. Let Λ be a Lagrangian submanifold of M.

The first subsection is devoted to the construction and the properties of a section F (cf. equation (4)) associated to Λ. Then we give a local definition of the Lagrangian sections associated to Λ and of their symbol. In the following subsections, we compute the norm of the Lagrangian sections and describe the action of the Toeplitz operators. Finally, we introduce Lagrangian sections associated to a fibration by Lagrangian tori, the motivation is to construct the quasimodes of a Toeplitz operator.

2.1. The section F.

Both of the next propositions give the main local properties of the section F associated to Λ. Observe that L restricts on Λ to a flat fiber bundle, that is the curvature of the induced connection vanishes.

Proposition 2.1. *Let $x \in \Lambda$. There exists a neighborhood $U \subset M$ of x and a section $F : U \rightarrow L$ such that $F|_{\Lambda \cap U}$ is flat with a constant norm equal to 1 and*

$$\nabla_{\bar{Z}} F \equiv 0 \mod \mathcal{I}^\infty(\Lambda \cap U), \quad \forall \text{ holomorphic vector field } Z.$$

If $F' : U' \rightarrow L$ satisfies the same assumption and $U \cap U'$ is connected, then there exists a real number a such that $e^{ia} F \equiv F'$ modulo $\mathcal{I}^\infty(\Lambda \cap U \cap U')$.

Proof. Since $L|_\Lambda$ is flat, there exists a flat section of $L|_\Lambda$ defined on a neighborhood of x with constant norm equal to 1. It is locally unique modulo a multiplicative constant of modulus 1. Extend this section to a local section s of L defined on a neighborhood of x in M. We look for a section F of the form $e^{i\varphi} s$, where φ vanishes over Λ. Write $\nabla s = -i\beta \otimes s$ where $\beta \in \Omega^1(U)$. $\bar{\partial}\beta^{0,1} = 0$ because $\omega = d\beta \in \Omega^{1,1}(M)$. So there exists $\rho \in C^\infty(U)$ such that $\bar{\partial}\rho + i\beta^{0,1} = 0$. Let us write $\varphi = i(\rho - \tilde{\rho})$. We have to solve

$$\tilde{\rho}|_\Lambda = \rho|_\Lambda \quad \text{and} \quad \bar{\partial}\tilde{\rho} \equiv 0 \mod \mathcal{I}^\infty(\Lambda).$$

These equations have a unique solution modulo $\mathcal{I}^\infty(\Lambda \cap U)$ by lemma 1.1, because the distribution $T^{0,1}M$ is integrable and $(T_x\Lambda \otimes \mathbb{C}) \oplus T_x^{0,1}M = T_xM \otimes \mathbb{C}$. Indeed, if $X \in (T_x\Lambda \otimes \mathbb{C}) \cap T_x^{0,1}M$, then $\omega(X, \bar{X}) = 0$ since $T_x\Lambda \otimes \mathbb{C}$ is a Lagrangian space, so $X = 0$ since J is positive. □

The Taylor expansion of F along Λ is determined by Λ and the Kählerian structure. We compute the first and second derivatives in terms of these data.

Proposition 2.2. *Let $F : U \rightarrow L$ be a section defined as in proposition 2.1. Denote by α_F the 1-form defined by $\nabla F = \alpha_F \otimes F$ and by δ the function $\delta = -2\ln|F|$.*

- *α_F vanishes at every $x \in \Lambda$ and its derivative $T\alpha_F : T_xM \rightarrow T_x^*M \otimes \mathbb{C}$ is given by*

$$\langle T_X \alpha_F, Y \rangle = -i\omega(q(X), Y), \quad \forall\, X, Y \in T_xM \tag{8}$$

  where q is the projection of $T_xM \otimes \mathbb{C}$ onto $T_x^{0,1}M$ whose kernel is $T_x\Lambda \otimes \mathbb{C}$.
- *δ vanishes along Λ with its first derivatives. Its Hessian is the bilinear symmetric form of T_xM whose kernel is $T_x\Lambda$ and which restricts on $JT_x\Lambda$ to $g|_{JT_x\Lambda}$.*

Hence $\delta = -2 \ln |F|$ is positive on a neighborhood of Λ minus Λ. By modifying F outside this neighborhood, we may assume that

$$|F|(x) < 1 \text{ if } x \notin \Lambda. \tag{9}$$

In the following we will always assume that the section F satisfies this condition even if we do not mention it.

Proof. Recall first that $(T_x\Lambda \otimes \mathbb{C}) \oplus T_x^{0,1}M = T_xM \otimes \mathbb{C}$ (cf. proof of proposition 2.1). If X is an anti-holomorphic vector field, then $\langle \alpha_F, X\rangle$ vanishes to order ∞ along Λ. If $X \in T_x\Lambda$, then $\langle \alpha_F, X \rangle = 0$ since $F|_\Lambda$ is flat. Consequently, α_F vanishes at $x \in \Lambda$ and the derivative $T\alpha_F$ is well-defined.

If $X \in T_x\Lambda$, the two sides of equation (8) vanish. Hence it suffices to prove equation (8) with $X \in T_x^{0,1}M$. Assume that X and Y are vector fields and X is anti-holomorphic. Using that α_F vanishes along Λ, we obtain on Λ

$$\nabla_X \nabla_Y F - \nabla_Y \nabla_X F - \nabla_{[X,Y]} F = \big(\langle T_X \alpha_F, Y \rangle - \langle T_Y \alpha_F, X \rangle \big) F.$$

The second term of the right side vanishes. Since the curvature of ∇ is $-i\omega$, we have $\langle T_X\alpha_F, Y\rangle = \frac{1}{i}\omega(X,Y)$, and this proves (8).

$|E||_\Lambda = 1$, so δ vanishes along Λ, and the same holds with its first derivatives since $d\delta = -\alpha_F - \bar{\alpha}_F$. So the Hessian of δ at $x \in \Lambda$ is well-defined. Its kernel contains $T_x\Lambda$. Furthermore (8) implies

$$\operatorname{Hess} \delta(X,Y) = -\, (2i)^{-1} \omega(q(X) - \overline{q(X)}, Y), \quad \forall\, X, Y \in T_xM$$

and if $X \in JT_x\Lambda$, then $q(X) = X + iJX$. So $\operatorname{Hess}\delta(X,Y) = -\omega(JX,Y) = g(X,Y)$. ☐

Remark 2.3. Let E be the section associated to the kernel of the Toeplitz operators (cf. (6)). If $\nabla E = \alpha_E \otimes E$ on a neighborhood of $\operatorname{diag} M$, we can prove that α_E vanishes at $(x,x) \in \operatorname{diag}(M)$ and its first derivative is given by

$$\langle T_{(X_1,X_2)} \alpha_E, (Y_1,Y_2) \rangle = \tfrac{1}{i}\omega(X_1^{0,1} - X_2^{0,1}, Y_1) + \tfrac{1}{i}\omega(X_1^{1,0} - X_2^{1,0}, Y_2) \tag{10}$$

where $X^{1,0} = \frac{1}{2}(X + iJX)$ is the holomorphic part of X and $X^{0,1} = \frac{1}{2}(X - iJX)$ its anti-holomorphic part. ☐

2.2. **Definition of Lagrangian sections.** Let U be an open set of M, such that there exists a section $F : U \to L$ which satisfies the assumptions of proposition 2.1 and condition (9). We are interested in admissible sequences $(u_\alpha, k_\alpha)_{\alpha\in\mathbb{N}}$ of the following form over U

$$u_\alpha = F^{k_\alpha} a(., k_\alpha) + O_\infty(k_\alpha^{-\infty}) \tag{11}$$

where $a(.,k)$ is a symbol of $S^0(U)$, whose asymptotic expansion $\sum k^{-l} a_l$ satisfies

$$\bar{Z}.a_l \equiv 0 \mod \mathcal{I}^\infty(U \cap \Lambda), \quad \forall \text{ holomorphic vector field } Z. \tag{12}$$

If moreover $u_\alpha \in \mathcal{H}_{k_\alpha}$ for every α, we will say that (u_α, k_α) is a *Lagrangian section over* U.

Proposition 2.4. *Let $(u_\alpha, k_\alpha)_\alpha$ be an admissible sequence of the form* (11) *over U. Then*

$$\Pi_{k_\alpha} u_\alpha = u_\alpha + O_\infty(k_\alpha^{-\infty}) \text{ over } U.$$

Let (u'_α, k_α) be an admissible sequence of the form (11) *over U with a section F' and a symbol $a'(.,k)$. Assume that $F|_{U\cap\Lambda} = F'|_{U\cap\Lambda}$. Then*

$$u_\alpha = u'_\alpha + O_\infty(k_\alpha^{-\infty}) \text{ over } U \quad \Leftrightarrow \quad a_l|_{U\cap\Lambda} = a'_l|_{U\cap\Lambda} \text{ for every } l.$$

We will call the formal series

$$\textstyle\sum_l \hbar^l f_l := \sum_l \hbar^l a_l|_{U\cap\Lambda}$$

the *symbol* of the Lagrangian section (u_α, k_α). The function f_0 is the *principal symbol*.

From the first assertion of the previous proposition, the existence of a Lagrangian section over U with an arbitrary symbol $\sum \hbar^l f_l$ is equivalent to the existence of an admissible sequence (u_α, k_α) of the form (11) where the asymptotic expansion of the symbol $a(., k)$ restricts to $\sum k^{-l} f_l$ over $U \cap \Lambda$. If (u_α, k_α) is a Lagrangian section over U with symbol $\sum \hbar^l f_l$, observe that (u_α) is $O_\infty(k_\alpha^{-\infty})$ over U if and only if its symbol vanishes. More precisely, we deduce from lemma 1.3 that

$$|u_\alpha| = O\big(k_\alpha^{-N}\big) \text{ over } U \quad \Leftrightarrow \quad f_0 = ... = f_{N-1} = 0.$$

To define a global Lagrangian section, we need a quantization condition. As instance assume that $(L|_\Lambda, \nabla)$ is trivial. Then there exists a flat section

$$t : \Lambda \to L$$

of constant norm equal to 1. Using a partition of unity, we can obtain a global section $F : M \to L$ which restricts to t over Λ and satisfies the assumptions of proposition 2.1 over a neighborhood of Λ. Define the space $\mathcal{S}(\Lambda, t)$ of Lagrangian sections (u_α) such that $u_\alpha \in \mathcal{H}_\alpha$ for every α, (u_α) is of the form (11) over a neighborhood of Λ with $k_\alpha = \alpha$ and is negligible outside this neighborhood. Then proposition 2.4 implies that the symbol map $\mathcal{S}(\Lambda, t) \to C^\infty(\Lambda)[[\hbar]]$,

$$(u_\alpha) \to \textstyle\sum_l \hbar^l f_l \text{ such that } u_k|_\Lambda = t^k \sum_l k^{-l} f_l + O(k^{-\infty}) \tag{13}$$

is onto. Its kernel consists of the negligible sequences. This quantization condition will be used to define the kernel of the quantum maps. To define the quasimodes we will need a more complicated condition (cf. section 2.6).

Proof of proposition 2.4. We begin with the second assertion. By proposition 2.1, F and F' are equal modulo $\mathcal{I}^\infty(\Lambda \cap U)$. Hence it follows from lemma 1.3 and the properties of the Hessian of $\ln |F|$ (cf. proposition 2.2) that $u_\alpha = u'_\alpha + O_\infty(k_\alpha^{-\infty})$ over U if and only if a_l and a'_l have the same Taylor expansion along $U \cap \Lambda$, for every l. By (12), this is satisfied if and only if a_l and a'_l are equal over $U \cap \Lambda$. The first assertion is a consequence of the following lemma.

Lemma 2.5. *Let (u_α, k_α) be an admissible sequence of the form (11) over U with a symbol $(b(., k))$. Then $(\Pi_{k_\alpha} u_\alpha, k_\alpha)$ is of the same form over U with a symbol $(c(., k))$ such that c_0 is equal to b_0 over $U \cap \Lambda$.*

Let $x \in \Lambda \cap U$. Applying the previous remarks, we can obtain from lemma 2.5 an admissible sequence (w_α, k_α) satisfying (11) and such that

$$\Pi_{k_\alpha} w_\alpha = u_\alpha + O(k_\alpha^{-\infty}) \tag{14}$$

on a neighborhood of x. Indeed we can construct the symbol of (w_α) by successive approximations. Now applying Π_{k_α} to (14), it follows that $\Pi_{k_\alpha} u_\alpha = u_\alpha + O(k_\alpha^{-\infty})$ on a neighborhood of x. □

Proof of lemma 2.5. We use the ansatz (6) for the Schwartz kernel of Π_k. Hence

$$\Pi_k\big(F^k b(., k)\big)(x_1) = \Big(\frac{k}{2\pi}\Big)^n \int_U E^k(x_1, x_2).F^k(x_2) f(x_1, x_2, k) b(x_1, k) \mu_M(x_2)$$

modulo $O_\infty(k^{-\infty})$, where U is an arbitrary small neighborhood of x_1. We compute this integral by applying stationary phase lemma. Introduce a section $s : U \to L$ with constant norm equal to 1 and such that $s|_\Lambda = F|_\Lambda$. Write

$$E(x_1,x_2).F(x_2) = e^{i\phi(x_1,x_2)} s(x_1), \qquad F(x_1) = e^{i\varphi(x_1)} s(x_1) \tag{15}$$

where φ vanishes along Λ and ϕ along $\mathrm{diag}(\Lambda)$. With these notations, we have to estimate

$$\int_U e^{ik\phi(x_1,x_2)}\ f(x_1,x_2,k) b(x_1,k)\ \big(\det[g_{jk}](x_2)\big)^{\frac{1}{2}} |dx_2|$$

as $k \to \infty$. First we prove that $d_{x_2}\phi$ vanishes on $\mathrm{diag}(\Lambda)$ and $d^2_{x_2}\phi$ is definite on $\mathrm{diag}(\Lambda)$. By proposition 2.2 and remark 2.3, α_E vanishes along $\mathrm{diag}(M)$ and α_F along Λ. So derivating the first equation in (15) we obtain $d_{x_2}\phi = 0$ on $\mathrm{diag}(\Lambda)$. Derivating again, we deduce from (8) and (10) that

$$d^2_{x_2}\phi\big(X,Y\big) = \omega(X^{1,0} - q(X), Y) \text{ on } \mathrm{diag}(\Lambda). \tag{16}$$

Hence $d^2_{x_2}\phi$ is definite on $\mathrm{diag}(\Lambda)$. Indeed $d^2_{x_2}\phi\big(X,.\big) = 0$ implies $X^{1,0} = q(X)$. Since $q(X) \in T^{0,1}M$, $X^{1,0} = 0$. So $X \in (T\Lambda \otimes \mathbb{C}) \cap T^{0,1}M = (0)$. Consequently we can apply the stationary phase lemma (chapter 7.7 of [8]).

Since the phase ϕ takes complex values, we do not consider its critical set, but the ideal generated by the family $(\partial_{x_2^k}\phi)_k$. Introduce a coordinates system (x_2^i) on the second factor of $U \times U$ and a complex coordinates system (z_1^i) on the first factor. Derivating $F^{-1}(x_1)E(x_1,x_2).F(x_2)$, we obtain

$$\partial_{\bar z_1^i}\phi(x_1,x_2) \equiv \partial_{\bar z_1^i}\varphi(x_1) \mod \mathcal{I}^\infty(\mathrm{diag}\,\Lambda). \tag{17}$$

Hence $\partial_{\bar z_1^i}\partial_{x_2^k}\phi$ vanishes to any order along $\mathrm{diag}(\Lambda)$. We will deduce from this that the ideal generated by the family $(\partial_{x_2^k}\phi)_k$ is the set $\mathcal{J}$ which consists of the functions $f(x_1,x_2)$ such that

$$f|_{\mathrm{diag}\,\Lambda} = 0, \qquad \partial_{\bar z_1^i} f \equiv 0 \mod \mathcal{I}^\infty(\mathrm{diag}\,\Lambda).$$

We consider the vector fields $\partial_{\bar z_1^i}, \partial_{x_2^k}$ ($1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant 2n$). They generate a distribution transversal to $\mathrm{diag}(\Lambda)$. Working as in lemma 1.1, we associate to them the functions $\underline{\bar Z}_1^i, \underline{X}_2^k$. We will prove that every function of $\mathcal{J}$ is a linear combination of the $\underline{X}_2^k$ with C^∞ coefficients and conversely. If $f \in \mathcal{J}$, then the formal series (cf. lemma 1.2) associated to the Taylor expansion of f belongs to the ideal generated by the X_2^k and consequently f is a linear combination of the $\underline{X}_2^k$ modulo a function of $\mathcal{I}^\infty(\mathrm{diag}\,\Lambda)$. We verify that

$$\langle d\underline{X}_2^1, ..., d\underline{X}_2^n, d\underline{\bar X}_2^1, ..., d\underline{\bar X}_2^n \rangle^\perp = \mathrm{diag}(T\Lambda) \otimes \mathbb{C}.$$

So $\sum \underline{X}_2^j \underline{\bar X}_2^j$ is transversally elliptic to $\mathrm{diag}(\Lambda)$, every function of $\mathcal{I}^\infty(\mathrm{diag}\,\Lambda)$ can be divided by $\sum \underline{X}_2^k \underline{\bar X}_2^k$ and can be written as a linear combination of the $\underline{X}_2^k$. The converse is easy since the $\underline{X}_2^k$ belong to $\mathcal{J}$.

The functions $\partial_{x_2^k}\phi$ belong to $\mathcal{J}$, so they are of the form

$$\partial_{x_2^k}\phi = \sum_j f_{kj} \underline{X}_2^j.$$

If $x \in \mathrm{diag}(\Lambda)$, we have

$$f_{kj}(x) = \partial_{x_2^j}\partial_{x_2^k}\phi(x).$$

Hence, f_{kj} is invertible on a neighborhood of $\mathrm{diag}(\Lambda)$, the $\underline{X}_2^j$ are linear combination of the $\partial_{x_2^k}\phi$. This proves that the ideal generated by the $\partial_{x_2^k}\phi$ is $\mathcal{J}$.

From (17) we deduce that $\phi(x_1,x_2) = \varphi(x_1) \mod \mathcal{J}$. We obtain

$$\Pi_k\big(F^k b(.,k)\big)(x) = F^k(x) c(x,k) + O(k^{-\infty})$$

where $(c(.,k))$ is a symbol of $S^0(M)$. Derivating the previous equality with respect to any antiholomorphic vector field, we deduce from lemma 1.3 that $c(.,k)$ satisfies (12). Furthermore if $x \in \Lambda$, since $f(x,x,k) = 1 + O(k^{-1})$ we have

$$c_0(x) = b_0(x).\big(\det[-i\partial_{x_2^j}\partial_{x_2^k}\phi](x,x)\big)^{-\frac{1}{2}}.\big(\det[g_{jk}](x)\big)^{\frac{1}{2}}$$

where the g_{jk} are the coefficient of the Riemannian metric $g = \sum_{j,k} g_{jk} dx_2^j \otimes dx_2^k$. Since $c_0(x)$ does not depend on the coordinates (x_2^j), we can choose them to compute easily the two determinants. If $(\partial_{x_2^i})_{i=1,\dots,n}$ is an orthonormal base of $T_x\Lambda$ and $\partial_{x_2^{i+n}} = J\partial_{x_2^i}$ at x, then $g_{jk}(x) = \delta_{jk}$. And it follows from (16) that the matrix $-i\partial_{x_2^j}\partial_{x_2^k}\phi(x,x)$ is :

$$\frac{1}{2}\begin{pmatrix} \mathrm{Id} & -i\,\mathrm{Id} \\ -i\,\mathrm{Id} & 3\,\mathrm{Id} \end{pmatrix}$$

We deduce from this that $c_0(x) = b_0(x)$. □

2.3. **Norm of the Lagrangian sections.** The following proposition is a consequence of the stationary phase lemma.

Proposition 2.6. *Let (u_α, k_α) and (v_α, k_α) be Lagrangian sections over U with the same section F and principal symbols f_0, $g_0 \in C^\infty(U \cap \Lambda)$. Then*

$$\int_U h(u_\alpha, v_\alpha)\mu_M = \Big(\frac{\pi}{k_\alpha}\Big)^{\frac{n}{2}} \int_{\Lambda\cap U} f_0.\bar{g}_0 \; \mu_\Lambda + O(k_\alpha^{-\frac{n}{2}-1})$$

where μ_Λ is the measure of Λ induced by the Riemannian structure g.

More generally, we can estimate the integral of $h(u_\alpha, v_\alpha)\mu_M$, where (u_α, k_α) and (v_α, k_α) are Lagrangian sections over U associated to Lagrangian manifolds Λ and Λ' respectively such that the intersection of Λ with Λ' is non-degenerate (cf. [2]). For example, when the dimension is $n = 1$, assume that $\Lambda \cap \Lambda' = \{x\} \subset U$ and that this intersection is transversal. If $F(x) = F'(x)$, we have

$$\int_U h(u_\alpha, v_\alpha)\mu_M = \Big(\frac{\pi}{k_\alpha}\Big) f_0(x).\bar{g}_0(x)\sqrt{1+ia} + O\big(k_\alpha^{-2}\big)$$

where f_0 and g_0 are the principal symbols of (u_α) and (v_α) and $a = \operatorname{cotan}\theta$ if θ is the angle between $T_x\Lambda$ and $T_x\Lambda'$. The square root is chosen so as to be continuous with respect to a and to take the value 1 when $a = 0$.

Proof of proposition 2.6. Recall the notation $|F|^2 = e^{-\delta}$ (cf. proposition 2.2). We estimate the integral

$$\int_U e^{-k\delta(x)} a(x,k)\bar{b}(x,k)\mu_M(x).$$

Choose a coordinates system (x^j, y^j) such that $\Lambda = \{y^1 = \dots = y^n = 0\}$. We may assume that the orthogonal set of $T_x\Lambda$ is $\langle \partial_{y^1}, \dots, \partial_{y^n}\rangle|_x$ for every $x \in \Lambda \cap V$. Then the metric is given along Λ by

$$g(x^l, 0) = g_{jk}(x^l, 0) dx^j \otimes dx^k + g'_{jk}(x^l, 0) dy^j \otimes dy^k.$$

Furthermore by proposition 2.2, $\delta \equiv g'_{jk}(x^l,0)y^jy^k \mod \mathcal{I}^3(\Lambda)$. We have

$$\int e^{-k\delta(x)}a(x,k)\bar{b}(x,k)\mu_M(x) = \int |dx^j| \int e^{-k\delta(x)} a(x,k)\bar{b}(x,k)\big(\det[g]\big)^{\frac{1}{2}}|dy^j|$$
$$= \Big(\frac{\pi}{k}\Big)^{\frac{n}{2}} \int f_0.\bar{g}_0\big(\det[g_{j,k}]\big)^{\frac{1}{2}}|dx^j| + O(k^{-\frac{n}{2}-1})$$

by the stationary phase lemma. □

2.4. **Action of the Toeplitz operators.** We consider now the action of the Toeplitz operators on the Lagrangian sections.

Proposition 2.7. *If (T_k) is a Toeplitz operator and (u_α, k_α) a Lagrangian section over U, then $(T_{k_\alpha}u_\alpha, k_\alpha)$ is a Lagrangian section over U. Furthermore, there exists a sequence of bilinear operators $L_l : C^\infty(M)\times C^\infty(\Lambda)\rightarrow C^\infty(\Lambda)$ such that the symbol of $(T_{k_\alpha}u_\alpha)$ is*

$$\sum \hbar^l \sum_{l_1+l_2+l_3=l} L_{l_1}(f_{l_2}, g_{l_3})$$

if $\sigma(T_k) = \sum \hbar^l f_l$ and the symbol of (u_α) is $\sum \hbar^l g_l$. The operators L_l depend only on Λ, M and its Kählerian structure, and

- *L_0 is the map which sends $f\in C^\infty(M)$, $g \in C^\infty(\Lambda)$ into $f|_\Lambda.g$.*
- *L_1, L_2, ... are locally on the form*

$$L_l(f,g)|_{\Lambda\cap V} = \sum_{|\alpha|+|\gamma|\leqslant 2l} a_{\alpha,\gamma}\partial^\alpha_{\bar{z}}f|_{\Lambda\cap V}.\partial^\gamma_x g, \qquad \forall f\in C^\infty(M),\ g\in C^\infty(\Lambda) \tag{18}$$

where V is an open set of M, (z^i) a complex coordinates system defined on V, (x^i) a coordinates system of Λ defined on $\Lambda\cap V$ and $a_{\alpha,\gamma}\in C^\infty(\Lambda\cap V)$.

Proof. The proof is the same as the proof of lemma 2.5 except that we replace (Π_k) with (T_k) and that we have to compute the full asymptotic expansion. As in the proofs of proposition 2.1 and lemma 2.5, we introduce a local section s with constant norm equal to 1 such that $F|_\Lambda = s|_\Lambda$ and a function ρ such that $\nabla s = (\bar\partial\rho - \partial\bar\rho)\otimes s$. We have to estimate

$$\Big(\frac{k}{2\pi}\Big)^n s^k(x_1)\int e^{ik\phi(x_1,x_2)} f(z_1,\bar{z}_2)g(z_2)d\mu_M(x_2) \tag{19}$$

as k tends to ∞. From the proof of proposition 2.1 and [5], the phase ϕ is given by

$$\phi(x_1,x_2) = i\big(\rho(x_1)+\bar\rho(x_2) - (\rho+\bar\rho)(z_1,\bar{z}_2) + \rho(x_2)-\rho(z_2)\big).$$

Let us explain the notations : if $f\in C^\infty(U)$, $f(z_1,\bar{z}_2)$ is a function $\tilde{f}$ defined on $U\times U$ such that $\tilde{f}(x,x)=f(x)$ and the derivatives $\partial_{\bar{z}_1^i}\tilde{f}$, $\partial_{z_2^i}\tilde{f}$ vanish to any order along $\operatorname{diag}(U)$. In the same way, if $g\in C^\infty(\Lambda)$ then $g(z)$ is a function $\tilde{g}$ defined on U which restricts on Λ to g and such that the derivatives $\partial_{\bar{z}^i}\tilde{g}$ vanish to any order along Λ.

We introduce some notations to handle the Taylor expansion along $\operatorname{diag}(\Lambda)$ and the Taylor expansion along Λ. Following lemmas 1.1 and 1.2, we identify the Taylor expansion along Λ of the functions of $C^\infty(U)$ with the formal series of $C^\infty(\Lambda)[[Z^i]]$ (the functions $\underline{Z}^i_1$ are associated to the vector fields ∂_{z^i}). In the same way we identify the Taylor expansion along $\operatorname{diag}(\Lambda)$ of the functions of $C^\infty(U\times U)$ with the formal series of $C^\infty(\Lambda)[[\bar{Z}^i_1, Z^i_2, \bar{Z}^i_2]]$, (the functions $\underline{\bar{Z}}^i_1, \underline{Z}^i_2, \underline{\bar{Z}}^i_2$ are associated to the vector fields $\partial_{\bar{z}^i_1}, \partial_{z^i_2}, \partial_{\bar{z}^i_2}$). Observe that the functions $\underline{\bar{Z}}^i_2$ are not conjugated to $\underline{Z}^i_2$.

As we saw in the proof of lemma 2.5, the ideal generated by the $\partial_{z_2^i}\phi$ and $\partial_{\bar{z}_2^i}\phi$ is the set $\mathcal{J}$ which consists of the functions whose Taylor expansion belongs to the ideal generated by the $Z_2^i, \bar{Z}_2^i$. If the Taylor expansion of $f \in C^\infty(U \times U)$ is

$$\sum f_{\alpha,\beta,\gamma} \bar{Z}_1^\alpha Z_2^\beta \bar{Z}_2^\gamma$$

then the Taylor expansion of a function $g \in C^\infty(U)$ such that $g(x_1) = f(x_1, x_2)$ mod $\mathcal{J}$ is

$$\sum f_{\alpha,0,0} \bar{Z}^\alpha.$$

We have $F(x) = e^{i\varphi(x)} s(x)$ with $\varphi(x) = i(\rho(x) - \rho(z))$. Let us compute the Taylor expansion of $\phi(x_1, x_2) - \varphi(x_1)$. Introduce the functions $G_{\alpha,\beta} = \partial_z^\alpha \partial_{\bar{z}}^\beta (\rho + \bar{\rho})|_\Lambda$. We have

$$\rho(x_2) + \bar{\rho}(x_2) \sim \sum_{\alpha,\beta} \frac{G_{\alpha,\beta}}{\alpha!\beta!} Z_2^\alpha \bar{Z}_2^\beta, \qquad \rho(z_1) \sim \rho_0,$$

$$(\rho + \bar{\rho})(z_1, \bar{z}_2) \sim \sum_\beta \frac{G_{0,\beta}}{\beta!} \bar{Z}_2^\beta, \qquad \rho(z_2) \sim \sum_\alpha \frac{\rho_\alpha}{\alpha!} Z_2^\alpha$$

where the ρ_α are the restrictions on Λ of the successive derivatives of $\rho(z)$ with respect to ∂_{z^i}. We deduce from this that

$$\phi(x_1, x_2) - \varphi(x_1) \sim i \sum_{|\alpha|>0, |\beta|>0} \frac{G_{\alpha,\beta}}{\alpha!\beta!} Z_2^\alpha \bar{Z}_2^\beta + i \sum_{|\alpha|>0} \frac{G_{\alpha,0} - \rho_\alpha}{\alpha!} Z_2^\alpha.$$

From the proof of lemma 2.5, $\phi(x_1, x_2) - \varphi(x_1)$ vanishes to the second order along $\operatorname{diag}(\Lambda)$, so $G_{i,0} = \rho_i$. This can also be directly checked using that $\langle X, \bar{\partial}\rho - \partial\bar{\rho} \rangle = 0$ if X is tangent to Λ. If $x \in \Lambda$, the matrice of $-id^2_{x_2}\phi(x, x)$ in the base $\partial_{z_2^i}, \partial_{\bar{z}_2^i}$ is

$$\begin{pmatrix} G_{ij,0} - \rho_{ij} & G_{i,j} \\ G_{j,i} & 0 \end{pmatrix}.$$

From (16), $G_{ij,0} - \rho_{ij} = i\omega(q(\partial_{z^i}), \partial_{z^j}) = q_i^l G_{j,l}$. The inverse of this matrix is

$$\begin{pmatrix} 0 & G^{i,j} \\ G^{j,i} & -q_l^i G^{l,j} \end{pmatrix}.$$

Applying theorem 7.7.12 of [8], we obtain that (19) is equal to

$$s^k(x_1) e^{ik\varphi(x_1)} h(x_1, k) + O_\infty(k^{-\infty})$$

where $(h(., k))$ admits an asymptotic expansion $\sum_l k^{-l} h_l$ for the C^∞ topology. Furthermore, the Taylor expansion along Λ of the coefficients is given by

$$h_l \sim [\det(G_{ij})]^{-1} \sum_{k=l}^{3l} \frac{(-1)^{l-k}}{k!(k-l)!} \left[\Delta^k (R^{k-l} F.G.D) \right]_{\substack{Z_2^i = \bar{Z}_2^i = 0 \\ \bar{Z}^i = \bar{Z}_1^i}}. \tag{20}$$

F, G are the formal series associated to the Taylor expansion of $f(z_1, \bar{z}_2)$ and $g(z_2)$:

$$F = \sum_\beta \frac{1}{\beta!} \partial_{\bar{z}}^\beta f|_\Lambda \bar{Z}_2^\beta, \qquad G = \sum_\alpha \frac{g_\alpha}{\alpha!} Z_2^\alpha$$

where the g_α are the restrictions on Λ of the successive derivatives of $g(z)$ with respect to ∂_{z^i}.

$$R = \sum_{\substack{|\alpha|>0, |\beta|>0, \\ |\alpha|+|\beta| \geqslant 3}} \frac{G_{\alpha,\beta}}{\alpha!\beta!} Z_2^\alpha \bar{Z}_2^\beta + \sum_{|\alpha| \geqslant 3} \frac{G_{\alpha,0} - \rho_\alpha}{\alpha!} Z_2^\alpha.$$

D is the formal series associated to the Taylor expansion of $\det[\partial_{z^i}\partial_{\bar{z}^j}(\rho+\bar{\rho})](x_2)$ and Δ is the operator

$$\Delta = \sum_{i,j} G^{i,j} \partial_{Z_2^i} \partial_{\bar{Z}_2^j} - \frac{1}{2} q_l^i G^{l,j} \partial_{\bar{Z}_2^i} \partial_{\bar{Z}_2^j}.$$

Since the formal variables $\bar{Z}_1^i$ do not enter in the computation, the derivatives with respect to $\partial_{\bar{z}^i}$ of the functions h_l vanish to every order along Λ. The restriction of the functions h_l to Λ is given by the above formula. To prove that the operators L_l are locally of the form (18), it suffices to prove that $g_\alpha = P.g$ where P is a differential operator $C^\infty(\Lambda) \to C^\infty(\Lambda)$. The differential of $g(z)$ at $x \in \Lambda$ vanishes on $T_x^{0,1}M$ and its restriction on $T_x\Lambda$ is dg. Consequently,

$$g_i = \big(\partial_{z^i} g(z)\big)|_\Lambda = q^c(\partial_{z^i}).g. \tag{21}$$

This gives the result for g_i. We generalize to the functions g_α by induction on $|\alpha|$ by using that $\partial_{z^i} g(z)$ is also a function whose derivatives with respect to $\partial_{\bar{z}^i}$ vanish to any order along Λ. The computation of $L_0(f,g)$ was done in the proof of lemma 2.5. □

2.5. **Subsymbolic calculus.** As we saw in the previous subsection, every Toeplitz operator induces a map $T : C^\infty(\Lambda)[[\hbar]] \to C^\infty(\Lambda)[[\hbar]]$ of the form

$$Th = f_0|_\Lambda.h + \hbar(f_1|_\Lambda.h + L_1(f_0,h)) + O(\hbar^2)$$

where $\sigma(T_k) = f_0 + \hbar f_1 + O(\hbar^2)$. The purpose of this section is to compute $L_1(f_0,h)$. From this result, we will deduce the following theorem that we will use to compute the Bohr-Sommerfeld conditions modulo $O(\hbar^2)$.

Theorem 2.8. *If the Weyl symbol of (T_k) is $f_0 + \hbar f_1^w + O(\hbar^2)$ and $\Lambda \subset \{f_0 = E\}$ where E is a real number, then*

$$Th = Eh - i\hbar\Big(\mathcal{L}_{X_{f_0}}.h + \big(if_1^w - \tfrac{i}{2}H.f_0 + \tfrac{1}{2}\operatorname{div}_{\Lambda_E}(X_{f_0})\big)\ h\Big) + O(\hbar^2)$$

where

- *X_{f_0} is the Hamiltonian vector field of f_0 (i.e. $df_0 + \iota_{X_{f_0}}\omega = 0$),*
- *$H \in C^\infty(\Lambda, JT\Lambda)$ is the mean curvature vector field of Λ,*
- *$div_\Lambda : C^\infty(\Lambda, T\Lambda) \to C^\infty(\Lambda)$ is the divergence with respect to the measure μ_Λ induced by the Riemannian metric.*

To state the result about $L_1(f_0,h)$, we need to define an operator $\Box : C^\infty(M) \to C^\infty(\Lambda)$. First let P^2_∇ be the operator

$$P^2_\nabla : C^\infty(M) \xrightarrow{\partial} C^\infty(M, \Lambda^{1,0}M) \xrightarrow{\nabla^{\Lambda^{1,0}M}} C^\infty(M, \Lambda^{1,0}M \otimes \Lambda M) \tag{22}$$

where $\nabla^{\Lambda^{1,0}M}$ is the covariant derivation of the holomorphic Hermitian bundle $\Lambda^{1,0}M$. Denote the conjugate operator by $\bar{P}^2_\nabla : C^\infty(M) \to C^\infty(M, \Lambda^{0,1}M \otimes \Lambda M)$. If $x \in \Lambda$, recall that $q|_x$ is the projection of $T_xM \otimes \mathbb{C}$ with image $T_x^{0,1}M$ and kernel $T_x\Lambda \otimes \mathbb{C}$. The restriction of q on $T^{1,0}M$ defines a tensor

$$q_k^j dz^k \otimes \partial_{\bar{z}^j} \in C^\infty(\Lambda, \Lambda^{1,0}M \otimes T^{0,1}M).$$

By contracting with $G^{-1} = G^{jk}\partial_{z^j} \otimes \partial_{\bar{z}^k}$, this gives the tensor

$$q_l^j G^{l,k} \partial_{\bar{z}^j} \otimes \partial_{\bar{z}^k} \in C^\infty(\Lambda, T^{0,1}M \otimes T^{0,1}M).$$

Finally we set

$$\Box f = q_l^j G^{l,k} f_{jk}|_\Lambda \text{ where } \bar{P}^2_\nabla f = f_{jk} d\bar{z}^j \otimes d\bar{z}^k.$$

Proposition 2.9. *If $f \in C^\infty(M)$, $g \in C^\infty(\Lambda)$ then*

$$L_1(f,g) = -\tfrac{1}{2}(r.f)|_\Lambda g - \tfrac{1}{2}(\Box f).g - i\mathcal{L}_{q^c(X_f)}.g$$

where r is the scalar curvature of M and $q^c|_x$ is the projection of $T_xM \otimes \mathbb{C}$ with image $T_x\Lambda \otimes \mathbb{C}$ and kernel $T_x^{0,1}M$.

Proof. We start from the proof of proposition 2.7. Choose a coordinates system on a neighborhood of $x \in M$, such that $G_{i,jk}(x) = G_{ij,k}(x) = 0$. From (20) a direct computation gives

$$h_1 = \frac{1}{2}G_{ij,ij}.f.g - \frac{1}{2}q_l^i G^{l,j}(\partial_{\bar{z}^i}\partial_{\bar{z}^j} f).g + G^{i,j}(\partial_{\bar{z}^i} f)g_j$$

at x where $g_i = \big(\partial_{z^i} g(z)\big)|_\Lambda$. We recognize the scalar curvature r and $\bar{P}^2_\nabla f$, which are given at x by

$$r = G_{ij,ij}, \qquad \bar{P}^2_\nabla f = (\partial_{\bar{z}^i}\partial_{\bar{z}^j} f) d\bar{z}^i \otimes d\bar{z}^j.$$

To recognize the last term of the sum, observe that

$$X_f = -iG^{j,k}(\partial_{z^j} f)\partial_{\bar{z}^k} + iG^{j,k}(\partial_{\bar{z}^k} f)\partial_{z^j}.$$

So $q^c(X_f) = iq^c(G^{j,k}\partial_{\bar{z}^k} f \partial_{z^j})$. Then the results follows from (21). □

We will give another formulation of this result when the Hamiltonian vector field of f is tangent to Λ. Recall that the second fundamental form of Λ is the section

$$\sigma \in C^\infty(\Lambda, T^*\Lambda \otimes T^*\Lambda \otimes JT\Lambda) \text{ such that } \sigma(X,Y) = \nabla^{TM}_X Y - \nabla^{T\Lambda}_X Y$$

where ∇^{TM} and $\nabla^{T\Lambda}$ are the Levi-Civita connections of (M,g) and $(\Lambda, g|_{T\Lambda})$. The mean curvature vector field is $H \in C^\infty(\Lambda, JT\Lambda)$ defined by $H = \operatorname{tr}\sigma$.

Proposition 2.10. *If the Hamiltonian vector field of $f \in C^\infty(M)$ is tangent to Λ, then*

$$(\Box f)|_\Lambda = (\Delta f)|_\Lambda + H.f + i\operatorname{div}_\Lambda(X_f)$$

and consequently

$$L_1(f,g) = -\tfrac{1}{2}(r.f + \Delta f)|_\Lambda g - \tfrac{i}{2}\operatorname{div}_\Lambda(X_f).g - \tfrac{1}{2}(H.f).g - i\mathcal{L}_{X_f}.g.$$

From the definition of the Weyl symbol (7), we obtain theorem 2.8 as a corollary of this proposition.

Proof. Let (X_j) be an orthonormal base of $T_x\Lambda$. Let $Y_j = JX_j$. So (X_j, Y_j) is an orthonormal base of T_xM. Let (ξ^j, η^j) be the dual base ($\eta^j = -J^t\xi^j$). The family $Z_j = \frac{1}{\sqrt{2}}(X_j - iY_j)$ is a base of $T_x^{1,0}M$ whose dual base is $\big(\zeta^j = \frac{1}{\sqrt{2}}(\xi^j + i\eta^j)\big)$. We have $G = \zeta^j \otimes \bar{\zeta}^j$, so $G^{-1} = Z_j \otimes \bar{Z}_j$. The restriction of q at $T^{1,0}M$ is $-\bar{\zeta}^j \otimes Z_j$. Contracting with G^{-1}, this gives

$$-Z_j \otimes Z_j = -\frac{1}{2}(X_j \otimes X_j - Y_j \otimes Y_j) + \frac{i}{2}(X_j \otimes Y_j + Y_j \otimes X_j).$$

Recall that on a Kähler manifold, the Levi-Civita connection ∇^{TM} preserves $T^{1,0}M$ and $T^{0,1}M$, is compatible with G and restricts on $T^{1,0}M$ to the covariant derivation of the holomorphic Hermitian bundle $T^{1,0}M$ (cf. [1]). So, ∇^{T^*M} preserves $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$, is compatible with G^{-1} and restricts on $\Lambda^{1,0}M$ to the covariant derivation of the holomorphic Hermitian bundle $\Lambda^{1,0}M$.

We extend the (X_j) on a neighborhood U of x so that they give an orthonormal base of $T_y\Lambda$ when $y \in \Lambda \cap U$. Define as above the vector and covector fields $Y_j, Z_j, \xi^j, \eta^j, \zeta^j$. Then we have on $U \cap \Lambda$

$$\Box f = -\langle \nabla^{T^*M} df, Z_j \otimes Z_j \rangle, \qquad \Delta f = \langle \nabla^{T^*M} df, Z_j \otimes \bar{Z}_j \rangle$$

$$\Rightarrow \Box f - \Delta f = -\langle \nabla^{T^*M} df, X_j \otimes X_j \rangle + i \langle \nabla^{T^*M} df, Y_j \otimes X_j \rangle. \tag{23}$$

Write $df = (X_j f)\xi^j + (Y_j f)\eta^j$, and observe that $X_j.f$ vanishes on Λ since X_f is tangent to Λ. We have then on $U \cap \Lambda$

$$\begin{aligned} \langle \nabla^{T^*M} df, X_j \otimes X_j \rangle &= (Y_k f) \langle \nabla^{T^*M}_{X_j} \eta^k, X_j \rangle \\ &= -(Y_k f) \langle \eta^k, \nabla^{TM}_{X_j} X_j \rangle \qquad \text{since } d\langle \eta^k, X_j \rangle = 0 \\ &= -(Y_k f) \langle \eta^k, \nabla^{TM}_{X_j} X_j - \nabla^{T\Lambda}_{X_j} X_j \rangle \end{aligned}$$

since $\nabla^{T\Lambda}_{X_j} X_j$ is tangent to Λ. Consequently, $\langle \nabla^{T^*M} df, X_j \otimes X_j \rangle = -H.f$. We treat now the second term of (23)

$$\begin{aligned} \langle \nabla^{T^*M} df, Y_j \otimes X_j \rangle &= X_j(Y_j f) + (Y_k f) \langle \nabla^{T^*M}_{X_j} \eta^k, Y_j \rangle \\ &= X_j(Y_j f) - (Y_k f) \langle \eta^k, \nabla^{TM}_{X_j} Y_j \rangle \end{aligned}$$

on $U \cap \Lambda$. Recall that $\operatorname{div}_\Lambda X = -\operatorname{tr} \nabla^{T\Lambda} Y$ (cf [1]). From $X_f = -(Y_k f) X_k$ on Λ, we deduce that

$$\begin{aligned} \operatorname{div}_\Lambda X_f &= X_j(Y_j f) + (Y_k f) \langle \xi^j, \nabla^{T\Lambda}_{X_j} X_k \rangle \\ &= X_j(Y_j f) - (Y_k f) \langle \xi^k, \nabla^{T\Lambda}_{X_j} X_j \rangle \end{aligned}$$

since $\langle \xi^j, \nabla^{T\Lambda}_{X_j} X_k \rangle = g(X_j, \nabla^{T\Lambda}_{X_j} X_k) = -g(\nabla^{T\Lambda}_{X_j} X_j, X_k) = -\langle \xi^k, \nabla^{T\Lambda}_{X_j} X_j \rangle$. But

$$\nabla^{T\Lambda}_{X_j} X_j - \nabla^{TM}_{X_j} X_j \in JT\Lambda.$$

We obtain that $\langle \nabla^{T^*M} df, Y_j \otimes X_j \rangle = \operatorname{div}_\Lambda X_f$. $\Box$

2.6. **Lagrangian sections associated to a fibration by Lagrangian Tori.** Let us consider an open set of M diffeomorphic to the product $B_r \times \mathbb{T}^n$, where $B_r \subset \mathbb{R}^n$ is the open ball of radius r with center 0 and $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Denote by ξ^i and x^i the usual coordinates of $B_r \subset \mathbb{R}^n$ and $\mathbb{T}^n$. We assume that

$$\omega|_{B_r \times \mathbb{T}^n} = d\xi^i \wedge dx^i. \tag{24}$$

We are interested in the family of Lagrangian submanifolds

$$\Lambda_\gamma = \{(\xi, x) / \xi^i = \gamma^i, \ \forall i\} \subset M$$

where γ belongs to $\Gamma = B_{r/2}$. Denote by π the projection $\Gamma \times B_r \times \mathbb{T}^n \to B_r \times \mathbb{T}^n$.

We will define Lagrangian section $(u_\alpha, k_\alpha, \gamma_\alpha)$, where $u_\alpha \in \mathcal{H}_{k_\alpha}$ is associated to Λ_{γ_α} for every α. Locally, they are of the following form

$$u_\alpha = \Big(\frac{k_\alpha}{\pi} \Big)^{\frac{n}{4}} F_V^{k_\alpha}(\gamma_\alpha, .) a_V(\gamma_\alpha, ., k_\alpha) + O_\infty(k_\alpha^{-\infty}) \text{ on } B_r \times V \tag{25}$$

where V is an open contractible set of $\mathbb{T}^n$ and

- F_V is a section of $\pi^\# L$ defined on $\Gamma \times B_r \times V$, such that $F_V(\gamma, .)$ is flat along Λ_γ and $\nabla_{\bar{Z}} F_V$ vanishes at every order along $\{(\gamma, \xi, x)/\gamma = \xi\}$ for every vector field $Z \in C^\infty(\Gamma \times B_r \times \mathbb{T}^n, T^{1,0}(B_r \times \mathbb{T}^n))$.
- a_V is a symbol of $S^0(\Gamma \times B_r \times V)$, with asymptotic expansion $\sum k^{-l} a_{V,l}$ such that $\bar{Z}.a_{V,l}$ vanishes at every order along $\{(\gamma, \xi, x)/\gamma = \xi\}$ for every vector field $Z \in C^\infty(\Gamma \times B_r \times \mathbb{T}^n, T^{1,0}(B_r \times \mathbb{T}^n))$.

Locally the symbol will be defined as the formal series $\sum \hbar^l f_{V,l}$ of $C^\infty(\Gamma \times V)[[\hbar]]$ such that

$$f_{V,l}(\gamma, x) = a_{V,l}(\gamma, \gamma, x). \tag{26}$$

Globally, it will be natural to consider it as a section of a flat $\mathbb{C}[[\hbar]]$-bundle.

First let us introduce the notion of a flat $\mathbb{C}[[\hbar]]$-bundle $K \to \mathbb{T}^n$ of rank 1 with structural group $\{e^{i\varphi(\hbar)} \ / \ \varphi(\hbar) \in \mathbb{R}[[\hbar]]\}$. Such a bundle is locally isomorphic to $K|_V \simeq V \times \mathbb{C}[[\hbar]]$ and the transition functions are of the form

$$(V \cap W) \times \mathbb{C}[[\hbar]] \to (V \cap W) \times \mathbb{C}[[\hbar]], \quad (x, c(\hbar)) \to (x, c(\hbar) e^{i\varphi(\hbar)})$$

where $\varphi(\hbar) \in \mathbb{R}[[\hbar]]$ and $e^{i\varphi(\hbar)}$ is defined by

$$e^{i\varphi(\hbar)} = e^{i\varphi_0} \sum_m \tfrac{1}{m!}\big(i \textstyle\sum_{l=1}^{l=\infty} \hbar^l \varphi_l\big)^m, \quad \text{ if } \varphi(\hbar) = \sum \hbar^l \varphi_l.$$

Using the flat structure we can introduce the parallel transport $K|_{\delta(0)} \to K|_{\delta(1)}$ along a path $\delta : [0,1] \to \mathbb{T}^n$. Let $\delta_1, ..., \delta_n$ be loops with the same base point x such that $([\delta_j])$ is a base of $H_1(\mathbb{T}^n, \mathbb{Z})$. The parallel transport along δ_j is a map $K|_x \to K|_x$ of the form $c(\hbar) \to e^{2\pi i \varphi^j(\hbar)} c(\hbar)$. The holonomy of the loop δ_j is by definition $\varphi^j(\hbar) \in \mathbb{R}[[\hbar]]/\mathbb{Z}$.

Now let $K \to \Gamma \times \mathbb{T}^n$ be a $\mathbb{C}[[\hbar]]$-bundle of rank one with transition functions of the form

$$\Gamma \times (V \cap W) \times \mathbb{C}[[\hbar]] \to \Gamma \times (V \cap W) \times \mathbb{C}[[\hbar]], \quad (\gamma, x, c(\hbar)) \to (\gamma, x, c(\hbar) e^{i\varphi(\gamma,\hbar)})$$

where $\varphi(\gamma, \hbar) \in C^\infty(\Gamma, \mathbb{R})[[\hbar]]$. For every $\gamma \in \Gamma$, this bundle restricts on $\{\gamma\} \times \mathbb{T}^n \simeq \Lambda_\gamma$ to a flat bundle K_γ as we considered below. A C^∞ section of K is map $f : \Gamma \times \mathbb{T}^n \to K$, locally of the form

$$\Gamma \times V \to \Gamma \times V \times \mathbb{C}[[\hbar]], \qquad (\gamma, x) \to (\gamma, x, f_V(\gamma, x, \hbar)) \tag{27}$$

where $f_V \in C^\infty(\Gamma \times V)[[\hbar]]$. These sections will be the symbols of the Lagrangian sections.

Finally let us give the quantization condition. Fix a base $([\delta_i])$ of $H_1(\mathbb{T}^n, \mathbb{Z})$ such that $\delta_1, ..., \delta_n$ have the same base point x. Denote by $\sum \hbar^l \varphi^i_l(\gamma)$ the holonomy of δ^i for the bundle $K|_\gamma$ and by $\varphi^i_{-1}(\gamma)$ the holonomy of δ^i for the bundle $L|_{\Lambda_\gamma}$. We assume that the sequence $(\gamma_\alpha, k_\alpha)$ of $\Gamma \times \mathbb{N}$ satisfies

$$\varphi_{-1}(\gamma_\alpha) - k_\alpha^{-1} \varphi(\gamma_\alpha, k_\alpha) \in k_\alpha^{-1} \mathbb{Z}^n + O(k_\alpha^{-\infty}) \tag{28}$$

where $\varphi(\gamma, k) = \sum_{l \geqslant 0} k^{-l} \varphi^i_l(\gamma) + O(k^{-\infty})$.

Let f be a section of K and let us define a sequence $(u_\alpha, k_\alpha, \gamma_\alpha)$ associated. Fix two sections $t_L : \Gamma \times \{0\} \to L$ and $t_K : \Gamma \times \{0\} \to K$ with constant norms equal to 1. If V is an open contractible set of $\mathbb{T}^n$, choose a path $\delta : [0,1] \to \mathbb{T}^n$ with $\delta(0) = 0$ and $\delta(1) \in V$. Then define the section F_V in (25) in such a way that $F_V(\gamma, \gamma, \delta(1))$ is the parallel transport of $t_L(\gamma, 0)$ along $\{\gamma\} \times \delta$. Consider the trivialization $K|_{\Gamma \times V} \simeq \Gamma \times V \times \mathbb{C}[[\hbar]]$ such that the parallel transport along $\{\gamma\} \times \delta$ sends $t_K(\gamma, 0)$ into $(\gamma, \delta(1), 1)$. This defines the formal series f_V by (27) and we can introduce a symbol $a_V(., k)$ which satisfies (26).

Hence we defined the left hand side of (25) for every open contractible set V of $\mathbb{T}^n$. The point is that these expressions patch together modulo $O_\infty(k_\alpha^{-\infty})$ because of the quantization condition (28). So using a partition of unity, we can introduce an admissible sequence (v_α, k_α) which restricts over every $B_r \times V$ to these local expressions. Then we set $u_\alpha = \Pi_{k_\alpha} v_\alpha$. By proposition 2.4 that we can generalize with parameters, we know that $u_\alpha = v_\alpha + O_\infty(k_\alpha^{-\infty})$.

It is straightforward to generalize the propositions 2.6 et 2.7. We just state the results. The norm $|f|^2$ is a formal series of $C^\infty(\Gamma \times \mathbb{T}^n, \mathbb{R})[[\hbar]]$. The norm of the section u_α is estimated by

$$(u_\alpha, u_\alpha) = \int_{\Lambda_{\gamma_\alpha}} g(\gamma_\alpha, x)\, \mu_{\Lambda_{\gamma_\alpha}}(x) + O(k_\alpha^{-1}) \tag{29}$$

with $|f|^2 = g + O(\hbar)$.

If (T_k) is a Toeplitz operator, we can describe the sequence $(T_{k_\alpha} u_\alpha)$ in the following way: as we saw in proposition 2.7, (T_k) induces an action on the space of symbol $C^\infty(\Lambda_\gamma)[[\hbar]]$, for every γ. Since K_γ is flat, this gives a map $T_\gamma : C^\infty(\Lambda_\gamma, K_\gamma) \to C^\infty(\Lambda_\gamma, K_\gamma)$, and consequently a map

$$T : C^\infty(\Gamma \times \mathbb{T}^n, K) \to C^\infty(\Gamma \times \mathbb{T}^n, K).$$

Applying this operator to the symbol f, we obtain a symbol Tf and so a Lagrangian section $(w_\alpha, k_\alpha, \gamma_\alpha)$ (we define it using the same sections t_L and t_K as we chose to define (u_α)). Then the result is that

$$w_\alpha = T_{k_\alpha} u_\alpha + O(k_\alpha^{-\infty}).$$

To end this section, we discuss the quantization condition (28). First, observe that it does not depend on the choice of the base $([\delta_i])$. Furthermore, using (24) and that the curvature of L is $\frac{1}{i}\omega$, we prove that the maps φ^j_{-1} are affine

$$\varphi^j_{-1}(\gamma) = \varphi^j_{-1}(0) + \frac{\gamma^j}{2\pi}.$$

Hence the map φ_{-1} is a diffeomorphism of B_r onto its image, the same holds with the functions $\varphi_{-1} - k^{-1}\varphi(.,k)$ if k is sufficiently large. The inverse $r(.,k)$ of $\varphi_{-1} - k^{-1}\varphi(.,k)$ is well-defined on $\varphi_{-1}(0) + (2\pi)^{-1}B_{r-\epsilon}$ and admits an asymptotic expansion $\sum_l k^{-l} r_l$ with $r_0(\bar{\gamma}) = 2\pi(\bar{\gamma} - \varphi^j_{-1}(0))$. We have

$$\bar{\gamma}_\alpha \in k_\alpha^{-1}\mathbb{Z}^n + O(k_\alpha^{-\infty}) \text{ iff } \gamma_\alpha = r(\bar{\gamma}_\alpha, k_\alpha) \text{ satisfies (28).}$$

Hence (28) says that γ_α takes its values in the deformed lattices $r(k_\alpha^{-1}\mathbb{Z}^n, k_\alpha) + O(k_\alpha^{-\infty})$.

3. The Bohr-Sommerfeld conditions

Let $(T_k^1), ..., (T_k^n)$ be Toeplitz operators which commute. The joint spectrum of these operators is the sequence of subsets of $\mathbb{R}^n$:

$$\operatorname{Sp}(T_k) = \big\{ (E^1, ..., E^n) \ / \ \exists \ v \in \mathcal{H}_k \text{ such that } v \neq 0 \text{ and } (T_k^i v = E^i v, \ \forall i) \big\}.$$

The multiplicity of $E \in \operatorname{Sp}(T_k)$ is the dimension of $\cap_i \operatorname{Ker}(T_k^i - E^i)$.

Let h_0^i and h_1^i be the principal and subprincipal Weyl symbols of (T_k^i). Denote by $h_0 : M \to \mathbb{R}^n$ the map whose components are the h_0^i. By assumption,

$$\{h_0^i, h_0^j\} = 0, \text{ for every } i \text{ and } j.$$

Let $E^0 \in \mathbb{R}^n$ be a regular value of h_0 such that $h_0^{-1}(E^0)$ is connected. From Arnold-Liouville theorem, there exists a neighborhood U of E^0 such that $h_0^{-1}(U)$ is diffeomorphic to $U \times \mathbb{T}^n$, with the level sets $h_0^{-1}(E)$ diffeomorphic to the Lagrangian tori $\{E\} \times \mathbb{T}^n$.

In the first subsection, we state the Bohr-Sommerfeld conditions and discuss them. The second subsection is devoted to the local solutions of $T_k^i u_k = E_k^i u_k$. In the third subsection, we construct global solutions modulo $O(k^{-\infty})$ and prove the Bohr-Sommerfeld conditions.

3.1. **Statement of the results.**

If $E \in U$, we denote the torus $h_0^{-1}(E)$ by Λ_E and the restriction of the Hamiltonian vector fields $X_{h_0^i}$ on Λ_E by X_E^i. We need also the following notations:

- $\beta_E \in \Omega^1(\Lambda_E)$ is the 1-form of Λ_E such that $\langle \beta_E, X_E^i \rangle = h_1^i$ for every i.
- $\delta_E \in \Omega^1(\Lambda_E)$ is the 1-form of Λ_E such that $\langle \delta_E, X \rangle = \omega(H_E, X)$ for every $X \in T\Lambda_E$, where H_E is the mean curvature vector field of Λ_E.

Choose a family of loops $l_E^1, ..., l_E^n$ in Λ_E which depends continuously of E and such that $([l_E^i])$ is a base of $H_1(\Lambda_E, \mathbb{Z})$.

Theorem 3.1. *There exists a formal series $\sum_{l \geqslant -1} \hbar^l g_l$, with coefficients g_l in $C^\infty(U, \mathbb{R}^n)$ such that :*

for every open set $O \subset \mathbb{R}^n$ with compact closure $\bar{O} \subset U$ and for every sequences $\big(k_\alpha, E_\alpha\big)$, $\big(k_\alpha, E'_\alpha\big)$ of $\mathbb{N} \times O$, we have

i. $E_\alpha \in \operatorname{Sp}(T_{k_\alpha}) + O(k_\alpha^{-\infty}) \Longleftrightarrow g(E_\alpha, k_\alpha) \in k_\alpha^{-1}\mathbb{Z}^n + O(k_\alpha^{-\infty})$.

ii. If $E_\alpha \in \operatorname{Sp}(T_{k_\alpha})$, $E'_\alpha \in \operatorname{Sp}(T_{k_\alpha})$ and $E_\alpha = E'_\alpha + O(k_\alpha^{-\infty})$, then when k_α is sufficiently large, $E_\alpha = E'_\alpha$ and the multiplicity of E_α is 1.

where $(g(.,k))$ is a sequence of maps $U \to \mathbb{R}^n$ such that

- $g(E,k) = k^{-1} \sum_{l \geqslant -1} k^{-l} g_l(E) + O(k^{-\infty})$
- *$g_{-1}^i(E)$ is the holonomy of l_E^i in L, that is the parallel transport in L along l_E^i is the multiplication by $\exp(2i\pi g_{-1}^i(E))$.*
- $g_0^i(E) = \frac{1}{2\pi} \int_{l_E^i} -\beta_E + \frac{1}{2}\delta_E$.

Let us precise the sense of the estimations: if (S_k) is a sequence of subsets of $\mathbb{R}^n$ and $(k_\alpha, E_\alpha)_{\alpha \in \mathbb{N}}$ a sequence of $\mathbb{N} \times \mathbb{R}^n$, the notation $E_\alpha \in S_{k_\alpha} + O(k_\alpha^{-\infty})$ means that for every N, there exists C such that

$$\operatorname{Inf}_{E \in S_k} |E - E_\alpha| \leqslant C k_\alpha^{-N}$$

when k_α is sufficiently large.

Remark 3.2. Assume that M is 2-dimensional. So we consider a unique Toeplitz operator T_k with Weyl symbol $h_0 + \hbar h_1 + O(\hbar^2)$. Then

$$\beta_E = h_1 \gamma_E,$$

where γ_E is the one-form of Λ_E such that $\langle \gamma_E, X \rangle = 1$ if X is the Hamiltonian vector field of h_0. Introduce a vector field t tangent to Λ_E such that $|t| = 1$ and a normal vector field n such that (t, n) is an oriented orthonormal base of $T_x M$ for every $x \in \Lambda_E$, that is $n = Jt$. The geodesic curvature is the function $\tau_E \in C^\infty(\Lambda_E)$ defined by

$$\tau_E = g(\nabla_t t, n).$$

The mean curvature vector field is

$$H_E = \tau_E n.$$

So if γ'_E is the one-form of Λ_E such that $\langle \gamma'_E, t \rangle = 1$, then $\delta_E = -\tau_E \gamma'_E$. Hence

$$g_0(E) = -\frac{1}{2\pi} \Big(\int_{\Lambda_E} h_1 \; \gamma_E + \tfrac{1}{2} \int_{\Lambda_E} \tau_E \; \gamma'_E \Big)$$

where the orientation of Λ_E is chosen as to compute the holonomy of L. Theorem 0.1 of the introduction follows. □

Remark 3.3. If $M = \mathbb{R}^2$ is endowed with the usual Riemann structure, then

$$\frac{1}{2\pi} \int_{\Lambda_E} \tau_E \gamma'_E$$

is the degree d of the tangent map

$$\Lambda_E \simeq S_1 \to S^1, \quad x \to t(x)$$

where we identify $T_x M$ with $\mathbb{R}^2$ and the set of vectors whose norm is equal to 1 with the circle S^1. Since $\Lambda_E \to M$ is an embedding, $d = \pm 1$ and this leads to $\pm \frac{1}{2}$ in the definition of g_1. Furthermore, the Maslov index of Λ_E is $2d$ and the function h_1, that we called the Weyl subsymbol of (T_k), is the usual Weyl subsymbol of

$$U^{-1} T_k U$$

where U is the Bargmann transform. Consequently, we obtain the usual Bohr-Sommerfeld condition. More generally, if $M = \mathbb{C}^n$ is endowed with the usual Riemannian structure, δ_E is closed and its cohomology class is the Maslov class (cf. [9]). □

Remark 3.4. The Maslov index and the integral of δ_E differ in some aspects. As instance, let M be the sphere ($M = \mathbb{C}P^1$) with volume 2π endowed with its metric of constant curvature. If $\Lambda \to M$ is an embedding, it is the boundary of a domain D and Gauss-Bonnet formula yields

$$\int_\Lambda \tau \gamma' = 2\pi - 2 \,\text{Area}(D).$$

In this example, it is clear that $\int_\Lambda \tau \gamma'$ is not constant when we deform Λ. To the contrary the Maslov index is locally constant. Furthermore, as we will see in the proof of proposition 3.5,

$$\beta_E + \tfrac{1}{2} \delta_E$$

is closed. But if the dimension of M is $\geqslant 4$, the 1-form δ_E is not necessarily closed. In the usual Bohr Sommerfeld conditions on a cotangent phase space, $\beta_E + \frac{1}{2}\delta_E$ is replaced by a sum of two closed forms, the first one is obtained as β_E from the subsymbols and the second one is the Maslov form (cf. theorem 4.5.8 of [10]). □

3.2. **Local solutions.** By theorem 2.8, the Toeplitz operators (T^i_k) induce operators $C^\infty(\Lambda_E)[[\hbar]] \to C^\infty(\Lambda_E)[[\hbar]]$ of the form

$$T^j_E f = E^j f - i\hbar\big(X^j_E.f + \big(ih^j_1 - \tfrac{i}{2}H_E.h^j_0 + \tfrac{1}{2}\operatorname{div}_{\Lambda_E}(X^j_E)\big)\ f\big) + \hbar^2 S^j_E f \tag{30}$$

where $S^j_E = \sum_{l\geqslant 0}\hbar^l S^j_{E,l}$ and the $S^j_{E,l}$ are differential operators which act on $C^\infty(\Lambda_E)$.

Proposition 3.5. *If V is an open contractible set of Λ_E, $x_0 \in V$ and $C(\hbar) \in \mathbb{C}[[\hbar]]$, then the equations*

$$\begin{cases} T^i_E f(.,\hbar) = E^i f(.,\hbar), \ \textit{for every } i = 1,...,n \\ f(x_0,\hbar) = C(\hbar) \end{cases} \tag{31}$$

admit a unique solution $f(.,\hbar) \in C^\infty(V)[[\hbar]]$. Furthermore there exists a formal series $\alpha_E = \sum \hbar^l \alpha_{E,l} \in \Omega^1(\Lambda_E)[[\hbar]]$, with

$$\alpha_{0,E} = \beta_E - \tfrac{1}{2}\delta_E$$

and whose coefficients $\alpha_{l,E}$ are closed 1-forms which depends in a C^∞ way of E and do not depend on V, such that the solution of (31) is given by

$$f(.,\hbar) = \tfrac{C(\hbar)}{a_E(x_0)}\ a_E e^{i\varphi_0} e^{i\sum_{l\geqslant 1}\hbar^l\varphi_l}$$

where the functions $\varphi_l \in C^\infty(V)$ are determined by $\varphi_l(x_0) = 0$ and $d\varphi_l = \alpha_l$, and $a_E \in C^\infty(\Lambda_E)$ is the positive function defined by

$$a_E^{-2} = \mu_{\Lambda_E}(X^1_E \wedge ... \wedge X^n_E).$$

Let $\bar{E} \in U$ and Γ be a sufficiently small neighborhood of $\bar{E}$. Identify $h_0^{-1}(\Gamma)$ with $\Gamma \times \mathbb{T}^n$ and introduce an open contractible set V of $\mathbb{T}^n$ and $x_0 \in V$. By choosing

$$C(\hbar) = a_E(x_0)$$

in the previous proposition, we obtain functions $f_{V,l} \in C^\infty(\Gamma \times V)$ such that $\sum \hbar^l f_{V,l}(E,.)$ is the solution of equations (31). Introduce as in the beginning of section 2.6 a section F_V and a symbol $a_V(.,k)$ defined on $U \times V$ and such that (26) is verified.

If (u_α, k_α) is an admissible sequence such that

$$u_\alpha = \Big(\frac{k_\alpha}{\pi}\Big)^{\frac{n}{4}} F_V^{k_\alpha}(E_\alpha,.) a_V(E_\alpha,.,k_\alpha) + O_\infty(k^{-\infty}) \text{ over } U \times V \tag{32}$$

where E_α takes its values in Γ, then

$$T^i_{k_\alpha} u_\alpha = E^i_\alpha u_\alpha + O_\infty(k^{-\infty})$$

over $U \times V$.

The following proposition is a converse of this. It will be proved at the end of section 4.4 by using microlocal equivalences.

Proposition 3.6. *Let (u_α, k_α) be an admissible sequence such that $u_\alpha \in \mathcal{H}_{k_\alpha}$ for every α and*

$$T^i_{k_\alpha} u_\alpha = E^i_\alpha u_\alpha + O_\infty(k_\alpha^{-\infty}) \ \textit{on } U \times V$$

where (E_α) takes its values in Γ. Then there exists a sequence (c_α) of complex numbers such that

$$u_\alpha = c_\alpha \Big(\frac{k_\alpha}{\pi}\Big)^{\frac{n}{4}} F_V^{k_\alpha}(E_\alpha,.) a_V(E_\alpha,.,k_\alpha) + O_\infty(k_\alpha^{-\infty}) \ \textit{on } U \times V.$$

Proof of proposition 3.5. First we prove that $\beta_E - \frac{1}{2}\delta_E$ is closed. Observe that

$$\langle \beta_E - \tfrac{1}{2}\delta_E, X_E^i \rangle = h_1^i - \tfrac{1}{2} H_E.h_0^i.$$

Since the vector fields X_E^i commutes and $(X_E^i|_x)$ is a base of $T_x\Lambda_E$ for every $x \in \Lambda_E$, it suffices to prove that

$$X_E^j.(h_1^i - \tfrac{1}{2}H_E.h_0^i) = X_E^i.(h_1^j - \tfrac{1}{2}H_E.h_0^j).$$

From (30), we deduce by using $[X_E^i, X_E^j] = 0$ that

$$[T_E^i, T_E^j]f = i\hbar^2\big(X_E^j.(h_1^i - \tfrac{1}{2}H_E.h_0^i) - X_E^i.(h_1^j - \tfrac{1}{2}H_E.h_0^j)\big)\, f + O(\hbar^3).$$

$[T_k^i, T_k^j] = 0$ implies $[T_E^i, T_E^j] = 0$, and this proves the result.

Consequently, if V is an open contractible set and $x_0 \in V$, there exists a function $\varphi_0 \in C^\infty(V)$ such that $\varphi_0(x_0) = 0$ and $d\varphi_0 = -\beta_E + \frac{1}{2}\delta_E$, that is

$$X_E^i \varphi_0 = -h_1^i + \tfrac{1}{2}H_E.h_0^i. \tag{33}$$

We have

$$X_E^i a_E = -\tfrac{1}{2} a_E \operatorname{div}_{\Lambda_E}(X_E^i).$$

So we deduce from (30) that

$$(T_E^i - E^i)a_E e^{i\varphi_0} f(.,\hbar) = a_E e^{i\varphi_0}(-i\hbar X_E^i + \hbar^2 R_E^i) f(.,\hbar) \tag{34}$$

where $R_E^i(f) = a_E^{-1} e^{-i\varphi_0} S_E^i(a_E e^{i\varphi_0} f)$.

Now we prove by induction that equations (31) with $C(\hbar) = a_E(x_0)$ admit a unique solution. From (34), we see that $a_E e^{i\varphi_0}$ is the unique function such that

$$(T_E^i - E^i)a_E e^{i\varphi_0} = 0 + O(\hbar^2).$$

Let N be a non negative integer. Assume that we have proved that equations (31) modulo $O(\hbar^{N+2})$ admit a unique solution modulo $O(\hbar^{N+1})$ and that this solution is $a_E e^{i(\varphi_0 + \hbar\varphi)}$ with

$$\varphi = \varphi_1 + \hbar\varphi_2 + ... + \hbar^{N-1}\varphi_N.$$

We have

$$(T_E^i - E^i)a_E e^{i(\varphi_0+\hbar\varphi)} h(.,\hbar) = a_E e^{i(\varphi_0+\hbar\varphi)}(-i\hbar X_E^i + \hbar^2 U_E^i) h(.,\hbar)$$

where $U_E^i(f) = (X_E^i.\varphi) f + e^{-i\hbar\varphi} R_E^i(e^{i\hbar\varphi} f)$. By assumption

$$U_E^i(1) = \hbar^N r^i + O(\hbar^{N+1}).$$

We look for a solution modulo $O(\hbar^{N+3})$ of the form

$$a_E e^{i(\varphi_0+\hbar\varphi)}(1 + i\hbar^{N+1}\varphi_{N+1}).$$

So we have to solve

$$X_E^i.\varphi_{N+1} + r^i = 0, \quad \varphi_{N+1}(x_0) = 0$$

These equation admit a unique solution because $X_E^i.r^j = X_E^j.r^i$. Indeed

$$[T_E^i - E^i, T_E^j - E^j] = 0$$

and we have

$$[T_E^i - E^i, T_E^j - E^j] a_E e^{i\varphi_0} e^{i\hbar\varphi} = -i a_E e^{i\varphi_0} e^{i\hbar\varphi} \hbar^{N+3}(X_E^i.r^j - X_E^j.r^i) + O(\hbar^{N+4}).$$

Consequently,

$$a_E e^{i(\varphi_0 + \hbar\varphi + \hbar^{N+1}\varphi_{N+1})}$$

is the unique solution modulo $O(\hbar^{N+2})$ of equations (31) modulo $O(\hbar^{N+3})$. By iterating this we obtain that (31) admit a unique solution and this solution is of the form $a_E e^{i\varphi_0} e^{i\sum_{l\geqslant 1}\hbar^l\varphi_l}$.

A solution of equations (31) with a general initial condition $C(\hbar)$ is given by

$$C(\hbar) a_E^{-1}(x_0) a_E e^{i\varphi_0} e^{i\sum_{l\geqslant 1}\hbar^l\varphi_l}.$$

It is unique because of the uniqueness of the solution with initial condition $a_E(x_0)$. Indeed it is clear if $C(\hbar)$ is invertible, i.e. $C(\hbar) = C_0 + O(\hbar^l)$ with $C_0 \neq 0$, because we can obtain a solution with initial condition $a_E(x_0)$ from a solution with initial condition $C(\hbar)$ by multiplying it by $(C(\hbar))^{-1}a_E(x_0)$. In the case

$$C(\hbar) = \hbar^m C_m + O(\hbar^{m+1})$$

with $C_m \neq 0$, we deduce from (34) that a solution with this initial condition is necessarily of the form $\hbar^m C_m a_E(x_0)^{-1} a_E e^{i\varphi_0} + O(\hbar^{m+1})$. So multiplying by $\hbar^{-m}$ we obtain a solution with initial condition $\hbar^{-m}C(\hbar)$ and we are in the previous case.

Finally the 1-forms $d\varphi_l \in \Omega^1(U)$ extend to global one-forms $\alpha_l \in \Omega^1(\Lambda_E)$, which do not depend on the choice of V. Indeed if we consider two open contractible set V and V' with solutions of equation (31) of the form

$$a_E e^{i\varphi_0} e^{i\sum_{l\geqslant 1}\hbar^l\varphi_l}, \quad a_E e^{i\varphi_0} e^{i\sum_{l\geqslant 1}\hbar^l\varphi'_l}.$$

Then we deduce form the uniqueness, that on each component of $V \cap V'$, $\varphi_l - \varphi'_l$ is a constant and so $d\varphi_l = d\varphi'_l$. □

3.3. **Quasimode.** For every positive integer l, we set

$$g_l^i(E) = \frac{1}{2\pi}\int_{l_E^i} \operatorname{Re}\alpha_{E,l}.$$

and this define the functions g_l in theorem 3.1. Concerning the imaginary part of $\alpha_{E,l}$, we have the following lemma.

Lemma 3.7. *The imaginary part of $\alpha_{E,l}$ is exact.*

Using this, we can construct a flat $\mathbb{C}[[\hbar]]$-bundle $K_E \to \Lambda_E$ of rank one with a section f_E such that

$$T_E^i f_E = E^i f_E, \quad \forall i \qquad \text{and} \qquad |f_E| = a_E + O(\hbar)$$

and the holonomy of the loop l_E^i in K_E is $-\sum_{l\geqslant 0}\hbar^l g_l^i(E)$. Then the equation

$$g(E_\alpha, k_\alpha) \in k_\alpha^{-1}\mathbb{Z}^n + O(k_\alpha^{-\infty}) \tag{35}$$

where the sequence $(g(.,k))$ is defined as in theorem 3.1, is the quantization condition (28). If (k_α, E_α) satisfies it and (E_α) takes its values in a compact set $C \subset U$, following section 2.6 we construct a Lagrangian section (u_α) with symbol f_E. We have

$$\begin{aligned} &T_{k_\alpha} u_\alpha = E_\alpha u_\alpha + O_\infty(k_\alpha^{-\infty}) \\ \text{and} \quad &(u_\alpha, u_\alpha) = \int_{\Lambda_{E_\alpha}} \nu_{E_\alpha} + O(k_\alpha^{-1}) \end{aligned} \tag{36}$$

where $\nu_E \in |\Omega|(\Lambda_E)$ is defined by

$$\nu_E(X_1 \wedge \ldots \wedge X_n) = 1.$$

So $\int \nu_E$ does not vanish. It follows that $E_\alpha \in \operatorname{Sp}(T_{k_\alpha}) + O(k_\alpha^{-\infty})$. Hence we have proved the converse of assertion *i.* of theorem 3.1.

In section 2.6, we assume that the parameter γ takes its values in a sufficiently small open set, to obtain a uniform control. Here, we can introduce a finite cover of C by arbitrary small open sets to apply the results of section 2.6.

Proof. We explain how we can construct the bundle $K_E \to \Lambda_E$ and compute its holonomy. Choose angle coordinates (x^i) on Λ_E such that

$$\int_{l^i_E} dx^j = \delta_{ij}.$$

We use these coordinates to identify Λ_E with $\mathbb{T}^n$. Let $p : \mathbb{R}^n \to \Lambda_E$ be the associated projection. Let $\varphi_l \in C^\infty(\mathbb{R}^n)$ be such that $d\varphi_l = p^*\alpha_{E,l}$ and $\varphi_l(0) = 0$. Consider the section $\tilde{f}_E$ of $\mathbb{R}^n \times \mathbb{C}[[\hbar]] \to \mathbb{R}^n$ defined by

$$\tilde{f}_E = (p^* a_E) e^{i \sum_{l \geqslant 0} \hbar^l \varphi_l}.$$

Now define the bundle $K_E \to \Lambda_E$ by dividing $\mathbb{R}^n \times \mathbb{C}[[\hbar]] \to \mathbb{R}^n$ by the action of $\mathbb{Z}^n$

$$\mathbb{Z}^n \times (\mathbb{R}^n \times \mathbb{C}[[\hbar]]) \to \mathbb{R}^n \times \mathbb{C}[[\hbar]], \quad (\epsilon, x, c(\hbar)) \to (x + \epsilon, c(\hbar) e^{i\epsilon^j . 2\pi\phi^j(E,\hbar)})$$

with $2\pi\phi^j(E,\hbar) = \sum_l \hbar^l \varphi_l(n_j)$ where $n^j = (\delta_{1j}, ..., \delta_{nj})$. By lemma 3.7, $\varphi_l(n_j)$ is real. We obtain the section f_E from the section $\tilde{f}_E$ and the holonomy of the loop l^j_E is $-\phi^j(E,\hbar)$. □

Proof of lemma 3.7. Assume that the imaginary parts of $\alpha_{1,E}$,...,$\alpha_{m,E}$ are exact. Define the real numbers

$$r^i(E) = \int_{l^i_E} \text{Im } \alpha_{m+1,E}.$$

Choose angle coordinates x^i as in the previous proof, define the 1-form

$$\alpha'_{m+1,E} = \alpha_{m+1,E} - i r^i(E) dx^i$$

whose imaginary part is exact. If V is a contractible set of Λ_E and $x \in V$, then define the functions φ_l such that $d\varphi_l = \alpha_{l,E}$ for $l = 0, ..., m$, $d\varphi_{m+1} = \alpha'_{m+1,E}$ and $\varphi_l(x) = 0$. As in the proof of proposition 3.5, we obtain that

$$T^i_E f = (E^i - i\hbar^{m+1} r^j(E) M^i_j(E)) f + O(\hbar^{m+2}), \qquad \text{if } f = a_E e^{i\varphi_0} e^{i \sum_{l=1}^{m+1} \hbar^l \varphi_l}$$

where $M^i_j(E) = \langle X^i_E, dx^j \rangle$. $(M^i_j(E))$ is invertible. As we did before, we can associate to this symbol a Lagrangian section (u_α) such that

$$T^i_{k_\alpha} u_\alpha = \big(E^i_\alpha - i k_\alpha^{-m-1} M^i_j(E_\alpha) r^j(E_\alpha)\big) u_\alpha + O(k_\alpha^{-m-2}).$$

Furthermore the estimate of (u_α, u_α) is the same as before. The previous equation implies

$$(T^i_{k_\alpha} u_\alpha, u_\alpha) - (u_\alpha, T^i_{k_\alpha} u_\alpha) = -2i k_\alpha^{-m-1} M^i_j(E_\alpha) r^j(E_\alpha) (u_\alpha, u_\alpha) + O(k_\alpha^{-m-2}).$$

Since the T^i_k are self-adjoint, we obtain by choosing various sequences (k_α, E_α) that r^i vanishes on a dense set. □

Proposition 3.8. *Let $(v_\alpha, k_\alpha, E_\alpha)$ be a sequence such that $v_\alpha \in \mathcal{H}_{k_\alpha}$ for every α and*

$$T^i_{k_\alpha} v_\alpha = E^i_\alpha v_\alpha + O(k_\alpha^{-\infty}), \quad (v_\alpha, v_\alpha) = 1. \tag{37}$$

Assume that (E_α) takes its values in a compact $C \subset U$, then (E_α) satisfies the quantization condition (35). Furthermore, if (u_α) is a Lagrangian section defined as in (36), then there exists a sequence (c_α) of complex numbers such that

$$v_\alpha = c_\alpha u_\alpha + O_\infty(k^{-\infty}).$$

Hence the Lagrangian sections we constructed approximate modulo $O(k^{-\infty})$ the eigenvectors. So they are rather modes than quasimodes. This comes from the assumption that $h_0^{-1}(E)$ is connected when $E \in U$.

As it is proved in section 5 of [5], the assumption $(v_\alpha, v_\alpha) = 1$ implies that (v_α) is an admissible sequence. In the same way a sequence of sections $O(k_\alpha^{-\infty})$ for the L^2 norm is negligible.

To prove the proposition we will use proposition 3.6 to determine (v_α) over $h_0^{-1}(U)$. Outside this domain, (v_α) is negligible. Indeed, we can prove that the microsupport of (v_α) is a subset of $h_0^{-1}(C)$ (cf. proposition 4.4.6 of [10] for a proof in the case of pseudodifferential operators with a small parameter that we can easily adapt to our situation).

Proof. Assume that $d(g(E_\alpha, k_\alpha), k_\alpha^{-1}\mathbb{Z}^n) \neq O(k_\alpha^{-\infty})$. By replacing $(v_\alpha, k_\alpha, E_\alpha)$ by a subsequence, we may assume that for some i_0 and positive integer N,

$$d(k_\alpha g^{i_0}(E_\alpha, k_\alpha), \mathbb{Z}) \geqslant k^{-N}.$$

So

$$\left| e^{ik_\alpha g^{i_0}(E_\alpha, k_\alpha)} - 1 \right| \geqslant C k_\alpha^{-N}. \tag{38}$$

We will prove that this leads to a contradiction. By replacing $(v_\alpha, k_\alpha, E_\alpha)$ by a subsequence, we may assume that $E_\alpha \to \tilde{E}$ as α tends to ∞. Let V and V' be two contractible sets such that $V \cap V' \neq \emptyset$. We may introduce as in (32) the sections $F_V(E, .)$, $F_{V'}(E, .)$ and the symbols $a_V(E, ., k)$, $a_{V'}(E, ., k)$. Furthermore if $\tilde{x} \in V \cap V'$, we can choose them so as to have

$$F_V(\tilde{E}, \tilde{E}, \tilde{x}) = F_{V'}(\tilde{E}, \tilde{E}, \tilde{x})$$

and

$$a_V(\tilde{E}, \tilde{E}, \tilde{x}, k) = a_{V'}(\tilde{E}, \tilde{E}, \tilde{x}, k) + O(k^{-\infty}).$$

By proposition 3.6,

$$\begin{aligned} v_\alpha &= c_\alpha \Big(\frac{k_\alpha}{2\pi}\Big)^{\frac{n}{4}} F_V^{k_\alpha}(E_\alpha, .) a_V(E_\alpha, ., k_\alpha) + O(k_\alpha^{-\infty}) \text{ on } U \times V \\ &= c'_\alpha \Big(\frac{k_\alpha}{2\pi}\Big)^{\frac{n}{4}} F_{V'}^{k_\alpha}(E_\alpha, .) a_{V'}(E_\alpha, ., k_\alpha) + O(k_\alpha^{-\infty}) \text{ on } U \times V'. \end{aligned}$$

By taking the limit at $(\tilde{E}, \tilde{x})$ as $\alpha \to \infty$, we obtain that $c_\alpha = c'_\alpha + O(k^{-\infty})$. Now applying this to an open covering of $l_{\tilde{E}}^{i_0}$, we obtain that

$$c_\alpha = c_\alpha e^{ik_\alpha g^{i_0}(E_\alpha, k_\alpha)} + O(k_\alpha^{-\infty}).$$

Using (38), it follows that $|c_\alpha| = O(k_\alpha^{-\infty})$. Using that $\mathbb{T}^n$ is connected, we deduce that the c_α associated to every V is $O(k_\alpha^{-\infty})$. Hence (v_α) is negligible on a neighborhood of $\Lambda_{\tilde{E}}$. By the remark before the proof, it is also negligible outside this neighborhood. Consequently

$$(v_\alpha, v_\alpha) = O(k_\alpha^{-\infty}),$$

a contradiction. We prove in the same way the second assertion by identifying locally the sequence (u_α) and (v_α). □

Proof of assertion ii. of theorem 3.1. Let $(v_\alpha, k_\alpha, E_\alpha)$ and $(v'_\alpha, k_\alpha, E'_\alpha)$ be sequences satisfying (37) and such that

$$E_\alpha = E'_\alpha + O(k_\alpha^{-\infty}).$$

Assume that $(v_\alpha, v'_\alpha) = 0$. From proposition 3.8 there exists a Lagrangian section (u_α) such that $v_\alpha = c_\alpha u_\alpha + O(k_\alpha^{-\infty})$ and $v'_\alpha = c'_\alpha u_\alpha + O(k_\alpha^{-\infty})$. Computing the norms, we obtain that

$$|c_\alpha|, |c'_\alpha| \geqslant C,$$

where C is positive constant. On the other hand,

$$(v_\alpha, v'_\alpha) = c_\alpha \overline{c}'_\alpha \int_{\Lambda_{E_\alpha}} \nu_{E_\alpha} + O(k_\alpha^{-1})$$

which contradicts $(v_\alpha, v'_\alpha) = 0$. □

4. Quantum maps

4.1. **Definitions and symbolic calculus.** Let (M,ω) be a compact symplectic manifold endowed with a prequantization bundle $L \to M$. Let us introduce two complex structures J^a and J^b of M which are integrable and compatible with ω. So we obtain two Kählerian structures and two quantizations $\mathcal{H}_k^a$ and $\mathcal{H}_k^b$.

We are interested in the operators $T_k : \mathcal{H}_k^b \to \mathcal{H}_k^a$. As we did with the Toeplitz operators, we identify them with the operators

$$T_k : C^\infty(M, L^k) \to C^\infty(M, L^k) \text{ such that } \Pi_k^a T_k \Pi_k^b = T_k.$$

Their Schwartz kernels are sections of $L^k \boxtimes L^{-k} \to M^2$. We will define them as Lagrangian sections.

Let $\varphi : M \to M$ be a symplectomorphism. A prequantization lift of φ is a lift $\tilde{\varphi} : L \to L$ of φ such that

i. $\tilde{\varphi}$ restricts on L_x to a unitary map $\tilde{\varphi}_x : L_x \to L_{\varphi(x)}$
ii. $\nabla \varphi^* s = \varphi^* \nabla s, \ \forall s \in C^\infty(M,L)$

where $\varphi^* s$ is the section of L defined by $(\varphi^* s)(x) = \tilde{\varphi}_x^{-1}.s(\varphi(x))$. Denote by Λ the Lagrangian submanifold $\{(\varphi(x), x) \ / \ x \in M\} \subset M^2$.

Definition 4.1. The set of quantum maps $\mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi})$ consists of the sequences (T_k) of operators such that $\Pi_k^a T_k \Pi_k^b = T_k$ for every k and

$$T_k(x_l, x_r) = \left(\frac{k}{2\pi}\right)^{\frac{n}{2}} E^k(x_l, x_r) a(x_l, x_r, k) + O(k^{-\infty})$$

where

- E is a section of $L \boxtimes L^{-1} \to M^2$ such that $E(\varphi(x), x) = \tilde{\varphi}_x$ and $\nabla_{\bar{Z}} E \equiv 0$ modulo $\mathcal{I}^\infty(\Lambda)$ for every holomorphic vector field Z of $(M^2, J_a \times -J_b)$.
- $(a(.,k))$ is a symbol of $S^0(M^2)$ whose coefficients of its asymptotic expansion $\sum k^{-l} a_l$ satisfy $\bar{Z}.a_l \equiv 0$ modulo $\mathcal{I}^\infty(\Lambda)$ for every holomorphic vector field Z of $(M^2, J_a \times -J_b)$.

Let us define the full symbol map

$$\sigma : \mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi}) \to C^\infty(M)[[\hbar]], \quad (T_k) \to \sum \hbar^l a_l(\varphi(x), x).$$

It is onto and its kernel consists of the smoothing operators.

Proof. Consider that M^2 is a Kähler manifold with the complex structure

$$J^a \times -J^b$$

and the fundamental 2-form

$$\pi_l^* \omega - \pi_r^* \omega,$$

where π_r and π_l are the projections $M^2 \to M$ on the first and second factor. We denote by

$$\Pi_k^{ab} : C^\infty(M^2, L^k \boxtimes L^{-k}) \to C^\infty(M^2, L^k \boxtimes L^{-k})$$

the associated Szegö projector and by $\mathcal{H}_k^{ab}$ its image. Consider an operator

$$T_k : C^\infty(M, L^k) \to C^\infty(M, L^k).$$

Then $\Pi_k^a T_k \Pi_k^b = T_k$ if and only if its Schwartz kernel $T_k(x_l, x_r) \in \mathcal{H}_k^{ab}$. Observe that Λ satisfies a quantization condition as in section 2.2. Indeed the section

$$t : \Lambda \to L \boxtimes L^{-1}$$

defined by $t(\varphi(x),x) = \tilde{\varphi}_x \in L_{\varphi(x)} \otimes L_x^{-1}$, is flat with constant norm equal to 1. So the kernels of the quantum maps are exactly the Lagrangian sections introduced in section 2.2 and the symbol map is the same as (13). □

If (T_k) belongs to $\mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi})$ with symbol $\sum_l \hbar^l f_l$, then the adjoint (T_k^*) is a quantum map of $\mathcal{F}(\varphi^{-1}, J_b, J_a, \tilde{\varphi}^{-1})$. Its symbol is $\sum_l \hbar^l (\varphi^{-1})^* \bar{f}_l$. The next proposition describes the product of two quantum maps.

Proposition 4.2. *The product of operators defines a bilinear map*

$$\mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi}) \times \mathcal{F}(\psi, J_c, J_d, \tilde{\psi}) \rightarrow \mathcal{F}(\varphi \circ \psi, J_a, J_d, \tilde{\varphi} \circ \tilde{\psi}).$$

This induces a products on the symbols $C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]$ *which is of the form*

$$B\big(\textstyle\sum \hbar^l f_l, \sum \hbar^l g_l\big) = \sum_l \hbar^l \sum_{l_1+l_2+l_3=l} B_{l_1}(\psi^* f_{l_2}, g_{l_3})$$

where

- *the* B_l *are bidifferential operators,*
- *if* $J_b = J_c$, *then* $B_0(f,g) = \psi^*\big(\det(q_{\varphi^{-1}(J_a),J_b} + \bar{q}_{\psi(J_d),J_c})\big)^{-\frac{1}{2}} fg.$

Let us explain the last notation: If J_a is a complex structure and φ a symplectomorphism, then $\varphi(J_a)$ is the complex structure

$$\varphi(J_a) := \varphi_* \circ J_a \circ \varphi_*^{-1}.$$

Furthermore, if J_a and J_b are two complex structures, then $q_{J_a,J_b}|_x$ is the projection of $T_xM \otimes \mathbb{C}$ onto $T_x^{(1,0)_b}M$ with kernel $T_x^{(0,1)_a}M$.

We can also consider the action of a quantum map on a Lagrangian section.

Proposition 4.3. *Let* (T_k) *be a quantum map of* $\mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi})$. *Let* Λ *be a Lagrangian manifold and* (u_α, k_α) *a Lagrangian section associated over an open set* U *such that* $u_\alpha \in \mathcal{H}^b_{k_\alpha}$. *Then* $(T_{k_\alpha}.u_\alpha)$ *is a Lagrangian section over* $\varphi(U)$ *associated to* $\varphi(\Lambda)$. *Furthermore, there exists a sequence of operators* $C_l : C^\infty(M) \times C^\infty(\Lambda) \rightarrow C^\infty(\varphi(\Lambda))$ *such that the symbol of* $(T_{k_\alpha} u_\alpha)$ *is*

$$C\big(\textstyle\sum \hbar^l f_l, \sum \hbar^l g_l\big) = \sum_l \hbar^l \sum_{l_1+l_2+l_3=l} C_{l_1}(f_{l_2}, g_{l_3})$$

if $\sum \hbar^l f_l$ *and* $\sum \hbar^l g_l$ *are the symbols of* (T_k) *and* (u_α). *The operators* C_l *depend only on* J_a, J_b, Λ *and* φ *and*

- *they are locally such that*

  $$\varphi^* C_l(f,g)|_{U\cap\Lambda} = \sum_{|\alpha|+|\gamma|\leqslant 2l} a_{\alpha,\gamma}.\partial_x^\alpha f|_{U\cap\Lambda}.\partial_y^\gamma g, \quad \text{with } a_{\alpha,\gamma} \in C^\infty(U\cap\Lambda)$$

  if (x^j) *is a coordinates system of* M *defined on an open set* U *and* (y^k) *a coordinates system of* Λ *defined on* $U \cap \Lambda$,
- *if* $J_b = J_c$, *then* C_0 *is given by*

  $$\varphi^* C_0(f,g) = \big(\det(q_{\varphi^{-1}(J_a),J_b} + q_{J_b})\big)^{-\frac{1}{2}}.f|_\Lambda.g$$

  where $q_{J_b}|_x$ *is the projection of* $T_xM \otimes \mathbb{C}$ *onto* $T_x^{(0,1)_b}M$ *with kernel* $T_x\Lambda$.

Let us specify that the section $F_{\varphi(\Lambda)}$ used to define $(T_{k_\alpha} u_\alpha)$ has to be chosen in such a way that $F_{\varphi(\Lambda)}(\varphi(x)) = \tilde{\varphi}_x.F_\Lambda(x)$ for every $x \in U \cap \Lambda$, if F_Λ is the section used to define (u_α).

4.2. **Proof of proposition 4.2.** Let $\phi : M \to M$ be a symplectomorphism of M and $\tilde{\phi}$ a prequantization lift. Define as above the maps

$$\phi^* : C^\infty(M, L^k) \to C^\infty(M, L^k).$$

If $(T_k) \in \mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi})$, then we have

$$(\phi^* \circ T_k) \in \mathcal{F}(\phi^{-1} \circ \varphi, \phi^{-1}(J_a), J_b, \tilde{\phi}^{-1} \circ \tilde{\varphi}), \quad \sigma(\phi^* \circ T_k) = \sum_l \hbar^l f_l,$$

$$(T_k \circ \Phi^*) \in \mathcal{F}(\varphi \circ \phi^{-1}, J_a, \phi(J_b), \tilde{\varphi} \circ \tilde{\phi}^{-1}), \quad \sigma(T_k \circ \phi^*) = \sum_l \hbar^l (\phi^{-1})^* f_l.$$

Using this, we have just to prove the proposition with $\varphi = \psi = \mathrm{Id}$ and $\tilde{\varphi} = \tilde{\psi} = \mathrm{Id}$, and then writing:

$$T_k U_k = (\varphi^{-1})^* \circ \big((\varphi^* \circ T_k) \circ (U_k \circ \psi^*) \big) \circ (\psi^{-1})^*$$

if $(T_k) \in \mathcal{F}(\varphi, J_a, J_b, \tilde{\varphi})$ and $(U_k) \in \mathcal{F}(\psi, J_c, J_d, \tilde{\psi})$. So assume that $\varphi = \psi = \mathrm{Id}$ and $\tilde{\varphi} = \tilde{\psi} = \mathrm{Id}$. The Schwartz kernel of $T_k U_k$ is of the form

$$(T_k U_k)(x_1, x_3) = \Big(\frac{k}{2\pi}\Big)^n \int_M E^k_{ab}(x_1, x_2) E^k_{cd}(x_2, x_3) \tilde{f}(x_1, x_2) \tilde{g}(x_2, x_3)\, \mu_M(x_2)$$

where E_{ab} and E_{cd} are sections of $L \boxtimes L^{-1} \to M \times M$ defined by proposition 2.1. Their norms are < 1 outside the diagonal, so we can localize the product on a neighborhood of $\mathrm{Trig}(M) = \{x_1 = x_2 = x_3\}$.

Let s be a local section of L defined on an open set U endowed with a complex coordinates system (z_1^i) (resp. (z_3^i)) associated to J_a (resp. J_d) and a real coordinates system (x_2^j). Write

$$E_{ab}(x_1, x_2).E_{cd}(x_2, x_3) = e^{i\phi(x_1, x_2, x_3)} s(x_1) \otimes s^{-1}(x_3),$$
$$E_{ad}(x_1, x_3) = e^{i\psi(x_1, x_3)} s(x_1) \otimes s^{-1}(x_3).$$

From $\nabla_{\partial_{\bar{z}_1^i}} E_{a,b} \equiv \nabla_{\partial_{\bar{z}_1^i}} E_{a,d} \equiv 0$ modulo $\mathcal{I}^\infty(\mathrm{Trig}(M))$, we deduce that

$$\partial_{\bar{z}_1^i}(\phi - \psi) \equiv 0 \mod \mathcal{I}^\infty(\mathrm{Trig}(M)).$$

In the same way, we prove that

$$\partial_{z_3^i}(\phi - \psi) \equiv 0 \mod \mathcal{I}^\infty(\mathrm{Trig}(M)).$$

Later we will prove that $\partial_{x_2^i} \phi$ vanishes along $\mathrm{Trig}(M)$ and that $\big(\partial_{x_2^i} \partial_{x_2^j} \phi\big)_{ij}$ is invertible along $\mathrm{Trig}(M)$. Then using the same method as in the proof of lemma 2.5, we obtain that the ideal $\mathcal{J}$ generated by the functions $\partial_{x_2^j} \phi$ consists of the functions $f(x_1, x_2, x_3)$ which satisfy

$$f|_{\mathrm{Trig}(M)} = 0, \qquad \partial_{\bar{z}_1^i} f \equiv \partial_{z_3^i} f \equiv 0 \mod \mathcal{I}^\infty(\mathrm{Trig}(M)).$$

By applying stationary phase lemma (cf. [8]), we obtain that

$$(T_k U_k)(x_1, x_3) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{2}} e^{ik\psi(x_1, x_3)}\, \tilde{h}(x_1, x_3, k) s^k(x_1) \otimes s^{-k}(x_3) + O_\infty(k^{-\infty})$$

where $(\tilde{h}(., k))$ is a symbol. For the details, let us precise that the computation can be done easily by writing the Taylor expansions of the functions $f(x_1, x_2, x_3)$ (resp. $f(x_1, x_3)$) along $\{x_1 = x_2 = x_3\}$ (resp. $\{x_1 = x_3\}$) as in the lemma 1.2 by using the vector fields $\partial_{\bar{z}_1^i}, \partial_{x_2^j}, \partial_{z_3^i}$ (resp.$\partial_{\bar{z}_1^i}, \partial_{z_3^i}$).

Let us prove that $d_{x_2} \phi$ vanishes along $\mathrm{Trig}(M)$ and compute $d^2_{x_2} \phi$. Let α_{ab} be the 1-form defined by

$$\nabla E_{ab} = \alpha_{ab} \otimes E_{ab}.$$

By proposition 2.2, α_{ab} vanishes along the diagonal, and the same holds with α_{cd}. From this we deduce that $d_{x_2}\phi$ vanishes along $\mathrm{Trig}(M)$. Let q_{ab} be the projection of $T_xM \otimes \mathbb{C}$ onto

$$T_x^{(1,0)_b}M = \{X - iJ_bX \ / \ X \in T_xM\}$$

with kernel

$$T_x^{(0,1)_a}M = \{X + iJ_aX \ / \ X \in T_xM\}.$$

Observe that $\bar{q}_{ba} + q_{ab} = \mathrm{Id}$. Let us write

$$\begin{aligned}(0, X_2) &= (-\bar{q}_{ba}(X_2), q_{ab}(X_2)) + (\bar{q}_{ba}(X_2), \bar{q}_{ba}(X_2)),\\ (X_1, 0) &= (\bar{q}_{ba}(X_1), -q_{ab}(X_1)) + (q_{ab}(X_1), q_{ab}(X_1)).\end{aligned}$$

Then we deduce from (8) that if $X_1, X_2, Y_1, Y_2 \in T_x(M)$,

$$\langle T_{(X_1,X_2)}\alpha_{ab}, (Y_1, Y_2)\rangle = \frac{1}{i}\omega(\bar{q}_{ba}(X_1 - X_2), Y_1) + \frac{1}{i}\omega(q_{ab}(X_1 - X_2), Y_2)$$

which generalizes equation (10). From this we obtain that for every $X, Y \in T_xM$

$$d^2_{x_2}\phi(X, Y) = \omega(q_{ab}(X) - \bar{q}_{dc}(X), Y). \tag{39}$$

Consequently, $d^2_{x_2}\phi(X, .) = 0$ implies that

$$q_{ab}(X) = \bar{q}_{dc}(X) = 0$$

because $T^{(1,0)_b}M \cap T^{(0,1)_c}M = (0)$. So $X \in T^{(0,1)_b}M \cap T^{(1,0)_d}M = (0)$, hence $X = 0$.

Finally assume that $J_b = J_c$ and let us compute $B_0(f, g)$. We have

$$B_0(f, g) = f_0(x).g_0(x). \left(\frac{\det[-i\partial_{x_2^j}\partial_{x_2^k}\phi](x, x, x)}{\det[g^b_{ij}](x)}\right)^{-\frac{1}{2}}$$

where we have used that $\mu_M = \frac{1}{n!}|\omega^n|$ is the measure induced by the Riemannian metric $g^b(X, Y) = \omega(X, J_bY)$. Indeed $g^b = g^b_{ij}dx^i \otimes dx^j$ implies

$$\mu_M = (\det[g^b_{ij}])^{\frac{1}{2}}|dx^1...dx^{2n}|.$$

Now the quotient of the determinants can be view as the determinant of

$$T_xM \xrightarrow{-id^2_{x_2}\phi} T^*_xM \xrightarrow{(g^b)^{-1}} T_xM$$

From (39), we deduce that

$$\begin{aligned}-id^2_{x_2}\phi(X, Y) =& \omega(-iq_{ab}(X) + i\bar{q}_{db}(X), Y)\\ =& \omega(-J_bq_{ab}(X) - J_b\bar{q}_{db}(X), Y)\end{aligned}$$

since the image of q_{ab} is $T^{(1,0)_b}M = \mathrm{Ker}(J_b - i)$ and that of $\bar{q}_{db}$ is $T^{(0,1)_b}M = \mathrm{Ker}(J_b + i)$. Finally we obtain that

$$-id^2_{x_2}\phi(X, Y) = g^b(q_{ab}(X) + \bar{q}_{db}(X), Y).$$

Consequently, $(g^b)^{-1} \circ -id^2_{x_2}\phi = q_{ab} + \bar{q}_{db}$. This completes the proof.

4.3. **Applications.** Following Kostant, Blattner and Sternberg, the quantization of M should not depend on the choice of the complex structure. First, by the Riemann-Roch-Hirzebruch theorem, the dimensions of $(\mathcal{H}^a_k)$ and $(\mathcal{H}^b_k)$ are equals when k is sufficiently large. So in these cases there exists a unitary operator

$$U_k : \mathcal{H}^b_k \to \mathcal{H}^a_k.$$

To obtain such an operator with good semi-classical properties, we may choose it in $\mathcal{F}(\mathrm{Id}, J_a, J_b, \mathrm{Id})$.

Proof. First consider an operator $(V_k) \in \mathcal{F}(\mathrm{Id}, J_a, J_b, \mathrm{Id})$ with non-vanishing principal symbol f_0. From proposition 4.2, $(V_k^* V_k)$ is a Toeplitz operator with principal symbol

$$g_0 = |f_0|^2 . \det^{-\frac{1}{2}} (q_{J_a,J_b} + \bar{q}_{J_a,J_b}).$$

g_0 takes real positive values. Hence if k is sufficiently large, the spectrum of $(V_k^* V_k)$ is a subset of (ϵ, ∞) where $\epsilon > 0$ does not depend on k. Applying proposition 12 of [5], we obtain that $(V_k^* V_k)^{-\frac{1}{2}}$ is a Toeplitz operator. Now

$$U_k := V_k (V_k^* V_k)^{-\frac{1}{2}}$$

belongs to $\mathcal{F}(\mathrm{Id}, J_a, J_b, \mathrm{Id})$. It satisfies $U_k^* U_k = \mathrm{Id}$ and $U_k U_k^* = \mathrm{Id}$, if k is sufficiently large. □

The semi-classical properties of (U_k) are consequences of propositions 4.2 and 4.3. Indeed (U_k) sends a Lagrangian section of $\mathcal{H}_k^b$ into a Lagrangian section of $\mathcal{H}_k^a$ associated to the same Lagrangian submanifold. Furthermore, sending (T_k) into $(U_k^* T_k U_k)$, we obtain an isomorphism between the algebra of Toeplitz operators of $(\mathcal{H}_k^b)$ and the algebra of Toeplitz operators of $(\mathcal{H}_k^a)$. This induces an equivalence of star-products.

Another application is the quantization of the symplectomorphisms. We consider only one complex structure. If $\tilde{\varphi} : L \to L$ is a prequantization lift of a symplectomorphism φ, we can show as above that there exist unitary operators in $\mathcal{F}(\varphi, J, J, \tilde{\varphi})$. We say that such an operator quantizes φ. In [11], Zelditch quantizes the data $(\varphi, \tilde{\varphi})$ in the following way. He consider first the operator

$$(\Pi_k (\varphi^{-1})^* \Pi_k),$$

which belongs to $\mathcal{F}(\varphi, \varphi(J), J, \tilde{\varphi})$. Then by the same method we used above, he constructs a unitary operator of the form $(\Pi_k (\varphi^{-1})^* T_k)$ where (T_k) is a Toeplitz operator. By proposition 4.2, this operator belongs to $\mathcal{F}(\varphi, J, J, \tilde{\varphi})$.

Finally in [6], it is proved that the quantum propagator $U_k(t) = e^{-iktT_k}$ of a self-adjoint Toeplitz operator (T_k) quantizes the Hamiltonian flow φ_t of the principal symbol of (T_k).

4.4. **Proof of proposition 3.6.** Consider n Toeplitz operators $(T_k^1, ..., T_k^n)$ which commute. Denote by h_0^i the principal symbol of T_k^i and assume that $h_0 : M \to \mathbb{R}^n$ has maximal rank at $\bar{y} \in M$.

Let M_t be the torus $\big(\mathbb{R}/2\pi\mathbb{Z}\big)^n \times \big(\mathbb{R}/\mathbb{Z}\big)^n \ni (\xi^i, x^i)$ with symplectic form

$$\omega = \sum d\xi^i \wedge dx^i$$

and complex coordinates $z^j = (\sqrt{2})^{-1}(\xi^j + i x^j)$. Introduce a prequantization bundle $L_t \to M_t$ and define the associated quantum spaces $\mathcal{H}_k^t$. Finally introduce n Toeplitz operators

$$S^1, ..., S^n$$

such that $\sigma(S^i) = \xi^i$ on a neighborhood of $0 \in M_t$.

Then there exists a symplectomorphism $\varphi : U \to U_t$, where U and U_t are neighborhood of $\bar{y}$ and 0, such that $\varphi(\bar{y}) = 0$ and

$$h_0^i = \varphi^* (\xi^i + h_0^i(\bar{y})).$$

Using a variant of the quantum maps, we may quantize this local equivalence.

Proposition 4.4. *There exists an admissible sequence of operators*

$$U_k : C^\infty(M_t, L_t^k) \to C^\infty(M, L^k)$$

such that $\Pi_k U_k \Pi_k^t = U_k$, $\mathrm{MS}(U_k) \subset \{(y, \varphi(y)) \ / \ y \in U\}$ *and*

$$U_k U_k^* \sim \Pi_k \qquad \text{on a neighborhood of } (\bar{y}, \bar{y}),$$

$$U_k^* U_k \sim \Pi_k^t \qquad \text{on a neighborhood of } (0,0),$$

$$U_k^* T_k^i U_k \sim S_k^i + h_0^i(\bar{y}) \Pi_k^t \qquad \text{on a neighborhood of } (\bar{y}, 0).$$

Proof. We assume that $M \times M_t$ is endowed with the complex structure $J \times -J_t$. On a neighborhood of $\bar{y}$ (resp. 0) we may introduce a local gauge s of L (resp. s_t of L_t) such that $\nabla s = i\varphi^* \alpha \otimes s$ if $\nabla s_t = i\alpha \otimes s_t$. Let us define a local section E of $L \boxtimes L_t^{-1}$ on $U \times U_t$ such that

$$E(x, \varphi(x)) = s(x) \otimes s_t^{-1}(\varphi(x))$$

and $\nabla_{\bar{Z}} E$ vanishes to any order along the graph of φ, if Z is a holomorphic vector field of $(M \times M_t, J \times -J_t)$. Consider the operators $\mathcal{H}_k^t \to \mathcal{H}_k$ whose Schwartz kernel are of the form

$$\Big(\frac{k}{2\pi}\Big)^n E^k(x_l, x_r) a(x_l, x_r, k) + O_\infty(k^{-\infty})$$

where $(a(.,k))$ is a symbol of $S^0(M \times M_t)$, whose coefficients of its asymptotic expansion have their support included in a fixed compact $K \subset U \times U_t$. All the properties of the quantum maps generalize to these operators by identifying U with U_t and s with s_t. Let (V_k) be such an operator with a principal symbol $a_0(x, \varphi(x))$ which does not vanish. Then $(V_k^* V_k)$ and $(V_k^* T_k^i V_k)$ are Toeplitz operators with principal symbols f_0 and $f_0(\xi^i + h_0^i(\bar{y}))$ where f_0 takes real positive values. Following a standard argument, we may choose a Toeplitz operator P_k such that $U_k = V_k P_k$ satisfies the assumptions of the proposition. Indeed the proof just uses the symbolic calculus which is the same as in the case of pseudodifferential operators with a small parameter. □

Furthermore, generalizing proposition 4.3, we may prove that (U_k) sends a Lagrangian section associated to the local fibration ξ =cst to a Lagrangian section associated to the local fibration h_0 =cst. So to prove proposition 3.6, we just need to check the results in the case of the torus.

Chose a section s of L_t defined on a neighborhood of $0 \in M_t$ and such that $|s|^2 = e^{-|z|^2}$ and $\nabla s = -\bar{z}^j dz^j \otimes s$. Consider the operator (R_k^i) defined by

$$f.s^k \to \tfrac{\phi}{\sqrt{2}}(z^i.f + k^{-1}\partial_{z^i} f) s^k$$

where $\phi \in C_o^\infty(U)$ is equal to 1 on a neighborhood of 0. Let us prove that there exists a neighborhood V of 0 such that the kernel of (S_k^i) restricts on $V \times M_t$ to the kernel of $(R_k^i \Pi_k^t)$ modulo a smoothing operator. Since the microsupports of (S_k^i) and (Π_k^t) are subsets of the diagonal, it suffices to prove this on a neighborhood of $(0,0)$. The kernel of (Π_k^t) is determined modulo a smoothing operator by the local data, so

$$\Pi_k^t(x_l, x_r) = \Big(\frac{k}{2\pi}\Big)^n e^{-k|z_r|^2 + k z_l^j . \bar{z}_r^j} \ s^k(x_l) \otimes s^{-k}(x_r) + O_\infty(k^{-\infty})$$

on a neighborhood of $(0,0)$. Consequently,

$$(R_k^i \Pi_k^t)(x_l, x_r) = \Big(\frac{k}{2\pi}\Big)^n e^{-k|z_r|^2 + k z_l^j . \bar{z}_r^j} \ \tfrac{\phi(x_l)}{\sqrt{2}}(z_l + \bar{z}_r) \ s^k(x_l) \otimes s^{-k}(x_r)$$

modulo $O_\infty(k^{-\infty})$. We recognize on a neighborhood of $(0,0)$ the kernel of S_k^i. Hence in the following we may replace the operator S_k^i with $R_k^i \Pi_k^t$.

Consider the family of Lagrangian tori $\Lambda_E = \{(\xi^i, x^i) \;/\; \xi^i = E^i, \forall i\}$. The associated section is

$$F(E,\xi,x) = e^{\frac{1}{4}\sum_i -(\sqrt{2}z^i - E^i)^2 - i2\sqrt{2}z^i E^i - (E^i)^2} s(\xi,x).$$

Indeed, we have

$$|F|^2 = e^{-\sum_i (\xi^i - E^i)^2} \quad \text{and} \quad \nabla F = \sqrt{2}(E^i - \xi^i) dz^i \otimes F.$$

We may check that $S^i_k F^k(E,.) = E^i F^k(E,.) + O_\infty(k^{-\infty})$ on a neighborhood of 0. Hence the Lagrangian sections solution of

$$S^i_{k_\alpha} v_\alpha = E^i_\alpha v_\alpha + O_\infty(k_\alpha^{-\infty})$$

on a neighborhood of 0 are of the form $v_\alpha = F^{k_\alpha}(E_\alpha,.)$.

Let $(u_\alpha, k_\alpha, E_\alpha)$ be a sequence such that $u_\alpha \in \mathcal{H}_{k_\alpha}$ for every α and

$$S^i_{k_\alpha} u_\alpha = E^i_\alpha u_\alpha + O_\infty(k_\alpha^{-\infty}) \tag{40}$$

on a neighborhood of 0. Let us prove that $u_\alpha = c_\alpha F^{k_\alpha}(E_\alpha,.) + O_\infty(k^{-\infty})$ on a neighborhood of 0. Define the complex numbers c_α to obtain an equality at $(\xi,x) = (E_\alpha, 0)$. Introduce the functions f_α such that

$$u_\alpha(\xi,x) - c_\alpha F^{k_\alpha}(E_\alpha,\xi,x) = f_\alpha(\xi,x) F^{k_\alpha}(E_\alpha,\xi,x)$$

So $f_\alpha(E_\alpha,0) = 0$. Furthermore $\partial_{\bar{z}^j} f_\alpha = 0$ since the u_α are holomorphic sections. From (40), it follows that

$$(\partial_{z^j} f_\alpha)(\xi,x) F^{k_\alpha}(E_\alpha,\xi,x) = O_\infty(k_\alpha^{-\infty})$$

on a neighborhood of 0. From all of this, we deduce that

$$f_\alpha(x,y) F^{k_\alpha}(E_\alpha,\xi,x) = O_\infty(k_\alpha^{-\infty})$$

on a neighborhood of 0 and this completes the proof.

References

[1] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators*, Springer Verlag, 1992.

[2] D. Borthwick, T. Paul and A. Uribe, Legendrian distributions with applications to relative Poincaré series, *Invent. Math.*, Vol. **122**, pp. 359-402, 1995.

[3] D. Borthwick, T. Paul and A. Uribe, Semiclassical spectral estimates for Toeplitz operators, *Ann. Inst. Fourier*, Vol. **48**, 1998.

[4] L. Boutet de Monvel, V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of math. stud. 99, Princeton University, 1981.

[5] L. Charles, Berezin-Toeplitz Operators, a semi-classical approach, *Commun. Math. Phys.*, Vol. **239** no. 1-2, pp. 1-28, 2003.

[6] L. Charles, *Aspects semi-classiques de la quantification géométrique*, PhD thesis, Université Paris IX-Dauphine, 2000.

[7] Y. Colin de Verdière, *Méthodes semi-classiques et théorie spectrale*, in progress.

[8] L. Hörmander, *The analysis of linear partial differential operators I,IV*, Springer Verlag, 1983.

[9] J.M. Morvan, Quelques invariants topologiques en géométrie symplectique, *Ann. Inst. Henri Poincaré*, Section A: Phys. théor., Vol. **38**, pp 349-370, 1983.

[10] S. Vu Ngoc, Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type, *Comm. Pure and Applied Math* , Vol. **53**, pp 143-217, 1999.

[11] S. Zelditch, Index and dynamics of quantized contact transformations, *Ann.Inst. Fourier*, Vol **47**, pp 305-363, 1997.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato, 5, 40127 Bologna, Italia