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Quasinormal Families of Meromorphic Functions

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Abstract

Let $\mathcal{F}$ be a family of functions meromorphic on the plane domain D, all of whose zeros are multiple. Suppose that $f'(z) \neq 1$ for all $f \in \mathcal{F}$ and $z \in D$. Then if $\mathcal{F}$ is quasinormal on D, it is quasinormal of order 1 there.

1. Introduction

In this paper, we are concerned with the order of quasinormality of families of meromorphic functions on plane domains, all of whose zeros are multiple.

Recall that a family $\mathcal{F}$ of functions meromorphic on a plane domain $D \subset \mathbb{C}$ is said to be quasinormal on D [2] if from each sequence $\{f_n\} \subset \mathcal{F}$ one can extract a subsequence $\{f_{n_k}\}$ which converges locally uniformly with respect to the spherical metric on $D \setminus E$, where the set E (which may depend on $\{f_{n_k}\}$) has no accumulation point in D. If E can always be chosen to satisfy $|E| \leq \nu$, $\mathcal{F}$ is said to quasinormal of order ν on D. Thus a family is quasinormal of order 0 on D if and only if it is normal on D. The family $\mathcal{F}$ is said to (quasi)normal at $z_0 \in D$ if it is (quasi)normal on some neighborhood of z_0; thus $\mathcal{F}$ is quasinormal on D if and only if it is quasinormal at each point $z \in D$. On the other hand, $\mathcal{F}$ fails to be quasinormal of order ν on D precisely when there exist points $z_1, z_2, \ldots, z_{\nu+1}$ in D and a sequence $\{f_n\} \subset \mathcal{F}$ such that no subsequence of $\{f_n\}$ is normal at z_j, $j = 1, 2, \ldots, \nu + 1$.

Our point of departure is the following classical result of Gu.

Theorem A ([3]). *Let $\mathcal{F}$ be a family of functions meromorphic on D. If for each $f \in \mathcal{F}$ and $z \in D$, $f(z) \neq 0$ and $f'(z) \neq 1$, then $\mathcal{F}$ is normal on D.*

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Theorem A has been generalized in a number of different directions; cf., for instance, [1], [4], [7], [8]. In the present work, we are concerned with the situation in which the condition $f \neq 0$ is replaced by the assumption that all zeros of f are multiple and $\mathcal{F}$ is assumed to be quasinormal on D. Our main result is that in this case, $\mathcal{F}$ must be quasinormal of order 1.

Theorem. *Let $\mathcal{F}$ be a quasinormal family of meromorphic functions on D, all of whose zeros are multiple. If for any $f \in \mathcal{F}$, $f'(z) \neq 1$ for $z \in D$, then $\mathcal{F}$ is quasinormal of order 1 on D.*

Corollary. *Let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros are multiple. Suppose that each $f \in \mathcal{F}$ has at most K zeros on D and that $f'(z) \neq 1$ on D. Then $\mathcal{F}$ is quasinormal of order 1 on D.*

Indeed, it follows easily from Theorem A that $\mathcal{F}$ is quasinormal of order no greater than K, so the hypotheses of our Theorem are satisfied. That $\mathcal{F}$ need not be normal on D is shown by the following example.

Example 1. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_\alpha\}$, where

$$f_\alpha(z) = \frac{(z+\alpha)^2}{z+2\alpha} = z + \frac{\alpha^2}{(z+2\alpha)}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Then all zeros of f_α are multiple and $f'_\alpha(z) \neq 1$. However, f_α takes on the values 0 and ∞ in any fixed neighborhood of 0 if α is sufficiently small, so $\mathcal{F}$ fails to be normal at 0.

In certain generalizations of Gu's Theorem, the requirement that $f'(z) \neq 1$ can be weakened to $f'(z) \neq a(z)$, where $a(z)$ is some fixed analytic function on D [4], [7], which in some cases may be required not to vanish on D. Unfortunately, no such extension of our theorem is available.

Example 2. Consider the family $\mathcal{F} = \{f_n\}$ on $D = \{z : |z| < 1\}$, where

$$f_n(z) = \frac{\left(z - \frac{n+2}{2n}\right)^2}{z - 1/2}.$$

Then $\mathcal{F}$ fails to be normal at $z = 1/2$ but is quasinormal of order 1 on D. Let $\varphi(z) = e^{(z+1)/(z-1)}$. Then $\varphi(D) \subset D$; $\varphi'(z) \neq 0$ on D; and, for each $w \in D \setminus \{0\}$, $\varphi^{-1}(w)$ consists of countably many points of D accumulating at $z = 1$. Consider the family $\tilde{\mathcal{F}} = \{F_n\}$ on D, where $F_n = f_n \circ \varphi$. Then $\tilde{\mathcal{F}}$ is a quasinormal family of meromorphic functions on D, all of whose zeros are multiple. Also, for any $F \in \tilde{\mathcal{F}}$, $F'(z) = f'(\varphi(z))\varphi'(z) \neq \varphi'(z)$ since $f'(z) \neq 1$ for any $f \in \mathcal{F}$. However, $\tilde{\mathcal{F}}$ is not quasinormal of any finite order on D as no subsequence of $\tilde{\mathcal{F}}$ is normal at any point of $\varphi^{-1}(1/2)$.

2. Notation and preliminary results

Let us set some notation. We denote by Δ the open unit disc in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z-z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z-z_0| < r\}$. We write $f_n \stackrel{\chi}{\Longrightarrow} f$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Longrightarrow f$ on D if the convergence is in the Euclidean metric.

We require the following known results.

Lemma 1. *Let $\mathcal{F}$ be a family of functions meromorphic on Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

a) points $z_n \in \Delta$, $z_n \longrightarrow z_0$;

b) functions $f_n \in \mathcal{F}$; and

c) positive numbers $\rho_n \longrightarrow 0$

such that

$$\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \stackrel{\chi}{\Longrightarrow} g(\zeta) \quad \text{on } \mathbb{C},$$

where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that

$$g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1.$$

In particular, g has order at most 2.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

This is the local version of [6, Lemma 2] (cf. [4, Lemma 1], [9, pp. 216-217]). The proof consists of a simple change of variable in the result cited from [6]; cf. [5, pp. 299-300].

Lemma 2. *Let $\mathcal{F}$ be a family of functions meromorphic on D, all of whose zeros and poles are multiple. If for each $f \in \mathcal{F}$, $f'(z) \neq 1$, $z \in D$, then $\mathcal{F}$ is normal on D.*

This is the case $n = 2$, $k = 1$ of Theorem 5 in [8].

Lemma 3. *Let f be a nonconstant meromorphic function of finite order on $\mathbb{C}$, all of whose zeros are multiple. If $f'(z) \neq 1$ on $\mathbb{C}$, then*

$$f(z) = \frac{(z-a)^2}{z-b}$$

for some a and b ($\neq a$) in $\mathbb{C}$.

This follows from Lemma 6 (with $j = 1$ and $k = 2$) and Lemma 8 (with $k = 1$) of [8].

3. Auxiliary lemmas

The proof of the theorem proceeds by a number of intermediate results.

Lemma 4. *Let $\{a_k\}$ be a sequence in Δ which has no accumulation points in Δ. Let $\{f_n\}$ be a sequence of functions meromorphic on Δ, all of whose zeros are multiple, such that $f_n'(z) \neq 1$ for all n and all $z \in \Delta$. Suppose that*

(a) no subsequence of $\{f_n\}$ is normal at a_1;

(b) there exists $\delta > 0$ such that each f_n has a single (multiple) zero on $\Delta(a_1, \delta)$; and

(c) $f_n \overset{\chi}{\Longrightarrow} f$ on $\Delta \setminus \{a_k\}_{k=1}^{\infty}$.

Then

(d) there exists $\eta_0 > 0$ such that for each $0 < \eta < \eta_0$, f_n has a single simple pole on $\Delta(a_1, \eta)$ for all sufficiently large n; and

(e) $f(z) = z - a_1$.

Proof. It suffices to prove that each subsequence of $\{f_n\}$ has a subsequence which satisfies (d) and (e). So suppose we have a subsequence of $\{f_n\}$, which (to avoid complication in notation) we again call $\{f_n\}$.

Since $\{f_n\}$ is not normal at a_1, it follows from Lemma 1 that we can extract a subsequence (which, renumbering, we continue to call $\{f_n\}$), points $z_n \longrightarrow a_1$, and positive numbers $\rho_n \longrightarrow 0$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n} \overset{\chi}{\Longrightarrow} g(\zeta), \tag{3.1}$$

where g is a nonconstant meromorphic function of finite order on $\mathbb{C}$, all of whose zeros are multiple. Since $g_n'(\zeta) = f_n'(z_n+\rho_n\zeta) \neq 1$ and $g_n' \Longrightarrow g'$ on the complement of the poles of g, either $g' \neq 1$ or $g' \equiv 1$, by Hurwitz' Theorem. In the latter case, $g(\zeta) = \zeta + c$, which does not have multiple zeros. Thus $g'(\zeta) \neq 1$ on $\mathbb{C}$; so by Lemma 3,

$$g(\zeta) = \frac{(\zeta - a)^2}{(\zeta - b)} \tag{3.2}$$

for distinct complex numbers a and b. It now follows from the argument principle that there exist sequences $\xi_n \longrightarrow a$ and $\eta_n \longrightarrow b$ such that, for sufficiently large n, $g_n(\xi_n) = 0$ and $g_n(\eta_n) = \infty$. Thus, writing

$$z_{n,0} = z_n + \rho_n \xi_n, \quad z_{n,1} = z_n + \rho_n \eta_n,$$

we have $z_{n,j} \longrightarrow a_1$ $(j = 0, 1)$, $f_n(z_{n,0}) = 0$ and $f_n(z_{n,1}) = \infty$.

Let us now assume that (d) has been shown to hold. It follows from Lemma 2 that the pole of f_n at $z_{n,1}$ is simple. The limit function f from (c) is either meromorphic on $\Delta \setminus \{a_k\}_{k=1}^{\infty}$ or identically infinite there. Suppose first that it is meromorphic on $\Delta \setminus \{a_k\}_{k=1}^{\infty}$. Then there exists $\delta_0 > 0$ such that f has no poles on $\Gamma = \{z : |z - a_1| = \delta_0\}$ and f'_n converges uniformly to f' on Γ. We claim that $f' \equiv 1$ on $\Delta'(a_1, \delta_0)$. Indeed, otherwise by Hurwitz' Theorem, $f' \neq 1$. Now $1/(f'_n - 1)$ is analytic on $\Delta(a_1, \delta_0)$ and converges uniformly on Γ to $1/(f' - 1)$. By the maximum principle, $1/(f'_n - 1)$ converges uniformly on $\Delta(a_1, \delta_0)$, so $\{f'_n\}$ is normal at a_1. However, since $f'_n(z_{n,0}) = 0$ and $f'_n(z_{n,1}) = \infty$ and $z_{n,j} \longrightarrow a_1$ $(j = 0, 1)$, $\{f'_n\}$ is not equicontinuous at a_1, a contradiction.

Thus f has no poles on $\Delta'(a_1, \delta_0)$ and $f'_n \Longrightarrow 1$ on $\Delta'(a_1, \delta_0)$. Hence for any $z, z_0 \in \Delta'(a_1, \delta_0)$

$$f_n(z) - f_n(z_0) = \int_{z_0}^{z} f'_n(\zeta)\, d\zeta \longrightarrow z - z_0.$$

Taking a subsequence if necessary, we may suppose that $f_n(z_0) - z_0 \longrightarrow \alpha$. We claim that $\alpha = -a_1$. For otherwise, taking $r < \min\{|\alpha + a_1|, \delta_0\}$, we have, for large n,

$$\frac{1}{2\pi i} \int_{|z-a_1|=r} \frac{f'_n(z)}{f_n(z)}\, dz = \frac{1}{2\pi i} \int_{|z-a_1|=r} \frac{dz}{z - a_1 + (f_n(z_0) - z_0 + a_1)} = 0.$$

However, by the argument principle, the left hand side is the number of zeros minus the number of poles (counting multiplicities) of f_n in $\Delta(a_1, r)$, which for large n is at least $2 - 1 = 1$. It follows that $f(z) = z - a_1$.

Suppose now that $f \equiv \infty$ on $\Delta \setminus \{a_k\}_{k=1}^{\infty}$. Let

$$F_n(z) = f_n(z) \frac{z - z_{n,1}}{(z - z_{n,0})^2}.$$

By (b), $F_n(z) \neq 0$ on $\Delta(a_1, \delta)$. Applying the maximum principle to the sequence $\{1/F_n\}$ of analytic functions, we see that $F_n \Longrightarrow \infty$ on $\Delta(a_1, \delta)$. We have

$$\begin{aligned} \frac{f_n(z_n + \rho_n\zeta)}{\rho_n} &= \frac{F_n(z_n + \rho_n\zeta)}{\rho_n} \frac{(\rho_n\zeta + z_n - z_{n,0})^2}{(\rho_n\zeta + z_n - z_{n,1})} \\ &= F_n(z_n + \rho_n\zeta) \frac{(\zeta - \xi_n)^2}{\zeta - \eta_n}. \end{aligned} \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that $F_n(z_n + \rho_n\zeta) \longrightarrow 1$, which contradicts $F_n \Longrightarrow \infty$ near a_1. Thus the possibility $f \equiv \infty$ may be ruled out.

We have shown that when (d) obtains, (e) does as well. Now let us show that (d) must hold. Suppose not. Then, taking a subsequence and renumbering, we may assume that on any neighborhood of a_1, f_n has at least two poles for sufficiently large n. Keeping the notation established above, let $z_{n,2} \neq z_{n,1}$ be such that $f_n(z_{n,2}) = \infty$ and f_n has no poles in $\Delta'(z_{n,1}, |z_{n,1} - z_{n,2}|)$. Write $z_{n,2} = z_n + \rho_n \eta_n^*$. Then $z_{n,2} \longrightarrow a_1$ but $\eta_n^* \longrightarrow \infty$ since the right hand side of (3.2) has but a single simple pole. Set

$$G_n(\zeta) = \frac{f_n(z_{n,1} + (z_{n,2} - z_{n,1})\zeta)}{z_{n,2} - z_{n,1}}.$$

Since $z_{n,2} - z_{n,1} \longrightarrow 0$, $G_n(\zeta)$ is defined for any $\zeta \in \mathbb{C}$ if n is sufficiently large; and $G_n'(\zeta) \neq 1$. Now

$$G_n(0) = \infty \qquad G_n\left(\frac{z_{n,0} - z_{n,1}}{z_{n,2} - z_{n,1}}\right) = 0$$

and

$$\frac{z_{n,0} - z_{n,1}}{z_{n,2} - z_{n,1}} = \frac{\xi_n - \eta_n}{\eta_n^* - \eta_n} \longrightarrow 0,$$

so $\{G_n\}$ is not normal at 0.

On the other hand, for n sufficiently large, G_n has only a single zero (which tends to 0 as $n \longrightarrow \infty$) on any compact subset of $\mathbb{C}$. Since $G_n'(\zeta) \neq 1$, it follows from Theorem A that $\{G_n\}$ is normal on $\mathbb{C} \setminus \{0\}$. Taking a subsequence and renumbering, we may assume that $G_n \overset{\chi}{\Longrightarrow} G$ on $\mathbb{C} \setminus \{0\}$. Since G has only a single pole on Δ, conditions (a), (b), (c), and (d) hold for the sequence $\{G_n\}$ (defined, say, on $\Delta(0,2)$) with $a_1 = 0$ and $\delta = 1$. Thus, by the first part of the proof, $G(\zeta) = \zeta$. But this contradicts $G(1) = \infty$. This completes the proof of Lemma 4. ■

Definition. Let $z_1, z_2 \in \mathbb{C}$ and put $\tilde{z} = (z_1 + z_2)/2$. We say that (z_1, z_2) is a nontrivial pair of zeros of f if

(i) $f(z_1) = f(z_2) = 0$ and

(ii) there exists z_3 such that $|z_3 - \tilde{z}| < |z_1 - z_2|$ and $|f'(z_3)| > 1$.

Note that (ii) is equivalent to

(ii′) there exists z^* such that $|z^*| < 1$ and $|h'(z^*)| > 1$, where

$$h(z) = \frac{f(\tilde{z} + (z_1 - z_2)z)}{z_1 - z_2}.$$

Since $|h'(z)| \geq h^{\#}(z)$, it suffices to have $h^{\#}(z^*) > 1$ in (ii′).

Our next result deals with the situation in which the functions f_n have more than a single zero in each neighborhood of a point of non-normality.

Lemma 5. *Let $\{f_n\}$ be a sequence of functions meromorphic on Δ, all of whose zeros are multiple, such that $f_n'(z) \neq 1$ for all n and all $z \in \Delta$. Suppose that*

(a) no subsequence of $\{f_n\}$ is normal at z_0, and

(b) for each $\delta > 0$, f_n has at least two distinct zeros on $\Delta(z_0, \delta)$ for sufficiently large n.

Then for each $\delta > 0$, f_n has a nontrivial pair (a_n, c_n) of zeros on $\Delta(z_0, \delta)$ for sufficiently large n, and

$$\left\{ \frac{f_n(d_n + (a_n - c_n)\zeta)}{a_n - c_n} \right\}$$

is not normal on Δ. Here $d_n = (a_n + c_n)/2$.

Proof. As in the proof of the previous lemma, it follows from (a) and Lemmas 1 and 3 that for each subsequence of $\{f_n\}$ there exists a (sub)subsequence (which, renumbering, we continue to denote by $\{f_n\}$), points $z_n \to z_0$, numbers $\rho_n \to 0^+$, and distinct $a, b \in \mathbb{C}$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n} \overset{\chi}{\Longrightarrow} g(\zeta) = \frac{(\zeta - a)^2}{\zeta - b} \quad \text{on} \quad \mathbb{C}. \tag{3.4}$$

Thus there exist $\xi_n \longrightarrow a$, $\eta_n \longrightarrow b$ so that $a_n = z_n + \rho_n \xi_n \longrightarrow z_0$, $b_n = z_n + \rho_n \eta_n \longrightarrow z_0$ and $g_n(\xi_n) = f_n(a_n) = 0$, $g_n(\eta_n) = f_n(b_n) = \infty$ for n sufficiently large.

By assumption, there also exists $c_n \neq a_n$, $c_n \longrightarrow z_0$, such that $f_n(c_n) = 0$. Thus $c_n = z_n + \rho_n \xi_n^*$ and $\xi_n^* \longrightarrow \infty$ by (3.4). Setting $d_n = (a_n + c_n)/2$, we see that the function

$$h_n(\zeta) = \frac{f_n(d_n + (a_n - c_n)\zeta)}{a_n - c_n}$$

is defined for any $\zeta \in \mathbb{C}$ if n is sufficiently large.

We claim that $\{h_n\}$ is not normal at $\zeta = 1/2$. Indeed, we have

$$\frac{a_n - d_n}{a_n - c_n} \longrightarrow \frac{1}{2}, \qquad \frac{b_n - d_n}{a_n - c_n} \longrightarrow \frac{1}{2},$$

$$h_n\left(\frac{a_n - d_n}{a_n - c_n}\right) = f_n(a_n) = 0, \qquad h_n\left(\frac{b_n - d_n}{a_n - c_n}\right) = f_n(b_n) = \infty,$$

so $\{h_n\}$ fails to be equicontinuous in a neighborhood of $1/2$.

It follows from Marty's Theorem that

$$\lim_{n\longrightarrow\infty}\sup_{|\zeta-\frac{1}{2}|\le\frac{1}{4}} h_n^{\#}(\zeta)=\infty.$$

Thus (a_n, c_n) is a nontrivial pair of zeros of f_n for n sufficiently large. ■

Lemma 6. *Let $\{f_n\}$ be a sequence of functions meromorphic on Δ, all of whose zeros are multiple, such that $f_n'(z)\neq 1$ for all n and all $z\in\Delta$. Suppose that*

(a) there exist $d\in\Delta$, $a_n\longrightarrow d$, $c_n\longrightarrow d$, and $z_0\in\mathbb{C}$ such that for every $\delta>0$,

$$h_n(z)=\frac{f_n(d_n+(a_n-c_n)z)}{a_n-c_n}$$

has at least two distinct zeros on $\Delta(z_0,\delta)$ for sufficiently large n, where $d_n=(a_n+c_n)/2$; and

(b) no subsequence of $\{h_n\}$ is normal at z_0.

Then for n sufficiently large, f_n has a nontrivial pair of zeros $(z^*_{n,1}, z^*_{n,2})$ such that $z^*_{n,j}\longrightarrow d$ $(j=1,2)$ and $|z^*_{n,1}-z^*_{n,2}|<|a_n-c_n|$.

Proof. As before, it follows from Lemmas 1 and 3 that to each subsequence of $\{h_n\}$ there corresponds a subsequence (which we continue to write as $\{h_n\}$), $z_n\longrightarrow z_0$, and $\rho_n\longrightarrow 0^+$ such that

$$g_n(\zeta)=\frac{h_n(z_n+\rho_n\zeta)}{\rho_n}\overset{\chi}{\Longrightarrow}\frac{(\zeta-a)^2}{\zeta-b}\quad\text{on}\quad\mathbb{C}.$$

Thus there exist $\xi_{n,0}\longrightarrow b$, $\xi_{n,1}\longrightarrow a$ so that

$$z_{n,j}=z_n+\rho_n\xi_{n,j}\longrightarrow z_0\quad(j=0,1)$$

and $g_n(\xi_{n,0})=h_n(z_{n,0})=\infty$, $g_n(\xi_{n,1})=h_n(z_{n,1})=0$. By (a), there exist $z_{n,2}\longrightarrow z_0$, $z_{n,2}\neq z_{n,1}$, such that $h_n(z_{n,2})=0$. Setting $z_{n,2}=z_n+\rho_n\xi_{n,2}$, we have $\xi_{n,2}\longrightarrow\infty$.

Now put

$$z^*_{n,j}=d_n+(a_n-c_n)z_n+\rho_n(a_n-c_n)\xi_{n,j}\quad j=0,1,2.$$

Clearly $z^*_{n,j}\longrightarrow d$, $j=0,1,2$. Define

$$G_n(\zeta)=\frac{f_n\left(\frac{z^*_{n,1}+z^*_{n,2}}{2}+(z^*_{n,1}-z^*_{n,2})\zeta\right)}{z^*_{n,1}-z^*_{n,2}}.$$

Then $\{G_n\}$ is not normal at $\zeta = 1/2$. Indeed,

$$G_n\left(\frac{2\xi_{n,0} - \xi_{n,1} - \xi_{n,2}}{2(\xi_{n,1} - \xi_{n,2})}\right) = \infty, \qquad G_n(1/2) = 0.$$

Since

$$\frac{2\xi_{n,0} - \xi_{n,1} - \xi_{n,2}}{2(\xi_{n,1} - \xi_{n,2})} \longrightarrow 1/2,$$

$\{G_n\}$ is not equicontinuous at $\zeta = 1/2$. As before, it follows from Marty's Theorem that $(z^*_{n,1}, z^*_{n,2})$ is a nontrivial pair of zeros of f_n. Now

$$|z^*_{n,1} - z^*_{n,2}| = |a_n - c_n|\ |z_{n,1} - z_{n,2}|;$$

therefore, since $z_{n,j} \longrightarrow z_0 \quad (j = 1, 2)$, we have $|z^*_{n,1} - z^*_{n,2}| < |a_n - c_n|$ for large enough n, as required. ■

Lemma 7. *Let $\{f_n\}$ be a sequence of functions meromorphic on Δ, all of whose zeros are multiple, such that $f'_n(z) \neq 1$ for all n and all $z \in \Delta$. Suppose that*

(a) $\{f_n\}$ is normal on $\Delta'(0,1)$, but no subsequence of $\{f_n\}$ is normal at 0;
(b) there exists $\delta > 0$ such that f_n has a single (multiple) zero on $\Delta(0,\delta)$ for all sufficiently large n.

Then there exists a subsequence of $\{f_n\}$ (which we continue to call $\{f_n\}$) such that for any $a \in \mathbb{C}$, $f_n - a$ has at most two zeros (counting multiplicity) on $\Delta(0, 1/2)$.

Proof. Taking a subsequence and renumbering, we may assume that

$$f_n \overset{\chi}{\Longrightarrow} f \quad \text{on } \Delta'(0,1).$$

By Lemma 4, $f(z) = z$. Suppose that $|a| \leq 2/3$. Taking Γ to be the circle $\{|z| = 3/4\}$ traversed once in the positive direction, we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'_n(z)}{f_n(z) - a}\, dz \longrightarrow \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a} dz = 1.$$

However, the left hand side is the number of a-points of f_n minus the number of poles of f_n inside Γ, counting multiplicities. By Lemma 4, there exists $0 < \delta < 3/4$ such that f_n has a single simple pole on $\Delta(0,\delta)$ for n sufficiently large.

Since f_n converges uniformly to z on $\{z : \delta \leq |z| \leq 3/4\}$, there exists N_1 such that if $n \geq N_1$ f_n has a single simple pole in $\Delta(0, 3/4)$. Hence for $n \geq N_1$, f_n takes on the value a (counting multiplicities) exactly twice on $\Delta(0, 3/4)$.

Suppose now that $|a| > 2/3$. Let Γ' be the circle $\{|z| = 5/9\}$ traversed in the positive direction. Then

$$\frac{1}{2\pi i}\int_{\Gamma'}\frac{f_n'(z)}{f_n(z)-a}dz \longrightarrow \frac{1}{2\pi i}\int_{\Gamma'}\frac{1}{z-a} = 0,$$

so the number of a-points minus the number of poles of f_n (counting multiplicity) inside Γ' is 0 for large n. It follows as before that there exists N_2 such that f_n takes on the value a exactly once (counting multiplicities) on $\Delta(0, 5/9)$ if $n \geq N_2$. Dropping the elements f_n with $n < \max(N_1, N_2)$ and renumbering, we obtain the desired sequence. ■

Lemma 8. *Let f be a meromorphic function on $\mathbb{C}$, all of whose zeros are multiple, such that $f'(z) \neq 1$, $z \in \mathbb{C}$. Then either*

(i) f is rational; or

(ii) there exist nontrivial pairs (a_n, c_n) of zeros of f such that $|a_n - c_n| \longrightarrow 0$ and a sequence of functions

$$h_n(\zeta) = \frac{f(d_n + (a_n - c_n)\zeta)}{a_n - c_n}$$

which is not normal on Δ; here $d_n = (a_n + c_n)/2$.

Proof. Suppose f is not rational. Then by Lemma 3, f has infinite order, so there exist $z_n \to \infty$ and $\varepsilon_n \to 0$ such that

$$S(\Delta(z_n, \varepsilon_n), f) = \frac{1}{\pi}\iint_{|z-z_n|\leq\varepsilon_n}[f^{\#}(z)]^2 dxdy \longrightarrow \infty. \tag{3.5}$$

Indeed, otherwise there would exist $\varepsilon > 0$ and $M > 0$ such that

$$S(\Delta(\zeta, \varepsilon), f) \leq M$$

for all $\zeta \in \mathbb{C}$. From this follows

$$S(r) = \frac{1}{\pi}\iint_{|z|<r}[f^{\#}(z)]^2\, dxdy = O(r^2),$$

so that (cf. [9, p. 217]) f would have order at most 2, a contradiction. In particular, there exist $z_n^* \in \Delta(z_n, \varepsilon_n)$ such that $f^{\#}(z_n^*) \longrightarrow \infty$. Let $f_n(z) = f(z + z_n^*)$. Then no subsequence of $\{f_n\}$ is normal at 0.

Suppose there exists $\delta > 0$ such that f_n has only a single (multiple) zero ξ_n on $\Delta(0, \delta)$. Since no subsequence of $\{f_n\}$ is normal at 0, $\xi_n \longrightarrow 0$ by Theorem A. Thus, again by Theorem A, $\{f_n\}$ is normal on $\Delta'(0, \delta)$. It follows from Lemma 7 that there exist $n_1 < n_2 < \cdots$ such that for any $a \in \mathbb{C}$, $f_{n_k} - a$ has at most two zeros (counting multiplicity) on $\Delta(0, \delta/2)$. Thus, for large enough k,

$$S(\Delta(z_{n_k}, \varepsilon_{n_k}), f) \leq S(\Delta(0, \delta/2), f_{n_k}) \leq 2$$

which contradicts (3.5).

Thus, for each $\delta > 0$, f_n has at least two distinct zeros on $\Delta(0, \delta)$ for sufficiently large n. The result now follows immediately from Lemma 5. ■

4. Proof of the Theorem

Suppose the Theorem is false. Then there exists a sequence $\{a_k^*\} \subset D$ with no accumulation point in D and such that $a_1^* \neq a_2^*$ and a sequence $\{f_n\} \subset \mathcal{F}$ such that $f_n \overset{\chi}{\Longrightarrow} f$ on $D \setminus \{a_k^*\}$ but no subsequence of $\{f_n\}$ is normal at a_1^* or a_2^*. We may assume that $a_1^* = 0$ and $D = \Delta$. The argument given in the proof of Lemma 4 shows that $f_n' \Longrightarrow 1$ on $\Delta \setminus \{a_k^*\}$, so $f \not\equiv 0$.

If there exists $\delta > 0$ such that f_n has only a single (multiple) zero on each $\Delta(a_j^*, \delta)$ $(j = 1, 2)$ for large enough n, it follows from Lemma 4 that $f(z) = z - a_j^*$ $(j = 1, 2)$ on $\Delta \setminus \{a_k^*\}$. Thus $a_1^* = a_2^*$, a contradiction.

Therefore, one may suppose that for any $\delta > 0$, f_n has at least two distinct zeros on $\Delta(0, \delta)$ for sufficiently large n. By Lemma 5, f_n has a nontrivial pair of zeros in $\Delta(0, \delta)$ for n large enough. Therefore, some subsequence of $\{f_n\}$ (which, as usual, we continue to call $\{f_n\}$) has a nontrivial pair of zeros (z_n, w_n) such that $|z_n| < 1/n$, $|w_n| < 1/n$. There exist $\delta_0 > 0$ and $1 < s < 2$ such that $f_n \overset{\chi}{\Longrightarrow} f$ on $\Delta'(0, 2\delta_0)$ and f does not vanish for $\delta_0 \leq |z| \leq s\delta_0$. For $1/n < \delta_0$, let (a_n, c_n) be a nontrivial pair of zeros of f_n in $\Delta(0, \delta_0)$ whose distance is minimal. Clearly, $a_n - c_n \longrightarrow 0$. Set $d_n = (a_n + c_n)/2$. Then $d_n \in \Delta(0, \delta_0)$; and, passing to a subsequence, we may assume that $d_n \longrightarrow a$, so $|a| \leq \delta_0$. Since f and f_n have no zeros on $\{z : \delta_0 \leq |z| \leq s\delta_0\}$ if n is large enough, (a_n, c_n) is a nontrivial pair of zeros of f_n on $\Delta(0, s\delta_0)$ whose distance is minimal.

Set

$$h_n(\zeta) = \frac{f_n(d_n + (a_n - c_n)\zeta)}{a_n - c_n}.$$

Then for each $\zeta \in \mathbb{C}$, $h_n(\zeta)$ is defined if n is sufficiently large. Clearly, all zeros of h_n are multiple and $h_n'(\zeta) \neq 1$. We claim that no subsequence of $\{h_n\}$ is normal on $\mathbb{C}$. Otherwise, taking a subsequence and renumbering, we would have $h_n \overset{\chi}{\Longrightarrow} h$ on $\mathbb{C}$. Since (a_n, c_n) is a nontrivial pair of zeros of f_n,

$$h_n(\pm 1/2) = h_n'(\pm 1/2) = 0 \qquad \text{and} \qquad \sup_{\Delta} |h_n'(z)| > 1.$$

It follows easily that $h'(\zeta) \neq 1$ on $\mathbb{C}$ and that h is nonconstant. Since all zeros of h are multiple, Lemma 3 shows that h must be transcendental. It then follows from Lemma 8 that there exist infinitely many nontrivial pairs (ξ_k, η_k) of zeros of h such that $\xi_k \longrightarrow \infty$ and $\xi_k - \eta_k \longrightarrow 0$, and z_k^* with

$$\left| z_k^* - \frac{\xi_k + \eta_k}{2} \right| < |\xi_k - \eta_k| \qquad \text{and} \qquad h^{\#}(z_k^*) \longrightarrow \infty.$$

Fix k such that $h^{\#}(z_k^*) \geq 2$ and $|\xi_k - \eta_k| < 1$. Then there exist $\xi_{n,k} \longrightarrow \xi_k$ and $\eta_{n,k} \longrightarrow \eta_k$ such that for n sufficiently large,

$$h_n(\xi_{n,k}) = h_n(\eta_{n,k}) = 0$$

and

$$|z_k^* - (\xi_{n,k} + \eta_{n,k})/2| < |\xi_{n,k} - \eta_{n,k}|.$$

Put

$$\begin{aligned} \xi_{n,k}^* &= d_n + (a_n - c_n)\xi_{n,k} \\ \eta_{n,k}^* &= d_n + (a_n - c_n)\eta_{n,k} \\ z_{n,k}^* &= d_n + (a_n - c_n)z_k^*. \end{aligned}$$

Then

$$\begin{aligned} \left| z_{n,k}^* - \frac{\xi_{n,k}^* + \eta_{n,k}^*}{2} \right| &= |a_n - c_n| \left| z_k^* - \frac{\xi_{n,k} + \eta_{n,k}}{2} \right| \\ &< |a_n - c_n| \, |\xi_{n,k} - \eta_{n,k}| = |\xi_{n,k}^* - \eta_{n,k}^*|, \end{aligned}$$

where $\xi_{n,k}^* \longrightarrow a$, $\eta_{n,k}^* \longrightarrow a$ and $|a| < s\delta_0$; also, for n sufficiently large,

$$|f_n'(z_{n,k}^*)| = |h_n'(z_k^*)| \geq h_n^{\#}(z_k^*) > 1.$$

We conclude that $(\xi_{n,k}^*, \eta_{n,k}^*)$ is a nontrivial pair of zeros of f_n on $\Delta(0, s\delta_0)$. However,

$$|\xi_{n,k}^* - \eta_{n,k}^*| = |a_n - c_n| \, |\xi_{n,k} - \eta_{n,k}| < |a_n - c_n|$$

if n is sufficiently large. This contradicts the fact that (a_n, c_n) is a nontrivial pair of zeros of f_n in $\Delta(0, s\delta_0)$ whose distance is minimal.

Thus no subsequence of $\{h_n\}$ is normal on $\mathbb{C}$. Let E be the set on which $\{h_n\}$ is not normal. Suppose that for each $\zeta \in E$, there is a neighborhood on which h_n has only a single (multiple) zero for sufficiently large n. Then by Theorem A, $\{h_n\}$ is quasinormal at each point of E and hence on all of $\mathbb{C}$. Let $\zeta_0 \in E$. Taking a subsequence, we may assume that no subsequence of $\{h_n\}$ is normal at ζ_0 and that $\{h_n\}$ converges locally spherically uniformly on $\mathbb{C} \setminus E_0$, where $E_0 \subset E$ is a discrete set containing ζ_0. By Lemma 4,

$$h_n \overset{\chi}{\Longrightarrow} \zeta - \zeta_0 \quad \text{on} \quad \mathbb{C} \setminus E_0.$$

Taking additional subsequences and diagonalizing, we may assume that no subsequence of $\{h_n\}$ is normal at any point of E_0. We claim that $E_0 = \{\zeta_0\}$. Indeed, otherwise there exists $\zeta_1 \in E_0$, $\zeta_1 \neq \zeta_0$; then, as before, it follows from Lemma 4 that

$$h_n(\zeta) \overset{\chi}{\Longrightarrow} \zeta - \zeta_1 \quad \text{on} \quad \mathbb{C} \setminus E_0,$$

so that $\zeta_1 = \zeta_0$, $E_0 = \{\zeta_0\}$, and

$$h_n(\zeta) \overset{\chi}{\Longrightarrow} \zeta - \zeta_0 \quad \text{on} \quad \mathbb{C} \setminus \{\zeta_0\}.$$

But this contradicts $h_n(\pm 1/2) = 0$. Hence there exists $\zeta_0 \in E$ such that for each $\delta > 0$, there is a subsequence of $\{h_n\}$ (which we continue to call $\{h_n\}$) such that each h_n has at least two distinct zeros in $\Delta(\zeta_0, \delta)$ for sufficiently large n. Then by Lemma 6, for n sufficiently large, f_n has a nontrivial pair of zeros $(w^*_{n,1}, w^*_{n,2})$ such that

$$w^*_{n,j} \longrightarrow a \;\; (j = 1, 2) \qquad \text{and} \qquad |w^*_{n,1} - w^*_{n,2}| < |a_n - c_n|.$$

This contradicts the fact that (a_n, c_n) is a nontrivial pair of zeros of f_n in $\Delta(0, s\delta_0)$ whose distance is minimal.

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