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Quasiprobability distributions for the cavity-damped Jaynes-Cummings model with an additional Kerr medium

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A cavity-damped Jaynes-Cummings model with a Kerr-like medium filling the cavity is investigated in the rotating-wave approximation. We introduce six operators with respect to the light field whose equations of motion are transformed to six coupled partial differential equations using the s-parametrized quasiprobability distributions of Cahill and Glauber [Phys. Rev. **177**, 1882 (1969)]. Equations of motion for expansion coefficients of the distribution functions are solved by a Runge-Kutta procedure for vector tridiagonal relations. Starting with an initial coherent state for the cavity field and the atom in its upper state, we find that revivals of the atomic inversion are more pronounced for a given damping constant compared to the case of no Kerr medium. Also, quadrature squeezing is less affected by weak cavity damping and thermal noise compared to the standard Jaynes-Cummings model. The effect of damping on interesting non-Gaussian structures is also discussed.

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I. INTRODUCTION

Real systems are often modeled by simple systems that have analytic solutions. One example is the Jaynes-Cummings (JC) model [1], which has received much attention in the past, and new, interesting results are still appearing in the literature [2]. Predictions from this model have been verified by experiment [3, 4]. Many predictions have also been made concerning systems containing a Kerr medium that has an effective Hamiltonian of an anharmonic oscillator [5]. The statistical properties of an anharmonic oscillator, and in particular the Q function, was investigated by Milburn some time ago [6]. It was shown that an initial coherent state spreads as it rotates in the complex α plane and that for certain rational normalized times symmetric peaked distributions appear. These are the so-called Schrödinger catlike states described by Yurke and Stoler [7].

The simple quantum-optical model of a single-mode field interacting with a single atom described by Jaynes and Cummings has several interesting features. One of these is the collapse and revival of the atomic inversion [8–12] due to the atom interacting with the quantized electromagnetic field. Other interesting features such as squeezing also appear [13]. Eiselt and one of us [14] have investigated the cavity-damped JC model and have shown that the Q function splits into two counter-rotating distributions that undergo collisions coinciding with the revivals of the Rabi oscillations. With damping the distribution functions spiral into the origin. No analytic solution is known for the latter system and this is normally the case for damped quantum systems. However, a solution has been obtained for the Q function of the damped anharmonic oscillator using an initial Gaussian distribution [15] or a more general initial condition [16]. Here we consider a combination of these models with an initial Gaussian distribution for the field and the atom in the upper state [17, 18].

The quasiprobability distributions of the cavity mode are calculated. We show that the method described by Eiselt and Risken can also be used here where an interaction with a nonlinear Kerr medium is included. This may be used to solve other problems in nonlinear quantum optics where a Kerr medium arises. An important example is the common silica fiber used in communications [19].

II. DAMPED JC PLUS KERR MODEL AND ITS EQUATION OF MOTION

As already mentioned the Jaynes-Cummings model consists of a single two-level atom, coupled to a single-cavity mode. The Hamiltonian of this system reads in the rotating-wave approximation [1] (see also Ref. [20]),

$$H/\hbar = \omega_c a^\dagger a + \omega_a \sigma_z/2 + g(a\sigma^+ + a^\dagger \sigma^-) \ , \qquad (2.1)$$

where $\sigma_z, \sigma^+, \sigma^-$ are the Pauli spin matrices; $a^\dagger$, a are the creation and annihilation operators of the light mode; ω_a and ω_c are the frequencies of the atom and of the cavity mode, respectively; and g is the coupling constant. The coupling to the Kerr medium is modeled by an anharmonic oscillator of strength χ so that the total Hamiltonian is

$$H/\hbar = \omega_c a^\dagger a + \chi (a^\dagger)^2 a^2 + \omega_a \sigma_z/2 + g(a\sigma^+ + a^\dagger \sigma^-) \ . \qquad (2.2)$$

In the presence of cavity damping with a decay rate κ the equation of motion for the density operator $\rho = \rho(t)$ of the system takes the form

$$\dot{\rho} = -i[H/\hbar, \rho] + \kappa L_{\rm ir}(\rho) \ , \qquad (2.3)$$

where $L_{\rm ir}$, which describes the irreversible motion caused by cavity damping, is given by

$$L_{\rm ir}(\rho) = 2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a\rho + 2n_{\rm th}[[a,\rho],a^\dagger] = (n_{\rm th}+1)(2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a \rho) + n_{\rm th}(2a^\dagger \rho a - \rho a a^\dagger - a a^\dagger \rho) \ . \quad (2.4)$$

In (2.4) $n_{\rm th} = 1/\{\exp[\hbar\omega_c/(kT)] - 1\}$ is the number of thermal quanta. In the interaction picture

$$\exp[i\omega_c(a^\dagger a + \sigma_z/2)t]\ \rho\ \exp[-i\omega_c(a^\dagger a + \sigma_z/2)t] \Rightarrow \rho \quad (2.5)$$

we obtain the same equation of motion (2.3) with (2.4) unchanged, but with the transformed Hamilton operator

$$H/\hbar = \Delta\sigma_z/2 + \chi(a^\dagger)^2 a^2 + g(a\sigma^+ + a^\dagger\sigma^-) \ , \quad (2.6)$$

where $\Delta = \omega_a - \omega_c$ is the detuning.

Introducing matrix elements with respect to the two atomic states $|\uparrow\rangle, |\downarrow\rangle$ (denoting $\langle\uparrow|\rho|\uparrow\rangle$ by $\rho_{\uparrow\uparrow}$, etc.) the equations of motion read

$$\dot\rho_{\uparrow\uparrow} = -i\chi[(a^\dagger)^2a^2,\rho_{\uparrow\uparrow}] + ig(\rho_{\uparrow\downarrow}a^\dagger - a\rho_{\downarrow\uparrow}) + \kappa L_{\rm ir}(\rho_{\uparrow\uparrow}), \quad \dot\rho_{\downarrow\downarrow} = -i\chi[(a^\dagger)^2a^2,\rho_{\downarrow\downarrow}] + ig(\rho_{\downarrow\uparrow}a - a^\dagger\rho_{\uparrow\downarrow}) + \kappa L_{\rm ir}(\rho_{\downarrow\downarrow}), \quad (2.7)$$

$$\dot\rho_{\uparrow\downarrow} = -i\chi[(a^\dagger)^2a^2,\rho_{\uparrow\downarrow}] - i\Delta\rho_{\uparrow\downarrow} + ig(\rho_{\uparrow\uparrow}a - a\rho_{\downarrow\downarrow}) + \kappa L_{\rm ir}(\rho_{\uparrow\downarrow}),$$

$$\dot\rho_{\downarrow\uparrow} = -i\chi[(a^\dagger)^2a^2,\rho_{\downarrow\uparrow}] + i\Delta\rho_{\downarrow\uparrow} + ig(\rho_{\downarrow\downarrow}a^\dagger - a^\dagger\rho_{\uparrow\uparrow}) + \kappa L_{\rm ir}(\rho_{\downarrow\uparrow}).$$

These equations of motion for atomic matrix elements are coupled due to the electric dipole interaction and are still operators with respect to the light mode.

We introduce the following six combinations of the matrix elements:

$$\rho_1 = \rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}, \rho_2 = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}, \rho_3 = i(a\rho_{\downarrow\uparrow} - \rho_{\uparrow\downarrow}a^\dagger)/2,$$
$$\rho_4 = i(\rho_{\downarrow\uparrow}a - a^\dagger\rho_{\uparrow\downarrow})/2, \rho_5 = (a\rho_{\downarrow\uparrow} + \rho_{\uparrow\downarrow}a^\dagger)/2, \rho_6 = (\rho_{\downarrow\uparrow}a + a^\dagger\rho_{\uparrow\downarrow})/2. \quad (2.8)$$

From the equations of motion of the four matrix elements we obtain the following closed system of equations of motion for the combinations $\rho_1,\ldots,\rho_6$:

$$\dot\rho_1 = -i\chi[(a^\dagger)^2a^2,\rho_1] - 2g\rho_3 + 2g\rho_4 + \kappa L_{\rm ir}(\rho_1), \quad \dot\rho_2 = -i\chi[(a^\dagger)^2a^2,\rho_2] - 2g\rho_3 - 2g\rho_4 + \kappa L_{\rm ir}(\rho_2),$$

$$\dot\rho_3 = -i\chi[(a^\dagger)^2a^2,\rho_3] - i\chi[a^\dagger a,\rho_3] + \chi(a^\dagger a\rho_5 + \rho_5 a^\dagger a) - \Delta\rho_5 + (g/4)(\rho_1 aa^\dagger + aa^\dagger\rho_1 - 2a\rho_1a^\dagger + \rho_2 aa^\dagger + aa^\dagger\rho_2 + 2a\rho_2a^\dagger) + \kappa[L_{\rm ir}(\rho_3) - (2n_{\rm th}+1)\rho_3 + 2n_{\rm th}\rho_4],$$

$$\dot\rho_4 = -i\chi[(a^\dagger)^2a^2,\rho_4] + i\chi[a^\dagger a,\rho_4] - 2\chi\rho_6 + \chi(a^\dagger a\rho_6 + \rho_6 a^\dagger a) - \Delta\rho_6 + (g/4)(2a^\dagger\rho_1 a - a^\dagger a\rho_1 - \rho_1 a^\dagger a + 2a^\dagger\rho_2 a + a^\dagger a\rho_2 + \rho_2 a^\dagger a) + \kappa[L_{\rm ir}(\rho_4) + (2n_{\rm th}+1)\rho_4 - 2(n_{\rm th}+1)\rho_3], \quad (2.9)$$

$$\dot\rho_5 = -i\chi[(a^\dagger)^2a^2,\rho_5] - i\chi[a^\dagger a,\rho_5] - \chi(a^\dagger a\rho_3 + \rho_3 a^\dagger a) + \Delta\rho_3 + i(g/4)(\rho_1 aa^\dagger - aa^\dagger\rho_1 + \rho_2 aa^\dagger - aa^\dagger\rho_2) + \kappa[L_{\rm ir}(\rho_5) - (2n_{\rm th}+1)\rho_5 + 2n_{\rm th}\rho_6],$$

$$\dot\rho_6 = -i\chi[(a^\dagger)^2a^2,\rho_6] + i\chi[a^\dagger a,\rho_6] - \chi(a^\dagger a\rho_4 + \rho_4 a^\dagger a) + 2\chi\rho_4 + \Delta\rho_4 + i(g/4)(\rho_1 a^\dagger a - a^\dagger a\rho_1 - \rho_2 a^\dagger a + a^\dagger a\rho_2) + \kappa[L_{\rm ir}(\rho_6) + (2n_{\rm th}+1)\rho_6 - 2(n_{\rm th}+1)\rho_5].$$

Notice that the operators $\rho_1, \ldots, \rho_6$ appear only in bilinear products with the field annihilation and creation operators apart from the photon number conserving terms $[(a^\dagger)^2 a^2, \rho_i]$. This results in equations of motion for expansion coefficients of the quasiprobability distributions being tridiagonally coupled in only one index (see Sec. III for details).

The properties of the light field (F) are obtained by performing a trace with respect to the atom (A), i.e.,

$$\mathrm{Tr}_A(\rho) = \rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow} = \rho_1 . \tag{2.10}$$

Since this only involves one of the combinations ρ_1, only one distribution function need be considered for calculating expectation values of field operators. The same can be said for the atomic inversion D where we have

$$D = \mathrm{Tr}_{AF}(\sigma_z \rho) = \mathrm{Tr}_F(\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}) = \mathrm{Tr}_F(\rho_2) . \tag{2.11}$$

Initial values

In this paper we assume that initially the atom is in its upper state and that the cavity mode is in the coherent state $|\alpha_0\rangle$. (For the sake of simplicity we further assume that α_0 is real.) The initial condition for the density operator of the system reads

$$\rho(0) = |\uparrow\rangle\langle\uparrow| \otimes |\alpha_0\rangle\langle\alpha_0| . \tag{2.12}$$

The initial conditions for the six combinations $\rho_1, \ldots, \rho_6$ then take the form

$$\begin{aligned} &\rho_1(0) = \rho_2(0) = |\alpha_0\rangle\langle\alpha_0|, \\ &\rho_3(0) = \rho_4(0) = \rho_5(0) = \rho_6(0) = 0. \end{aligned} \tag{2.13}$$

III. EQUATION OF MOTION FOR THE QUASIPROBABILITY DISTRIBUTIONS

In order to solve (2.9) we introduce the quasiprobability distributions of Cahill and Glauber [21]. These distributions are c-number representations of the density operator, which contain the usual quasiprobability functions (P, Wigner, Q function) as special cases. In the present case we use for each of the six $\rho_i(t)$ the distributions $W_i(\alpha, s; t)$. They may be defined as Fourier transforms of the characteristic functions (our definitions deviate from those of Ref. [21] by a factor of $1/\pi$)

$$\chi_i(\xi, s; t) = \mathrm{Tr}_F[\exp(\xi a^\dagger - \xi^* a + s|\xi|^2/2)\rho_i(t)] , \tag{3.1}$$

i.e.,

$$W_i(\alpha, s; t) = \frac{1}{\pi^2} \int \chi_i(\xi, s; t) \exp(\alpha\xi^* - \alpha^*\xi) d^2\xi . \tag{3.2}$$

With these distribution functions, s-ordered products $\langle\{(a^\dagger)^n a^m\}_s\rangle_i$ can be obtained by proper integration with weight $W_i(\alpha, s; t)$ in the complex α plane. For the special choices of the parameter $s = 1, 0, -1$, the s-ordered products are the normal, symmetric, and antinormal ordered products and the quasiprobability distributions are the P, Wigner, and Q functions. For a definition of s-ordered products for arbitrary s, see Ref. [21].

The system of equations of motion (2.9) transforms into a system of partial differential equations for the distributions W_i. Applying the relations in Table I of Ref. [22] this system is obtained by a simple though lengthy calculation. For further considerations it is advantageous to use intensity I and phase ϕ variables defined by

$$\alpha = \sqrt{I}\ \exp(i\phi). \tag{3.3}$$

The following identities are used:

$$\begin{aligned} \frac{\partial}{\partial\alpha} &= \Big(\frac{\partial}{\partial I} - \frac{i}{2I}\frac{\partial}{\partial\phi}\Big)\sqrt{I}\, e^{-i\phi} \\ &= \sqrt{I}\, e^{-i\phi}\Big(\frac{\partial}{\partial I} - \frac{i}{2I}\frac{\partial}{\partial\phi}\Big), \\ &\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* = 2\frac{\partial}{\partial I}I, \\ &\frac{\partial^2}{\partial\alpha\partial\alpha^*} = \frac{\partial}{\partial I}I\frac{\partial}{\partial I} + \frac{1}{4I}\frac{\partial^2}{\partial\phi^2}, \\ &\frac{\partial^2}{\partial(\alpha^*)^2}(\alpha^*)^2 - \frac{\partial^2}{\partial\alpha^2}\alpha^2 = i\left(2\frac{\partial}{\partial I}I + 1\right)\frac{\partial}{\partial\phi}. \end{aligned} \tag{3.4}$$

In this way we obtain from (2.9) the following system for the quasiprobabilities $W_i(I, \phi, s; t)$:

$$\begin{aligned} \dot{W}_1 = \chi &\left[2[I - (1-s)]\frac{\partial}{\partial\phi} - s\Big(2\frac{\partial}{\partial I}I + 1\Big)\frac{\partial}{\partial\phi} - \frac{1}{2}(1-s^2)\Delta_2\frac{\partial}{\partial\phi}\right] W_1 \\ &-2gW_3 + 2gW_4 + \kappa\Big(2\frac{\partial}{\partial I}I + (2n_{\mathrm{th}} + 1 - s)\Delta_2\Big)W_1, \end{aligned}$$

$$\begin{aligned} \dot{W}_2 = \chi &\left[2[I - (1-s)]\frac{\partial}{\partial\phi} - s\Big(2\frac{\partial}{\partial I}I + 1\Big)\frac{\partial}{\partial\phi} - \frac{1}{2}(1-s^2)\Delta_2\frac{\partial}{\partial\phi}\right] W_2 \\ &-2gW_3 - 2gW_4 + \kappa\Big(2\frac{\partial}{\partial I}I + (2n_{\mathrm{th}} + 1 - s)\Delta_2\Big)W_2, \end{aligned}$$

$$\begin{aligned}
\dot W_3 = \chi\Bigg(&\left\{2\left[I-\left(\tfrac{1}{2}-s\right)\right]\frac{\partial}{\partial\phi}-s\Big(2\frac{\partial}{\partial I}I+1\Big)\frac{\partial}{\partial\phi}-\tfrac{1}{2}(1-s^2)\Delta_2\frac{\partial}{\partial\phi}\right\}W_3\\
&+\Big(2I-2s\frac{\partial}{\partial I}I+\frac{(s^2-1)}{2}\Delta_2+(s-1)\Big)W_5\Bigg)\\
&-\Delta W_5+\frac{g}{2}\Big(1-\frac{\partial}{\partial I}I+\frac{s-1}{2}\Delta_2\Big)W_1+\frac{g}{2}\Big(2I+s+(1-2s)\frac{\partial}{\partial I}I+\frac{s(s-1)}{2}\Delta_2\Big)W_2\\
&+\kappa\left[\Big(2\frac{\partial}{\partial I}I-(2n_{\rm th}+1)+(2n_{\rm th}+1-s)\Delta_2\Big)W_3+2n_{\rm th}W_4\right],\\
\dot W_4 = \chi\Bigg(&\left\{2\left[I-\left(\tfrac{3}{2}-s\right)\right]\frac{\partial}{\partial\phi}-s\Big(2\frac{\partial}{\partial I}I+1\Big)\frac{\partial}{\partial\phi}-\tfrac{1}{2}(1-s^2)\Delta_2\frac{\partial}{\partial\phi}\right\}W_4\\
&+\Big(2I-2s\frac{\partial}{\partial I}I+\frac{(s^2-1)}{2}\Delta_2+(s-3)\Big)W_6\Bigg)\\
&-\Delta W_6+\frac{g}{2}\Big(1-\frac{\partial}{\partial I}I+\frac{s+1}{2}\Delta_2\Big)W_1+\frac{g}{2}\Big(2I+s-(1+2s)\frac{\partial}{\partial I}I+\frac{s(s+1)}{2}\Delta_2\Big)W_2\\
&+\kappa\left[\Big(2\frac{\partial}{\partial I}I+(2n_{\rm th}+1)+(2n_{\rm th}+1-s)\Delta_2\Big)W_4-2(n_{\rm th}+1)W_3\right],\\
\dot W_5 = \chi\Bigg(&\left\{2\left[I-\left(\tfrac{1}{2}-s\right)\right]\frac{\partial}{\partial\phi}-s\Big(2\frac{\partial}{\partial I}I+1\Big)\frac{\partial}{\partial\phi}-\tfrac{1}{2}(1-s^2)\Delta_2\frac{\partial}{\partial\phi}\right\}W_5\\
&-\Big(2I-2s\frac{\partial}{\partial I}I+\frac{(s^2-1)}{2}\Delta_2+(s-1)\Big)W_3\Bigg)\\
&+\Delta W_3+\frac{g}{4}\Big(\frac{\partial W_1}{\partial\phi}+\frac{\partial W_2}{\partial\phi}\Big)+\kappa\left[\Big(2\frac{\partial}{\partial I}I-(2n_{\rm th}+1)+(2n_{\rm th}+1-s)\Delta_2\Big)W_5+2n_{\rm th}W_6\right],\\
\dot W_6 = \chi\Bigg(&\left\{2\left[I-\left(\tfrac{3}{2}-s\right)\right]\frac{\partial}{\partial\phi}-s\Big(2\frac{\partial}{\partial I}I+1\Big)\frac{\partial}{\partial\phi}-\tfrac{1}{2}(1-s^2)\Delta_2\frac{\partial}{\partial\phi}\right\}W_6\\
&-\Big(2I-2s\frac{\partial}{\partial I}I+\frac{(s^2-1)}{2}\Delta_2+(s-3)\Big)W_4\Bigg)\\
&+\Delta W_4+\frac{g}{4}\Big(\frac{\partial W_1}{\partial\phi}-\frac{\partial W_2}{\partial\phi}\Big)+\kappa\left[\Big(2\frac{\partial}{\partial I}I+(2n_{\rm th}+1)+(2n_{\rm th}+1-s)\Delta_2\Big)W_6-2(n_{\rm th}+1)W_5\right].
\end{aligned} \tag{3.5}$$

Here Δ_2 is an abbreviation for

$$\Delta_2=\frac{\partial}{\partial I}I\frac{\partial}{\partial I}+\frac{1}{4I}\frac{\partial^2}{\partial\phi^2}, \tag{3.6}$$

which is one-fourth of the two-dimensional Laplace operator. Because of (2.10), the Cahill-Glauber distribution for the light field is given by $W_1(I,\phi,s;t)$. In the intensity and phase variables the integration with respect to α has to be replaced according to

$$\int\cdots\, d^2\alpha \;\Rightarrow\; \tfrac{1}{2}\int_0^\infty\int_0^{2\pi}\cdots\, dI\, d\phi\,. \tag{3.7}$$

Therefore the inversion (2.11) now takes the form

$$D(t)=\tfrac{1}{2}\int_0^\infty\int_0^{2\pi}W_2(I,\phi,s;t)\,dI\,d\phi\,. \tag{3.8}$$

The initial conditions for the distributions $W_i(\alpha,s;t)$ are easily obtained by inserting (2.13) into (3.1) and evaluating the integral in (3.2). At $t=0$ the W_1, W_2 are shifted Gaussians in the α variable. With $\alpha_0=\sqrt{I_0}$ real, the W_i read in intensity and phase variables for $t=0$,

$$\begin{aligned}
&W_1(I,\phi,s;0)\\
&\quad=W_2(I,\phi,s;0)\\
&\quad=\frac{2}{\pi(1-s)}\exp\left(-\frac{2}{(1-s)}(I-2\sqrt{II_0}\cos\phi+I_0)\right),\\
&W_i(I,\phi,s;0)=0 \text{ for } i=3,\ldots,6.
\end{aligned} \tag{3.9}$$

IV. EQUATIONS OF MOTION FOR THE EXPANSION COEFFICIENTS

In order to handle the coupled partial differential equations (3.5) we expand the distributions W_i into two complete sets. Because the $W_i(I,\phi,s;t)$ are periodic in ϕ and only defined for $I \geq 0$ we use a Fourier series with respect to ϕ and Laguerre functions with respect to I, i.e.,

$$W_i(I,\phi,s;t) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} c_{n,m}^{(i)}(t) e^{in\phi} (I/\tilde{I})^{|n|/2} \times L_m^{(|n|)}(I/\tilde{I}) \exp(-I/\tilde{I}). \tag{4.1}$$

Here $L_m^{(|n|)}$ are the generalized Laguerre polynominals and $\tilde{I}$ is an arbitrary scaling intensity, which is chosen such that good numerical convergence is achieved. Inserting the expansion (4.1) into (3.5) and using the recurrence relations and orthogonality relations for the generalized Laguerre polynominals [23], we obtain the following equation of motion for the expansion coefficients ($m \geq 0$, coefficients with a negative index m formally occurring for $m = 0$ can be omitted because of the prefactor m):

$$\dot{c}_{n,m}^{(1)} = i\chi n\big(f_1 c_{n,m}^{(1)} + m f_2 c_{n,m-1}^{(1)} - f_3 c_{n,m+1}^{(1)}\big) - 2g c_{n,m}^{(3)} + 2g c_{n,m}^{(4)} + \kappa[a m c_{n,m-1}^{(1)} - (2m+|n|) c_{n,m}^{(1)}],$$

$$\dot{c}_{n,m}^{(2)} = i\chi n\big(f_1 c_{n,m}^{(2)} + m f_2 c_{n,m-1}^{(2)} - f_3 c_{n,m+1}^{(2)}\big) - 2g c_{n,m}^{(3)} - 2g c_{n,m}^{(4)} + \kappa[a m c_{n,m-1}^{(2)} - (2m+|n|) c_{n,m}^{(2)}],$$

$$\begin{aligned}
\dot{c}_{n,m}^{(3)} = {}& i\chi n[(f_1+1) c_{n,m}^{(3)} + m f_2 c_{n,m-1}^{(3)} - f_3 c_{n,m+1}^{(3)}] + \chi[(f_1+1) c_{n,m}^{(5)} + m f_2 c_{n,m-1}^{(5)} - f_3 c_{n,m+1}^{(5)}] \\
& - \Delta c_{n,m}^{(5)} + g\{(1+m+|n|/2) c_{n,m}^{(1)} - m[1-(1-s)/(2\tilde{I})] c_{n,m-1}^{(1)} \\
& + [f_1 + 2 - (m+|n|/2)] c_{n,m}^{(2)} - b^{+} m c_{n,m-1}^{(2)} - f_3 c_{n,m+1}^{(2)}\}/2 \\
& + \kappa[a m c_{n,m-1}^{(3)} - (2m+|n|+2n_{\text{th}}+1) c_{n,m}^{(3)} + 2n_{\text{th}} c_{n,m}^{(4)}],
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\dot{c}_{n,m}^{(4)} = {}& i\chi n\big[(f_1-1) c_{n,m}^{(4)} + m f_2 c_{n,m-1}^{(4)} - f_3 c_{n,m+1}^{(4)}\big] + \chi\big[(f_1-1) c_{n,m}^{(6)} + m f_2 c_{n,m-1}^{(6)} - f_3 c_{n,m+1}^{(6)}\big] \\
& - \Delta c_{n,m}^{(6)} + g\{(1+m+|n|/2) c_{n,m}^{(1)} - m[1+(1+s)/(2\tilde{I})] c_{n,m-1}^{(1)} \\
& + (f_1 + 2 + m + |n|/2) c_{n,m}^{(2)} - b^{-} m c_{n,m-1}^{(2)} - f_3 c_{n,m+1}^{(2)}\}/2 \\
& + \kappa\big[a m c_{n,m-1}^{(4)} - (2m+|n|-2n_{\text{th}}-1) c_{n,m}^{(4)} - 2(n_{\text{th}}+1) c_{n,m}^{(3)}\big],
\end{aligned}$$

$$\begin{aligned}
\dot{c}_{n,m}^{(5)} = {}& i\chi n\big[(f_1+1) c_{n,m}^{(5)} + m f_2 c_{n,m-1}^{(5)} - f_3 c_{n,m+1}^{(5)}\big] - \chi\big[(f_1+1) c_{n,m}^{(3)} + m f_2 c_{n,m-1}^{(3)} - f_3 c_{n,m+1}^{(3)}\big] \\
& + \Delta c_{n,m}^{(3)} + i g n (c_{n,m}^{(1)} + c_{n,m}^{(2)})/4 + \kappa\big[a m c_{n,m-1}^{(5)} - (2m+|n|+2n_{\text{th}}+1) c_{n,m}^{(5)} + 2n_{\text{th}} c_{n,m}^{(6)}\big],
\end{aligned}$$

$$\begin{aligned}
\dot{c}_{n,m}^{(6)} = {}& i\chi n\big[(f_1-1) c_{n,m}^{(6)} + m f_2 c_{n,m-1}^{(6)} - f_3 c_{n,m+1}^{(6)}\big] - \chi\big[(f_1-1) c_{n,m}^{(4)} + m f_2 c_{n,m-1}^{(4)} - f_5 c_{n,m+1}^{(4)}\big] \\
& + \Delta c_{n,m}^{(4)} + i g n (c_{n,m}^{(1)} - c_{n,m}^{(2)})/4 + \kappa\big[a m c_{n,m-1}^{(6)} - (2m+|n|-2n_{\text{th}}-1) c_{n,m}^{(6)} - 2(n_{\text{th}}+1) c_{n,m}^{(5)}\big].
\end{aligned}$$

The constants a, $b^{\pm}$, and f_i are defined by

$$\begin{aligned}
a &= 2 - (2n_{\text{th}} + 1 - s)/\tilde{I}, \\
b^{\pm} &= 2\tilde{I} + 2s \mp 1 + s(s \mp 1)/(2\tilde{I}), \\
f_1 &= 2\tilde{I}(2m+|n|+1) + s - 2 \\
&\quad +2s(m+|n|/2), \\
f_2 &= \frac{(1-s^2)}{2\tilde{I}} - 2(s+\tilde{I}), \\
f_3 &= 2\tilde{I}(m+|n|+1).
\end{aligned} \tag{4.3}$$

with initial values given by [14]

$$\begin{aligned}
c_{n,m}^{(1)}(0) &= c_{n,m}^{(2)}(0) \\
&= \frac{m!}{\pi\tilde{I}(m+|n|)!} \Big(\frac{I_0}{\tilde{I}}\Big)^{|n|/2} \Big(\frac{2\tilde{I}+s-1}{2\tilde{I}}\Big)^m \\
&\quad \times L_m^{(|n|)}\Big(\frac{2I_0}{2\tilde{I}+s-1}\Big),
\end{aligned} \tag{4.4}$$

$$c_{n,m}^{(i)}(0) = 0 \text{ for } i = 3, \ldots, 6.$$

In the system (4.2) of ordinary differential equations a tridiagonal coupling occurs in only the second index m;

the first index n appears only as a parameter. Because of this simple coupling, it is quite easy to solve (4.2).

A. Expectation values

Expectation values are usually expressed by some of the first few coefficients. For instance, the averaged intensity, the averaged complex amplitude, averaged squared complex amplitude, and the inversion are given by

$$\bar{I}(t) = \langle I \rangle = \mathrm{Tr}_F[a^\dagger a \rho_1(t)] = \tfrac{1}{2}\int_0^\infty \int_0^{2\pi} I\, W_1(I,\phi,s;t)\, dI\, d\phi - (1-s)/2 = \pi \tilde{I}^{\,2}\, [c^{(1)}_{0,0}(t) - c^{(1)}_{0,1}(t)] - (1-s)/2, \tag{4.5}$$

$$\bar{a}(t) = \mathrm{Tr}_F[a\rho_1(t)] = \tfrac{1}{2}\int_0^\infty \int_0^{2\pi} \sqrt{I}\, e^{i\phi} W_1(I,\phi,s;t)\, dI\, d\phi = \pi \tilde{I}^{\,3/2}\, c^{(1)}_{-1,0}(t), \tag{4.6}$$

$$\bar{a^2}(t) = \mathrm{Tr}_F[a^2\rho_1(t)] = \tfrac{1}{2}\int_0^\infty \int_0^{2\pi} I\, e^{i2\phi} W_1(I,\phi,s;t)\, dI\, d\phi = 2\pi \tilde{I}^{\,2}\, c^{(1)}_{-2,0}(t), \tag{4.7}$$

$$D(t) = \langle \sigma_z \rangle = \mathrm{Tr}_F[\rho_2(t)] = \tfrac{1}{2}\int_0^\infty \int_0^{2\pi} W_2(I,\phi,s;t)\, dI\, d\phi = \pi \tilde{I}\, c^{(2)}_{0,0}(t). \tag{4.8}$$

(To derive (4.5) we have used the relation $a^\dagger a = \{a^\dagger a\}_s - (1-s)/2$ for the s-ordered product, we have expressed I by $\tilde{I}\,[L_0^{(0)}(I/\tilde{I}) - L_1^{(0)}(I/\tilde{I})]$ and applied the orthogonality relations for the Laguerre polynomials.) Thus, in order to obtain these expectation values we need only solve (4.2) for $n = 0, 1$, or 2 separately.

B. Numerical method

The system (4.2) with given initial conditions such as (4.4) can be integrated by any numerical method for solving systems of ordinary differential equations. For these numerical methods the infinite system has to be truncated. The truncation indices M and $\pm N$ at which the infinite sums in (4.1) are truncated must be chosen such that a further increase of M and N does not change the results within a given accuracy. The truncation indices increase for increasing initial photon intensities. The addition of a Kerr medium results in the necessity to use smaller time steps for given truncation indices. This extra computational expense is negligible for quantities like average intensity and atomic inversion where only one value of n is required. But it becomes significant when computing quasiprobability functions for values of χ/g of

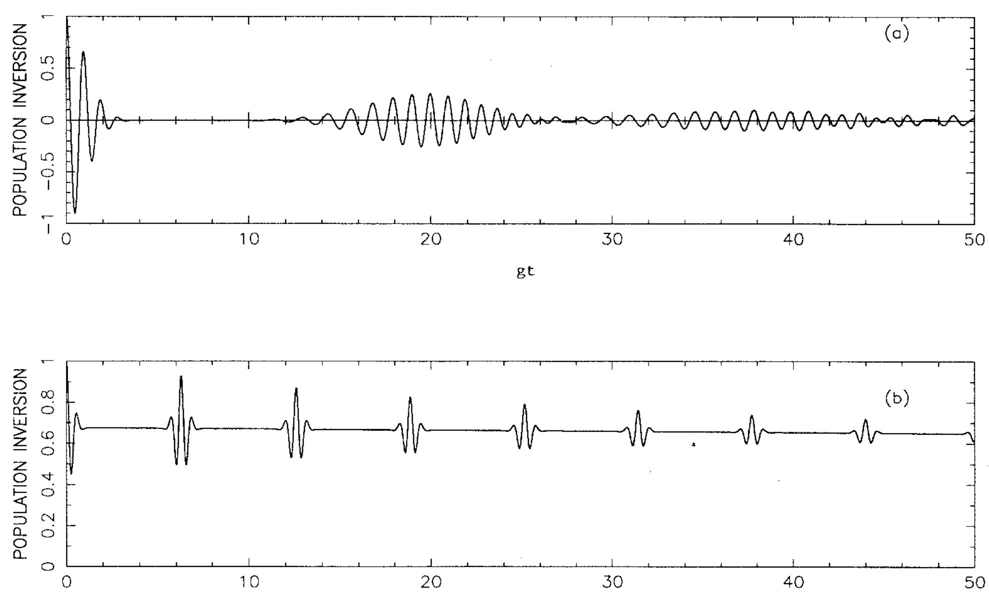

FIG. 1. The atomic inversion $D(t)$ as a function of gt with no detuning and $I_0 = 10$, $n_{\rm th} = 0, \kappa/g = 0.002$ for (a) $\chi = 0$, (b) $\chi/g = 0.5$.

$O(1)$. For $I_0 = 10$ we obtain results with an accuracy sufficient for the plots with $M = 100$ and $N = 20$. Because the expansion coefficients are not coupled in the first index n, we can integrate the system for each n separately. As already mentioned the ρ_i in (2.6) are Hermitian operators. Therefore the W_i are always real, leading for the expansion coefficients in (4.1) to the relation

$$c^{(i)}_{-n,m}(t) = [c^{(i)}_{n,m}(t)]^* . \tag{4.9}$$

Thus we generally need to integrate the system (4.2) only for $n \geq 0$.

For the integration we have used a fourth-order Runge-Kutta method [24]. With the Runge-Kutta method the next time step follows through some intermediate steps explicitly from the previous one. This procedure has the advantage that the previous coefficients can be overwritten. Therefore, besides 6×12 coefficients needed for the intermediate steps, we only need to store $6 \times M$ complex numbers. Even for $M = 100$ a PC can easily handle these numbers, although, use of a PC to calculate quasiprobability functions is only practical for small values of χ/g. The value of the time step h has to be chosen in such a way that a further decrease does not change the final result. If h is too large, numerical instabilities usually occur.

V. RESULTS

We first discuss the Q function for vanishing damping and detuning. As shown in the Appendix without damping the Q function is easily evaluated. Coupling the field to an atom or to a nonlinear medium has different effects on the Q function. Initially, the atom splits the Q function into two counter-rotating peaks that are individually squeezed and collide at times corresponding to revivals in the atomic inversion. Quadrature squeezing occurs initially and for those times when the two peaks collide. The Kerr medium alone results in a rotational sheer of the Q function [6], causing spreading along the circle $|\alpha| \approx \alpha_0$. Also, superpositions of coherent states appear at rational normalized times [7]. Other interesting non-Gaussian structures can also appear if the atomic coupling is included as well [18].

Collapse and revivals in the Jaynes-Cummings model have been known for some time. With the addition of a Kerr medium the revival time decreases along with the peak-to-peak amplitude of the oscillations. In the usual model the revivals tend to be irregular and indistinguishable after several sequences, but with a Kerr medium included many more distinct revival-collapse sequences occur. In a real experiment where cavity damping and thermal noise are important it can be seen in Figs. 1 and 2 that for a given damping constant the system with a Kerr medium has many more observable collapse-revival sequences.

Quadrature squeezing has been observed in many systems but not in single-atom single-mode cavity experiments. One reason for this is that the predicted squeezing is small. For low photon numbers the best squeezing occurs when the atom and cavity are detuned. In this case the initial transient squeezing is less than that which occurs at the first revival. The effect of damping and thermal photons is catastrophic since without damping the noise reduction occurs after the separated peaks

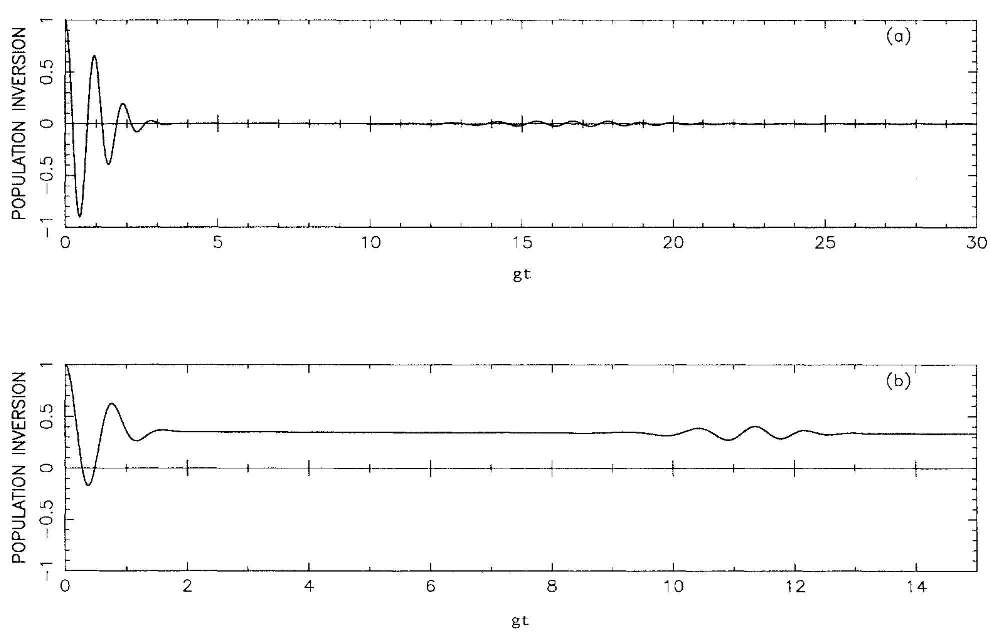

FIG. 2. The atomic inversion $D(t)$ as a function of gt with no detuning and $I_0 = 10$, $n_{\rm th} = 0.5$, $\kappa/g = 0.005$ for (a) $\chi = 0$, (b) $\chi/g = 0.5$.

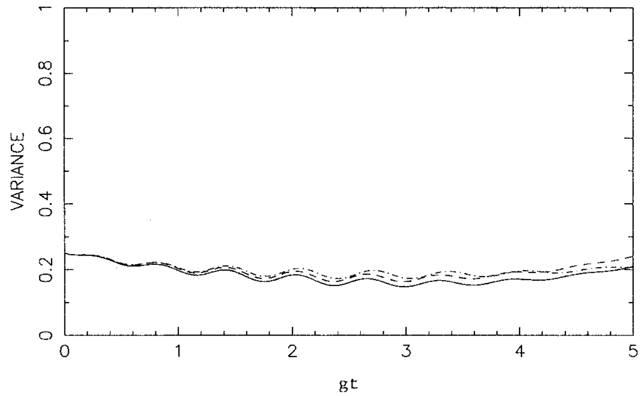

FIG. 3. The minimum eigenvalue of the variance matrix of the quadrature phases as a function of gt with $\chi = 0$, $I_0 = 10$, and $\Delta/g = 8$ for (i) (full line) $\kappa/g = 0.005$, $n_{\rm th} = 0.1$; (ii) (dashed line) $\kappa/g = 0.005$; $n_{\rm th} = 1.5$, (iii) (dot-dash line) $\kappa/g = 0.05$, $n_{\rm th} = 0.1$.

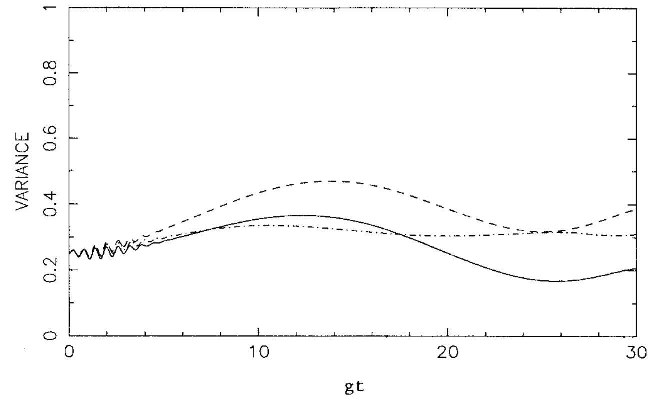

FIG. 4. The same parameters as in Fig. 3 except $\chi/g = 0.01$.

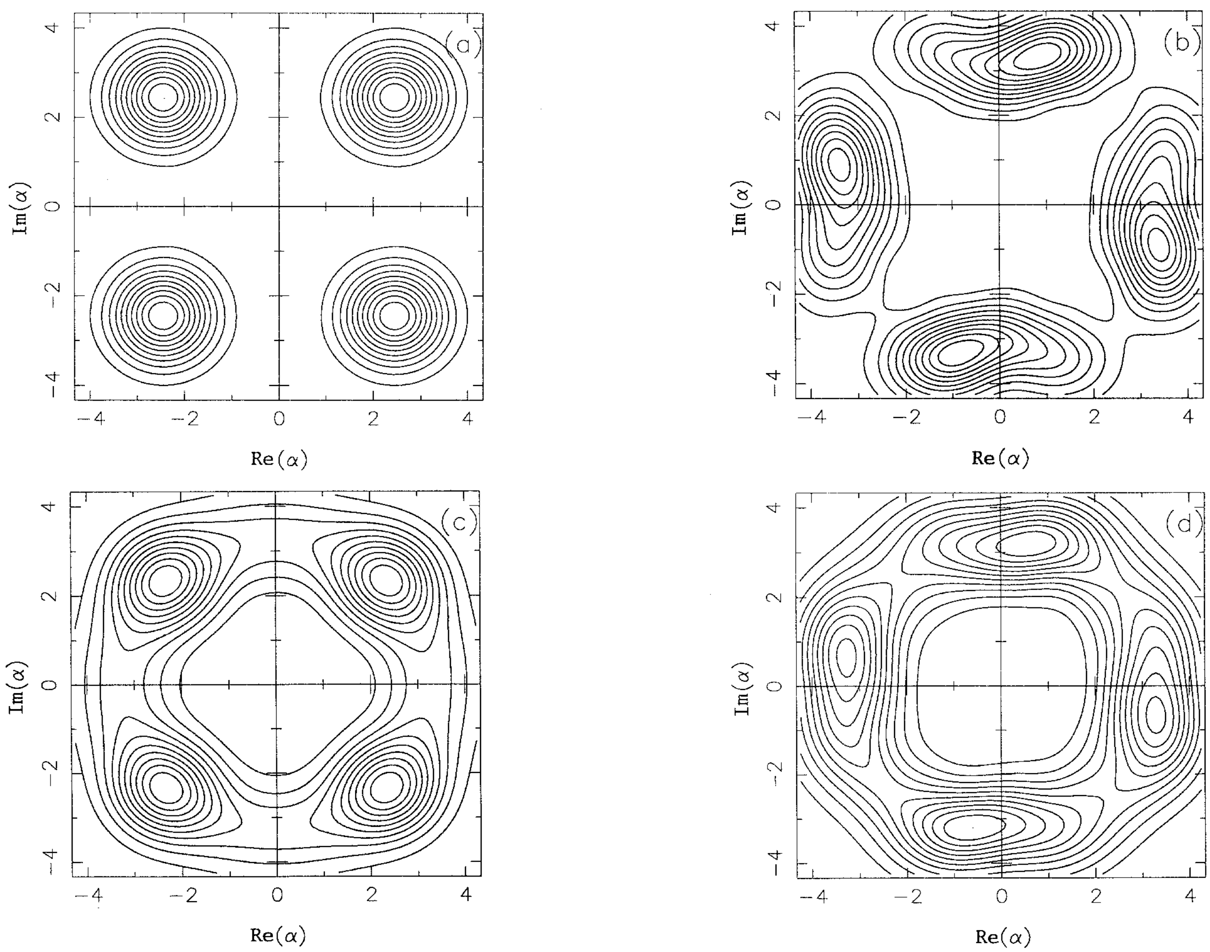

FIG. 5. Q function at time $\chi t = \pi/4$, $I_0 = 12$, $\Delta = n_{\rm th} = 0$ for (a) $g = 0$, $\kappa = 0$; (b) $g/\chi = 10$, $\kappa/\chi = 0$; (c) $g = 0$, $\kappa/\chi = 0.05$; (d) $g/\chi = 10$, $\kappa/\chi = 0.05$.

of the Q function have traversed π radians in the complex α plane. Hence, with damping there has been sufficient time for the rephasing to be affected by the noise introduced by the reservoir (see Fig. 3). Adding a weak Kerr medium tends to destroy the squeezing at the first revival but increases the initial transient squeezing (see Fig. 4). This destruction of squeezing after longer times is due to the Kerr medium introducing phase noise. The initial transient squeezing is not affected to the same degree by the external reservoir for a given damping constant simply due to the time scales involved. Hence, one would expect to observe more squeezing in an experiment using a cavity filled with a Kerr medium.

Superpositions of coherent states are formed for rational normalized times by the Kerr medium alone acting on an initial coherent state. The effect of damping has been calculated by Milburn and Holmes [15], where it was shown that the destructive effect on coherence was stronger for a given initial amplitude than contraction of the dynamics in phase space. When the atomic coupling is included, the Q function contains non-Gaussian structures at rational normalized times [18]. The atomic coupling causes the Gaussians to split into two peaks that counter-rotate in the complex α plane as the coupling is increased. The effect of damping on these structures appears to be the same as for superpositions of coherent states as seen in Fig. 5. There is a contraction of the radius of the circle upon which the centers of the peaks lie, and there is a spreading of the probability resulting in the maximum of the peaks in the Q function becoming diminished, although this has not been indicated in the figure.

VI. SUMMARY

The quasiprobability distributions for the cavity-damped Jaynes-Cummings model with an additional Kerr medium have been obtained. Following the work of Eiselt and Risken we introduced six combinations of the matrix elements of the density operator. Using the s-parametrized quasiprobability distribution functions we obtained six coupled partial differential equations. By expanding the quasiprobability functions in two complete sets (Laguerre polynomials and Fourier series), we obtained six coupled ordinary differential equations tridiagonally coupled in only one index for the expansion coefficients that were solved using a fourth-order Runge-Kutta integration procedure. As shown earlier the time evolution of the expansion coefficients are closely related to the moments of the field operators. This makes calculation of intensity, atomic inversion, and second-order moments of the field straightforward. A set of quasiprobability functions can be calculated and in particular the Q function has been used here to show the effect of damping on structures in phase space.

It has been shown that the addition of a Kerr medium into the cavity-damped Jaynes-Cummings model results in more distinct revivals and collapses of the atomic inversion and the perseverance of squeezing levels predicted for the detuned JC model even with damping and thermal noise.

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APPENDIX: Q FUNCTION WITHOUT DAMPING

Starting with an initially coherent state $|\alpha_0\rangle$ ($|\alpha_0| = \sqrt{I_0}$) in the upper state of the atom, the solution of the Schrödinger equation with the Hamiltonian (2.6) takes the form [1, 18, 20]

$$|\psi(t)\rangle = \exp(-I_0/2) \sum_{n=0}^{\infty} \frac{\alpha_0{}^n}{\sqrt{n!}} [\, A_n(t)|n\rangle|\uparrow\rangle + B_n(t)|n+1\rangle|\downarrow\rangle], \qquad \text{(A1)}$$

where the functions $A_n(t)$ and $B_n(t)$ are

$$A_n(t) = \exp(-i\chi n^2 t)\left(\cos(\lambda_n t) - i\frac{(\Delta - 2\chi n)}{2\lambda_n}\sin(\lambda_n t)\right), \qquad \text{(A2)}$$

$$B_n(t) = -i\frac{\sqrt{n+1}\, g}{\lambda_n}\exp(-i\chi n^2 t)\sin(\lambda_n t),$$

with λ_n given by

$$\lambda_n = \sqrt{(\Delta/2 - \chi n)^2 + (n+1)g^2}\,. \qquad \text{(A3)}$$

The Q function of the light field is defined by

$$Q(\alpha, t) = W_1(\alpha, -1; t) = \mathrm{Tr}_A[\langle\alpha|\rho(t)|\alpha\rangle]/\pi, \qquad \text{(A4)}$$

where the density operator is expressed by the wave function according to $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. Insertion of (A1) leads to

$$Q(\alpha, t) = (|V_A|^2 + |V_B|^2)/\pi, \qquad \text{(A5)}$$

with V_A and V_B given by

$$V_A = \exp\left(-\frac{I_0}{2} - \frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\sqrt{I_0}\ \alpha^*)^n}{n!} A_n(t),$$

$$V_B = \exp\left(-\frac{I_0}{2} - \frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\sqrt{I_0}\ \alpha^*)^n}{n!} \frac{\alpha^*}{\sqrt{n+1}} B_n(t). \qquad \text{(A6)}$$

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