Acta Math., 170 (1993), 29–81

Quasiregular mappings in even dimensions

by

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To Professor F. W. Gehring on the occasion of his 65th birthday

Contents

0. Introduction

This paper grew out of our study of the paper of S. Donaldson and D. Sullivan "Quasiconformal 4-Manifolds" [DS]. In that paper the authors develop a quasiconformal Yang–Mills theory by studying elliptic index theory with measurable coefficients in four dimensions via the Calderón–Zygmund theory of singular integrals. Other approaches to index theory on quasiconformal manifolds have been found by N. Teleman [T] and there are related results due to D. Freed and K. Uhlenbeck [FU] and others. We soon realized that there were many other applications of these ideas to the general theory of quasiconformal mappings. In this paper we present just a few of them.

[1] Research supported in part by a grant from the U.S. National Science Foundation, DMS-9007946

A principal feature of our methods is that they are quite explicit and closely mimic the two dimensional theory as developed by Ahlfors [A1], Boyarski [B] and Lehto [Le]. (Because of this we review a little of that theory.) Moreover, precise calculations can be made. Indeed we have found integral formulas for all the operators involved and given estimates of their norms. The case of dimension 4 is especially interesting and we have dealt with it in some detail in an appendix. Indeed it was the calculations we first made there that led to our results below.

In many respects quasiregular mappings are a part of PDE theory. This is again borne out in our paper which in many ways unifies some earlier approaches [BI2] [R3]. Let us recall some basic definitions and give a brief sketch of the theory leading to the governing equations. We refer the reader to §1 for the notation we use. For an extended survey, see [I1].

Let Ω be an open subset of $\mathbf{R}^n$ and $f:\Omega\to\mathbf{R}^n$, $f=(f^1,f^2,\dots,f^n)$, be a mapping of Sobolev class $W^1_{p,\mathrm{loc}}(\Omega,\mathbf{R}^n)$. Its differential $Df\in L^p_{\mathrm{loc}}(\Omega,GL(n))$ is given by the Jacobian matrix

$$Df(x)=\left(\frac{\partial f^i}{\partial x^j}\right)_{i,j}.$$

We define $D^tf(x)$ as the transpose of $Df(x)$ and $J(x,f)$ is the Jacobian determinant.

Definition 1. A mapping $f:\Omega\to\mathbf{R}^n$ is said to be K-*quasiregular*, $1\leqslant K<\infty$, if

(i) $f\in W^1_{n,\mathrm{loc}}(\Omega,\mathbf{R}^n)$,

(ii) $J(x,f)\geqslant 0$ a.e. or $J(x,f)\leqslant 0$ a.e.,

(iii) $\max_{|h|=1}|Df(x)h|\leqslant K\min_{|h|=1}|Df(x)h|$ a.e.

The number K is called the *dilatation* of f. If in addition f is a homeomorphism, then f is called K-*quasiconformal.*

Development of the analytic theory of quasiregular mappings depends upon advances in PDE's, harmonic analysis and (in dimension 2) complex function theory. The first equation of particular relevance to the theory of quasiregular mappings is the n-dimensional Beltrami system

$$D^tf(x)Df(x)=J(x,f)^{2/n}G(x) \tag{0.1}$$

for mappings f with non-negative Jacobian. The matrix function $G:\Omega\to S(n)$ is symmetric and, in view of the dilatation condition (iii) above, satisfies

$$K^{2-2n}\leqslant\langle G(x)\zeta,\zeta\rangle^n\leqslant K^{2n-2} \tag{0.2}$$

for $(x,\zeta)\in\Omega\times S^{n-1}$ and $\det G(x)=1$. In general G need not be continuous, the case when G is only assumed measurable is the most important for quasiconformal analysis.

If $G(x)=\mathrm{Id}$, the identity matrix, then

$$D^t f(x) Df(x) = J(x,f)^{2/n}\,\mathrm{Id} \tag{0.3}$$

is called the Cauchy–Riemann system. Recall that a Möbius transformation of $\bar{\mathbf{R}}^n$ is the finite composition of reflections in spheres or hyperplanes. Then we have the well known

LIOUVILLE THEOREM. *Every 1-quasiregular mapping of a domain $\Omega\subset\mathbf{R}^n$, $n\geqslant 3$, is either constant or the restriction to Ω of a Möbius transformation of $\bar{\mathbf{R}}^n$.*

Thus the only conformal mappings of subdomains of $\bar{\mathbf{R}}^n$, $n\geqslant 3$, are the Möbius transformations. This theorem was first proved by Liouville [L] in 1850 for diffeomorphisms of class $C^4(\Omega)$ using differential geometric techniques. For general 1-quasiregular mappings the result was established by Gehring [G1], Boyarski and Iwaniec [BI1] and Reshetnyak [R1]. We give examples to show that $W^1_{p,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ solutions of equation (0.3) need not be Möbius for $1<p<n/2$. Actually we show in even dimensions that $p=n/2$ is the critical exponent for the regularity theory associated with this equation, see Theorem 1.

In dimension 2, the equation (0.1) reduces to the linear complex Beltrami equation

$$\frac{\partial f}{\partial \bar z} = \mu(z)\frac{\partial f}{\partial z} \tag{0.4}$$

where μ is referred to as the complex dilatation of f. It is a measurable function satisfying

$$|\mu(z)|\leqslant \frac{K-1}{K+1}<1. \tag{0.5}$$

Many analytic problems of the two dimensional theory of quasiconformal mappings eventually lead to the study of the singular integral

$$S\omega(z) = -\frac{1}{2\pi i}\iint_{\mathbf{C}} \frac{\omega(\zeta)\,d\zeta\wedge d\bar\zeta}{(z-\zeta)^2} \tag{0.6}$$

which is known as the complex Hilbert transform or the Beurling–Ahlfors transform. One can characterise this operator by the symbolic equation

$$\frac{\partial}{\partial z} = S\circ\frac{\partial}{\partial \bar z} \tag{0.7}$$

connecting the Cauchy–Riemann derivatives. Like other Calderón–Zygmund type operators the complex Hilbert transform is bounded in $L^p(\mathbf{C})$ for all $1<p<\infty$. Many regularity results for quasiconformal mappings depend on the p-norms of S. A pressing task is to identify these norms. Simple examples show

$$\|S\|_p\geqslant \max\left\{\frac{1}{p-1}, p-1\right\},\quad 1<p<\infty. \tag{0.8}$$

It is conjectured that (0.8) holds with equality. An affirmative answer would imply (in 2 dimensions) Gehring's conjecture [G3]:

CONJECTURE 0.9. *If f is a K-quasiconformal mapping, then $f \in W^1_{p,\mathrm{loc}}(\Omega, \mathbf{R}^n)$ for every $p < nK/(K-1)$.*

In the plane it was shown by Boyarski [B] that the condition

$$\frac{K-1}{K+1}\|S\|_p < 1, \quad p > 2 \tag{0.10}$$

is sufficient for f to be in the class $W^1_{p,\mathrm{loc}}(\Omega, \mathbf{R}^2)$.

In the other direction, one might consider the relationship between regularity and dilatation. We therefore make a definition.

Definition 2. A mapping $f \in W^1_{p,\mathrm{loc}}(\Omega, \mathbf{R}^n)$ is said to be *weakly K-quasiregular* if f satisfies the conditions (ii) and (iii) of Definition 1.

In 2 dimensions Lehto [Le] has proven that if $f \in W^1_{q,\mathrm{loc}}(\Omega, \mathbf{R}^2)$ is weakly K-quasiregular and if $1 \leqslant q < 2$ is such that

$$\frac{K-1}{K+1}\|S\|_q < 1 \tag{0.11}$$

then f is K-quasiregular.

The other major type of equation considered in the theory of quasiconformal mappings is the second order Lagrange–Euler system.

$$\operatorname{div} A(x, Df) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} A^{ij}(x, Df) \right)_{i=1,2,\ldots,n} = 0 \tag{0.12}$$

where $A\colon \Omega \times \mathbf{R}^n \to GL(n)$ is defined by

$$A(x, M) = \langle G^{-1}(x)M, M\rangle^{(n-2)/2} G^{-1}(x)M$$

for $(x, M) \in \Omega \times GL(n)$. This equation is in fact the variational equation of the conformally invariant integral

$$I(f) = \int_\Omega \langle G^{-1}(x)D^t f(x), D^t f(x)\rangle^{n/2}\, dx.$$

Each component $u = f^i(x)$ of f satisfies a single equation of degenerate elliptic type

$$\operatorname{div} A(x, \nabla u) = 0 \tag{0.13}$$

which in the case of a conformal mapping ($K=1$) reduces to the well known n-harmonic equation

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = 0, \tag{0.14}$$

see for instance [BI2], [R3], and [GLM].

We shall see that all of these equations are special cases of many more equations related to quasiregular mappings which we shall derive in §4. Some of these equations arise in dimension $n=2l$ as linear relations (with measurable coefficients) between the determinants of $l\times l$ minors of the differential $Df(x)$ of a quasiregular mapping. For instance if f is 1-quasiregular, then $Df(x)$ is pointwise a scalar multiple of an orthogonal matrix

$$Df(x)=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, and D are the $l\times l$ submatrices. When $J(x,f)\geqslant 0$, these relations are

$$\det A(x)=\det D(x)$$

$$\det B(x)=(-1)^l \det C(x)$$

which generalize the Cauchy–Riemann equations. (It is an interesting exercise to directly verify these equations for orthogonal matrices!) The other identities are obtained from these by permuting the rows and columns of $Df(x)$. It is at this point that our theory really begins.

Previously, in order to get the integrability and regularity theory of quasiregular mappings off the ground it was necessary to integrate the determinant of the Jacobian matrix, thus the usual hypothesis $f\in W^1_{2l,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$. These identities (and the fact that there are so many and that they are linear!) show that it is really only necessary to integrate determinants of the $l\times l$ minors, thus reducing the necessary integrability assumptions (determinants are null-Lagrangians and their integrability theory is well understood). This enables us to establish the following Caccioppoli type estimate for weakly quasiregular mappings in dimension $2l$.

$$\int_\Omega \eta(x)^s|Df(x)|^s\,dx\leqslant C(n,K)\int_\Omega |\nabla\eta(x)|^s|f(x)-f_0|^s\,dx \tag{0.15}$$

for some $s\in[l,2l)$. Here $\eta\in C_0^\infty(\Omega)$ is a test function and $f_0\in\mathbf{R}^{2l}$. As far as we are aware, this is the first time integral estimates have been obtained with $s<n$ for quasiregular mappings. This estimate is enough to derive higher integrability results for the differential of f. We shall use (0.15) to prove removable singularity theorems for quasiregular mappings.

For the basic geometric properties of quasiregular mappings we refer to the foundational papers of Martio, Rickman and Väisälä [MRV 1–3], the books of Reshetnyak [R3] and Vuorinen [V], and the forthcoming book of Rickman [Ri4].

Statement of results. Our first main theorem is the following sharp generalization of the Liouville Theorem, see §5.

THEOREM 1. *Every weakly* 1*-quasiregular mapping* f *of Sobolev class* $W^1_{l,\mathrm{loc}}(\Omega, \mathbf{R}^{2l})$, $l>1$, *is either constant or the restriction to* Ω *of a Möbius transformation of* $\mathbf{R}^{2l}$. *The exponent* l *of the Sobolev class is the lowest possible for the theorem to be true.*

Our Theorem 1 is part of a more general spectrum of results concerning the integrability theory of quasiregular mappings. The precise results are formulated in terms of the p-norms of a singular operator S which, because of the strong analogy with the two dimensional case, see [AB], we call the Beurling–Ahlfors operator, see too [IM2]. Formally S is the operator

$$S\omega = (d\delta - \delta d)\circ\Delta^{-1}\omega$$

where d is the exterior derivative and δ is the Hodge operator. Thus S maps m-forms to m-forms for all m. In spin geometry this operator is called the signature operator, see for example [LM]. In §8 we give the explicit forms of the Fourier multiplier and the convolution formula for this operator. This is possible because the entries of the multiplier of S are spherical harmonics of degree 2 and we can apply the Hecke identities as in [S] to produce this formula. This operator plays an essential role in what follows and is an extension of an operator introduced by Donaldson and Sullivan. Our next main result (cf. (0.11)) is the following (see §7 for the definition of Beltrami coefficient).

THEOREM 2. *Let* f *be a weakly quasiregular mapping of Sobolev class* $W^1_{pl,\mathrm{loc}}(\Omega, \mathbf{R}^{2l})$ *with* $1<p\leqslant 2$. *Let* $\mu_f(x)$ *be the Beltrami coefficient of* f, $|\mu_f|<1$. *If*

$$|\mu_f|\,\|S\|_p < 1,$$

then f *is quasiregular. That is* $f\in W^1_{2l,\mathrm{loc}}(\Omega)$.

Again we show by example that Theorem 2 is qualitatively best possible. Indeed we conjecture Theorem 2 is sharp. Because of Theorem 2, the calculation of the p-norms $\|S\|_p$ seems an important problem in higher dimensions. We obtain the estimates $\|S\|_2=1$ and

$$\max\left\{\frac{1}{p-1}, p-1\right\} \leqslant \|S\|_p \leqslant C(n)\max\left\{\frac{1}{p-1}, p-1\right\}, \quad 1<p<\infty,$$

where $C(n)$ is a constant which depends only on the dimension (it comes from the weak L^1 norm). We give other estimates for $\|S\|_p$ in §8. We point out that good estimation of the p-norms provides interesting geometric facts about quasiregular mappings. Our conjecture regarding the sharpness of Theorem 2 would follow from the conjecture that the lower bound we gave above for $\|S\|_p$ is sharp. (More recently we have established the estimate $\|S\|_p \leqslant (n+1)A_p$, where A_p denotes the p-norm of the two dimensional Beurling–Ahlfors operator [IM2].) In even dimensions $n=2l$ the Beltrami coefficient μ can be estimated in terms of the linear dilatation K. In particular a sharp estimate (independent of the mapping) is

$$|\mu| \leqslant \frac{K^l-1}{K^l+1}$$

and so we could have phrased our result in terms of K. However in that case it is unlikely to be sharp (the reason being that we expect the dilatation of the radial mapping $x \to x|x|^{-1+1/K}$ to be extremal for many of these problems. As for extremal Teichmüller mappings, the radial mapping has constant norm $|\mu(x)|=(K-1)/(K+1)$ and so the estimate above is not sharp for this mapping when $l>1$). Next, because $\|S\|_2=1$, Theorem 2 gives us (via an interpolation argument) an estimate for Gehring's Integrability Theorem [G2] (cf. (0.9) and (0.10)).

THEOREM 3. *Let f be a quasiregular mapping with Beltrami coefficient μ_f. Then $f \in W^1_{pl,\mathrm{loc}}(\Omega, \mathbf{R}^{2l})$ for all $p>2$ such that $|\mu_f|\,\|S\|_p<1$.*

Related to Theorems 2 and 3 we shall outline a considerably simpler proof of the following result in a remark in §9 (see [I2], [LF], [Ma] and [R2] for a proof of this result in all dimensions).

THEOREM 4. *Let f be a quasiregular mapping with Beltrami coefficient*

$$\mu_f \in C^{k,\alpha}(\Omega, \mathbf{R}^{2l}).$$

Then $f \in C^{k+1,\alpha}(\Omega, \mathbf{R}^{2l})$.

Perhaps one of our most important results concerns the removable singularity theorems for quasiregular mappings. A closed set $E \subset \mathbf{R}^n$ is said to be *removable under bounded K-quasiregular mappings* if for every open set $\Omega \subset \mathbf{R}^n$, any bounded K-quasiregular mapping $f: \Omega \setminus E \to \mathbf{R}^n$ extends to a K-quasiregular mapping $f: \Omega \to \mathbf{R}^n$. We stress here that the mapping f need not be (even locally) injective. The simplest result of this type is the classical result of Painlevé that any set E of linear measure zero is removable under bounded holomorphic mappings (thus $K=1$ and $n=2$). The p-norms of our operator S give a sufficient condition for removability.

THEOREM 5. *Each closed set $E \subset \mathbf{R}^{2l}$ of lp-capacity zero, $1 \leqslant p < 2$, is removable under bounded quasiregular mappings f whenever $|\mu_f|\,\|S\|_p<1$.*

It is important to notice that as $\|S\|_p$ is continuous, $\|S\|_2=1$ and $|\mu_f|<1$, Theorem 5 implies that there are nontrivial removable sets *whatever the dilatation of f may be.*

We again show these results to be qualitatively best possible and we actually prove a slightly better result by weakening the hypothesis that f is bounded to an integrability condition.

Notice too that as a consequence of Theorem 5 closed sets of Hausdorff dimension k are removable under bounded quasiregular mappings as soon as $|\mu_f|<\delta=\delta(k,l)$, a constant depending only on k and the dimension. We show too that $\delta(k,l)\to 1$ for fixed k as $l\to\infty$. Indeed it is immediate that sets of Hausdorff dimension zero are always removable for bounded quasiregular mappings in even dimensions (actually all dimensions, see [I5]).

There have been no earlier results along the lines of Theorem 2 and Theorem 5 in dimension greater than 2 (except for the semiclassical fact that sets of n-capacity zero are removable [MRV2](2)).

Qualitatively our Theorems 2 and 5 amount to the principle

$$\text{smaller dilatation} \Longrightarrow \text{better regularity} \Longrightarrow \text{larger removable sets.}$$

Indeed our results suggest the following:

CONJECTURE. *Sets of Hausdorff d-measure zero, $d=n/(K+1)\leqslant n/2$, are removable under bounded K-quasiregular mappings (defined in subdomains of $\mathbf{R}^n$).*

For related questions, see [AB], [Tu] and [P].

There are of course many related results of interest which we shall have to postpone. For instance obtaining better and dimension free estimates for the p-norms of the operator S. The operator S seems closely linked with the index theory of even dimensional manifolds (it permutes the signature operators d^+ and d^-) and should be a useful tool in that study (as realized by Donaldson and Sullivan [DS, Appendix 2] who give an alternative (and simpler) proof of Teleman's main results [T]). Our results also have applications to the study of quasiconformal structures on even dimensional manifolds.

There is also the nagging question of odd dimensions. Despite some efforts we have not yet been able to extend our results even to dimension 3 (though of course we conjecture that all of our results hold, with obvious modifications, in all dimensions). It seems entirely new methods are necessary to establish the highly non-trivial estimates that stand in the way. There have been earlier attempts to find higher dimensional forms

(2)Some related results have been communicated to us by M. Vuorinen and P. Järvi concerning the removability of certain selfsimilar Cantor sets [JV], and also independent results by P. Koskela and O. Martio [KM] concerning removability of certain Cantor sets of Hausdorff dimension zero. Both of these approaches are in all dimensions. In response to this work, S. Rickman has constructed Cantor sets E in $\mathbf{R}^3$ of arbitrarily small Hausdorff dimension and bounded quasiregular mappings $f:\mathbf{R}^3\setminus E\to\mathbf{R}^3$. The set E is necessarily a nonremovable set for such f [Ri3].

of the Cauchy–Riemann operators. For instance the Linear Cauchy–Riemann operators introduced by Ahlfors [A2]

$$Sf=\frac{1}{2}(Df+D^tf)-\frac{1}{n}(\operatorname{Tr}Df)\,\mathrm{Id} \quad\text{and}\quad Af=\frac{1}{2}(Df-D^tf)+\frac{1}{n}(\operatorname{Tr}Df)\,\mathrm{Id}$$

have found applications to linear elasticity theory, quasiconformal semiflows and stability estimates for quasiregular mappings with dilatation close to 1 [Sa]. However as first degree (linear) approximations of the nonlinear system of equations for conformal mappings, the Ahlfors operators are rather difficult to use. Some new ideas in the nonlinear theory of elasticity have produced differential equations which although of a formal nature may be useful in generalizing our results to odd dimensions [I3]. Perhaps even nonlinear potential theory via the p-harmonic operator (see [GLM] and [I4]) will lead to interesting results in odd dimensions.(3)

With these future developments in mind we have tried as much as possible to give a reasonably detailed and accessible account. Thus various parts of this paper can be viewed as an elaboration, refinement or extension of aspects of the paper of Donaldson and Sullivan. Besides the interest of the results we obtain we hope too that our paper partly serves as a complement to their beautiful paper.

Acknowledgement. We would like to thank the people at the Institut Mittag-Leffler for their hospitality during our stay there, during which time this research was completed, and in particular to Seppo Rickman for his advice and encouragement.

1. Notation

In the sequel we shall be concerned with the following spaces of functions and distributions defined on an open subset Ω of $\mathbf{R}^n$.

$L^p(\Omega)$ and $L^p_{\mathrm{loc}}(\Omega)$: For $1\leqslant p\leqslant\infty$, the usual L^p spaces with respect to Lebesgue measure.

$C^\infty(\Omega)$: The complex space of infinitely differentiable functions.

$C_0^\infty(\Omega)$: The subset of $C^\infty(\Omega)$ consisting of functions whose support is compact in Ω.

$\mathcal{D}'(\Omega)$: The dual space to $C_0^\infty(\Omega)$, that is the complex space of Schwartz distributions.

$L^p_m(\Omega)$: For $1\leqslant p<\infty$, $m=0,1,\ldots$, the subspace of $\mathcal{D}'(\Omega)$ consisting of all distributions whose mth order derivatives are in $L^p(\Omega)$. This space is equipped with the semi-norm

(3)In fact very recently the first author, using the p-harmonic operator, has extended many of the results herein (in a qualitative form) to all dimensions [I5].

$$\|u\|_{m,p}=\left(\int_\Omega\left(\sum_{|\nu|=m}|D^\nu u(x)|^2\right)^{p/2}dx\right)^{1/p}$$

The regular distributions are those which are represented by locally integrable functions on Ω. That is $L^1_{\mathrm{loc}}(\Omega)\subset\mathcal{D}'(\Omega)$.

$GL(n)$: The space of $n\times n$ matrices with real entries and non zero determinant.

$S(n)$: The subset of $GL(n)$ consisting of the positive definite, symmetric matrices whose determinant is equal to one.

We recall two results. The first is classical, the second can be found for instance in [M].

WEYL'S LEMMA. *If $\Delta u=0$ for some $u\in\mathcal{D}'(\Omega)$, then u is harmonic in the usual sense.*

LEMMA 1.1. *$C^\infty(\Omega)\cap L^p_m(\Omega)$ is dense in $L^p_m(\Omega)$.*

The *Sobolev space* is then defined as

$$W^m_p(\Omega)=\bigcap\{L^p_k(\Omega):k=0,1,2,...,m\}.$$

We also use the subscript loc in the obvious fashion as for instance in $W^1_{p,\mathrm{loc}}(\Omega)$.

If X is a linear space, then the symbol $\mathcal{D}'(\Omega,X)$ is used to denote those distributions on Ω which take their values in X. Similar notation is used for the other classes of function spaces. If X is a normed space, then $L^p_m(\Omega,X)$ has an obvious semi-norm which is a norm for $L^p(\Omega,X)$.

2. Some exterior algebra

Let $e^1,e^2,...,e^n$ denote the *standard* basis of $\mathbf{R}^n$. For each $l=0,1,2,...,n$ denote by $\bigwedge^l=\bigwedge^l(\mathbf{R}^n)$ the complex space of l-vectors on $\mathbf{R}^n$, $\bigwedge^0=\mathbf{C}$, $\bigwedge^1=\mathbf{C}^n$. Then $\bigwedge^l$ consists of linear combinations of exterior products

$$e^I=e^{i_1}\wedge e^{i_2}\wedge...\wedge e^{i_l}, \tag{2.1}$$

where $I=(i_1,i_2,...,i_l)$ is any l-tuple. The *standard basis* of $\bigwedge^l$ is $\{e^I\}$ where I is an ordered l-tuple, $1\leqslant i_1<i_2<...<i_l\leqslant n$. The complex dimension of the space $\bigwedge^l$ is

$$\dim\bigwedge^l=\binom{n}{l}, \tag{2.2}$$

see [F, §II]. For $\mu=\sum_I \mu^I e^I$ and $\lambda=\sum_I \lambda^I e^I$ in $\bigwedge^l$, the *scalar product* of λ and μ is

$$\langle \lambda, \mu\rangle = \sum_I \lambda_I \bar{\mu}_I, \tag{2.3}$$

where of course the summation is over all ordered l-tuples. We denote the *volume form* on $\mathbf{R}^n$ by

$$\mathrm{Vol} = e^1 \wedge e^2 \wedge \ldots \wedge e^n \in \textstyle\bigwedge^n(\mathbf{R}^n). \tag{2.4}$$

To simplify notation it is often convenient to speak of $\bigwedge^l(\mathbf{R}^n)$ for each integer l. Thus for $l \notin \{0,1,\ldots,n\}$ we set $\bigwedge^l(\mathbf{R}^n)=\{0\}$. The exterior product of $\alpha\in\bigwedge^l$ and $\beta\in\bigwedge^k$ is

$$\alpha\wedge\beta = (-1)^{l\cdot k}\beta\wedge\alpha \in \textstyle\bigwedge^{l+k}. \tag{2.5}$$

Let $A\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation, i.e. an $n\times n$ matrix $A=(a^i_j)$. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in \bigwedge^1(\mathbf{R}^n)=\mathbf{C}^n$, then (see [F, §II, 2.2])

$$A\alpha_1 \wedge A\alpha_2 \wedge \ldots \wedge A\alpha_n = (\det A)\alpha_1\wedge\alpha_2\wedge\ldots\wedge\alpha_n. \tag{2.6}$$

The lth *exterior power* of A is the linear operator

$$A_\#\colon \textstyle\bigwedge^l \to \bigwedge^l$$

defined by

$$A_\#(\alpha_1\wedge\alpha_2\wedge\ldots\wedge\alpha_l) = A\alpha_1\wedge A\alpha_2\wedge\ldots\wedge A\alpha_l, \tag{2.7}$$

for $\alpha_1,\alpha_2,\ldots,\alpha_n\in\bigwedge^1$ and then extended linearly to all of $\bigwedge^l$. In particular, for an ordered l-tuple $j=(j_1,j_2,\ldots,j_l)$

$$\begin{aligned} A_\#(e^J) &= Ae^{j_1}\wedge Ae^{j_2}\wedge\ldots\wedge Ae^{j_l} \\ &= \Big(\sum_{i_1=1}^n a^{i_1}_{j_1} e^{i_1}\Big)\wedge\Big(\sum_{i_2=1}^n a^{i_2}_{j_2} e^{i_2}\Big)\wedge\ldots\wedge\Big(\sum_{i_l=1}^n a^{i_l}_{j_l} e^{i_l}\Big) \\ &= \sum a^{i_1}_{j_1} a^{i_2}_{j_2}\ldots a^{i_l}_{j_l} e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_l} \\ &= \sum_I A^I_J e^I \end{aligned} \tag{2.8}$$

where the summations are over all ordered l-tuples $I=(i_1,i_2,\ldots,i_l)$ and A^I_J is the determinant of the $l\times l$ minor obtained from A by deleting all the ith rows with $i\notin I$ and all the jth columns with $j\notin J$. Thus $A_\#$ is represented by the $\binom{n}{l}\times\binom{n}{l}$ matrix of minors

$$A_\# = (A^I_J). \tag{2.9}$$

The following lemma is immediate (see also [F, §II, 2.4]).

LEMMA 2.10. *For matrices $A, B \in GL(n)$,*

(i) $(AB)_\# = A_\# B_\#$,

(ii) $(A^{-1})_\# = (A_\#)^{-1} = A_\#^{-1}$ *if* $\det(A) \neq 0$,

(iii) $(A^t)_\# = (A_\#)^t = A_\#^t$ *(where A^t is the transpose of A)*,

(iv) $a_\#(\omega \wedge \eta) = (A_\# \omega) \wedge (A_\# \eta)$ *for* $\omega \in \bigwedge^p$, $\eta \in \bigwedge^q$.

It then follows that if the matrix A is the identity, orthogonal, symmetric, diagonal or invertible, then so too is the matrix $A_\#$.

The *Hodge star operator* is a linear operator $*\colon \bigwedge^l \to \bigwedge^{n-l}$ defined by the rule

$$\bar{\mu} \wedge * \lambda = \langle \lambda, \mu \rangle \,\mathrm{Vol} \quad \text{for all } \mu, \lambda \in \textstyle\bigwedge^l. \tag{2.11}$$

In order to aid calculation we need to introduce the notion of complementary index. For an l-tuple $I = (i_1, i_2, \ldots, i_l)$ the *complementary index* is the $(n-l)$-tuple $N - I$ obtained from $N = (1, 2, \ldots, n)$ by deleting those entries $i_k \in I$. Then of course

$$* e^I = \sigma(I, N-I) e^{N-I},$$

where $\sigma(I, N-I) \in \{-1, 1\}$ is the sign of the induced permutation which is either odd or even. As $\sigma(I, N-I) = (-1)^{l(n-l)} \sigma(N-I, I)$ we see

$$** = (-1)^{l(n-l)} \quad \text{on } \textstyle\bigwedge^l. \tag{2.12}$$

LEMMA 2.13. *For any matrix $A \in GL(n)$*

$$A_\#^t * A_\# = (\det A) * \colon \textstyle\bigwedge^l \to \bigwedge^{n-l}.$$

Proof. Let $\mu \in \bigwedge^{n-l}$ and $\lambda \in \bigwedge^l$. We compute

$$\begin{aligned}
\langle A_\#^t * A_\# \lambda, \mu \rangle \,\mathrm{Vol} &= \langle * A_\# \lambda, A_\# \mu \rangle \,\mathrm{Vol} \\
&= A_\# \bar{\mu} \wedge ** A_\# \lambda = A_\# \bar{\mu} \wedge A_\# ** \lambda \\
&= A_\# (\bar{\mu} \wedge ** \lambda) = A_\# \langle * \lambda, \mu \rangle \,\mathrm{Vol} \\
&= (\det A) \langle * \lambda, \mu \rangle \,\mathrm{Vol}.
\end{aligned}$$

Hence $(A_\#^t * A_\#)\lambda = (\det A) * \lambda$ and the lemma is proved.

Next we want to discuss those linear mappings which are conformal from the usual structure on $\mathbf{R}^n$ to another given conformal structure on $\mathbf{R}^n$ defined by $G \in S(n)$. Here G is a symmetric, positive definite $n \times n$ matrix with $\det G = 1$. A linear mapping $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is said to be *G-conformal* if

$$AA^t = |\det(A)|^{2/n} G \tag{2.14}$$

(compare with (0.1)). For $m=1,2,\dots,n-1$ we introduce the notion of *m-dimensional sectional distortion* $K_{\mathrm{sec}}(m)$ of a G-conformal mapping. Let E be the ellipsoid $E=\langle G\xi,\xi\rangle=1$,

$$K_{\mathrm{sec}}(m)=\frac{\max\{\mathrm{Vol}_m(E\cap\Pi_m):\Pi_m \text{ an } m\text{-dimensional hyperplane}\}}{\min\{\mathrm{Vol}_m(E\cap\Pi_m):\Pi_m \text{ an } m\text{-dimensional hyperplane}\}}.$$

Here we choose the hyperplanes to pass through the origin and so $K_{\mathrm{sec}}(m)$ is the ratio of the largest and smallest volumes of m-dimensional sections of E (see Figure 1). One can easily derive the formula

$$K_{\mathrm{sec}}(m)=\frac{\gamma_n\gamma_{n-1}\cdots\gamma_{n-m+1}}{\gamma_1\gamma_2\cdots\gamma_m} \tag{2.15}$$

where $\gamma_1^2\leqslant\gamma_2^2\leqslant\dots\leqslant\gamma_n^2$ are the eigenvalues of G. $K=K_{\mathrm{sec}}(1)$ is commonly referred to as the *dilatation* of a G-conformal map,

$$K=\frac{\max\{|Ah|:|h|=1\}}{\min\{|Ah|:|h|=1\}}.$$

The two estimates

$$K_{\mathrm{sec}}(m)\leqslant K^m \quad\text{and}\quad K_{\mathrm{sec}}(k)\leqslant K^{k-m}K_{\mathrm{sec}}(m) \quad \text{for } m\leqslant k$$

follow directly. (In practice A will be the transpose Jacobian matrix of a mapping f which is conformal in some measurable structure $G(x)$ and we will define the pointwise m-dimensional sectional distortion in the obvious manner.) It is a simple, but interesting, fact that if $[n/2]$ denotes the integer part of $n/2$, then $K_{\mathrm{sec}}([n/2])$ is the largest of all the sectional dilatations. Thus, in even dimensions, control of the middle dimensional sectional distortion gives good geometric information.

Next we introduce the operator μ (which will play the role of the complex dilatation, see §7) by the definition

$$\mu=\frac{G_\#-\mathrm{Id}}{G_\#+\mathrm{Id}}:\textstyle\bigwedge^l\to\bigwedge^l, \tag{2.16}$$

where Id is the identity on $\bigwedge^l$. Notice that since $G+\mathrm{Id}$ is symmetric, positive definite and hence invertible, so too is $G_\#+\mathrm{Id}$. As a consequence of Lemmas 2.10, 2.13 and (2.14) one easily derives

$$G_\# * A_\# = |\det A|^{2(l-n)/n}(\det A)A_\# *$$

on $\bigwedge^l$. Consequently we have the following

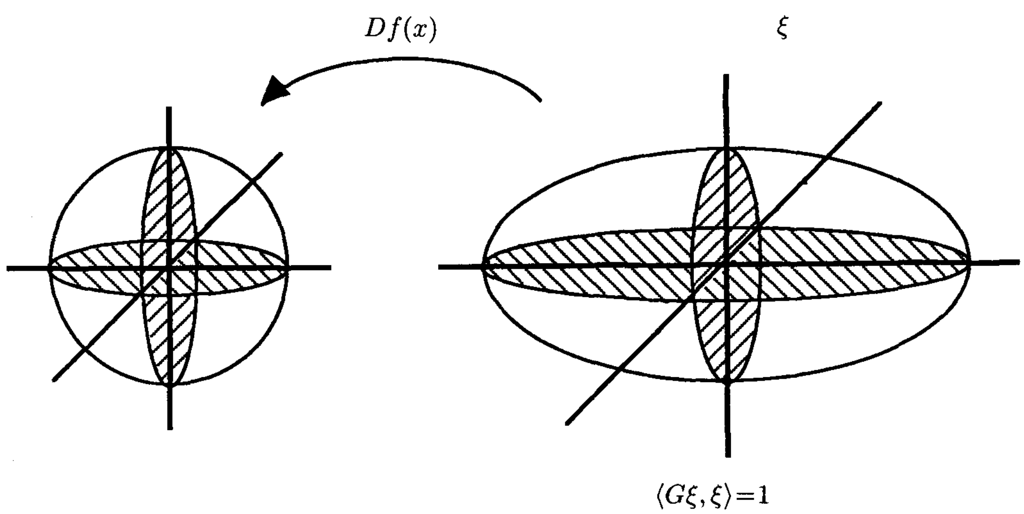

Fig. 1

LEMMA 2.17. *Let A be a G-conformal transformation of* $\mathbf{R}^{2l}$. *Then* $A_\#: \bigwedge^l \to \bigwedge^l$ *satisfies*

$$G_\# * A_\# = A_\# * \quad \text{if } \det A \geqslant 0$$

and

$$G_\# * A_\# = -A_\# * \quad \text{if } \det A \leqslant 0.$$

In particular, if O is conformal (so $G=\mathrm{Id}$), then

$$*O_\# = O_\# * \quad \text{if } \det O \geqslant 0$$

and

$$*O_\# = -O_\# * \quad \text{if } \det O \leqslant 0.$$

We next want to discuss the + and − eigenspaces of the Hodge $*$ operator. In this case we must restrict the dimension to $n=2l$. Then, in view of the identity $**=(-1)^l$, the eigenvalue problem $*\omega = c\omega$, $0\neq\omega\in\bigwedge^l$, $c\in\mathbf{C}$, has a solution only for $c^2=(-1)^l$. That is $c=\pm i^l$, $i=\sqrt{-1}$. The *positive and negative eigenspaces* of $\bigwedge^l$ are defined by

$$\textstyle\bigwedge^+ = \{\omega\in\bigwedge^l : *\omega = i^l\omega\} \quad \text{and} \quad \bigwedge^- = \{\omega\in\bigwedge^l : *\omega = -i^l\omega\}. \tag{2.18}$$

Let us remark that for $l=2$ our choice of $\bigwedge^+$ and $\bigwedge^-$ is opposite to that chosen by Donaldson and Sullivan [DS].

We have the natural orthogonal decomposition

$$\bigwedge^l = \bigwedge^+ \oplus \bigwedge^-$$

and

$$\dim \bigwedge^+ = \dim \bigwedge^- = \frac{1}{2}\binom{2l}{l} = \binom{2l-1}{l}.$$

For $\omega \in \bigwedge^l$ we denote by ω_+ and ω_- respectively, the *positive* and *negative components*

$$\begin{aligned} \omega_+ &= \tfrac{1}{2}(\omega + (-i)^l * \omega) \\ \omega_- &= \tfrac{1}{2}(\omega - (-i)^l * \omega). \end{aligned} \tag{2.19}$$

Thus $\omega = \omega_+ + \omega_-$. It is clear that $\bigwedge^+$ is spanned by $\{e_+^I\}$ and that $\bigwedge^-$ is spanned by $\{e_-^I\}$. We remark that if $\mathcal{F}$ is a maximal family of ordered l-tuples such that $I \in \mathcal{F}$ implies $N - I \notin \mathcal{F}$, then the vectors $\{e_+^I : I \in \mathcal{F}\}$ and $\{e_-^I : I \in \mathcal{F}\}$ are orthogonal bases for $\bigwedge^+$ and $\bigwedge^-$ respectively (each vector has the constant length $1/\sqrt{2}$).

The action of the operator μ on the eigenspaces of the Hodge star will be of particular interest to us.

THEOREM 2.20. *In dimension $n=2l$, the operator $\mu: \bigwedge^l \to \bigwedge^l$ permutes the spaces $\bigwedge^+$ and $\bigwedge^-$. Its norm is*

$$|\mu| = \frac{K_{\sec}(l) - 1}{K_{\sec}(l) + 1} \leqslant \frac{K^l - 1}{K^l + 1}.$$

Proof. To compute the norm of the operator μ we diagonalize. Let $G = O\Gamma^2 O^t$ where O is an orthogonal matrix and

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & 0 & \dots & 0 \\ 0 & \gamma_2 & 0 & \dots & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \dots & 0 & \gamma_{2l} \end{pmatrix}$$

with $\gamma_i > 0$, $i = 1, 2, \dots, 2l$ and $\gamma_1 \gamma_2 \dots \gamma_{2l} = 1$. We may then diagonalize μ as

$$\mu = O_\# \frac{\Gamma_\#^2 - \mathrm{Id}}{\Gamma_\#^2 + \mathrm{Id}} O_\#^t.$$

As $O_\#$ and $O_\#^t$ are isometries which either both preserve the spaces $\bigwedge^+$ and $\bigwedge^-$ or both permute these spaces (Lemma 2.17), we may simply assume that

$$\mu = \frac{\Gamma_\#^2 - \mathrm{Id}}{\Gamma_\#^2 + \mathrm{Id}}$$

and so μ is diagonal. On the orthonormal basis

$$\{e^I : I=(i_1,i_2,\ldots,i_l),\ 1\leqslant i_1<i_2<\ldots<i_l\leqslant 2l\},$$

μ acts by

$$\mu(e^I)=\frac{\gamma_{i_1}^2\gamma_{i_2}^2\cdots\gamma_{i_l}^2-1}{\gamma_{i_1}^2\gamma_{i_2}^2\cdots\gamma_{i_l}^2+1}\,e^I=\frac{\gamma_I^2-1}{\gamma_I^2+1}\,e^I,$$

where $\gamma_I=\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_l}$. Then the coefficient of e^I above is

$$\frac{\gamma_I^2-1}{\gamma_I^2+1}=\frac{\gamma_I-\gamma_{N-I}}{\gamma_I+\gamma_{N-I}}=\frac{\gamma_I/\gamma_{N-I}-1}{\gamma_I/\gamma_{N-I}+1}.$$

If we now look for the maximum over all multi-indices I we obtain the formula for the norm. Next, given an l-tuple I as above, let J be the l-tuple so that $*e^I=e^J$. Then

$$\mu(*e^I)=\mu(e^J)=\frac{\gamma_J^2-1}{\gamma_J^2+1}\,e^J=\frac{1-\gamma_I^2}{1+\gamma_I^2}\,(*e^I)=-*\mu(e^I),$$

because $\gamma_I\gamma_J=1$. Hence from (2.19) we find that

$$\mu(e_\pm^I)=\frac{\gamma_I^2-1}{\gamma_I^2+1}\,e_\mp^I$$

which shows μ permutes the spaces $\bigwedge^+$ and $\bigwedge^-$.

We next define the operators $A_\#^+$ and $A_\#^-:\bigwedge^l\to\bigwedge^l$ by the rules

$$A_\#^+\omega=(A_\#\omega)_+\quad\text{and}\quad A_\#^-\omega=(A_\#\omega)_-.$$

LEMMA 2.21. *Let $A\in GL(2l)$ be G-conformal. Then*
(i) $A_\#^+=\mu A_\#^-$ *on* $\bigwedge^-$ *if* $\det(A)\geqslant 0$,
(ii) $A_\#^-=\mu A_\#^+$ *on* $\bigwedge^-$ *if* $\det(A)\leqslant 0$,
(iii) $A_\#^-=\mu A_\#^+$ *on* $\bigwedge^+$ *if* $\det(A)\geqslant 0$,
(iv) $A_\#^+=\mu A_\#^-$ *on* $\bigwedge^+$ *if* $\det(A)\leqslant 0$.

Proof. Suppose $\det A\geqslant 0$. From Lemma 2.17 we have $G_\# * A_\# = A_\# *$ as operators in $\bigwedge^l$. Thus

$$\begin{aligned}(G_\#+\mathrm{Id})(A_\#\omega)_+-(G_\#-\mathrm{Id})(A_\#\omega)_-&\\ &=\tfrac12(G_\#+\mathrm{Id})\big(A_\#\omega+(-i)^l*A_\#\omega\big)-\tfrac12(G_\#-\mathrm{Id})\big(A_\#\omega-(-i)^l*A_\#\omega\big)\\ &=A_\#\omega+(-i)^lG_\#*A_\#\omega=A_\#\omega+(-i)^lA_\#*\omega\\ &=2A_\#\omega_+=0,\quad\text{for } \omega\in{\textstyle\bigwedge}^-.\end{aligned}$$

This proves (i). The case $\det A\leqslant 0$ is similar. The other identities are also clear.

As a point of interest we observe the following corollary.

COROLLARY 2.22. *If O is a $2l\times 2l$ orthogonal matrix, the operator $O_\#\colon \bigwedge^l \to \bigwedge^l$ is an isometry which preserves $\bigwedge^+$ and $\bigwedge^-$ if* $\det O=1$ *and permutes them if* $\det O=-1$.

We conclude this section with the following lemma.

LEMMA 2.23. *Let $A\colon \mathbf{R}^{2l}\to\mathbf{R}^{2l}$ be G-conformal. Then*

$$|A|^l|\omega| \leqslant K^l|A_\#\omega|$$

for all $\omega\in\bigwedge^l$.

Proof. By homogeneity we may assume that $\det A=1$ and $|\omega|=1$. Then $AA^t=G$ and $A_\# A_\#^t=G_\#$. We diagonalise G to Γ^2 as in Theorem 2.20. Then

$$\begin{aligned}|A|^l = \gamma_{\max}^l &\leqslant K^l \min\{\gamma_I : I \text{ any } l\text{-tuple}\}\\ &\leqslant K^l|\Gamma_\#\omega| = K^l|A_\#\omega|,\end{aligned}$$

where $\gamma_{\max}$ denotes the largest of the numbers $\gamma_1,\gamma_2,\dots,\gamma_{2l}$.

3. Differential forms in $L^p_m(\Omega,\bigwedge^n)$

Let $\Omega\subset\mathbf{R}^n$ be an open set and let $\bigwedge^k(\Omega)=\mathcal{D}'(\Omega,\bigwedge^k)$. Then $\alpha\in\bigwedge^k(\Omega)$ has the representation

$$\alpha=\sum_I \alpha^I\,dx^I \tag{3.1}$$

with coefficients which are complex valued distributions $\alpha^I\in\mathcal{D}'(\Omega)$ and I is an ordered k-tuple. If $\alpha^I\in L^p_{\mathrm{loc}}(\Omega)$ for each I, then the exterior algebra of k-vectors applies at almost every point $x\in\Omega$. There are the corresponding subspaces $C_0^\infty(\Omega,\bigwedge^k)$ and $L^p_m(\Omega,\bigwedge^k)$. This latter space has the semi-norm

$$\|\alpha\|_{m,p}=\left(\int_\Omega\left(\sum_I\left\{\sum_{|\nu|=m}|D^\nu\alpha^I(x)|^2\right\}\right)^{p/2}dx\right)^{1/p}.$$

We shall make extensive use of the *exterior derivative operator*

$$d\colon \textstyle\bigwedge^{k-1}(\Omega)\to\bigwedge^k(\Omega)$$

for $k=1,2,\dots,n+1$ and the *Hodge operator*

$$\delta\colon \textstyle\bigwedge^{k}(\Omega)\to\bigwedge^{k-1}(\Omega)$$

for $k=0,1,\dots,n$ defined by $\delta=(-1)^{n(n-k)}*d*$. The Hodge operator δ is the formal adjoint of d. More precisely, if $\alpha\in C^\infty(\Omega,\bigwedge^k)$ and $\beta\in C^\infty(\Omega,\bigwedge^{k+1})$, then

$$\int_\Omega \langle\alpha,\delta\beta\rangle\,dx=-\int_\Omega\langle d\alpha,\beta\rangle\,dx \tag{3.2}$$

provided that one of these forms has compact support. A calculation shows that the *Laplacian*

$$\Delta=d\delta+\delta d\colon \textstyle\bigwedge^k(\Omega)\to\bigwedge^k(\Omega) \tag{3.3}$$

acts only on coefficients of the k-form α. That is if $\alpha=\sum_I\alpha^I dx^I$, then

$$\Delta\alpha=\sum_I\Delta\alpha^I\,dx^I$$

where $\Delta\alpha^I$ is the usual Laplacian on functions. We shall need the following lemma.

LEMMA 3.4. *Let* $\alpha\in L_1^p(\mathbf{R}^n,\bigwedge^{k-1})$ *and* $\beta\in L_1^q(\mathbf{R}^n,\bigwedge^{k+1})$ *with* $1<p,q<\infty$ *and* $1/p+1/q=1$. *Then*

$$\int_{\mathbf{R}^n}\langle d\alpha,\delta\beta\rangle\,dx=0.$$

Proof. Using a standard modifying argument we find that $L_1^p(\mathbf{R}^n)\cap C^\infty(\mathbf{R}^n)$ and $L_1^q(\mathbf{R}^n)\cap C^\infty(\mathbf{R}^n)$ are dense in $L_1^p(\mathbf{R}^n)$ and $L_1^q(\mathbf{R}^n)$ respectively. Thus we may assume that α and β are smooth. Then for each test function $\varphi\in C_0^\infty(\mathbf{R}^n)$ we have

$$\varphi\langle\delta\beta,d\alpha\rangle\,dx=\varphi(d\bar\alpha\wedge*\delta\beta)=d(\varphi\bar\alpha\wedge*\delta\beta)-d\varphi\wedge\bar\alpha\wedge*\delta\beta.$$

Thus by Stokes' Theorem

$$\left|\int_{\mathbf{R}^n}\varphi\langle d\alpha,\delta\beta\rangle\,dx\right|=\left|\int_{\mathbf{R}^n}\bar\alpha\wedge d\varphi\wedge d*\beta\right|.$$

Let $R>0$ and $\varphi\in C_0^\infty(B(0,2R)), 0\leqslant\varphi\leqslant1, \varphi\equiv1$ on $B(0,R)$ and

$$|\nabla\varphi|\leqslant\frac{2}{R}.$$

Clearly we may replace α by $\alpha-\alpha_0$, where α_0 is any constant coefficient form $\alpha_0\in\bigwedge^{k-1}$. Now by Hölder's Inequality and the usual Poincaré Lemma we have

$$\begin{aligned}\left|\int_{\mathbf{R}^n}\varphi\langle d\alpha,\delta\beta\rangle\,dx\right|&\leqslant\frac{2}{R}\|\beta\|_{L_1^q(R<|x|<2R)}\|\alpha-\alpha_0\|_{L^p(|x|<2R)}\\&\leqslant C(n,p,k)\|\beta\|_{L_1^q(R<|x|<2R)}\|\alpha\|_{L_1^p(\mathbf{R}^n)}.\end{aligned}$$

Now let $R \to \infty$. The lemma follows from the Lebesgue Convergence Theorem which, in this case, implies $\|\beta\|_{L_1^q(R<|x|<2R)} \to 0$.

Next let Ω be a domain in $\mathbf{R}^n$ and $f : \Omega \to \mathbf{R}^n$, $f = (f^1, f^2, \dots, f^n)$, be a mapping of Sobolev class $W^1_{lp,\mathrm{loc}}(\Omega)$ with $p \geqslant 1$. Then f induces a homomorphism

$$f^* : C^\infty(\mathbf{R}^n, \Lambda^{l-1}) \to L^p_{\mathrm{loc}}(\Omega, \Lambda^{l-1})$$

called the pull back. More precisely, let $\alpha \in C^\infty(\mathbf{R}^n, \Lambda^{l-1})$, $\alpha = \sum_I \alpha^I dx^I$. Then

$$(f^*\alpha)(x) = \sum_I \alpha^I(f(x))\, df^{i_1} \wedge df^{i_2} \wedge \dots \wedge df^{i_{l-1}}.$$

If α has linear coefficients the exterior derivative of α is the l-form $d\alpha = \beta = \sum_J \beta^J dx^J$. This form has constant coefficients and so the induced l-form

$$(f^*\beta)(x) = \sum_J \beta^J df^{j_1} \wedge df^{j_2} \wedge \dots \wedge df^{j_l}$$

has measurable coefficients which are linear combinations of the $l \times l$ minors of the Jacobian matrix $Df(x)$ and so

$$f^*\beta \in L^p_{\mathrm{loc}}(\Omega, \Lambda^l).$$

As we have mentioned, the exterior algebra applies pointwise a.e. in Ω. The operator f^* on l-forms with constant coefficients is easily recognized as the lth exterior power of the linear transformation $D^t f(x)$. That is

$$(f^* d\alpha)(x) = [D^t f(x)]_\# \, d\alpha. \tag{3.5}$$

We shall need the following identity.

LEMMA 3.6. *For $\alpha \in \Lambda^{l-1}(\mathbf{R}^n)$ with linear coefficients and $f \in W^1_{pl,\mathrm{loc}}(\Omega, \mathbf{R}^n)$, $p \geqslant 1$, we have*

$$d(f^*\alpha) = f^*(d\alpha)$$

where the left hand side is understood in the sense of distributions.

Proof. We use a simple approximation argument. Let $f_\nu \in C^\infty(\Omega, \mathbf{R}^n)$, $\nu = 1, 2, \dots$ be a sequence of mappings converging to f in the topology of $W^1_{pl,\mathrm{loc}}(\Omega, \mathbf{R}^n)$. Then we obviously have $d(f_\nu^*\alpha) = f_\nu^*(d\alpha)$, $\nu = 1, 2, \dots$. But now

$$f_\nu^*\alpha \to f^*\alpha \quad \text{and} \quad f_\nu^*(d\alpha) \to f^*(d\alpha)$$

in $L^p_{\mathrm{loc}}(\Omega, \Lambda^{l-1})$ and $L^p_{\mathrm{loc}}(\Omega, \Lambda^l)$ respectively. Hence

$$d(f_\nu^*\alpha) \to d(f^*\alpha)$$

in $\mathcal{D}'(\Omega, \Lambda^l)$ which implies the lemma.

4. Differential systems for quasiregular mappings

In this section we formulate and unify a fairly complete set of second order differential equations for quasiregular mappings. These equations are all of divergence form and so it is possible to state and derive them for the larger class of weakly quasiregular mappings. We begin by recalling the algebraic identity of Lemma 2.13

$$A^t_\# * A_\# = (\det A) *: \bigwedge^l \to \bigwedge^{n-l}$$

for each $A\in GL(n)$. Let $0\leqslant l\leqslant n$ and let us suppose that $f\in W^1_{n-l,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ is a weakly 1-quasiregular mapping with non-negative Jacobian. Then f satisfies the Cauchy–Riemann system

$$D^t f(x) Df(x) = J(x,f)^{2/n}\,\mathrm{Id}\,.$$

For simplicity of notation, let $A=A(x)=D^t f(x)$. Then $AA^t=(\det A)^{2/n}\,\mathrm{Id}$ and for each l-form Ω with constant coefficients we have

$$A_\#\omega = f^*\omega.$$

Consequently

$$A_\# A^t_\# = (\det A)^{2(n-l)/n}\,\mathrm{Id}_\#\colon \bigwedge^{n-l}\to\bigwedge^{n-l}$$

and the identity of Lemma 2.13 gives

$$(\det A)^{(n-2l)/n} * A_\# = A_\# *\colon \bigwedge^l \to \bigwedge^{n-l}. \tag{4.1}$$

Applying this identity to an $\omega\in\bigwedge^l$ we obtain

$$J(x,f)^{(n-2l)/n} * f^*\omega = f^*(*\omega). \tag{4.2}$$

We can now differentiate both sides of this equation in the distributional sense. Because of the identity

$$df^*(*\omega) = f^*(d{*\omega}) = f^*(0) = 0$$

which follows from Lemma 3.6, we conclude that

$$d[J(x,f)^{(n-2l)/n} * f^*\omega] = 0. \tag{4.3}$$

Also, from the Cauchy–Riemann system above we have

$$J(x,f)^{1/n} = |Df(x)| = |\nabla f^i|, \quad i=1,2,\dots,n. \tag{4.4}$$

Now let us choose $\omega=dy^I=dy^{i_1}\wedge dy^{i_2}\wedge\ldots\wedge dy^{i_l}$, $I=(i_1,i_2,\ldots,i_l)$. Evidently we have the equation

$$d\big(|Df|^{n-2l}*df^{i_1}\wedge df^{i_2}\wedge\ldots\wedge df^{i_l}\big)=0 \tag{4.5}$$

for each l-tuple I. In the simplest case $l=1$ this system reduces to the familiar uncoupled system of n-harmonic equations

$$\operatorname{div}(|\nabla f^i|^{n-2}\nabla f^i)=0, \quad i=1,2,\ldots,n.$$

Here f is assumed to be in the Sobolev class $W^1_{n-1,\mathrm{loc}}(\Omega,\mathbf{R}^n)$. To get an idea of the algebraic structure of the equations of (4.5) let us consider the case $l=2$. Thus we suppose that $f\in W^1_{n-2,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ is a weak 1-quasiregular mapping. Choose

$$\omega=dy^\alpha\wedge dy^\beta, \quad \alpha,\beta=1,2,\ldots,n.$$

A direct calculation leads to the equations

$$\sum_{k=1}^n \frac{\partial}{\partial x^k}\left[|Df|^{n-4}\left(\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\beta}{\partial x^j}-\frac{\partial f^\alpha}{\partial x^j}\frac{\partial f^\beta}{\partial x^k}\right)\right]=0 \tag{4.6}$$

for each $j=1,2,\ldots,n$ and any pair $\alpha,\beta\in\{1,2,\ldots,n\}$. Equivalently we have the equations

$$\operatorname{div}\big(f^\beta_{x^j}|\nabla f^\alpha|^{n-4}\nabla f^\alpha\big)=\operatorname{div}\big(f^\alpha_{x^j}|\nabla f^\beta|^{n-4}\nabla f^\beta\big). \tag{4.7}$$

A similar situation arises for the general case of weak K-quasiregular mappings. Thus let $f\in W^1_{n-l,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ be a solution to the Beltrami system

$$D^tf(x)Df(x)=J(x,f)^{2/n}G(x), \quad J(x,f)\geqslant 0,$$

where $G:\Omega\to S(n)$ is measurable. Again for simplicity we denote $D^tf(x)$ by A, so that $AA^t=(\det A)^{2/n}G$. From the identity of Lemma 2.13 and proceeding as above we find

$$(\det A)^{(n-2l)/n}G_\# * A_\# = A_\# *\colon \textstyle\bigwedge^l\to\bigwedge^{n-l}.$$

We apply this to a constant coefficient l-form ω and because of the relation $A_\#\omega=f^*\omega$ we find

$$J(x,f)^{(n-2l)/n}G_\# * f^*\omega=f^*(*\omega)$$

and then differentiating as before we obtain

$$d\big[J(x,f)^{(n-2l)/n}G_\# * f^*\omega\big]=0. \tag{4.8}$$

The first order Beltrami system gives

$$\frac{1}{n}\langle DfG^{-1}, Df\rangle = J(x,f)^{2/n} = \langle G^{-1}\nabla f^i, \nabla f^i\rangle$$

for $1=1,2,...,n$. Hence

$$d\big[\langle DfG^{-1}, Df\rangle^{(n-2l)/n} G_{\#} * df^{i_1}\wedge df^{i_2}\ldots\wedge df^{i-l}\big] = 0 \tag{4.9}$$

for each l-tuple $I=(i_1,i_2,...,i_l)$. Setting $l=1$ we obtain the uncoupled system of divergence type

$$\operatorname{div}\big[\langle G^{-1}(x)\nabla f^i, \nabla f^i\rangle^{(n-2)/2} G^{-1}(x)\nabla f^i\big] = 0, \tag{4.10}$$

for $i=1,2,...,n$ and $f\in W^1_{n-1,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ weakly K-quasiregular.

Finally, we wish to record one more equation concerning quasiregular mappings related to the above. If the matrix $G(x)=J(x,f)^{-2/n}D^t f(x)Df(x)$ of f is twice continuously differentiable, then after differentiation of the Beltrami equation we obtain the linear elliptic equation for the function

$$U(x) = J(x,f)^{(n-2)/(2n)}$$

$$\sum_{i,j} G^{ij}(x)U_{ij} - \sum_{i,j,k} G^{ij}(x)\Gamma^k_{ij}U_k + \frac{n-2}{4(n-1)}R(x)U = 0. \tag{4.11}$$

where here Γ^k_{ij} are the Christoffel symbols and $R(x)$ is the scalar curvature of the metric tensor $G(x)$ on Ω. In the conformal case, this reduces to the Laplacian and we find U is harmonic. Solutions of linear elliptic equations have special properties and it would be of great value to give a meaning to (4.11) if G is only assumed measurable.

5. Liouville Theorem in even dimensions

A rather interesting situation arises in the system of equations (4.9) when $n=2l$. We are then assuming $f\in W^1_{l,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ is weakly K-quasiregular and we have the system of equations

$$d\big[G_{\#} * df^{i_1}\wedge df^{i_2}\wedge\ldots\wedge df^{i_l}\big] = 0 \tag{5.1}$$

for each l-tuple $I=(i_1,i_2,...,i_l)$. If f is weakly 1-quasiregular, then for each l-tuple

$$\delta\big[df^{i_1}\wedge df^{i_2}\wedge\ldots\wedge df^{i_l}\big] = 0. \tag{5.2}$$

In other words the form $df^{i_1}\wedge df^{i_2}\wedge\ldots\wedge df^{i_l}$ is exact and coexact. This implies that in the distributional sense the Laplace equation

$$\Delta(df^{i_1}\wedge df^{i_2}\wedge\ldots\wedge df^{i_l}) = (d\delta+\delta d)(df^{i_1}\wedge df^{i_2}\wedge df^{i_l}) = 0.$$

Hence by Weyl's Lemma, the l-form $df^{i_1}\wedge df^{i_2}\wedge\ldots\wedge df^{i_l}$ is harmonic in the usual sense. In particular it is C^∞ smooth. So too is the Jacobian determinant because

$$J(x,f)\,\mathrm{Vol}=df^1\wedge df^2\wedge\ldots\wedge df^l\wedge df^{l+1}\wedge df^{l+2}\wedge\ldots\wedge df^n.$$

At this point the first part of Theorem 1 follows from the earlier proof of the Liouville Theorem for $f\in W^1_{n,\mathrm{loc}}(\Omega,\mathbf{R}^n)$.

The sharpness of Theorem 1 is proved by example in §12.

Remark 5.3. In fact we have shown that every determinant of an $l\times l$ minor of the Jacobian matrix is harmonic. On the other hand, from the equation $AA^t=(\det A)^{2/n}\,\mathrm{Id}$ we find that

$$(\det A)\delta^I_J=\sum_K A^K_I A^K_J$$

where I,J and K are ordered l-tuples. If we put $I=J$ and sum we find

$$\binom{2l}{l}J(x,f)=\sum_{J:K}\left|\frac{\partial f^K}{\partial x^J}\right|^2.$$

This shows that not only is $J(x,f)\in C^\infty(\Omega)$ but also $\sqrt{J(x,f)}$ is locally a Lipschitz function. These additional regularity results considerably simplify the existing proofs of the Liouville Theorem, cf. [BI].

6. Hodge theory in $L^p(\Omega)$

In order to extend the approach to integrability theory which is suggested in the previous section to quasiregular mappings we need to develop some Hodge theory in $L^p(\Omega)$. In particular we first need the following version of the Hodge Decomposition Theorem.

THEOREM 6.1. *If $\omega\in L^p(\mathbf{R}^n,\bigwedge^k)$, $1<p<\infty$, then there is a $(k-1)$-form α and a $(k+1)$-form β such that*

$$\omega=d\alpha+\delta\beta$$

and $d\alpha,\delta\beta\in L^p(\mathbf{R}^n,\bigwedge^k)$. Moreover the forms $d\alpha$ and $\delta\beta$ are unique and

$$\alpha\in\operatorname{Ker}\delta\cap L^p_1(\mathbf{R}^n,\textstyle\bigwedge^{k-1}) \tag{6.2}$$

$$\beta\in\operatorname{Ker}d\cap L^p_1(\mathbf{R}^n,\textstyle\bigwedge^{k+1}) \tag{6.3}$$

and we have the uniform estimate

$$\|\alpha\|_{L^p_1(\mathbf{R}^n)}+\|\beta\|_{L^p_1(\mathbf{R}^n)}\leqslant C_p(k,n)\|\omega\|_p \tag{6.4}$$

for some constant $C_p(k,n)$ independent of ω.

Proof. Let us first prove uniqueness. Assume that

$$d\alpha+\delta\beta=0 \quad \text{with } d\alpha,\delta\beta\in L^p(\mathbf{R}^n,\textstyle\bigwedge^k).$$

Differentiating this equation we find that

$$\delta d\alpha=0 \quad \text{and} \quad d\delta\beta=0$$

at least in a distributional sense. Hence $d\delta(d\alpha)=0$ and $\delta d(\delta\beta)=0$. We also have the trivial identities $\delta d(d\alpha)=0$ and $d\delta(\delta\beta)=o$ from which we find

$$\Delta d\alpha=0 \quad \text{and} \quad \Delta\delta\beta=0.$$

That is the forms $d\alpha$ and $\delta\beta$ are harmonic and L^p integrable. By Weyl's Lemma it follows that $d\alpha$ and $\delta\beta$ are harmonic in the usual sense and since they are L^p integrable we find that $d\alpha=0$ and $\delta\beta=0$.

To prove existence, we first solve the Poisson equation

$$\omega=\Delta\varphi.$$

There is a solution φ expressed by the Riesz potential

$$\varphi(x)=\frac{1}{2\pi}\int\omega(y)\log|x-y|\,dy \quad \text{in dimension 2}$$

and

$$\varphi(x)=-\frac{\Gamma(n/2-1)}{4\pi^{n/2}}\int\frac{\omega(y)\,dy}{|x-y|^{n-2}} \quad \text{in dimension } n\geqslant 3$$

see [S, §V]. We write $\varphi=\mathfrak{R}(\omega)$. The second order derivatives of the coefficients of φ are found from the formula

$$\frac{\partial^2\varphi}{\partial x^i\partial x^j}=-\mathfrak{R}_i\mathfrak{R}_j(\Delta\varphi)=-\mathfrak{R}_i\mathfrak{R}_j(\omega)$$

for $i=1,2,\dots,n$, where $\mathfrak{R}_i$ are the Riesz transforms

$$\mathfrak{R}_i(f)(x)=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\int_{\mathbf{R}^n}\frac{(x_i-y_i)f(y)\,dy}{|x-y|^{n+1}}. \tag{6.5}$$

From the standard L^p theory of Riesz transforms [S, §V] it follows that not only is

$$\varphi\in L_2^p(\mathbf{R}^n,\textstyle\bigwedge^k)$$

but also

$$\|\varphi\|_{L_2^p(\mathbf{R}^n)} \leqslant C_p(k,n)\|\omega\|_p. \tag{6.6}$$

We now find that

$$\omega = (d\delta+\delta d)\varphi = d\alpha+\delta\beta$$

where $\alpha\in\operatorname{Ker}\delta\cap L_1^p(\mathbf{R}^n,\bigwedge^{k-1})$ and $\beta\in\operatorname{Ker}d\cap L_1^p(\mathbf{R}^n,\bigwedge^{k+1})$. The estimate (6.4) follows directly from (6.6).

Next, in dimension $n=2l$ the decomposition of $\bigwedge^l=\bigwedge^+\oplus\bigwedge^-$ applies pointwise to differential forms in the spaces $L_m^p(\Omega,\bigwedge^l)$. It is immediate that for an open subset Ω of $\mathbf{R}^{2l}$

$$L_m^p(\Omega,\textstyle\bigwedge^l) = L_m^p(\Omega,\bigwedge^+)\oplus L_m^p(\Omega,\bigwedge^-).$$

The exterior derivative $d: L_1^p(\Omega,\bigwedge^{l-1})\to L^p(\Omega,\bigwedge^l)$ naturally splits as $d=d^++d^-$ via composition with the obvious projection. More precisely, for $\alpha\in L_1^p(\Omega,\bigwedge^{l-1})$,

$$d^+\alpha = (d\alpha)_+ = \tfrac{1}{2}(\mathrm{Id}+(-i)^l*)\,d\alpha \quad\text{and}\quad d^-\alpha = (d\alpha)_- = \tfrac{1}{2}(\mathrm{Id}-(-i)^l*)\,d\alpha. \tag{6.7}$$

We will make use of the following lemma.

LEMMA 6.8. *Let $\alpha\in L_1^2(\mathbf{R}^{2l},\bigwedge^{l-1})$. Then*

$$\int_{\mathbf{R}^{2l}} |d^+\alpha|^2\,dx = \int_{\mathbf{R}^{2l}} |d^-\alpha|^2\,dx.$$

Proof. From Lemma 3.4

$$\int_{\mathbf{R}^{2l}} \langle d\alpha, *d\alpha\rangle\,dx = 0.$$

Hence

$$\begin{aligned}\int\big(|d^+\alpha|^2-|d^-\alpha|^2\big) &= \frac{1}{4}\int\big(|d\alpha+(-i)^l*d\alpha|^2-|d\alpha-(-i)^l*d\alpha|^2\big)\\ &= \mathrm{Re}\left\{i^l\int\langle d\alpha,*d\alpha\rangle\right\} = 0\end{aligned}$$

7. The Beltrami equation in even dimensions

Let $f\in W_{pl}^1(\Omega,\mathbf{R}^{2l})$, $1\leqslant p<\infty$, be weakly K-quasiregular and suppose $J(x,f)\geqslant 0$. Then we can define a measurable mapping $G:\Omega\to S(n)$ by the rule

$$G(x) = J(x,f)^{-1/l}D^tf(x)Df(x).$$

Pointwise a.e. the transpose differential $D^t f$ of f is G-conformal and so we can apply pointwise a.e. the exterior algebra developed in §2. In particular from (2.16) we define the *Beltrami coefficient* of f as the bounded operator with measurable coefficients

$$\mu_f: L^p(\Omega, \bigwedge^l) \to L^p(\Omega, \bigwedge^l)$$

given by

$$\mu_f(x) = \frac{G_\#(x) - \mathrm{Id}}{G_\#(x) + \mathrm{Id}}.$$

We immediately obtain from Theorem 2.20, Lemma 2.21 and Lemma 2.23, the following useful facts about the Beltrami coefficient.

THEOREM 7.1. *Let $f \in W^1_{pl}(\Omega, \mathbf{R}^{2l})$, $1 \leqslant p < \infty$, be weakly K-quasiregular and let μ_f be the matrix dilatation of f. Then μ_f permutes the spaces $L^p(\Omega, \bigwedge^+)$ and $L^p(\Omega, \bigwedge^-)$ and*

$$|\mu_f| = \frac{K_{\mathrm{sec}}(f) - 1}{K_{\mathrm{sec}}(f) + 1} \leqslant \frac{K^l - 1}{K^l + 1} < 1.$$

Also if $J(x,f) \geqslant 0$, then

(i) $(f^*\omega)_+ = \mu_f (f^*\omega)_-$, $\omega \in \bigwedge^-$,

(ii) $(f^*\omega)_- = \mu_f (f^*\omega)_+$, $\omega \in \bigwedge^+$,

and if $J(x,f) \leqslant 0$, then

(iii) $(f^*\omega)_+ = \mu_f (f^*\omega)_-$, $\omega \in \bigwedge^+$,

(iv) $(f^*\omega)_- = \mu_f (f^*\omega)_+$, $\omega \in \bigwedge^-$,

and we have the pointwise a.e. estimates

(v) $|Df(x)|^l |\omega| \leqslant K^l |(f^*\omega)(x)|$ *for all* $\omega \in \bigwedge^l$.

Next, we obtain as a consequence of Theorem 7.1, Lemma 3.6 and the decomposition of the exterior derivative $d = d^+ + d^-$ (see (6.7)) the following theorem which will be quite important in what follows.

THEOREM 7.2. *Let $f \in W^1_{pl,\mathrm{loc}}(\Omega, \mathbf{R}^{2l})$, $1 \leqslant p < \infty$, be weakly K-quasiregular and let μ_f be the Beltrami coefficient of f. Then for every $(l-1)$-form α with linear coefficients and such that*

$$d^+ \alpha = 0$$

we have

$$d^+(f^*\alpha) = \mu_f d^-(f^*\alpha) \tag{7.3}$$

if $J(x,f) \geqslant 0$ and

$$d^-(f^*\alpha) = \mu_f d^+(f^*\alpha) \tag{7.4}$$

if $J(x,f)\leqslant 0$.

We call the equation (7.3) the *Beltrami equation in even dimensions.* Compare this equation with (0.4).

We remark that there is a little subtlety involved in Theorem 7.2. If α is any form with say $C^\infty(\Omega)$ coefficients we can easily make sense of $d^\pm f^*\alpha$ as a distribution. But then it is impossible to multiply this by the (at best measurable) matrix dilatation μ_f. However, if α has linear coefficients, then actually $d(f^*\alpha)=f^*(d\alpha)$ is a function and so multiplication by μ_f presents no difficulties.

8. The Beurling–Ahlfors operator

Let $\omega\in L^p(\mathbf{R}^n,\bigwedge^k)$, $1<p<\infty$. We recall the Hodge decomposition of ω from §6 as

$$\omega=d\alpha+\delta\beta,$$

where $\alpha\in\operatorname{Ker}\delta\cap L^p_1(\mathbf{R}^n,\bigwedge^{k-1})$ and $\beta\in\operatorname{Ker}d\cap L^p_1(\mathbf{R}^n,\bigwedge^{k+1})$. We define an operator

$$S\colon L^p(\mathbf{R}^n,\textstyle\bigwedge^k)\to L^p(\mathbf{R}^n,\bigwedge^k)$$

by the rule

$$S\omega=d\alpha-\delta\beta.$$

Because of the uniform estimate on the Hodge decomposition (6.4) we find that S is bounded in all the spaces $L^p(\mathbf{R}^n,\bigwedge^k)$, $1<p<\infty$, $k=0,1,\dots,n$,

$$\|S\omega\|_p\leqslant A_p(n,k)\|\omega\|_p.$$

Because of the strong analogy with the planar case (compare (0.6) and (0.7) with what follows below) we call S the *Beurling–Ahlfors* operator, see [BA]. From the construction of the Hodge decomposition we find that formally

$$S=(d\delta-\delta d)\circ\Delta^{-1}$$

and as Δ^{-1} is expressible in terms of the Riesz potential we see that S is a singular integral operator of a rather natural type. We record the following simple facts about S.

THEOREM 8.1. *The operator S has the following properties*:

(i) *S acts as the identity on exact forms,*

$$S(d\alpha)=d\alpha$$

for all α *with* $d\alpha \in L^p(\mathbf{R}^n, \bigwedge^k)$.

(ii) *S acts as minus the identity on coexact forms,*

$$S(\delta\alpha) = -\delta\alpha$$

for all α *with* $\delta\alpha \in L^p(\mathbf{R}^n, \bigwedge^k)$.

(iii) *S is selfadjoint*

$$S \circ S = \mathrm{Id} \quad \text{and} \quad S = S^{-1}.$$

(iv) *S anticommutes with the Hodge star*

$$S* = -*S.$$

(v) *On functions* $S = -\mathrm{Id}$ *and on n-forms* $S = \mathrm{Id}$.

(vi) *In dimension* $n = 2l$, *S permutes the spaces* $L^p(\mathbf{R}^{2l}, \bigwedge^+)$ *and* $L^p(\mathbf{R}^{2l}, \bigwedge^-)$. *Thus*

$$S \circ d^+ = d^- \quad \text{and} \quad S \circ d^- = d^+$$

for all $(l-1)$*-forms* α *with* $d\alpha \in L^p(\mathbf{R}^{2l}, \bigwedge^l)$.

Proof. The identities (i)–(iii) follow from the uniqueness of the Hodge decomposition. The identity (v) is clear and (vi) follows from (iv) and (6.7). Thus we prove (iv). Let $\omega \in L^p(\mathbf{R}^n, \bigwedge^k)$, $\omega = d\alpha + \delta\beta$. As $*d\alpha$ is coexact and $*\delta\beta$ is exact, by (i) and (ii) we have

$$\begin{aligned}(S*)\omega &= S(*d\alpha) + S(*\delta\beta) = -*d\alpha + *\delta\beta \\ &= -*(d\alpha - \delta\beta) = -*S(\omega).\end{aligned}$$

Let us now choose $\omega \in L^2(\mathbf{R}^n, \bigwedge^k)$ with $\omega = d\alpha + \delta\beta$; $\alpha \in L_1^2(\mathbf{R}^n, \bigwedge^{k-1})$ and $\beta \in L_1^2(\mathbf{R}^n, \bigwedge^{k+1})$. We compute

$$\begin{aligned}\int |S\omega|^2 &= \int |d\alpha - \delta\beta|^2 = \int \left(|d\alpha|^2 + |\delta\beta|^2\right) - 2\,\mathrm{Re} \int \langle \delta\beta, d\alpha \rangle \\ &= \int \left(|d\alpha|^2 + |\delta\beta|^2\right) + 2\,\mathrm{Re} \int \langle \delta\beta, d\alpha \rangle = \int |d\alpha + \delta\beta|^2 = \int |\omega|^2\end{aligned}$$

where we have once again used Lemma 3.4. We therefore find that we have proven

THEOREM 8.2. *The operator* $S: L^2(\mathbf{R}^n, \bigwedge^k) \to L^2(\mathbf{R}^n, \bigwedge^k)$ *is an isometry,* $\|S\|_2 = 1$.

Thus for all n and k,

$$A_2(n, k) = 1.$$

From this we also find that for $\omega\in L^p(\mathbf{R}^n,\bigwedge^k)$ and $\gamma\in L^q(\mathbf{R}^n,\bigwedge^k)$ with $1/p+1/q=1$ (again from Lemma 3.4)

$$\int\langle\omega,\gamma\rangle=\int\langle S\omega,S\gamma\rangle$$

and so from Theorem 8.1 (iii)

$$\int\langle S\omega,\gamma\rangle=\int\langle\omega,S\gamma\rangle\leqslant\|\omega\|_p\|S\gamma\|_q\leqslant A_q(n,k)\|\omega\|_p\|\gamma\|_q.$$

As $\|S\omega\|_p=\sup\{\int\langle S\omega,\gamma\rangle:\|\gamma\|_q=1\}\leqslant A_q(n,k)\|\omega\|_p$, we conclude that $A_p(n,k)\leqslant A_q(n,k)$. Interchanging the roles of p and q we additionally find

LEMMA 8.3.

$$A_p(n,k)=A_q(n,k)\quad \textit{for}\ \frac{1}{p}+\frac{1}{q}=1.$$

As S can be defined in terms of Riesz transforms, it is of weak $(1,1)$-type. From [S, pp. 5–7] and a standard interpolation argument, together with the duality above, we find

THEOREM 8.4. *There is a constant $C(n)$ such that*

$$A_p(n,k)\leqslant C(n)\max\left\{\frac{1}{p-1},p-1\right\},\quad 1<p<\infty.$$

CONJECTURE 8.5. $C(n)=1$ *for each* n.

The operator $S\colon L^2(\mathbf{R}^n,\bigwedge^l)\to L^2(\mathbf{R}^n,\bigwedge^l)$ commutes with translations and thus it has a multiplier. The multiplier of S is a linear transformation $M(\xi)\colon\bigwedge^l(\mathbf{R}^n)\to\bigwedge^l(\mathbf{R}^n)$ defined for each $\xi\in\mathbf{R}^n\setminus\{0\}$. It has a matrix representation whose entries are linear combinations of the multipliers $-\xi_j\xi_k|\xi|^{-2}$ of the Riesz transforms $\mathfrak{R}_j\mathfrak{R}_k$, $j,k=1,2,...,n$. There is however an elegant description of $M(\xi)$ as the lth exterior power of an orthogonal transformation of $\mathbf{R}^n$ (the Jacobian matrix of the inversion in the unit sphere!). To compute this multiplier we begin with an l-form $\varphi=\sum_I\varphi^I dx^I$ say of class $C_0^\infty(\mathbf{R}^n,\bigwedge^l)$. Let

$$\omega=(d\delta+\delta d)\varphi=\Delta\varphi.$$

To simplify the following calculations we need to introduce some notation. Given a multiindex $I=(i_1,i_2,...,k,...,i_l)$ we denote by $I-k+j$ the multi-index $(i_1,i_2,...,j,...,i_l)$. That is we replace k by j in the same position in the index. It is a calculation, which we leave the reader to verify, that

$$S\omega=(d\delta-\delta d)\varphi=\sum_I\left[\sum_{j\notin I}\varphi^I_{jj}-\sum_{k\in I}\varphi^I_{kk}\right]dx^I-2\sum_I\left[\sum_{\substack{j\notin I\\ k\in I}}\varphi^I_{jk}\,dx^{I-k+j}\right].\tag{8.6}$$

As $\varphi^I_{jk}=-\mathfrak{R}_j\mathfrak{R}_k(\omega^I)$ we can express $S\omega$ in terms of the second order Riesz transforms

$$S\omega=\sum_I\left[\sum_{k\in I}\mathfrak{R}_k\mathfrak{R}_k-\sum_{j\notin I}\mathfrak{R}_j\mathfrak{R}_j\right]\omega^I\,dx^I+2\sum_I\left[\sum_{\substack{j\notin I\\ k\in I}}\mathfrak{R}_j\mathfrak{R}_k\omega^I\,dx^{I-k+j}\right].$$

Whence

$$|\xi|^2\widehat{S\omega}(\xi)=\sum_I\left[\sum_{j\notin I}\xi_j^2-\sum_{k\in I}\xi_k^2\right]\widehat{\omega}^I\,dx^I-2\sum_I\left[\sum_{\substack{j\notin I\\ k\in I}}\xi_j\xi_k\widehat{\omega}^I\,dx^{I-k+j}\right]. \tag{8.7}$$

Let us first suppose that ω is a 1-form. The expression (8.7) then reduces to

$$|\xi|^2\widehat{S\omega}(\xi)=\sum_{k=1}^n|\xi|^2\widehat{\omega}^k\,dx^k-2\sum_{j,k=1}^N\xi_j\xi_k\widehat{\omega}^j\,dx^k.$$

This gives us the multiplier of S on one forms

$$\widehat{S\omega}(\xi)=M(\xi)\widehat{\omega}(\xi) \tag{8.8}$$

where $M(\xi)=(m_{i,j})$ is the $n\times n$ matrix with entries homogeneous of degree 0 which in tensor notation we can simply write as

$$M(\xi)=\mathrm{Id}-2|\xi|^{-2}(\xi\otimes\xi)\quad\text{for }\xi\neq0. \tag{8.9}$$

For ξ fixed, $M(\xi)$ is easily seen to be an orthogonal transformation with determinant equal to -1. Next, for each $l=1,2,3,\dots,n$, we consider the lth exterior power of M, (see (2.7) for the definition) $M_\#:\bigwedge^l\to\bigwedge^l$. This induces a map of the Fourier images of the l-forms. More precisely, given $\omega\in L^2(\mathbf{R}^n,\bigwedge^l)$ we define

$$\widehat{\omega}(\xi)=\sum_I\widehat{\omega}^I(\xi)\,dx^I.$$

For ξ fixed this can be thought of as an l-vector and so $M_\#(\xi)\widehat{\omega}(\xi)$ is defined. This then produces a rather nice formula for the multiplier of S.

THEOREM 8.10. *The Fourier multiplier of the operator* $S\colon L^2(\mathbf{R}^n,\bigwedge^l)\to L^2(\mathbf{R}^n,\bigwedge^l)$ *is the orthogonal transformation* $M_\#(\xi)\colon\bigwedge^l\to\bigwedge^l$. *That is*

$$\widehat{S\omega}(\xi)=M_\#(\xi)\widehat{\omega}(\xi)$$

for each $\omega\in L^2(\mathbf{R}^n,\bigwedge^l)$.

Proof. We need only verify the above formula on forms of the type $\widehat{\omega}(\xi)\,dx^I$ for I and ordered l-tuple. For simplicity and without loss of generality, we assume that

$$\omega(x)=a(x)\,dx^1\wedge dx^2\wedge\dots\wedge dx^l=a(x)\,dx^I.$$

Then

$$|\xi|^2\widehat{S\omega}(\xi)=\Big(\sum_{l<j}\xi_j^2-\sum_{k\leqslant l}\xi_k^2\Big)\widehat{a}(\xi)\,dx^I-2\sum_{k\leqslant l<j}\xi_j\xi_k\widehat{a}(\xi)\,dx^{I-k+j}.$$

On the other hand assuming $|\xi|=1$, we have by (8.9)

$$M(\xi)\,dx^k=dx^k-2\xi_k\xi$$

where

$$\xi=\xi_1\,dx^1+\xi_2\,dx^2+\ldots+\xi_n\,dx^n.$$

Then by the definition of $M_\#$,

$$\begin{aligned}M_\#(\xi)(\widehat{a}(\xi)\,dx^I)&=\widehat{a}(\xi)(M dx^1\wedge M dx^2\wedge\ldots\wedge M dx^l)\\&=\widehat{a}(\xi)(dx^1-2\xi_1\xi)\wedge(dx^2-2\xi_2\xi)\wedge\ldots\wedge(dx^l-2\xi_l\xi).\end{aligned}$$

We now expand this exterior product taking into account $\xi\wedge\xi=0$. We find

$$\begin{aligned}M_\#(\xi)(\widehat{a}(\xi)\,dx^I)&=\widehat{a}(\xi)\Big[dx^I-2\sum_{k=1}^{l}\xi_k\,dx^1\wedge\ldots\wedge dx^{k-1}\wedge\xi\wedge dx^{k+1}\wedge\ldots\wedge dx^l\Big]\\&=\widehat{a}(\xi)\Big[dx^I-2\sum_{k=1}^{l}\xi_k^2\,dx^I-2\sum_{k\leqslant l<j}\xi_j\xi_k\,dx^1\wedge\ldots\wedge dx^{k-1}\wedge dx^j\wedge dx^{k+1}\wedge\ldots\,dx^l\Big]\end{aligned}$$

which agrees with (8.7) for $|\xi|=1$, and therefore proves the theorem.

Remark 8.11. Because of the orthogonality of the multiplier we have that $|\widehat{S\omega}(\xi)|=|\widehat{\omega}(\xi)|$ pointwise, which implies that S is an isometry in $L^2(\mathbf{R}^n,\bigwedge^l)$. As $M(\xi)$ is symmetric and orthogonal, so too is $M_\#(\xi)$. Then $M_\#(\xi)M_\#(\xi)=\mathrm{Id}$, which implies $S\circ S=\mathrm{Id}$. Since the determinant of $M(\xi)$ is equal to -1, we find from Lemma 2.17, $*M_\#=-*M_\#$ which implies $*S=-S*$.

We can now give an explicit convolution formula for the operator S.

THEOREM 8.12. *For each $1<p<\infty$, the operator $S\colon L^p(\mathbf{R}^n,\bigwedge^p)\to L^p(\mathbf{R}^n,\bigwedge^l)$ is represented by the following singular integral*

$$(S\omega)(x)=\Big(1-\frac{2l}{n}\Big)\,\omega(x)-\frac{\Gamma(1+n/2)}{\pi^{n/2}}\int_{\mathbf{R}^n}\frac{\Omega(x-y)\omega(y)}{|x-y|^n}\,dy$$

where

$$\Omega(\xi)=M_\#(\xi)+\Big(\frac{2l}{n}-1\Big)\,\mathrm{Id}\colon\textstyle\bigwedge^l\to\bigwedge^l.$$

Furthermore $\Omega(\xi)$ *is a homogeneous matrix function of degree zero whose mean value on the unit sphere is zero. The entries of* $\Omega(\xi)$ *are spherical harmonics of degree* 2, *that is* $|\xi|^2\Omega(\xi)$ *is a matrix of harmonic polynomials of degree* 2.

Proof. It follows from formula (8.7) that any off diagonal entry of the matrix $|\xi|^2\Omega(\xi)$ has the form $\pm 2\xi_j\xi_k$ with $j\neq k$ and so is clearly harmonic. The diagonal entry with index $I=(i_1,i_2,\dots,i_l)$ can be written as

$$m_I^I=\sum_{j\notin i}\xi_j^2-\sum_{k\in I}\xi_k^2+\left(\frac{2l}{n}-1\right)|\xi|^2.$$

Thus $\Delta m_I^I=2(n-l)-2l+2n(2l/n-1)=0$. This also shows that

$$\int_{S^{n-1}}\Omega(\xi)\,d\xi=0.$$

We now decompose the multiplier of S as

$$M_\#(\xi)=\frac{n-2l}{n}\,\mathrm{Id}+\Omega(\xi).$$

Then according to [S, Theorem 5, pp. 73] the kernel of the transformation corresponding to the multiplier $\Omega(\xi)$ is

$$K(x-y)=-\frac{\Gamma(1+n/2)}{\pi^{n/2}}\,\frac{\Omega(x-y)}{|x-y|^n}$$

and hence the formula of Theorem 8.12 follows.

An especially interesting situation arises when $n=2l$. Then we find

THEOREM 8.13. *The operator* $S\colon L^p(\mathbf{R}^{2l},\bigwedge^l)\to L^p(\mathbf{R}^{2l},\bigwedge^l)$ *is represented by the singular integral*

$$(S\omega)(x)=-\frac{l!}{\pi^l}\int_{\mathbf{R}^{2l}}\frac{M_\#(x-y)\omega(y)}{|x-y|^{2l}}\,dy.$$

We point out here that $M(\xi)=|\xi|^2D\Phi(\xi)$, where $D\Phi$ is the Jacobian matrix of the inversion Φ in the unit sphere of $\mathbf{R}^{2l}$.

$$\Phi(\xi)=\frac{\xi}{|\xi|^2}.$$

This again reinforces the analogy with the two-dimensional case, see (0.6). Then the kernel,

$$K(\xi)=|\xi|^{-2l}M_\#(\xi)\colon\textstyle\bigwedge^l\to\bigwedge^l,$$

is induced by the differential $D\Phi: \mathbf{R}^{2l} \to \mathbf{R}^{2l}$. More precisely

$$K(\xi)(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_l) = D\Phi(\xi)\alpha_1 \wedge D\Phi(\xi)\alpha_2 \wedge \ldots \wedge D\Phi(\xi)\alpha_l$$
$$= \Phi^*(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_l).$$

In other words the entries of the matrix $K(\xi)$ are the $l \times l$ minors of the Jacobian matrix of Φ. As such $K(\xi)$ is harmonic in $\mathbf{R}^{2l}$, is orthogonal and symmetric and permutes the spaces $\bigwedge^+$ and $\bigwedge^-$. Note too that exactly half the eigenvalues of $K(\xi)$ are $+1$ and the other half are -1.

It seems to us that the formulae above will be important for future calculations of the p-norms of the operator S. There is considerable information, both geometric and analytic, to be obtained for quasiregular mappings if one finds reasonable estimates for the norms $\|S\|_p$, see §9. It is for this reason we now give such an estimate. A problem with the Beurling–Ahlfors operator S, and other operators with even kernels, is due in part to limitations of the presently available methods from probabalistic and harmonic analysis. It seems a little too optimistic to find the norm $A_p(n,l)$ of $S: L^p(\mathbf{R}^n, \bigwedge^l) \to L^p(\mathbf{R}^n, \bigwedge^l)$. A simpler problem is to show that $A_p(n,l)$ actually depends only on p. But for our purposes, we need a bound which depends at most polynomially on n. This problem is still nontrivial, the reason being that the obvious estimate leads to exponential growth of $A_p(n,l)$ in n, because the number of terms involved in an explicit formula for S is of order 2^n. We prove

THEOREM 8.14. *For each $l=1,2,\ldots,n$ and $p \geqslant 2$*

$$A_p(n,l) \leqslant (np)^2.$$

Proof. The operator S depends linearly on the second order Riesz transforms $\mathfrak{R}_i\mathfrak{R}_j$, $i,j \in \{1,2,\ldots,n\}$. We denote by M_p the number

$$M_p = \max\{\|\mathfrak{R}_i\mathfrak{R}_j\|_p; i,j = 1,2,\ldots,n\}, \quad 1 < p < \infty.$$

The method of rotations, see [GR], and the fact that the p-norm of the one dimensional Hilbert transform is $\|H\|_p = \cot(\pi/2p) \leqslant 2p/\pi$, for $p \geqslant 2$, implies that $\|\mathfrak{R}_j\|_p \leqslant (\pi/2)\|H\|_p \leqslant p$ for all $j=1,2,\ldots,n$.(4) We then see

$$M_p \leqslant p^2, \quad p \geqslant 2. \tag{8.15}$$

Now let $\omega \in L^p(\mathbf{R}^n, \bigwedge^l)$. We solve the Poisson equation $\Delta\varphi = \omega$. So the coefficients satisfy $\omega^I = \Delta\varphi^I$ and $\varphi^I \in L_2^p(\mathbf{R}^n)$ for each l-tuple I. From (8.6) we have

$$S\omega = \sum_I \Big\{ \sum_{j=k} \pm\varphi^I_{j,k}\, dx^I \Big\} + 2 \sum_{k \in I} \Big\{ \sum_{j \notin I} \pm\varphi^I_{j,k}\, dx^{I-k+j} \Big\}.$$

(4)Added in proof: We have recently shown that $\|\mathfrak{R}_j\|_p = \|H\|_p$, see [IM2].

We do not specify the signs $\pm$ because we are not going to use any cancellation of terms. Now fix an l-tuple K and compute the Kth coordinate of $S\omega$. We find

$$(S\omega)^K\,dx^K=\left\{\sum_{j=k}\pm\varphi^K_{j,k}+2\sum_I\pm\varphi^I_{K-I,I-K}\right\}dx^K,$$

where the summation is over all l-tuples I such that $\#\{I\cup K\}=l-1$. There are at most $n+2(n-l)n\leqslant n^2$ terms in this sum. We therefore have the pointwise estimate

$$\left|(S\omega)^K\right|^2\leqslant n^2\left\{\sum_{j=k}|\varphi^K_{j,k}|^2+2\sum_I|\varphi^I_{K-I,I-K}|^2\right\},\tag{8.16}$$

where the index I is restricted as above. We now sum (8.16) over all ordered l-tuples K and interchange the order of summation to obtain

$$\begin{aligned}|S\omega|^2&\leqslant n^2\left\{\sum_K\left\{\sum_{j=k}|\varphi^K_{j,k}|^2\right\}+2\sum_K\left\{\sum_{\substack{j\in K\\ k\notin K}}|\varphi^K_{j,k}|^2\right\}\right\}\\&=n^2\sum_{j,k=1}^n\left\{\sum_K|\varphi^K_{j,k}|^2\right\}\\&=n^2\sum_{j,k=1}^n\left\{\sum_K|\mathfrak{R}_j\mathfrak{R}_k(\omega^K)|^2\right\}.\end{aligned}\tag{8.17}$$

Now we shall apply the following lemma. For a more general result concerning tensor products of operators acting on L^p spaces see [FIP], Corollary 1.2.

LEMMA 8.18. *Let $T\colon L^p(\Omega,d\lambda)\to L^p(\Omega,d\lambda)$ be a bounded linear operator with norm $\|T\|_p$, $1<p<\infty$. For $h\in L^p(\Omega,\mathbf{R}^m)$, $h=(h^1,h^2,\dots,h^m)$ define the tensor product*

$$\mathcal{T}=\mathrm{Id}\otimes T\colon L^p(\Omega,\mathbf{R}^m)\to L^p(\Omega,\mathbf{R}^m)$$

by

$$\mathcal{T}h=(Th^1,Th^2,\dots,Th^m).$$

Then

$$\|\mathcal{T}h\|_p\leqslant\|T\|_p\|h\|_p.$$

Proof. In view of the linearity of T we have for each $s=(s_1,s_2,\dots,s_m)\in S^{m-1}$

$$\langle s,\mathcal{T}h\rangle=T\langle s,h\rangle$$

and hence

$$\|\langle s,\mathcal{T}h\rangle\|_p\leqslant\|T\|_p\|\langle s,h\rangle\|_p.$$

We raise this inequality to the power p, integrate with respect to s and use Fubini's Theorem to obtain

$$\int_\Omega \left\{ \int_{S^{m-1}} |\langle s, Th\rangle|^p \, ds \right\} d\lambda \leqslant \|T\|_p^p \int_\Omega \left\{ \int_{S^{m-1}} |\langle s, h\rangle|^p \, ds \right\} d\lambda.$$

On the other hand, because of the invariance of the spherical Haar measure ds under the rotation group of S^{m-1} we have the identity

$$\int_{S^{m-1}} |\langle s, x\rangle|^p \, ds = |x|^p \int_{S^{m-1}} |s_1|^p \, ds$$

for each vector $x \in \mathbf{R}^m$. Therefore

$$\int_\Omega |Th|^p \, d\lambda \leqslant \|T\|_p^p \int_\Omega |h|^p \, d\lambda,$$

which proves the lemma.

Remark. It follows from these arguments that the lemma remains valid in the case of a complex space $L^p(\Omega, d\lambda)$ provided the operator T is the complexification of a real operator. That is if $T(\mathrm{Re}\{f\}) = \mathrm{Re}\{T(f)\}$. Notice that this is the case for the operators $\mathfrak{R}_j \mathfrak{R}_k$.

Lemma 8.18 applied to the equation (8.17) for each of the operators $\mathfrak{R}_j \mathfrak{R}_k$ separately gives via the triangle inequality (so $p \geqslant 2$)

$$\begin{aligned} \|S\omega\|_p^2 = \left\| |S\omega|^2 \right\|_{p/2} &\leqslant n^2 \sum_{j,k}^{n} \left\| \left(\sum_K |\mathfrak{R}_j \mathfrak{R}_k(\omega^K)|^2 \right)^{1/2} \right\|_p^2 \\ &\leqslant n^2 \sum_{j,k}^{n} \|\mathfrak{R}_j \mathfrak{R}_k\|_p^2 \left\| \left(\sum_K |\omega^K|^2 \right)^{1/2} \right\|_p^2 \\ &\leqslant n^4 M_p^2 \|\omega\|_p^2. \end{aligned}$$

Thus

$$\|S\omega\|_p \leqslant n^2 M_p \|\omega\|_p$$

which establishes the theorem in view of our estimate for M_p.

Next, a standard interpolation argument (via the Riesz–Thorin Convexity Theorem) implies

COROLLARY 8.19. *For all* $2 \leqslant p < q < \infty$

$$A_p(n,k) \leqslant (n^2 q^2)^{\frac{p-2}{q-2} \cdot \frac{q}{p}}.$$

There is a corresponding result for the conjugate index $1 < p \leqslant 2$. *As a particular case*

$$A_p(n,k) \leqslant n^{4|2-p|}$$

for all $\frac{3}{2} \leqslant p \leqslant 3$.

Finally notice that the Convexity Theorem also implies

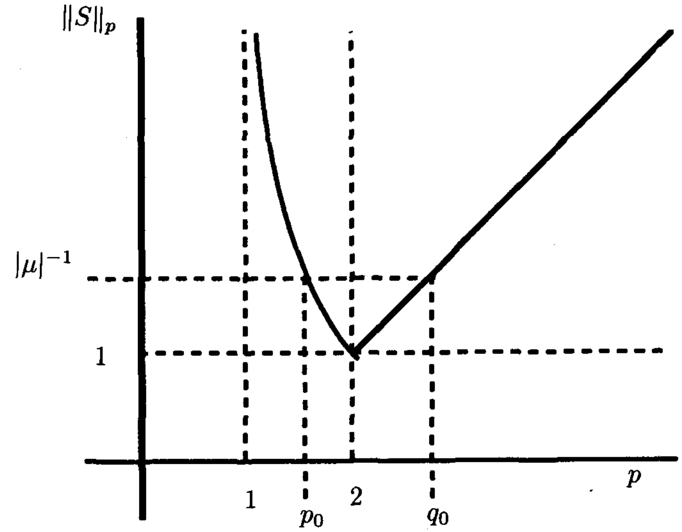

Fig. 2

PROPOSITION 8.20. *The function* $p \to A_p(n,k)$ *is continuously decreasing for* $1<p\leqslant 2$ *and continuously increasing for* $2\leqslant p<\infty$.

9. Regularity theorems for quasiregular mappings

In this section we give proofs of both Theorem 2 and Theorem 3. Thus let

$$f\in W^1_{pl,\mathrm{loc}}(\Omega,\mathbf{R}^{2l}),\quad p\geqslant 1,$$

be a weakly quasiregular mapping. Denote by μ the Beltrami coefficient μ_f of f. Define $|\mu|=\|\mu_f\|_\infty<1$. We shall investigate the L^{pl}_{loc}-integrability of the Jacobian Df for $p\in(p_0,q_0)$ where $1<p_0<2<q_0<\infty$ are the *critical exponents* of μ implicity defined by the equation

$$|\mu|\,\|S\|_{p_0}=|\mu|\,\|S\|_{q_0}=1$$

see Figure 2. Obviously Lemma 8.3 implies $1/p_0+1/q_0=1$, and Proposition 8.20 implies $|\mu|\,\|S\|_p<1$, for all $p\in(p_0,q_0)$. In particular then, the operator

$$\mathrm{Id}-\mu S\colon L^p(\mathbf{R}^{2l},\textstyle\bigwedge^l)\to L^p(\mathbf{R}^{2l},\textstyle\bigwedge^l)$$

is invertible and

$$\|(\mathrm{Id}-\mu S)^{-1}\|_p\leqslant(1-|\mu|\,\|S\|_p)^{-1}. \tag{9.1}$$

Our theorems are derived by an obvious induction using the following lemma.

LEMMA 9.2. *If* $f\in W^1_{pl,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$ *for some* $p\in(p_0,q_0)$, *then* $f\in W^1_{ql,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$ *for all* q *such that*

$$p_0<q<\min\left\{\frac{2l}{2l-1}p,q_0\right\}.$$

Proof. Let us assume that $J(x,f)\geqslant 0$ a.e. Then the Beltrami equation for f takes the form

$$d^+(f^*\alpha)=\mu d^-(f^*\alpha), \tag{9.3}$$

where we choose

$$\alpha=y^1dy^2\wedge dy^3\wedge\ldots\wedge dy^l-(-i)^ly^{l+1}dy^{l+2}\wedge dy^{l+3}\wedge\ldots\wedge dy^{2l}.$$

Then $d\alpha$ has constant coefficients,

$$d\alpha=dy^1\wedge dy^2\wedge dy^3\wedge\ldots\wedge dy^l-(-i)^ldy^{l+1}\wedge dy^{l+2}\wedge dy^{l+3}\wedge\ldots\wedge dy^{2l}\neq 0$$

$|d\alpha|=\sqrt{2}$ and

$$d^+\alpha=0.$$

The induced $(l-1)$-form

$$f^*\alpha=f^1df^2\wedge df^3\wedge\ldots\wedge df^l-(i)^lf^{l+1}df^{l+2}\wedge df^{l+3}\wedge\ldots\wedge df^{2l}$$

can be estimated pointwise a.e. as follows;

$$\begin{aligned}|f^*\alpha|^2&\leqslant(|f^1df^2\wedge\ldots\wedge df^l|+|f^{l+1}df^{l+2}\wedge df^{l+3}\wedge\ldots\wedge df^{2l}|)^2\\&\leqslant|f^1|^2|Df|^{2l-2}+|f^{l+1}|^2|Df|^{2l-2}\leqslant|f|^2|Df|^{2l-2},\end{aligned}$$

and hence we obtain the estimate

$$|f^*\alpha(x)|\leqslant|f(x)|\,|Df(x)|^{l-1}. \tag{9.4}$$

From the Imbedding Theorems we obtain

$$f^*\alpha\in L^q_{\mathrm{loc}}(\Omega,\textstyle\bigwedge^{l-1}). \tag{9.5}$$

To see this, first notice that $f\in L^{2pl}_{\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$. Indeed, if $1\leqslant p<2$, then by the Sobolev Imbedding Theorem, $f\in L^s_{\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$, with $s=(2llp)/(2l-lp)\geqslant 2lp$. If $p\geqslant 2$, then $f\in W^1_{2l,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$ which implies the integrability of f with arbitrary exponent. Now (9.5) follows from (9.4) by Hölder's inequality. More precisely,

$$|f|\,|Df|^{l-1}\in L^r_{\mathrm{loc}}(\Omega)$$

for any $r\geqslant 1$ satisfying
$$\frac{1}{r}\geqslant\frac{1}{2lp}+\frac{l-1}{lp}=\frac{2l-1}{2lp}$$
and so in particular with $r=q$.

In order to apply the Beurling–Ahlfors operator we should multiply $f^*\alpha$ by a test function, say $\eta\in C_0^\infty(\Omega)$. Obviously $\eta f^*\alpha\in L^q(\mathbf{R}^{2l},\bigwedge^l)$ and

$$d(\eta f^*\alpha)=\eta f^*(d\alpha)+d\eta\wedge f^*\alpha \tag{9.6}$$

is a compactly supported regular distribution of class $L^p(\mathbf{R}^{2l},\bigwedge^l)$. This then justifies the following permutation formula (cf. (0.7)),

$$d^-(\eta f^*\alpha)=S[d^+(\eta f^*\alpha)]. \tag{9.7}$$

On the other hand we have the Beltrami equation and so we find that

$$d^+(\eta f^*\alpha)-\mu d^-(\eta f^*\alpha)=(d\eta\wedge f^*\alpha)_+-\mu(d\eta\wedge f^*\alpha)_-=\omega,$$

say. From (9.4) we have pointwise a.e.

$$|\omega(x)|\leqslant\sqrt{2}(1+|\mu|)|d\eta\wedge f^*\alpha|\leqslant 4|\nabla\eta|\,|f|\,|Df|^{l-1} \tag{9.8}$$

and then by (9.5), $\omega\in L^q(\mathbf{R}^{2l},\bigwedge^l)\cap L^p(\mathbf{R}^{2l},\bigwedge^l)$. Now (9.7) reduces to the integral equation
$$d^+(\eta f^*\alpha)=(\mathrm{Id}-\mu S)^{-1}\omega.$$

Using (9.7) again we obtain

$$d(\eta f^*\alpha)=(d^++d^-)(\eta f^*\alpha)=(\mathrm{Id}+S)(\mathrm{Id}-\mu S)^{-1}\omega. \tag{9.9}$$

This shows that $d(\eta f^*\alpha)\in L^q(\mathbf{R}^{2l},\bigwedge^l)$ and that moreover we have the uniform estimate

$$\|d(\eta f^*\alpha)\|_q\leqslant\frac{1+\|S\|_q}{1-|\mu|\,\|S\|_q}\|\omega\|_q\leqslant\frac{8\|S\|_q}{1-|\mu|\,\|S\|_q}\,\big\|\,|\nabla\eta|\,|f|\,|Df|^{l-1}\big\|_q$$

which follows from (9.8). This together with the estimate of (9.4) and the identity (9.6) imply
$$\begin{aligned}\|\eta f^*(d\alpha)\|_q&\leqslant\|d(\eta f^*\alpha)\|_q+\|d\eta\wedge f^*\alpha\|_q\\&\leqslant\left(\frac{8\|S\|_q}{1-|\mu|\,\|S\|_q}+\sqrt{2}\right)\big\||\nabla\eta|\,|f|\,|Df|^{l-1}\big\|_q.\end{aligned} \tag{9.10}$$

Finally, we have the pointwise a.e. estimate

$$\sqrt{2}\,|Df(x)|^l\leqslant K^l|f^*(d\alpha)|, \tag{9.11}$$

by Theorem 7.1. Then from (9.10) we conclude that $Df \in L^{lq}_{\mathrm{loc}}(\Omega, GL(n))$. This completes the proof of the lemma.

As we have mentioned, an easy induction now establishes Theorems 2 and 3. We want to point out that we have also proven the following uniform estimate

$$\left\|\eta|Df(x)|^l\right\|_q \leqslant \frac{8K^l\|S\|_q}{1-|\mu|\,\|S\|_q}\left\|\,|\nabla\eta|\,|f|\,|Df|^{l-1}\right\|_q. \tag{9.12}$$

Remark 9.13. In [IM1] we have proved that each K-quasiregular mapping $f\colon\Omega\to\mathbf{R}^2$ belongs to $W^1_{p,\mathrm{loc}}(\Omega)$ with $p=2K^a/(K^a-1)$ for some a, $1\leqslant a\leqslant 7.283$.

Remark 9.14 (Hölder regularity). Equation (5.1) can be used (as in the proof of Liouville's Theorem) to derive the $C^{k+1,\alpha}$ regularity properties of a quasiregular mapping that we spoke of in Theorem 4. However, a simpler method can be found using the Beltrami equation (7.3). Thus suppose $f\in W^1_{2l,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$ and that the Beltrami coefficient $\mu=\mu_f$ belongs to the Hölder class $C^{k,\alpha}_{\mathrm{loc}}(\Omega, GL(2l))$ for some α and k with $0<\alpha\leqslant 1$ and $k\in\mathbf{N}$.

We shall outline the idea of the proof, which is simply to apply the operator S to (7.3). Consider the equation (9.9) above with $\omega=(d\eta\wedge f^*\alpha)_+-\mu(d\eta\wedge f^*\alpha)_-$. Let Ω' be an arbitrary open set compactly contained in Ω. We choose the compactly supported test function η so that $\eta\equiv 1$ on Ω'. It is important to notice the following property of the pseudo-differential operator

$$(\mathrm{Id}+S)(\mathrm{Id}-\mu S)^{-1}\colon L^2(\mathbf{R}^{2l},\textstyle\bigwedge^l)\to L^2(\mathbf{R}^{2l},\textstyle\bigwedge^l).$$

Namely that if $\omega\in L^2(\mathbf{R}^{2l},\bigwedge^l)$, then $(\mathrm{Id}+S)(\mathrm{Id}-\mu S)^{-1}\omega$ is of class $C^{k,\alpha}$ outside the support of ω. This can be viewed as a hypoellipticity type property of the operator $(\mathrm{Id}-\mu S)$ with $\mu\in C^{k,\alpha}$, $\|\mu\|_\infty<1$. Accordingly,

$$f^*(d\alpha)\in C^{k,\alpha}_{\mathrm{loc}}(\Omega',\textstyle\bigwedge^l)$$

because $\omega(x)$ vanishes on Ω' and $d(\eta f^*\alpha)=f^*(d\alpha)$ on Ω'. As Ω' is arbitrary we may set Ω in place of Ω' above. Then the l-vector $d\alpha\in\bigwedge^-$ (in view of the definition following (9.3)) satisfies

$$d\bar{\alpha}\wedge d\alpha=-2(-i)^l\,\mathrm{Vol}.$$

Applying f^* we obtain

$$-2(-i)^lJ(x,f)\,\mathrm{Vol}=f^*(\overline{d\alpha})\wedge f^*(d\alpha)\in C^{k,\alpha}_{\mathrm{loc}}(\Omega).$$

Hence the Jacobian determinant belongs to the class $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$. Now in much the same way we can show that $f^*(d\beta)\in C^{k,\alpha}_{\mathrm{loc}}(\Omega,\bigwedge^l)$ for each l-vector $d\beta\in\bigwedge^+$. In conclusion the $l\times l$ minors of the differential $Df(x)$ are of class $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$. This improved regularity considerably simplifies the remainder of the existing proofs. See [I2] for further details in this setting, and [Ma] for another approach to proving Theorem 4.

10. The Caccioppoli type estimate

Apriori estimates in quasiconformal analysis have been extensively used from the early days of the theory. Much of the information about the differential of a quasiregular mapping is contained in certain Caccioppoli type estimates. These estimates are derived from the fact that every quasiregular mapping of $\mathbf{R}^n$ achieves the minimum value of its own associated Dirichlet integral. This integral unfortunately requires the L^n-integrability of the differential. From these estimates, via Gehring's Lemma [G2], higher integrability results follow. However to remove nontrivial singularities of a quasiregular mapping one needs L^p estimates for p less than the dimension of the space. Some previous efforts have been made to achieve such results but have been largely unsuccessful. The following result is perhaps the first of its kind.

THEOREM 10.1. *Let $f\in W^1_{pl,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$, $1<p<2$ be weakly quasiregular with Beltrami coefficient μ_f and where p is such that*

$$|\mu_f|\,\|S\|_p<1.$$

Then

$$\int_\Omega |\varphi(x)Df(x)|^{pl}\,dx\leqslant C_p(l,K)\int_\Omega |\nabla\varphi(x)|^{pl}|f(x)-f_0|^{pl}\,dx$$

for each test function $\varphi\in C_0^\infty(\Omega)$ and $f_0\in\mathbf{R}^{2l}$.

Proof. We apply (9.12) to the function $\eta(x)=\varphi^l(x)$. Then by Hölder's inequality

$$\begin{aligned}\big\||\varphi Df(x)|^l\big\|_p&\leqslant\frac{8lK^l\|S\|_p}{1-|\mu_f|\,\|S\|_p}\,\big\||\nabla\varphi|\,|f|\,|\varphi Df|^{l-1}\big\|_p\\&\leqslant C_p(l,K)^{1/(pl)}\,\big\||\nabla\varphi|^l|f|^l\big\|_p^{1/l}\,\big\||\varphi Df|^l\big\|_p^{1-1/l},\end{aligned}$$

where

$$C_p(l,K)=\left(\frac{8lK^l\|S\|_p}{1-|\mu_f|\,\|S\|_p}\right)^{pl}.$$

The result now follows for $f_0=0$, by dividing out the common factor. In general, of course, we may replace f by $f-f_0$.

11. Removability theorems for quasiregular mappings

In this section we generalize the classical removable singularity theorems for holomorphic functions to quasiregular mappings in all even dimensions. Thus let Ω be a domain in $\mathbf{R}^n$

and E a compact subset of Ω. For all $1<s<\infty$, the *s-capacity* of the condenser (E,Q) is defined as

$$s\text{-Cap}(E,\Omega)=\inf\left\{\int_\Omega |\nabla\varphi(x)|^s\,dx\right\}$$

where the infimum is taken over all functions $\varphi\in C_0^\infty(\Omega)$ which are identically equal to one in a neighbourhood of E. Given $E\subset\mathbf{R}^n$, if $s\text{-Cap}(E,\Omega)=0$ for some bounded Ω (equivalently for every Ω) we say E has *zero s-capacity*. A closed set E has zero s-capacity if every compact subset of E has zero s-capacity.

Notice that E has zero s-capacity, $1<s\leqslant n$, implies that the d-dimensional Hausdorff measure of E, $H_d(E)=0$, for all $d>n-s$. Also, if E is a closed subset of $\mathbf{R}^n$ of Hausdorff dimension less than $k<n-1$, then E has zero $(n-k)$-capacity, and finally the countable union of sets of zero s-capacity has zero s-capacity, provided this union is closed.

Proof of Theorem 5. Let E be a closed subset of $\mathbf{R}^{2l}$ with pl-capacity zero, $1\leqslant p<2$ and suppose that $f:\Omega\setminus E\to\mathbf{R}^{2l}$ is a bounded quasiregular mapping with Beltrami coefficient μ_f such that $|\mu_f|\,\|S\|_p<1$. As $H_{2l}(E)=0$ it remains only to prove that $f\in W^1_{pl,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$. For then Theorem 2 implies $f\in W^1_{2l,\mathrm{loc}}(\Omega,\mathbf{R}^{2l})$, so f is quasiregular on Ω. Let $\eta\in C_0^\infty(\Omega)$ be an arbitrary test function, we may assume (without loss of generality) that E is compact, but not necessarily contained in Ω. Then there exists a sequence of functions $\varphi_j\in C_0^\infty(\mathbf{R}^{2l})$ such that for each j,

(i) $0\leqslant\varphi_j\leqslant 1$,

(ii) $\varphi_j=1$ on some neighbourhood U_j of E,

(iii) $\lim\varphi_j(x)=0$, for all $x\in\mathbf{R}^{2l}\setminus E$,

(iv) $\lim\|\nabla\varphi_j\|_{pl}=0$.

We set

$$\Psi_j=(1-\varphi_j)\eta\in C_0^\infty(\Omega\setminus E).$$

It is quite clear that

$$|\nabla\Psi_j|=|(1-\varphi_j)\nabla\eta-\eta\nabla\varphi_j|\leqslant|\nabla\eta|+|\eta|\,|\nabla\varphi_j|$$

and

$$D(\Psi_jf)=\Psi_jDf+f\otimes\nabla\Psi_j\in L^{pl}(\mathbf{R}^{2l},GL(2l)).$$

By the Caccioppoli type estimate Theorem 10.1 we obtain

$$\begin{aligned}\|D(\Psi_jf)\|_{pl}&\leqslant\|f\otimes\nabla\Psi_j\|_{pl}+\|\Psi_jDf\|_{pl}\\&\leqslant C(p,l,K)\|\,|f|\,|\nabla\Psi_j|\,\|_{pl}\\&\leqslant C(p,l,K)\left\{\|\,|f|\,|\nabla\eta|\,\|_{pl}+\|\,|\eta f|\,|\nabla\varphi_j|\,\|_{pl}\right\}.\end{aligned}\tag{11.1}$$

Now let $j\to\infty$. We have $\Psi_j f\to\eta f$ in $L^{pl}(\mathbf{R}^{2l},\mathbf{R}^{2l})$. Also, as f is assumed bounded, the right hand side of (11.1) remains bounded. Therefore $\Psi_j f\to\eta f$ in the weak topology of $W^1_{pl}(\mathbf{R}^{2l},\mathbf{R}^{2l})$. In particular $\eta f\in W^1_{pl}(\mathbf{R}^{2l},\mathbf{R}^{2l})$ and as a limit case of (11.1) we have the uniform estimate

$$\|D(\eta f)\|_{pl}\leqslant C(p,l,K)\|\,|f|\,|\nabla\eta|\,\|_{pl},$$

which completes the proof.

Remark 11.2. The assumption that f is bounded is of course much more than we really need. Actually, all we require is that the sequence

$$\|\,|\eta f|\,|\nabla\varphi_j|\,\|_{pl},\quad j=1,2,\dots$$

remains bounded. Thus for instance it is clear that Theorem 5 can be extended to the case

$$f\in L^s(\Omega\backslash E,\mathbf{R}^{2l}),\quad s\geqslant\frac{2pl}{2-p}.$$

This however requires the stronger restriction that the set E has zero $spl/(s-pl)$-capacity. This follows simply from Hölder's inequality.

Theorem 5, together with the estimate of Corollary 8.14 has the following consequence concerning removability near half the dimension.

THEOREM 11.3. *Let $0<\varepsilon<1$ and let $E\subset\mathbf{R}^{2l}$ be either a closed set of $(1+\varepsilon)l$-capacity zero or a closed set of Hausdorff dimension $d<(1-\varepsilon)l$. Then E is removable under all bounded quasiregular mappings $f:\Omega\backslash E\to\mathbf{R}^{2l}$ provided that*

$$C(2l)|\mu_f|<\varepsilon.$$

Indeed Conjecture 8.5 would imply that the condition $|\mu_f|<\varepsilon$ would suffice. Another interesting consequence arises from the estimate of Corollary 8.19, here we have p close to two.

THEOREM 11.4. *Let $\mu\in[0,1)$ and let $E\subset\mathbf{R}^n$, $n=2,4,6,\dots$, be a closed subset of Hausdorff dimension*

$$d\leqslant\frac{n(1-\mu)}{6\log(n)}.$$

Then every bounded quasiregular mapping $f:\Omega\backslash E\to\mathbf{R}^n$ with Beltrami coefficient $|\mu_f|<\mu$ extends to a quasiregular mapping of Ω.

Proof. E has $(n-d)$-capacity zero. Thus the condition for the removability of E is

$$|\mu_f|\,\|S\|_{2-d/l}<1,\quad\text{for } p=2\left(1-\frac{d}{n}\right).$$

In view of Corollary 8.19

$$\|S\|_p \leqslant n^{4(2-p)}$$

because $\frac{3}{2} \leqslant p \leqslant 3$. Hence a sufficient condition is easily seen to be

$$\mu n^{4(2-p)} = \mu n^{4d/n} < 1,$$

that is $d < -n \log \mu / 4 \log n$. This, by elementary considerations, is satisfied for all $d \leqslant n(1-\mu)/6 \log n$.

12. Some examples

In this section we give a number of examples. We show firstly that the regularity assumptions in our version of the Liouville Theorem cannot be improved at all; secondly, that the Regularity Theorem cannot be qualitatively improved and finally we give some examples concerning removable singularities for quasiregular mappings.

THEOREM 12.1. *Let Ω be a domain in $\mathbf{R}^n$ and let $K \geqslant 1$. Then for all $p \in [1, \frac{nK}{K+1})$ there is a weak K-quasiregular mapping $f \in W^1_{p,\mathrm{loc}}(\Omega, \mathbf{R}^n)$ which is not quasiregular.*

Before proving this theorem we need to make some preliminary remarks and formulate a couple of lemmas. Let us first fix $K \geqslant 1$.

The K-quasiconformal inversion in the sphere $S^{n-1}(a,r)$ is defined as

$$\Phi(x) = a + (x-a)\left(\frac{r}{|x-a|}\right)^{1+1/K}.$$

For $K=1$, $r=1$ and $a=0$ we obtain the usual inversion in the unit sphere. The following lemma is easy.

LEMMA 12.2. *With the notation above*

$$D\Phi(x) = \left(\frac{r}{|x-a|}\right)^{1+1/K} \left[\mathrm{Id} - \frac{K+1}{K} \frac{(x-a)\otimes(x-a)}{|x-a|^2}\right].$$

We shall be interested in the restriction of Φ to the ball $\mathbf{B} = B(a,r)$. There we have the inequality

$$|\Phi(x) - x|^K \leqslant \frac{r^{K+1}}{|x-a|}.$$

We therefore obtain the estimate, for each $1 \leqslant q < Kn$,

$$\int_{\mathbf{B}} |\Phi(x)-x|^q \, dx \leqslant r^{q+q/K} \int_{\mathbf{B}} \frac{dx}{|x-a|^{q/K}} = \frac{nKr^q}{nK-q} |\mathbf{B}|. \tag{12.3}$$

We now want to show that $\Phi\in W_p^1(\mathbf{B})$ for every $1\leqslant p<nK/(K+1)$. To do this we compute the Hilbert–Schmidt norm of $D\Phi(x)$ using the formula of Lemma 12.2,

$$D^t\Phi(x)D\Phi(x)=\left(\frac{r}{|x-a|}\right)^{2+2/K}\left[\mathrm{Id}-\left(1-\frac{1}{K^2}\right)\frac{(x-a)\otimes(x-a)}{|x-a|^2}\right].$$

Whence

$$|D\Phi|^2=\mathrm{Trace}\{D^t\Phi D\Phi\}\leqslant n\left(\frac{r}{|x-a|}\right)^{2+2/K}.$$

And thus

$$\begin{aligned}\int_{\mathbf{B}}|D\Phi|^p\,dx&\leqslant n^{p/2}r^{p+p/K}\int_{\mathbf{B}}\frac{dx}{|x-a|^{p+p/K}}\\&=\frac{n^{p/2+1}}{n-p-p/K}|\mathbf{B}|\leqslant\frac{Kn^n}{nK-p(K+1)}|\mathbf{B}|.\end{aligned}\tag{12.4}$$

Finally, since $\Phi(x)-x$ vanishes on $\partial\mathbf{B}$ we obtain from integration by parts

$$\int_{\mathbf{B}}D^t\eta(x)\,[\Phi(x)-x]\,dx=-\int_{\mathbf{B}}[D^t\Phi-\mathrm{Id}]\eta(x)\,dx\tag{12.5}$$

for any test mapping $\eta\in C^\infty(\mathbf{R}^n,\mathbf{R}^n)$.

This shows that $\Phi-\mathrm{Id}$ belongs to $W_p^1(\mathbf{B})$ for all $p<nK/(K+1)$.

We construct our example by gluing a number of these reflections together in a careful way. To do this we need the following notion. Let Ω be a domain in $\mathbf{R}^n$. Then an *exact packing of* Ω *by balls* is a disjoint family $\mathcal{F}=\{B_j:1\leqslant j<\infty\}$ of open balls $B_j\subset\Omega$ such that $\Omega\neq B_j$ for any j and

$$\left|\Omega-\bigcup_1^\infty B_j\right|=0.$$

The existence of such a packing for an arbitrary domain Ω follows from Vitali's Covering Theorem. We omit the proof.

LEMMA 12.6. *Every open subset* Ω *of* $\mathbf{R}^n$ *has an exact packing by balls of radius less than* 1.

Proof of Theorem 12.1. We assume for convenience that $|\Omega|<\infty$. Let $\mathcal{F}=\{\mathbf{B}_j\}$ be an exact packing of Ω. It is clear that $\mathcal{F}$ has infinitely many elements. Let Φ_j be the K-quasiconformal reflection in the spheres $\partial\mathbf{B}_j$. We define a function $F:\Omega\to\mathbf{R}^n$ piecewise as

$$F(x)=\Phi_j(x)\text{ if }x\in\mathbf{B}_j\text{ and }F(x)=x\text{ otherwise.}$$

Inequality (12.3) immediately implies that

$$\int_\Omega|F(x)-x|^q\,dx=\sum_{j=1}^\infty\left\{\int_{\mathbf{B}_j}|\Phi_j(x)-x|^q\,dx\right\}\leqslant\frac{nK}{nK-q}\sum_{j=1}^\infty|\mathbf{B}_j|=\frac{nK}{nK-q}|\Omega|<\infty,$$

for every $q\in[1,nK)$. Thus $F\in L^q(\Omega,\mathbf{R}^n)$. Next define a matrix function $A(x)$ as

$$A(x)=D\Phi_j(x) \text{ if } x\in\mathbf{B}_j \text{ and } A(x)=\mathrm{Id} \text{ otherwise.}$$

From inequality (12.4) we have

$$\int_\Omega |A(x)|^p\,dx\leqslant \frac{Kn^n}{nK-p(K+1)}|\Omega|<\infty,$$

for every $1\leqslant p<nK/(K+1)$. Therefore to prove that $F\in W_p^1(\Omega,\mathbf{R}^n)$ we need only verify that $A(x)$ coincides with $DF(x)$ in the sense of distributions. Thus let $\eta\in C_0^\infty(\Omega,\mathbf{R}^n)$ be a test mapping. Integration by parts and identity (12.5) applied to Φ_j gives

$$\begin{aligned}\int_\Omega D^t\eta(x)F(x)\,dx &= \int_\Omega D^t\eta(x)\,[F(x)-x]\;dx+\int_\Omega D^t\eta(x)x\,dx\\ &=\sum_{j=1}^\infty\left\{\int_{\mathbf{B}_j} D^t\eta(x)\,[\Phi_j(x)-x]\;dx\right\}-\int_\Omega\eta(x)\,dx\\ &=-\sum_{j=1}^\infty\left\{\int_{\mathbf{B}_j}\left[D^t\Phi_j(x)-\mathrm{Id}\right]\eta(x)\,dx\right\}-\int_\Omega\eta(x)\,dx\\ &=-\sum_{j=1}^\infty\left\{\int_{\mathbf{B}_j} D^t\Phi_j(x)\eta(x)\,dx\right\}=-\int_\Omega A^t(x)\eta(x)\,dx.\end{aligned}$$

It is readily seen from Lemma 12.2 that for $K=1$, $F\in W_p^1(\Omega,\mathbf{R}^n)$, $1\leqslant p<n/2$, is weakly 1-quasiregular and that more generally $F\in W_p^1(\Omega,\mathbf{R}^n)$, $1\leqslant p<nK/(K+1)$ is weakly K-quasiregular.

However, it is easy to see $J(x,F)\leqslant -1$ a.e. in Ω. To obtain a map whose Jacobian is positive almost everywhere, compose F with an orientation reversing conformal mapping. It remains only to observe that F is not quasiregular as it is not even bounded near the boundaries of the balls in the exact packing $\mathcal{F}$.

Remark. Recall that our condition for improved regularity (in even dimensions $n=2l$) was that

$$|\mu|\,\|S\|_{pl}<1.$$

From Lemma 12.2 and a little calculation we find that in our example

$$|\mu|=\frac{K-1}{K+1}.$$

As $pl<nK/(K+1)$ we obtain the following theorem

COROLLARY 12.8. *In dimension 2l we have the following estimate for the p-norms of the Beurling–Ahlfors operator*:

$$\|S\|_p \geqslant \frac{1}{p-1}, \quad 1<p\leqslant 2.$$

By duality (*see Lemma* 8.3) *we also have*

$$\|S\|_p \geqslant p-1, \quad 2\leqslant p<\infty.$$

We have conjectured these bounds to be sharp in Conjecture 8.5.

This suggests that our example in Theorem 12.1 might be best possible (it is for $K=1$ and n even because of our sharp version of Liouville's Theorem). Indeed we make the following conjecture.

CONJECTURE 12.9. *Every weakly K-quasiregular mapping $f\in W^1_{p,\mathrm{loc}}(\Omega,\mathbf{R}^n)$ with*

$$p\geqslant \frac{nK}{K+1}$$

is K-quasiregular.

Perhaps even the weaker assumption that $f\in L^{nK}_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$ is weakly K-quasiregular implies f is K-quasiregular. Notice that in the example above we did actually show that $F\in L^p(\Omega,\mathbf{R}^n)$ for all $p<nK$.

The following theorem (which seems well known) shows that (at least in the plane) Theorem 5 is qualitatively best possible.

THEOREM 12.9. *For each $0<\varepsilon<1$, there is a bounded K-quasiregular mapping $f:\mathbf{R}^2\setminus E\to\mathbf{R}^2$, where $H_{1-\varepsilon}(E)=0$, and f does not extend over E. Moreover as $\varepsilon\to 0$, $K\to 1$.*

Proof. Recall a theorem of Tukia which states that for each $\varepsilon>0$ there is a quasisymmetric map $g:\mathbf{R}\to\mathbf{R}$ and a set E of Hausdorff dimension less than $1-\varepsilon$, with the linear measure of $g(E)>0$ [Tu]. Moreover, as $\varepsilon\to 0$ the quasisymmetry constant tends to 1. We can then extend this map to a quasiconformal map of the entire plane with the same properties, say via the Beurling–Ahlfors extension. Given such a set E and quasiconformal g there is a bounded holomorphic function h defined in the complement of $g(E)$ which is not extendable over $g(E)$. Here it is important that $g(E)$ lies on the line, because it implies that the analytic capacity is positive (the existence of such h does not follow for general sets of positive linear measure [Ga]). The composition $h\circ g$ defined in the complement of E is bounded and quasiregular and E is not a removable set. Notice too that the dilatation tends to 1 as the Hausdorff dimension of E approaches 1.

The following example gives a general method in all dimensions to construct sets E and quasimeromorphic mappings defined in the complement of E which cannot be extended over any point of E. Here by *quasimeromorphic* we simply mean that the mapping may assume the value infinity. That is we consider the mapping to be quasiregular and valued in the sphere S^n.

THEOREM 12.10. *There are sets E of arbitrarily small Hausdorff dimension (E can even be a point) and quasimeromorphic mappings $S^n\setminus E\to S^n$ which cannot be extended continuously to any point of E.*

Proof. Let Γ be a compact type, torsion free group of conformal transformations acting on S^n and $E=L(\Gamma)$ the limit set of Γ. That is $(S^n\setminus L(\Gamma))/\Gamma=M^n$ is a compact orientable manifold. Such a group may be obtained as an index two subgroup of a group generated by reflections in a collection of disjoint round spheres with the property that no sphere separates the collection. It is well known that the Hausdorff dimension of $L(\Gamma)>0$, but can be arbitrarily small. (We could just take the Poincaré extension to the appropriate dimension of a Fuchsian group of the second kind. The dimension could also be quite large, for instance any round sphere of codimension 2 or more is possible.) If we want E to be one or two points, we take Γ to be a (Euclidean) crystallographic group or simply the group generated by a dilation $\Gamma=\langle x\to\lambda x\rangle$, where $\lambda>1$ (in these latter two cases we view S^n as the one point compactification of $\mathbf{R}^n$). As M is a compact n-manifold (it will be a simple handle body if one uses the sphere construction, possibly an n-torus in the Euclidean crystallographic case, or $S^{n-1}\times S^1$ in the case of a dilation) we can use the Whitney trick as follows: take any triangulation of M, take the barycentric subdivision, decompose S^n as the union of two simplices $\Delta_1,\Delta_2,\operatorname{int}(\Delta_1)\cap\operatorname{int}(\Delta_2)=\varnothing$, and then identify each simplex of the subdivision with a Δ_i via a piecewise linear map, this produces a piecewise linear map $M\to S^n$. Now the projection $S^n/L(\Gamma)\to M$ is locally conformal and then it is easy to see the composition

$$S^n/L(\Gamma)\to M\to S^n$$

is quasimeromorphic. (The branch set of the mapping is just the codimension two skeleton of Δ_1.) This map is automorphic with respect to Γ and not constant. Since the orbit of any point under the group Γ accumulates at every point of $L(\Gamma)$, the image of any neighbourhood of any limit point covers the sphere. It is quite clear that this map has no continuous extension.

We point out that the idea of using the Whitney trick to construct quasimeromorphic mappings is not new. For related constructions see for instance [MS], [Tu2] and [Pe].

Remark 12.11. The above example raises a couple of questions. By refining the triangulation of M can we make the dilatation smaller? In particular how close to 1 can we get? Notice that in dimension 2 there is a meromorphic map $M^2 \to S^2$! The point here is that by the Stability Theorem for quasiregular mappings [R3], dilatation close to 1 implies local injectivity (which is not the case for our examples). Quasiregular (and even quasimeromorphic) mappings which are locally injective and defined in the complement of a nice set are injective if the dilatation is small enough [MSa]. For instance if the complement is a uniform domain. Thus in dimension $n \geqslant 3$ it may be that any set of dimension $d \leqslant n-2$ is removable for quasimeromorphic mappings with dilatation close enough to 1 (it is a conjecture, attributed to O. Martio, that there is no branching if $K<2$). This at least shows that a codimension 2 (or larger) set which is not removable for all bounded K-quasiregular mappings, $K>1$ must be quite wild.

Also, we could arrange that M is noncompact by introducing parabolic elements into Γ (or even M could be $(S^1)^m \times \mathbf{R}^{n-m}$, $m \geqslant 0$ in the Euclidean case). Is it possible that the noncompact manifold $M' = (S^n \setminus L(\Gamma'))/\Gamma'$ admits a quasiregular mapping into $\mathbf{R}^n$ (bounded?)? This is the case in the plane (take a Riemann surface F, delete a Cantor set of large enough capacity so that $F \setminus E$ has bounded holomorphic functions, and take a Schottky uniformisation of $F \setminus E$. One may do this by assuming that F is the double of say F_1, then uniformise $F_1 \setminus E$ via a Fuchsian group (of the second kind) and extend the group to the Riemann sphere by reflection. This is a Schottky uniformisation of F minus two copies of E). The problem with compact M is that quasiregular mappings are open and so the image must be onto S^n. We also point out that one may produce more exotic examples by looking at the orbit spaces of uniformly quasiconformal groups.

Related and perhaps the most interesting examples of quasiregular mappings, are due to S. Rickman in his study of the Big Picard Theorem in higher dimensions, see for instance [Ri1,2]. We have also very recently been informed by Rickman that he has a construction of a bounded quasiregular mapping defined in the complement of certain Cantor sets in $\mathbf{R}^n$. This Cantor set is then not removable. This example is very important as it is the only known example in higher dimensions of a bounded quasiregular mapping defined in the complement of *any* set of dimension less than $n-1$ which is not removable, see [Ri3].

Appendix: The 4-dimensional case

It is both rewarding and illuminating to discuss what we have done and to make some explicit calculations in the special case of dimension 4. Let us first observe that in the two dimensional case, the multiplier of $S\colon L^2(\mathbf{R}^2, \bigwedge^1) \to L^2(\mathbf{R}^2, \bigwedge^1)$ can be factored into the two odd multipliers

$$|\xi|^2 M_{\#}(\xi) = \begin{pmatrix} \xi_2^2-\xi_1^2 & -2\xi_1\xi_2 \\ -2\xi_1\xi_2 & \xi_1^2-\xi_2^2 \end{pmatrix} = -\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}.$$

In dimension 4, the corresponding 6×6 matrix $|\xi|^2 M_{\#}(\xi)$ of quadratic polynomials is not factorisable in such a fashion. However, to obtain such a decomposition we factor the multiplier of the operator

$$S\otimes \mathrm{Id}: L^2(\mathbf{R}^4, \textstyle\bigwedge^-)\otimes L^2(\mathbf{R}^4) \to L^2(\mathbf{R}^4, \textstyle\bigwedge^+)\otimes L^2(\mathbf{R}^4).$$

(Here the tensor product is defined by $S\otimes\mathrm{Id}(\omega,\lambda)=(S\omega,\lambda)$ for $\omega\in L^2(\mathbf{R}^4,\bigwedge^-)$ and $\lambda\in L^2(\mathbf{R}^4)$.) The following three 2-forms are a basis for $\bigwedge^-$:

$$\sigma^1 = dx^1\wedge dx^2 + dx^3\wedge dx^4, \quad \sigma^2 = dx^1\wedge dx^3 - dx^2\wedge dx^4 \quad \text{and} \quad \sigma^3 = dx^1\wedge dx^4 + dx^2\wedge dx^3.$$

The corresponding basis in $\bigwedge^+$ is

$$\tau^1 = dx^1\wedge dx^2 - dx^3\wedge dx^4, \quad \tau^2 = dx^1\wedge dx^3 + dx^2\wedge dx^4 \quad \text{and} \quad \tau^3 = dx^1\wedge dx^4 - dx^2\wedge dx^3.$$

We then write $\omega = a_1\sigma^1 + a_2\sigma^2 + a_3\sigma^3$ and $S\omega = b_1\tau^1 + b_2\tau^2 + b_3\tau^3$. To express the coefficients $\{b_i\}$ in terms of the $\{a_i\}$ we solve the differential system

$$d^-\alpha = \omega \quad \text{and} \quad \delta\alpha = \lambda \tag{A1}$$

for $\alpha = \alpha^1 dx^1 + \alpha^2 dx^2 + \alpha^3 dx^3 + \alpha^4 dx^4 \in L^2(\mathbf{R}^4, \bigwedge^1)$. Notice that we have introduced the auxiliary equation $\delta\alpha = \lambda$ to make the system well determined. Our system is equivalent to (subscripts as usual denote differentiation)

$$\begin{aligned} a_1 &= -\alpha_2^1 + \alpha_1^2 + \alpha_4^3 - \alpha_3^4 \\ a_2 &= -\alpha_3^1 - \alpha_4^2 + \alpha_1^3 + \alpha_2^4 \\ a_3 &= -\alpha_4^1 + \alpha_3^2 - \alpha_2^3 + \alpha_1^4 \\ \lambda &= \alpha_1^1 + \alpha_2^2 + \alpha_3^3 + \alpha_4^4. \end{aligned}$$

In the same manner, the equations

$$d^+\alpha = S\omega \quad \text{and} \quad \delta\alpha = \lambda \tag{A2}$$

lead to

$$\begin{aligned} b_1 &= \alpha_2^1 + \alpha_1^2 - \alpha_4^3 + \alpha_3^4 \\ b_2 &= -\alpha_3^1 + \alpha_4^2 + \alpha_1^3 - \alpha_2^4 \\ b_3 &= -\alpha_4^1 - \alpha_3^2 + \alpha_2^3 + \alpha_1^4 \\ \lambda &= \alpha_1^1 + \alpha_2^2 + \alpha_3^3 + \alpha_4^4. \end{aligned}$$

Taking Fourier transforms, the corresponding matrices of the systems are the quaternions

$$\begin{pmatrix} -\xi_2 & \xi_1 & \xi_4 & -\xi_3 \\ -\xi_3 & -\xi_4 & \xi_1 & \xi_2 \\ -\xi_4 & \xi_3 & -\xi_2 & \xi_1 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \alpha^4 \end{pmatrix}^{\wedge} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \lambda \end{pmatrix}^{\wedge} \tag{A3}$$

and

$$\begin{pmatrix} -\xi_2 & \xi_1 & -\xi_4 & \xi_3 \\ -\xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ -\xi_4 & -\xi_3 & \xi_2 & \xi_1 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \alpha^4 \end{pmatrix}^{\wedge} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \lambda \end{pmatrix}^{\wedge}. \tag{A4}$$

Then the Fourier multiplier for the operator $S\otimes\mathrm{Id}$ is obtained by eliminating $\alpha^1, \alpha^2, \alpha^3$ and α^4 in the above. We find

$$\begin{aligned} &|\xi|^{-2}P(\xi)Q(\xi) \\ &= |\xi|^{-2} \begin{pmatrix} \xi_1^2+\xi_2^2-\xi_3^2-\xi_4^2 & -2\xi_1\xi_4+2\xi_2\xi_3 & 2\xi_2\xi_4+2\xi_1\xi_3 & 0 \\ 2\xi_1\xi_4+2\xi_2\xi_3 & \xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2 & 2\xi_3\xi_4-2\xi_1\xi_2 & 0 \\ 2\xi_2\xi_4-2\xi_1\xi_3 & 2\xi_3\xi_4+2\xi_1\xi_2 & \xi_1^2-\xi_2^2-\xi_3^2+\xi_4^2 & 0 \\ 0 & 0 & 0 & \xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2 \end{pmatrix}, \end{aligned}$$

where $P(\xi)$ is the quaternion in (A4) and $Q(\xi)$ is the transpose of the quaternion in (A3). This is very suggestive as to the form in the general case. Our calculations also show that the multiplier is the product of two odd multipliers (both of which are orthogonal matrices). The point is that $S\otimes\mathrm{Id}$ can be factored into two singular integral operators with odd kernels. These are analogues of the complex Riesz tranforms in the plane

$$T_1 f(z) = \frac{i}{4\pi}\int_{\mathbf{C}} \frac{f(\zeta)\, d\zeta\wedge d\bar{\zeta}}{|z-\zeta|(z-\zeta)}$$

and

$$T_2 f(z) = \frac{i}{4\pi}\int_{\mathbf{C}} \frac{f(\zeta)\, d\zeta\wedge d\bar{\zeta}}{|z-\zeta|(\bar{z}-\bar{\zeta})}.$$

Now we can apply the method of rotations to each of these operators individually to get an estimate of the p-norm of $S\colon L^2(\mathbf{R}^4, \bigwedge^2) \to L^2(\mathbf{R}^4, \bigwedge^2)$. We hope that these ideas will produce a general method for obtaining a dimension free estimate of $\|S\|_p$. Finally notice that the L^p-estimates for S of §8 mean in this case that

$$\int_{\mathbf{R}^4} (|a_1|^2+|a_2|^2+|a_3|^2)^q \leqslant [A_p(4)]^{2q} \int_{\mathbf{R}^4} (|b_1|^2+|b_2|^2+|b_3|^2)^q$$

$q > \frac{1}{2}$, for arbitrary C_0^∞ functions $\alpha^1, \alpha^2, \alpha^3, \alpha^4$ where a_1, a_2, a_3 are given by (A1) and b_1, b_2, b_3 are given by (A2). The uniform estimate implies the same inequality is true in $W^1_{2q}(\mathbf{R}^4)$.

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Received April 3, 1990