# QUASISTATIC EVOLUTION FOR CAM-CLAY PLASTICITY: A WEAK FORMULATION VIA VISCOPLASTIC REGULARIZATION AND TIME RESCALING

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Abstract. Cam-Clay nonassociative plasticity exhibits both hardening and softening behaviour, depending on the loading. For many initial data the classical formulation of the quasistatic evolution problem has no smooth solution. We propose here a notion of generalized solution, based on a viscoplastic approximation. To study the limit of the viscoplastic evolutions we rescale time, in such a way that the plastic strain is uniformly Lipschitz with respect to the rescaled time. The limit of these rescaled solutions, as the viscosity parameter tends to zero, is characterized through an energy-dissipation balance, that can be written in a natural way using the rescaled time. As shown in [4] and [6], the proposed solution may be discontinuous with respect to the original time. Our formulation allows to compute the amount of viscous dissipation occurring instantaneously at each discontinuity time.

**Keywords:** Cam-Clay plasticity, softening behaviour, nonassociative plasticity, pressure-sensitive yield criteria, quasistatic evolution, rate independent dissipative processes, vanishing viscosity limit.
**2000 Mathematics Subject Classification:** 74C05, 74L10, 74G65, 49L25, 35K55.

## Contents



## 1. Introduction

Cam-Clay is a plasticity model giving the conceptual framework to analyse the inelastic behaviour of fine grained soils. Some of its interesting features are its nonassociative character, and that it may lead to both hardening and softening behaviour, depending on the loading conditions. The variables considered in the model are the displacement $u(t,x)$, the elastic and plastic strains $e(t,x)$ and $p(t,x)$, the stress $\sigma(t,x)$, and the internal variables

Preprint SISSA 36/2009/M (July 2009).

$z(t,x)$ and $\zeta(t,x)$. All these functions are defined for positive time $t$ and for $x$ in the reference configuration $\Omega$, a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. As it is typical in plasticity, the stress is constrained to lie in a compact convex set $K(\zeta)$ of the space $\mathbb{M}^{n\times n}_{sym}$ of symmetric $n\times n$ matrices, whose size is controlled by a scalar parameter $\zeta$ and whose boundary represents the yield surface. Given a time-dependent body force $f(t,x)$, and denoting the normal cone to $K(\zeta)$ at $\sigma$ by $N_{K(\zeta)}(\sigma)$, the equations summarising the model are

(a) constitutive equations: $\sigma(t,x) = \mathbb{C}e(t,x)$ and $\zeta(t,x) = V(z(t,x))$,
(b) additive decomposition: $Eu(t,x) = e(t,x) + p(t,x)$,
(c) equilibrium condition : $-\mathrm{div}\,\sigma(t,x) = f(t,x)$,
(d) stress constraint: $\sigma(t,x) \in K(\zeta(t,x))$,
(e) flow rule: $\dot p(t,x) \in N_{K(\zeta(t,x))}(\sigma(t,x))$,
(f) evolution law for the internal variable: $\dot z(t,x) = \varrho \star [(\varrho \star \mathrm{tr}\,\sigma(t,\cdot))\,\mathrm{tr}\,\dot p(t,\cdot)](x)$,

accompanied by suitable boundary conditions. In the previous equations, $\mathbb{C}$ is the isotropic elasticity tensor (see (2.8)), $V$ is a nondecreasing globally Lipschitz function (see (2.45)), div is the divergence operator with respect to the space variable $x$, and $\varrho$ is a smooth convolution kernel (see (2.47)). In the typical applications, $\partial K(\zeta)$ are homothetic ellipsoids passing through the origin in the space $\mathbb{M}^{n\times n}_{sym}$; more in general we assume that $0 \in K(\zeta)$ and

$$K(\zeta) := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : (\sigma,\zeta) \in K\},$$

where $K$ is a convex cone in $\mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ with nonempty interior.

The above formulation contains two differences with respect to the classical one, where $V(z) = z$ and the convolution kernel is not present in the evolution law for the internal variable. The main reason for introducing the convolution is technical: it ensures that a very weak convergence of $\sigma$ and $\dot p$ implies strong convergence of the corresponding $\boldsymbol{z}$. From the point of view of mechanics, the convolution gives a nonlocal character to the evolution law for the internal variable: it implies that the size of the yield surface at a point $x$ is affected by pressure and volumetric plastic strain rate in a small neighborhood of $x$, which is not physically implausible.

The function $V$ is assumed to satisfy the condition

$$V(z) \geq \zeta_{min} > 0 \quad \text{for every } z \in \mathbb{R},$$

which implies that $\zeta(t,x) \geq \zeta_{min} > 0$ and prevents that the set $K(\zeta(t,x))$ shrinks to the origin. The classical case $V(z) = z$ is recovered whenever the solution $z(t,x)$ to the evolution law is positive and bounded away from $0$.

The study of the spatially homogeneous case (see [4] and [6]) shows that, for many initial data, the problem has no smooth solutions. The aim of this paper is then to introduce a notion of generalized solution, based on a viscoplastic approximation of Perzyna-type. Given a viscosity parameter $\varepsilon > 0$, the corresponding viscoplastic evolution $u_\varepsilon(t,x)$, $e_\varepsilon(t,x)$, $p_\varepsilon(t,x)$, $z_\varepsilon(t,x)$, $\sigma_\varepsilon(t,x)$, $\zeta_\varepsilon(t,x)$ satisfies (a), (b), (c), and (f); condition (d) is dropped, while (e) is replaced by

($\mathrm{e}_\varepsilon$) regularized flow rule: $\dot p(t,x) = N^\varepsilon_{K(\zeta(t,x))}(\sigma(t,x))$,

where $N^\varepsilon_K(\sigma,\zeta) := \frac{1}{\varepsilon}\big(\sigma - \pi_{K(\zeta)}(\sigma)\big)$ and $\pi_{K(\zeta)}$ is the projection onto $K(\zeta)$. In Section 3 we prove the existence of such an evolution. We first prove (Theorem 3.3) that for every function $\zeta(t,x)$ in a suitable function space there exists a solution $u^\zeta_\varepsilon(t,x)$, $e^\zeta_\varepsilon(t,x)$, $p^\zeta_\varepsilon(t,x)$, $\sigma^\zeta_\varepsilon(t,x)$ of (a), (b), (c), and ($\mathrm{e}_\varepsilon$), adapting a result obtained by Suquet [29]. Then we prove the existence of a viscoplastic evolution by a fixed point argument (Theorem 3.5).

An energy estimate (Theorem 3.4) allows to prove the existence of change of variables $t = t^\circ_\varepsilon(s)$, uniformly Lipschitz with respect to $s$, such that the rescaled functions $p^\circ_\varepsilon(s,x) := p_\varepsilon(t_\varepsilon(s),x)$ are uniformly Lipschitz with respect to $s$, in a suitable function space. This idea has already been used in [8, 17, 18] for rate independent dissipative problems in finite dimension. The authors of the last two papers have used the same idea to study a similar problem in infinite dimension [22].

The Ascoli-Arzelà Theorem provides the existence of a subsequence (not relabelled), such that

$$t^\circ_\varepsilon(s) \to t^\circ(s) \quad \text{and} \quad p^\circ_\varepsilon(s,\cdot) \rightharpoonup p^\circ(s,\cdot)\,,$$

the latter in a weak topology. A further argument, based on the uniqueness of the solution to an auxiliary variational problem, shows that

$$e^\circ_\varepsilon(s,\cdot) \rightharpoonup e^\circ(s,\cdot)\,, \quad u^\circ_\varepsilon(s,\cdot) \rightharpoonup u^\circ(s,\cdot)\,, \quad \sigma^\circ_\varepsilon(s,\cdot) \rightharpoonup \sigma^\circ(s,\cdot)\,.$$

The compactness ensured by the presence of the convolutions in the evolution law for the internal variable allows to prove that

$$z^\circ_\varepsilon(s,x) \to z^\circ(s,x) \quad \text{and} \quad \zeta^\circ_\varepsilon(s,x) \to \zeta^\circ(s,x)\,,$$

uniformly with respect to $x$. It is then easy to see that (a), (b), (c) are satisfied by the limit functions (Section 5). As for (f), it holds only in a weak form since, in general, the limit $p^\circ(s,\cdot)$ is just a measure and this requires an *ad-hoc* definition for the derivative (Section 6).

Condition (d) is satisfied in the limit for those values of $s$ for which $t^\circ(s)$ is not locally constant. Condition (e) is replaced by an energy-dissipation balance (see (4.8)) and a partial flow rule (see (4.9)). The former is similar to the energy-dissipation balance of perfect plasticity [5] with two main differences: first, the set $K$, and hence the plastic dissipation, depend now on $\zeta^\circ(s,x)$; second, there is an additional dissipative term,

$$\int_0^S \int_\Omega \big(\sigma^\circ(s,x) - \pi_{K(\zeta^\circ(s,x))}(\sigma^\circ(s,x))\big) : \dot p^\circ(s,x)\,dx\,ds\,, \tag{1.1}$$

which accounts for viscous dissipation in those intervals where $t^\circ(s)$ is locally constant (the colon denotes the scalar product between matrices). A similar term appears in [17], where an evolution problem with nonconvex energy is studied through a viscosity approximation and time rescaling.

To understand the meaning of this term, we observe that the convergence properties listed above allow to prove (Lemma 5.3) that

$$u_\varepsilon(t,\cdot) \rightharpoonup u(t,\cdot)\,, \quad e_\varepsilon(t,\cdot) \rightharpoonup e(t,\cdot)\,, \quad p_\varepsilon(t,\cdot) \rightharpoonup p(t,\cdot)\,, \quad z_\varepsilon(t,\cdot) \to z(t,\cdot)\,, \tag{1.2}$$

for all $t$ except for a countable subset, with $u^\circ(s,\cdot)) = u(t^\circ(s),\cdot), \dots, z^\circ(s,\cdot) = z(t^\circ(s),\cdot)$. The intervals where $t^\circ(s)$ is locally constant correspond to times $t$ where the limit evolution $u(t,\cdot)$, $e(t,\cdot)$, $p(t,\cdot)$, $z(t,\cdot)$ may be discontinuous and (1.1) measures the sum of the instantaneous dissipations due to viscous effects, which survive in the limit as the viscosity parameter $\varepsilon$ tends to zero.

The partial flow rule (4.9) shows that the rate of plastic strain is parallel to $\sigma^\circ(s,x) - \pi_{K(\zeta^\circ(s,x))}(\sigma^\circ(s,x))$ for those values of $s$ where the stress constraint (d) is not satisfied for a.e. $x \in \Omega$. The proof of the energy-dissipation balance (4.8) and of the partial flow rule (4.9) is given in the last three sections of the paper. One inequality (see (6.1)) is proved in Section 6 passing to the limit in the energy balance (3.25) for the viscoplastic evolutions by means of a lower semicontinuity argument. The opposite inequality (see (8.2)) is proved in Section 8 using the properties of the limit functions $p^\circ(s,\cdot)$, $e^\circ(s,\cdot)$, and $\zeta^\circ(s,\cdot)$ and some technical approximation arguments developed in Section 7.

In this paper we do not consider the following problems.

- Deduce from the energy-dissipation balance a weak formulation of the flow rule $\dot p^\circ(s,x) \in N_{K(\zeta^\circ(s,x))}(\sigma^\circ(s,x))$ for almost all $s$ where $t^\circ(s)$ is not locally constant.
- Characterize the limit functions $u(t,\cdot)$, $e(t,\cdot)$, $p(t,\cdot)$, $z(t,\cdot)$ defined by (1.2) through an energy-dissipation balance in terms of the original variable $t$.

We plan to address these interesting issues in a forthcoming paper.

## 2. Preliminaries

**Mathematical preliminaries.** The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. If $X \subset \mathbb{R}^n$ is locally compact and $\Xi$ is a finite dimensional Hilbert space, the space of bounded $\Xi$-valued Radon measures on $X$ is denoted by $M_b(X;\Xi)$. When $\Xi = \mathbb{R}$, it is omitted from the notation. The space $M_b(X;\Xi)$ is endowed with the norm $\|\mu\|_1 := |\mu|(X)$, where $|\mu| \in M_b(X)$ is the variation of the measure $\mu$. By the Riesz Representation Theorem (see, e.g., [23, Theorem 6.19]) $M_b(X;\Xi)$ is identified with the dual of $C_0^0(X;\Xi)$, the space of continuous functions $\varphi\colon X \to \Xi$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. This defines the weak$^*$ topology in $M_b(X;\Xi)$.

The space $L^1(X;\Xi)$ of $\xi$-valued $\mathcal{L}^n$-integrable functions is regarded as a subspace of $M_b(X;\Xi)$, with the induced norm. The $L^p$ norm, $1 \leq p \leq \infty$ is denoted by $\|\cdot\|_p$. We adopt the convention

$$\|v\|_p = +\infty \quad \text{whenever } v \notin L^p\,. \tag{2.1}$$

The brackets $\langle\cdot,\cdot\rangle$ denote the duality product between conjugate $L^p$ spaces, as well as between other pairs of spaces, according to the context.

The space of *symmetric* $n\times n$ *matrices* is denoted by $\mathbb{M}^{n\times n}_{sym}$; it is endowed with the euclidean scalar product $\xi\colon\eta := \sum_{ij}\xi_{ij}\eta_{ij}$ and with the corresponding euclidean norm $|\xi| := (\xi\colon\xi)^{1/2}$. The *symmetrized tensor product* $a \odot b$ of two vectors $a$, $b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j + a_jb_i)/2$.

For every $u \in L^1(U;\mathbb{R}^n)$, with $U$ open in $\mathbb{R}^n$, let $Eu$ be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on $U$ whose components are defined by $E_{ij}u = \frac{1}{2}(D_ju_i + D_iu_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u \in L^1(U;\mathbb{R}^n)$ such that $Eu \in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm $\|u\|_1 + \|Eu\|_1$. It is possible to prove that $BD(U)$ is the dual of a normed space (see [14] and [26]), and this defines the weak$^*$ topology of $BD(U)$. A sequence $u_k$ converges to $u$ weakly$^*$ in $BD(U)$ if and only if $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$ and $Eu_k \overset{*}{\rightharpoonup} Eu$ weakly$^*$ in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. For the general properties of $BD(U)$ we refer to [25]. If $U$ is a bounded open set with Lipschitz boundary, then

$$u \in L^1(U;\mathbb{R}^n) \text{ and } Eu \in L^2(U;\mathbb{M}^{n\times n}_{sym}) \quad\Longrightarrow\quad u \in H^1(U;\mathbb{R}^n)\,. \tag{2.2}$$

This can be obtained arguing as in the proof of [25, Chapter I, Proposition 1.1].

We will use boldface letters to denote functions defined in an interval $[a,b] \subset \mathbb{R}$ and with values in a possibly infinite dimensional Banach space $Y$. We recall that a function $\boldsymbol{f}\colon [a,b] \to Y$ is said to be absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|\boldsymbol{f}(t_i) - \boldsymbol{f}(s_i)\|_Y < \varepsilon$, whenever $a \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_k < t_k \leq b$ and $\sum_i(t_i - s_i) < \delta$. The space of these functions is denoted by $AC([0,T];Y)$. For the general properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [1, Appendix]. When $Y$ is the dual of a separable Banach space, one can prove (see [5, Theorem 7.1]) that the weak$^*$ limit

$$\dot{\boldsymbol{f}}(t) := w^*\text{-}\lim_{s\to t}\frac{\boldsymbol{f}(s) - \boldsymbol{f}(t)}{s-t} \tag{2.3}$$

exists for $\mathcal{L}^1$-a.e. $t \in [a,b]$. Note that in this general situation it may happen that $\dot{\boldsymbol{f}}$ is not Bochner integrable. If $\varphi\colon [c,d] \to [a,b]$ is nondecreasing and absolutely continuous, then the function $\boldsymbol{g}(s) := \boldsymbol{f}(\varphi(s))$ is absolutely continuous and

$$\dot{\boldsymbol{g}}(s) = \hat{\boldsymbol{f}}(\varphi(s))\dot{\varphi}(s) \quad \text{for } \mathcal{L}^1\text{-a.e.} \quad s \in [c,d], \tag{2.4}$$

where $\hat{\boldsymbol{f}}(t) = \dot{\boldsymbol{f}}(t)$ if the derivative (2.3) exists, while $\hat{\boldsymbol{f}}(t) = 0$ otherwise. It follows that

$$\int_c^d \boldsymbol{h}(\varphi(s))\dot{\varphi}(s)\,ds = \int_{\varphi(c)}^{\varphi(d)} \boldsymbol{h}(t)\,dt \tag{2.5}$$

for every $\boldsymbol{h} \in L^1([a,b];Y)$. Indeed, the derivatives with respect to $d$ of both sides in (2.5) coincide $\mathcal{L}^1$-a.e. by (2.4).

**The reference configuration.** Throughout the paper the *reference configuration* $\Omega$ is a *bounded connected open set* in $\mathbb{R}^n$, $n \geq 2$, with *Lipschitz boundary* $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup N$. We assume that $\Gamma_0$ and $\Gamma_1$ are relatively open, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, and $\mathcal{H}^{n-1}(N) = 0$.

On $\Gamma_0$ we will prescribe a Dirichlet boundary condition. This will be done by assigning a function $w \in H^{1/2}(\partial\Omega;\mathbb{R}^n)$, or, equivalently, a function $w \in H^1(\Omega;\mathbb{R}^n)$, whose trace on $\Gamma_0$ (also denoted by $w$) is the prescribed boundary value. The set $\Gamma_1$ will be the part of the boundary on which the traction is prescribed.

We shall frequently use the following closed linear subspace of $H^1(\Omega;\mathbb{R}^n)$:

$$H^1_{\Gamma_0}(\Omega;\mathbb{R}^n) := \{u \in H^1(\Omega;\mathbb{R}^n) : u = 0 \ \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_0\}\,. \tag{2.6}$$

**Stress and strain.** For a given displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\Omega;\mathbb{R}^n)$, the *elastic* and *plastic strains* $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_{sym})$ satisfy the *weak kinematic admissibility condition*

$$\begin{gathered} Eu = e + p \text{ in } \Omega\,, \\ p = (w-u) \odot \nu\, \mathcal{H}^{n-1} \text{ on } \Gamma_0\,, \end{gathered} \tag{2.7}$$

where $\nu$ is the outer unit normal to $\partial\Omega$. The stress $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is defined by

$$\sigma := \mathbb{C}e\,, \tag{2.8}$$

where $\mathbb{C}$ is the *elasticity tensor*, considered as a symmetric positive definite linear operator $\mathbb{C}\colon \mathbb{M}^{n\times n}_{sym} \to \mathbb{M}^{n\times n}_{sym}$. We assume that $\mathbb{C}$ is isotropic, so that we have $\mathbb{C}\xi = 2\mu\xi + \lambda(\mathrm{tr}\xi)I$, where $\lambda$ and $\mu$ are the Lamé constants. Let $Q\colon \mathbb{M}^{n\times n}_{sym} \to [0,+\infty)$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi$$

It turns out that there exist two constants $\alpha_Q$ and $\beta_Q$, with $0 < \alpha_Q \leq \beta_Q < +\infty$, such that

$$\alpha_Q|\xi|^2 \leq Q(\xi) \leq \beta_Q|\xi|^2 \tag{2.9}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$. These inequalities imply

$$|\mathbb{C}\xi| \leq 2\beta_Q|\xi|\,. \tag{2.10}$$

The stored elastic energy $\mathcal{Q} : L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$ is given by

$$\mathcal{Q}(e) = \int_\Omega Q(e(x))\,dx = \tfrac{1}{2}\langle\sigma, e\rangle\,.$$

It is well known that $\mathcal{Q}$ is lower semicontinuous on $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ with respect to weak convergence.

If $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^n)$, then we can define a distribution $[\sigma\nu]$ on $\partial\Omega$ by

$$\langle[\sigma\nu],\psi\rangle_{\partial\Omega} := \langle\operatorname{div}\sigma,\psi\rangle_\Omega + \langle\sigma, E\psi\rangle_\Omega \tag{2.11}$$

for every $\psi \in H^1(\Omega;\mathbb{R}^n)$. It turns out that $[\sigma\nu] \in H^{-1/2}(\partial\Omega;\mathbb{R}^n)$ (see, e.g., [25, Chapter I, Theorem 1.2]). If, in addition, $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, then (2.11) holds for $\psi \in W^{1,1}(\Omega;\mathbb{R}^n)$. By Gagliardo's extension result [10, Theorem 1.II], it is easy to see that in this case $[\sigma\nu] \in L^\infty(\partial\Omega;\mathbb{R}^n)$ and that

$$[\sigma_k\nu] \rightharpoonup [\sigma\nu] \quad \text{weakly}^* \text{ in } L^\infty(\partial\Omega;\mathbb{R}^n), \tag{2.12}$$

whenever $\sigma_k \rightharpoonup \sigma$ weakly$^*$ in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^n)$.

Let $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ such that (2.7) holds. According to [3, Section 3], for every $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ with $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$ we can define the distribution $[\sigma : p]$ on $\Omega$ by setting

$$\langle[\sigma : p], \varphi\rangle := -\langle\varphi u, \operatorname{div}\sigma\rangle - \langle\sigma, u \odot \nabla\varphi\rangle - \langle\sigma, \varphi e\rangle$$

for every $\varphi \in C^\infty_c(\Omega)$. It turns out that $[\sigma : p]$ does not depend on the functions $u$, $w$, $e$ satisfying (2.7), and that $[\sigma : p]$ is a bounded Radon measure on $\Omega$. As in [3, Section 3] we extend the definition of $[\sigma : p]$ by setting

$$[\sigma : p] := [\sigma\nu]\cdot(w-u)\,\mathcal{H}^{n-1} \quad \text{on } \Gamma_0\,,$$

so that $[\sigma : p] \in M_b(\Omega\cup\Gamma_0)$, and we define

$$\langle\sigma, p\rangle := [\sigma : p](\Omega\cup\Gamma_0). \tag{2.13}$$

It follows from [3, formula (3.9)] that

$$|\langle\sigma, p\rangle| \le \|\sigma\|_\infty\|p\|_1. \tag{2.14}$$

The following proposition provides a useful integration-by-parts formula.

**Proposition 2.1.** *Let $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$, $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$. Assume that* (2.7) *holds, $-\operatorname{div}\sigma = f$ in $\Omega$, and that $[\sigma\nu] = g$ on $\Gamma_1$. Then*

$$\langle[\sigma : p], \varphi\rangle + \langle\varphi\sigma, e - Ew\rangle + \langle\sigma, (u-w)\odot\nabla\varphi\rangle = \langle f, \varphi(u-w)\rangle_\Omega + \langle g, \varphi(u-w)\rangle_{\Gamma_1} \tag{2.15}$$

*for every $\varphi \in C^1(\overline{\Omega})$. Moreover*

$$\langle\sigma, p\rangle + \langle\sigma, e - Ew\rangle = \langle f, u-w\rangle_\Omega + \langle g, u-w\rangle_{\Gamma_1}. \tag{2.16}$$

*Proof.* See [3, Proposition 3.2]. □

The following closed linear subspace of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ will be used in our proofs:

$$\Sigma_0(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \operatorname{div}\sigma = 0 \text{ in } \Omega,\ [\sigma\nu] = 0 \text{ on } \Gamma_1\}. \tag{2.17}$$

It turns out that

$$\Sigma_0(\Omega)^\perp = \{E\varphi : \varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)\}\,, \tag{2.18}$$

since the latter space is closed in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ as a consequence of Poincaré's and Korn's inequalities.

**The constraint set and its support function.** Let $K$ be a closed convex cone in $\mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ with nonempty interior. For every $\zeta \in [0,+\infty)$ we define the closed convex set $K(\zeta)$ by

$$K(\zeta) := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : (\sigma,\zeta) \in K\}\,. \tag{2.19}$$

When $\zeta > 0$ the set $K(\zeta)$ has nonempty interior and

$$K(\zeta) = \zeta\, K(1)\,. \tag{2.20}$$

We assume that $0 \in K(1)$, hence

$$0 \in K(\zeta) \qquad \text{for every } \zeta \in [0,+\infty)\,, \tag{2.21}$$

and that

$$|\sigma| \le M_K\zeta \qquad \text{for every } (\sigma,\zeta) \in K \tag{2.22}$$

for a suitable constant $M_K < +\infty$. Since $K$ is a convex cone, (2.21) implies that

$$0 \le \zeta_1 \le \zeta_2 \quad\Longrightarrow\quad K(\zeta_1) \subset K(\zeta_2)\,. \tag{2.23}$$

For every closed convex set $C \subset \mathbb{M}^{n\times n}_{sym}$ let $\pi_C\colon \mathbb{M}^{n\times n}_{sym} \to C$ be the minimal distance projection onto $C$. It follows from (2.20) that

$$\pi_{K(\zeta)}(\sigma) = \zeta\pi_{K(1)}\big(\frac{\sigma}{\zeta}\big) \tag{2.24}$$

for every $\zeta > 0$ and every $\sigma \in \mathbb{M}^{n\times n}_{sym}$.

**Lemma 2.2.** *The map* $(\sigma,\zeta) \mapsto \pi_{K(\zeta)}(\sigma)$ *from* $\mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ *into* $\mathbb{M}^{n\times n}_{sym}$ *satisfies the Lipschitz estimate*

$$|\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_1)}(\sigma_1)| \le |\sigma_2 - \sigma_1| + 2M_K|\zeta_2 - \zeta_1| \tag{2.25}$$

*for every* $(\sigma_1,\zeta_1)$, $(\sigma_2,\zeta_2) \in \mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$.

*Proof.* See [4, Lemma 2.1]. □

Let $H\colon \mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ be defined by

$$H(\xi,\zeta) = \sup_{\sigma\in K(\zeta)} \sigma:\xi\,, \tag{2.26}$$

so that $H(\cdot,\zeta)$ is the support function of $K(\zeta)$. By (2.20) for every $(\xi,\zeta) \in \mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ we have

$$H(\xi,\zeta) = \zeta H(\xi,1)\,. \tag{2.27}$$

For every $\zeta \in [0,+\infty)$ the function $\xi \mapsto H(\xi,\zeta)$ is convex and positively one-homogeneous on $\mathbb{M}^{n\times n}_{sym}$. In particular, it satisfies the triangle inequality

$$H(\xi_1+\xi_2,\zeta) \le H(\xi_1,\zeta) + H(\xi_2,\zeta) \tag{2.28}$$

for every $\xi_1,\xi_2 \in \mathbb{M}^{n\times n}_{sym}$ and every $\zeta \in [0,+\infty)$. From (2.21), (2.22), and (2.27) it follows that

$$0 \le H(\xi,\zeta) \le M_K\zeta|\xi|\,, \tag{2.29}$$

$$|H(\xi_2,\zeta) - H(\xi_1,\zeta)| \le M_K\zeta|\xi_2-\xi_1|\,, \tag{2.30}$$

$$|H(\xi,\zeta_2) - H(\xi,\zeta_1)| \le M_K|\xi||\zeta_2-\zeta_1|\,, \tag{2.31}$$

for every $\xi,\xi_1,\xi_2 \in \mathbb{M}^{n\times n}_{sym}$ and every $\zeta,\zeta_1,\zeta_2 \in [0,+\infty)$.

**Stress constraint and plastic dissipation.** Given $\zeta \in C^0(\overline{\Omega})^+$, we define

$$\mathcal{K}(\zeta) := \{\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}) : \sigma(x) \in K(\zeta(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x\in\Omega\}\,. \tag{2.32}$$

When $\zeta$ is the internal variable, $\mathcal{K}(\zeta)$ is the corresponding stress constraint. For every closed convex set $\mathcal{C} \subset L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, let $\pi_{\mathcal{C}}\colon L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to \mathcal{C}$ be the minimal distance projection onto $\mathcal{C}$. For every $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ we define

$$d_2(\sigma,\mathcal{C}) := \|\sigma - \pi_{\mathcal{C}}(\sigma)\|_2\,, \tag{2.33}$$

the $L^2$-distance from $\sigma$ to $\mathcal{C}$. It is easy to see that, if $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, then

$$\hat\sigma = \pi_{\mathcal{K}(\zeta)}\sigma \quad\iff\quad \hat\sigma(x) = \pi_{K(\zeta(x))}(\sigma(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x\in\Omega\,. \tag{2.34}$$

Using the theory of convex functions of measures developed in [11], we introduce the functional $\mathcal{H}\colon M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})\times C^0(\overline{\Omega})^+ \to \mathbb{R}$ defined by

$$\mathcal{H}(p,\zeta) := \int_{\Omega\cup\Gamma_0} H(\tfrac{dp}{d\lambda}(x),\zeta(x))\,d\lambda(x)\,, \tag{2.35}$$

where $\lambda \in M_b(\Omega\cup\Gamma_0)^+$ is any measure such that $p \ll \lambda$; note that the homogeneity of $H$ with respect to $\xi$ implies that the integral does not depend on $\lambda$. In particular, if $p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, we have

$$\mathcal{H}(p,\zeta) = \int_\Omega H(p(x),\zeta(x))\,dx\,.$$

When $p$ is the rate of plastic strain and $\zeta$ the internal variable, $\mathcal{H}$ represents the rate of plastic dissipation.

For every $p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\zeta \in C^0(\overline{\Omega})^+$ the symbol $\partial_p\mathcal{H}(p,\zeta)$ denotes the subdifferential in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ of $\mathcal{H}(\cdot,\zeta)$ at $p$. Using [21, Corollary 23.5.3] and [9, Proposition IX.2.1] it is easy to show that

$$\partial_p\mathcal{H}(0,\zeta) = \mathcal{K}(\zeta)\,. \tag{2.36}$$

As $\mathcal{H}(p,\zeta)$ is positively homogeneous with respect to $p$ we have

$$\partial_p\mathcal{H}(p,\zeta) \subset \partial_p\mathcal{H}(0,\zeta) = \mathcal{K}(\zeta) \tag{2.37}$$

for every $p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and every $\zeta \in C^0(\overline{\Omega})^+$.

To prepare our treatment of the viscoplastic approximation, for every $\varepsilon > 0$ we introduce the function $H_\varepsilon \colon \mathbb{M}^{n\times n}_{sym}\times[0,+\infty) \to \mathbb{R}$ defined as

$$H_\varepsilon(\xi,\zeta) = H(\xi,\zeta) + \tfrac{\varepsilon}{2}|\xi|^2, \tag{2.38}$$

and the corresponding integral functional $\mathcal{H}_\varepsilon \colon L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times C^0(\overline{\Omega})^+ \to \mathbb{R}$ defined by

$$\mathcal{H}_\varepsilon(p,\zeta) := \int_\Omega H_\varepsilon(p(x),\zeta(x))\,dx\,.$$

Its subdifferential $\partial_p\mathcal{H}_\varepsilon$ with respect to $p$ satisfies the equality

$$\partial_p\mathcal{H}_\varepsilon(p,\zeta) = \partial_p\mathcal{H}(p,\zeta) + \varepsilon p \tag{2.39}$$

for every $(p,\zeta) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times C^0(\overline{\Omega})^+$.

The convex conjugate $H^*_\varepsilon \colon \mathbb{M}^{n\times n}_{sym}\times[0,+\infty) \to \mathbb{R}$ of $H_\varepsilon$ with respect to $\xi$ is defined by

$$H^*_\varepsilon(\sigma,\zeta) := \sup_{\xi\in\mathbb{M}^{n\times n}_{sym}} \{\sigma : \xi - H_\varepsilon(\xi,\zeta)\}\,.$$

Since the convex conjugate $H^*$ of $H$ with respect to $\xi$ satisfies $H^*(\sigma,\zeta) = 0$ for $\sigma \in K(\zeta)$ and $H^*(\sigma,\zeta) = +\infty$ for $\sigma \notin K$ (see [21, Theorem 13.2]), using [21, Theorem 16.4] one can prove that

$$H^*_\varepsilon(\sigma,\zeta) = \tfrac{1}{2\varepsilon}|\sigma - \pi_{K(\zeta)}(\sigma)|^2\,. \tag{2.40}$$

This implies that $H^*_\varepsilon$ is differentiable with respect to $\sigma$, and that its gradient is given by

$$\partial_\sigma H^*_\varepsilon(\sigma,\zeta) = N^\varepsilon_K(\sigma,\zeta) := \tfrac{1}{\varepsilon}\big(\sigma - \pi_{K(\zeta)}(\sigma)\big)\,. \tag{2.41}$$

Note that $N^\varepsilon_K(\sigma,\zeta)$ is Lipschitz continuous on $\mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ by Lemma 2.2

Let $\mathcal{H}^*_\varepsilon \colon L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times C^0(\overline{\Omega})^+ \to \mathbb{R}$ be the convex conjugate of $\mathcal{H}_\varepsilon$ with respect to $p$, and let $\mathcal{N}^\varepsilon_{\mathcal{K}} \colon L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times C^0(\overline{\Omega})^+ \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ be defined by

$$\mathcal{N}^\varepsilon_{\mathcal{K}}(\sigma,\zeta) := \tfrac{1}{\varepsilon}\big(\sigma - \pi_{\mathcal{K}(\zeta)}(\sigma)\big)\,. \tag{2.42}$$

It follows from (2.34) that

$$p = \mathcal{N}^\varepsilon_{\mathcal{K}}(\sigma,\zeta) \quad\Longleftrightarrow\quad p(x) = N^\varepsilon_K(\sigma(x),\zeta(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{2.43}$$

so that $\mathcal{N}^\varepsilon_{\mathcal{K}} \colon L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times C^0(\overline{\Omega})^+ \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is Lipschitz continuous. By a general property of integral functionals (see, e.g., [9, Proposition IX.2.1]) we have

$$\mathcal{H}^*_\varepsilon(\sigma,\zeta) = \int_\Omega H^*_\varepsilon(\sigma(x),\zeta(x))\,dx\,,$$

so that, by the Dominated Convergence Theorem and by (2.43), its gradient $\partial_\sigma\mathcal{H}^*_\varepsilon(\sigma,\zeta)$ with respect to $\sigma$ satisfies

$$\partial_\sigma\mathcal{H}^*_\varepsilon(\sigma,\zeta) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\sigma,\zeta) \qquad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{2.44}$$

**The internal variables.** There are two internal variables $z \in C^0(\overline{\Omega})$ and $\zeta \in C^0(\overline{\Omega})^+$. They are linked by the equality

$$\zeta := V(z)\,, \tag{2.45}$$

where $V\colon \mathbb{R} \to (0,+\infty)$ is a globally Lipschitz nondecreasing function. We assume that there exists a constant $\zeta_m > 0$ such that

$$V(z) \geq \zeta_m \quad \text{for every } z \in \mathbb{R}\,. \tag{2.46}$$

The evolution law for the internal variable is nonlocal and involves a convolution. Let $\rho \in C^1_c(\mathbb{R}^n)^+$ be a fixed kernel with

$$\int_{\mathbb{R}^n} \rho(x)\,dx = 1\,. \tag{2.47}$$

For $\mu \in M_b(\Omega \cup \Gamma_0)$, the convolution $\rho \star \mu$ is defined for every $x \in \overline{\Omega}$ by

$$(\rho \star \mu)(x) := \int_{\Omega\cup\Gamma_0} \rho(x-y)\,d\mu(y)\,. \tag{2.48}$$

It is clear that $\rho \star \mu \in C^1(\overline{\Omega})$ and that

$$\|\rho\star\mu\|_\infty \leq \|\rho\|_\infty\|\mu\|_1 \quad \text{and} \quad \|\nabla(\rho\star\mu)\|_\infty \leq \|\nabla\rho\|_\infty\|\mu\|_1\,, \tag{2.49}$$

hence the linear map $\mu \mapsto \rho\star\mu$ is continuous from $M_b(\Omega\cup\Gamma_0)$ to $C^1(\overline{\Omega})$.

**The data of the problem.** We assume that the *body force* $\boldsymbol{f}(t)$, the *surface force* $\boldsymbol{g}(t)$, and the prescribed *boundary displacement* $\boldsymbol{w}(t)$ satisfy the following assumptions:

$$\begin{aligned} &\boldsymbol{f} \in H^1_{loc}([0,+\infty); L^n(\Omega;\mathbb{R}^n))\,,\\ &\boldsymbol{g} \in H^1_{loc}([0,+\infty); L^\infty(\Gamma_1;\mathbb{R}^n))\,,\\ &\boldsymbol{w} \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^n))\,. \end{aligned} \tag{2.50}$$

For every $t \in [0,+\infty)$ the *total load* $\boldsymbol{L}(t) \in BD(\Omega)'$ applied at time $t$ is defined by

$$\langle \boldsymbol{L}(t), u\rangle = \langle \boldsymbol{f}(t), u\rangle_\Omega + \langle \boldsymbol{g}(t), u\rangle_{\Gamma_1} \quad \text{for every } u \in BD(\Omega)\,. \tag{2.51}$$

Under our assumptions $\boldsymbol{L}$ belongs to $H^1_{loc}([0,+\infty); BD(\Omega)')$ and its time derivative is given by

$$\langle \dot{\boldsymbol{L}}(t), u\rangle = \langle \dot{\boldsymbol{f}}(t), u\rangle_\Omega + \langle \dot{\boldsymbol{g}}(t), u\rangle_{\Gamma_1} \quad \text{for every } u \in BD(\Omega)\,. \tag{2.52}$$

Throughout the paper we will assume also the following *uniform safe-load condition*: there exist a function $\boldsymbol{\chi} \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ and a constant $r_0 > 0$ such that

$$-\mathrm{div}\,\boldsymbol{\chi}(t) = \boldsymbol{f}(t) \text{ in } \Omega \text{ and } [\boldsymbol{\chi}(t)\nu] = \boldsymbol{g}(t) \text{ on } \Gamma_1 \text{ for every } t \in [0,+\infty)\,, \tag{2.53}$$

$$B(\chi(t,x), r_0) \subset K(\zeta_m) \quad \text{for every } t \in [0,+\infty) \text{ and } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{2.54}$$

$$\dot{\boldsymbol{\chi}}(t) \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,+\infty)\,, \tag{2.55}$$

$$t \mapsto \|\dot{\boldsymbol{\chi}}(t)\|_\infty \quad \text{belongs to } L^1_{loc}([0,+\infty))\,, \tag{2.56}$$

where $\chi(t,x)$ denotes the value of $\boldsymbol{\chi}(t)$ at $x \in \Omega$, and $B(\sigma, r)$ denotes the open ball in $\mathbb{M}^{n\times n}_{sym}$ with centre $\sigma$ and radius $r$. By (2.23) inclusion (2.54) implies

$$H(\xi,\zeta) \geq \chi(t,x) : \xi + r_0|\xi| \tag{2.57}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$ and every $(\xi,\zeta) \in \mathbb{M}^{n\times n}_{sym} \times [\zeta_m, +\infty)$.

## 3. The viscoplastic solutions

In this section, given a viscosity paramater $\varepsilon > 0$, we study the existence of a solution to the Perzyna-type viscoplastic evolution problem (see, e.g., [29, Section 1.2.(ii)]) corresponding to Cam-Clay plasticity.

**Definition 3.1.** Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ be as in (2.50), let $u_0 \in H^1(\Omega;\mathbb{R}^n)$, $e_0 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_0 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $z_0 \in C^0(\overline{\Omega})$, and let $\varepsilon > 0$. An $\varepsilon$*-viscoplastic evolution* with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$, and initial condition $(u_0, e_0, p_0, z_0)$ is a function $(\boldsymbol{u}_\varepsilon, \boldsymbol{e}_\varepsilon, \boldsymbol{p}_\varepsilon, \boldsymbol{z}_\varepsilon)$, with

$$\begin{gathered} \boldsymbol{u}_\varepsilon \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^n))\,, \qquad \boldsymbol{e}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,, \\ \boldsymbol{p}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,, \qquad \boldsymbol{z}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega)), \\ \boldsymbol{z}_\varepsilon(t) \in C^0(\overline{\Omega}) \quad \text{for every } t \in [0,+\infty)\,, \end{gathered} \tag{3.1}$$

such that, setting

$$\boldsymbol{\sigma}_\varepsilon(t) := \mathbb{C}\boldsymbol{e}_\varepsilon(t) \quad \text{and} \quad \boldsymbol{\zeta}_\varepsilon(t) := V(\boldsymbol{z}_\varepsilon(t))\,, \tag{3.2}$$

the following conditions are satisfied:

$(\mathrm{ev0})_\varepsilon$ *initial condition:* $(\boldsymbol{u}_\varepsilon(0), \boldsymbol{e}_\varepsilon(0), \boldsymbol{p}_\varepsilon(0), \boldsymbol{z}_\varepsilon(0)) = (u_0, e_0, p_0, z_0)$;

$(\mathrm{ev1})_\varepsilon$ *kinematic admissibility:* for every $t \in [0,+\infty)$

$$\begin{gathered} E\boldsymbol{u}_\varepsilon(t) = \boldsymbol{e}_\varepsilon(t) + \boldsymbol{p}_\varepsilon(t) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \\ \boldsymbol{u}_\varepsilon(t) = \boldsymbol{w}(t) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0\,; \end{gathered} \tag{3.3}$$

$(\mathrm{ev2})_\varepsilon$ *equilibrium condition:* for every $t \in [0,+\infty)$

$$-\mathrm{div}\,\boldsymbol{\sigma}_\varepsilon(t) = \boldsymbol{f}(t) \quad \text{in } \Omega\,, \qquad [\boldsymbol{\sigma}_\varepsilon(t)\nu] = \boldsymbol{g}(t) \quad \text{on } \Gamma_1. \tag{3.4}$$

$(\mathrm{ev3})_\varepsilon$ *regularized flow rule:* for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$

$$\dot{\boldsymbol{p}}_\varepsilon(t) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{3.5}$$

where $\mathcal{N}^\varepsilon_{\mathcal{K}}$ is defined by (2.42).

$(\mathrm{ev4})_\varepsilon$ *evolution law for the internal variable:* for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$

$$\dot{\boldsymbol{z}}_\varepsilon(t) = \rho \star \big(\boldsymbol{a}_\varepsilon(t)\,\mathrm{tr}\,\dot{\boldsymbol{p}}_\varepsilon(t)\big) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{3.6}$$

where

$$\boldsymbol{a}_\varepsilon(t) := \rho \star \mathrm{tr}\,\boldsymbol{\sigma}_\varepsilon(t)\,. \tag{3.7}$$

**Remark 3.2.** Let us fix $t \in [0,+\infty)$ such that the derivatives $\dot{\boldsymbol{p}}_\varepsilon(t)$ exists. Then the following conditions are equivalent:

$$\dot{\boldsymbol{p}}_\varepsilon(t) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{3.8}$$

$$\boldsymbol{\sigma}_\varepsilon(t) \in \partial_p\mathcal{H}_\varepsilon(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{3.9}$$

$$\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \in \partial_p\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{3.10}$$

Indeed, by (2.44) we have $\partial_\sigma\mathcal{H}^*_\varepsilon(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t))$, so that (3.8) and (3.9) are equivalent by a standard property of conjugate functions (see, e.g., [9, Corollary I.5.2]). The equivalence between (3.9) and (3.10) follows immediately from (2.39).

To prove the existence of an $\varepsilon$-viscoplastic evolution we will use a fixed point argument. To this end, in the next theorem we prove existence and continous dependence on the data for a similar problem with prescribed $\boldsymbol{\zeta}$.

**Theorem 3.3.** *Let* $\boldsymbol{\zeta} \in C^0([0,+\infty); L^2(\Omega)^+)$ *and let* $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, $u_0$, $e_0$, $p_0$, $\varepsilon$ *be as in Definition 3.1. Assume that* $(u_0, e_0, p_0)$ *satisfies the kinematic admissibility condition at* $t = 0$*:*

$$\begin{gathered} Eu_0 = e_0 + p_0 \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \\ u_0 = \boldsymbol{w}(0) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0\,, \end{gathered} \tag{3.11}$$

*and that the safe load condition* (2.53)-(2.56) *holds. Then there exists a unique function* $(\boldsymbol{u}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{p}^{\boldsymbol{\zeta}}_\varepsilon)$, *with*

$$\begin{gathered} \boldsymbol{u}^{\boldsymbol{\zeta}}_\varepsilon \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^n))\,, \qquad \boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,, \\ \boldsymbol{p}^{\boldsymbol{\zeta}}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,, \end{gathered} \tag{3.12}$$

*such that setting*

$$\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t) = \mathbb{C}\boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t),$$

*the following conditions are satisfied:*

(ev0)$_\varepsilon^{\boldsymbol{\zeta}}$ *initial condition:* $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}(0), \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(0), \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}}(0)) = (u_0, e_0, p_0)$*;*

(ev1)$_\varepsilon^{\boldsymbol{\zeta}}$ *kinematic admissibility: for every* $t \in [0, +\infty)$

$$\begin{aligned} E\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}(t) = \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t) + \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}}(t) \quad \mathcal{L}^n\text{-}a.e.\ in\ \Omega\,, \\ \boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}(t) = \boldsymbol{w}(t) \quad \mathcal{H}^{n-1}\text{-}a.e.\ in\ \Gamma_0\,; \end{aligned} \tag{3.13}$$

(ev2)$_\varepsilon^{\boldsymbol{\zeta}}$ *equilibrium condition: for every* $t \in [0, +\infty)$

$$-\operatorname{div} \boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t) = \boldsymbol{f}(t) \quad in\ \Omega\,, \qquad [\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t)\nu] = \boldsymbol{g}(t) \quad on\ \Gamma_1. \tag{3.14}$$

(ev3)$_\varepsilon^{\boldsymbol{\zeta}}$ *regularized flow rule: for* $\mathcal{L}^1$*-a.e.* $t \in [0, +\infty)$

$$\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t) = \mathcal{N}_\mathcal{K}^\varepsilon(\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t), \boldsymbol{\zeta}(t)) \quad \mathcal{L}^n\text{-}a.e.\ in\ \Omega\,, \tag{3.15}$$

*where* $\mathcal{N}_\mathcal{K}^\varepsilon$ *is defined by* (2.42)*.*

*Since the right-hand side of* (3.15) *belongs to* $C^0([0, +\infty), L^2(\Omega, \mathbb{M}^{n\times n}_{sym}))$ *by* (2.25)*, it follows that*

$$\boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}} \in C^1([0, +\infty), L^2(\Omega, \mathbb{M}^{n\times n}_{sym}))\,. \tag{3.16}$$

*Moreover, for every* $T > 0$ *there exists a constant* $C_{\varepsilon,T}$ *such that*

$$\max_{t\in[0,T]} \|\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}_1}(t) - \boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}_2}(t)\|_2 \le C_{\varepsilon,T} \max_{t\in[0,T]} \|\boldsymbol{\zeta}_1(t) - \boldsymbol{\zeta}_2(t)\|_2 \tag{3.17}$$

$$\max_{t\in[0,T]} \|\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}_1}(t) - \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}_2}(t)\|_2 \le C_{\varepsilon,T} \max_{t\in[0,T]} \|\boldsymbol{\zeta}_1(t) - \boldsymbol{\zeta}_2(t)\|_2 \tag{3.18}$$

*for every* $\boldsymbol{\zeta}_1$*,* $\boldsymbol{\zeta}_2$ *in* $C^0([0, +\infty); L^2(\Omega)^+)$*.*

*Proof.* Let $\mathbb{A} := \mathbb{C}^{-1}$. If a triple $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}})$ satisfies conditions (ev0)$_\varepsilon^{\boldsymbol{\zeta}}$-(ev3)$_\varepsilon^{\boldsymbol{\zeta}}$, then for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$

$$E\dot{\boldsymbol{u}}_\varepsilon^{\boldsymbol{\zeta}}(t) - \mathbb{A}\dot{\boldsymbol{\sigma}}_\varepsilon^{\boldsymbol{\zeta}}(t) = \mathcal{N}_\mathcal{K}^\varepsilon(\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t), \boldsymbol{\zeta}(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{3.19}$$

Let us define $\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}} \in H^1_{loc}([0, +\infty); L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ by

$$\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}(t) := \boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t) - \boldsymbol{\chi}(t). \tag{3.20}$$

By (2.17), (2.53), and (3.14) we have $\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}(t) \in \Sigma_0(\Omega)$ for every $t \in [0, +\infty)$ and hence, integrating by parts, for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$ we obtain

$$\langle \mathbb{A}\dot{\boldsymbol{\tau}}_\varepsilon^{\boldsymbol{\zeta}}(t), \hat{\sigma}\rangle = -\langle \mathcal{N}_\mathcal{K}^\varepsilon(\boldsymbol{\chi}(t) + \boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}(t), \boldsymbol{\zeta}(t)), \hat{\sigma}\rangle + \langle E\dot{\boldsymbol{w}}(t) - \mathbb{A}\dot{\boldsymbol{\chi}}(t), \hat{\sigma}\rangle \tag{3.21}$$

for every $\hat{\sigma} \in \Sigma_0(\Omega)$. The initial condition for $\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}$ is given by

$$\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}(0) = \sigma_0 - \boldsymbol{\chi}(0)\,, \tag{3.22}$$

where $\sigma_0 := \mathbb{C}e_0$.

Conversely, assume that $\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}} \in H^1_{loc}([0, +\infty); \Sigma_0(\Omega))$ and that (3.21) holds for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$. If we define $\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t)$ through (3.20), then (3.14) follows from (2.53). Moreover, for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$, we obtain by (3.21) that $\mathbb{A}\dot{\boldsymbol{\sigma}}_\varepsilon^{\boldsymbol{\zeta}}(t) + \mathcal{N}_\mathcal{K}^\varepsilon(\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t), \boldsymbol{\zeta}(t)) - E\dot{\boldsymbol{w}}(t)$ are orthogonal to $\Sigma_0(\Omega)$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Therefore, by (2.18), for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$ there exists $\boldsymbol{v}_\varepsilon^{\boldsymbol{\zeta}}(t) \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ such that $E\boldsymbol{v}_\varepsilon^{\boldsymbol{\zeta}}(t) = \mathbb{A}\dot{\boldsymbol{\sigma}}_\varepsilon^{\boldsymbol{\zeta}}(t) + \mathcal{N}_\mathcal{K}^\varepsilon(\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t), \boldsymbol{\zeta}(t)) - E\dot{\boldsymbol{w}}(t)$ $\mathcal{L}^n$-a.e. in $\Omega$. If we define

$$\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}(t) := \boldsymbol{w}(t) + \int_0^t \boldsymbol{v}_\varepsilon^{\boldsymbol{\zeta}}(\tau)\,d\tau + u_0 - \boldsymbol{w}(0)\,, \quad \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t) = \mathbb{A}\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t)\,, \quad \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}}(t) := E\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}(t) - \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t)\,,$$

then the triple $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}})$ satisfies conditions (ev1)$_\varepsilon^{\boldsymbol{\zeta}}$-(ev3)$_\varepsilon^{\boldsymbol{\zeta}}$. If, in addition, (3.22) holds, then the initial condition (ev0)$_\varepsilon^{\boldsymbol{\zeta}}$ is satisfied.

We assume for the moment that

$$\boldsymbol{\zeta} \in \mathrm{Lip}_{loc}([0,+\infty); L^2(\Omega)^+) \quad \text{and} \quad \boldsymbol{\chi} \in \mathrm{Lip}_{loc}([0,+\infty); L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$$

(we do not need conditions (2.54)-(2.56)), and we apply the arguments of [29] with

$$\varphi(x,t,\sigma) := H^*_\varepsilon(\zeta(t,x), \sigma + \chi(t,x))\,,$$

where $H^*_\varepsilon$ is defined by (2.40), while $\zeta(t,x)$ and $\chi(t,x)$ denote the values of $\boldsymbol{\zeta}(t)$ and $\boldsymbol{\chi}(t)$ at $x \in \Omega$. Note that

$$\partial_\sigma \varphi(x,t,\sigma) = \tfrac{1}{\varepsilon}(\sigma + \chi(t,x) - \pi_{K(\zeta(t,x))}(\sigma + \chi(t,x)))\,.$$

The Lipschitz continuity conditions (22) and (24) of [29] follow from the Lipschitz continuity of $\boldsymbol{\zeta}$ and $\boldsymbol{\chi}$, thanks to Lemma 2.2. Following the arguments of [29, Section 1.4] it is possible to prove that there exists a unique solution of problem (3.21) with initial condition (3.22).

We now prove the existence result for a general $\boldsymbol{\zeta} \in C^0([0,+\infty); L^2(\Omega)^+)$ and $\boldsymbol{\chi} \in H^1_{loc}([0,+\infty); L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ satisfying (2.53). Let us fix $T > 0$. Consider two sequences $\boldsymbol{\zeta}_k \in \mathrm{Lip}([0,T]; L^2(\Omega)^+)$ and $\boldsymbol{\chi}_k \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ such that

$$\boldsymbol{\zeta}_k \to \boldsymbol{\zeta} \quad \text{in } C^0([0,T]; L^2(\Omega))\,, \qquad \boldsymbol{\chi}_k \to \boldsymbol{\chi} \quad \text{in } H^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))\,,$$

and $\boldsymbol{\chi}_k(0) = \boldsymbol{\chi}(0)$ for every $k$. Let $\boldsymbol{\sigma}_k$ be the corresponding solutions, with initial conditions $\boldsymbol{\sigma}_k(0) = \sigma_0$, and let $\boldsymbol{\tau}_k := \boldsymbol{\sigma}_k - \boldsymbol{\chi}_k$. Subtracting (3.21) corresponding to $\boldsymbol{\tau}_h$ and $\boldsymbol{\tau}_k$ and taking $\hat\sigma = \boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)$ as test function we obtain

$$\begin{aligned}\tfrac{d}{dt}\tfrac{1}{2}\|\boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\|^2_{\mathbb{A}} = -\langle \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_h(t), \boldsymbol{\zeta}_h(t)) - \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_k(t), \boldsymbol{\zeta}_k(t)), \boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\rangle - \\ - \langle \dot{\boldsymbol{\chi}}_h(t) - \dot{\boldsymbol{\chi}}_k(t), \boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\rangle_{\mathbb{A}},\end{aligned}$$

where $\langle \sigma, \tau\rangle_{\mathbb{A}} := \langle \mathbb{A}\sigma, \tau\rangle$ and $\|\cdot\|_{\mathbb{A}}$ is the corresponding norm. By (2.9) and Lemma 2.2

$$\begin{aligned}&\tfrac{d}{dt}\tfrac{1}{2}\|\boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\|^2_{\mathbb{A}} \le \\ &\le C\left[\|\boldsymbol{\chi}_h(t) - \boldsymbol{\chi}_k(t)\|_2 + \|\boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\|_{\mathbb{A}} + \|\boldsymbol{\zeta}_h(t) - \boldsymbol{\zeta}_k(t)\|_2\right]\|\boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\|_{\mathbb{A}} + \\ &\quad + \|\dot{\boldsymbol{\chi}}_h(t) - \dot{\boldsymbol{\chi}}_k(t)\|_{\mathbb{A}}\|\boldsymbol{\tau}_h(t) - \boldsymbol{\tau}_k(t)\|_{\mathbb{A}}\,,\end{aligned}$$

where $C$ is a constant depending on $\alpha_Q$, $\beta_Q$, and $M_K$. Taking into account the uniform convergence of $\boldsymbol{\zeta}_k$ and $\boldsymbol{\chi}_k$, and the $L^2$-convergence of $\dot{\boldsymbol{\chi}}_k$, by Gronwall inequality we get that $\boldsymbol{\sigma}_k(t)$ is a Cauchy sequence in $C^0([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. Thus there exists $\boldsymbol{\sigma} \in C^0([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ such that

$$\boldsymbol{\sigma}_k \to \boldsymbol{\sigma} \quad \text{in } C^0([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})). \tag{3.23}$$

To prove the convergence of the time derivatives $\dot{\boldsymbol{\sigma}}_k$ we subtract (3.21) corresponding to $\boldsymbol{\tau}_h$ and $\boldsymbol{\tau}_k$ and take $\hat\sigma = \dot{\boldsymbol{\tau}}_h(t) - \dot{\boldsymbol{\tau}}_k(t)$ as test function . We obtain

$$\begin{aligned}\|\dot{\boldsymbol{\tau}}_h(t) - \dot{\boldsymbol{\tau}}_k(t)\|^2_{\mathbb{A}} = -\langle \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_h(t), \boldsymbol{\zeta}_h(t)) - \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_k(t), \boldsymbol{\zeta}_k(t)), \dot{\boldsymbol{\tau}}_h(t) - \dot{\boldsymbol{\tau}}_k(t)\rangle - \\ -\langle \dot{\boldsymbol{\chi}}_h(t) - \dot{\boldsymbol{\chi}}_k(t), \dot{\boldsymbol{\tau}}_h(t) - \dot{\boldsymbol{\tau}}_k(t)\rangle_{\mathbb{A}}.\end{aligned}$$

By (2.9) and Lemma 2.2

$$\|\dot{\boldsymbol{\tau}}_h(t) - \dot{\boldsymbol{\tau}}_k(t)\|_{\mathbb{A}} \le \tfrac{\sqrt{2}\beta_Q}{\varepsilon\sqrt{\alpha_Q}}\left[\|\boldsymbol{\sigma}_h(t) - \boldsymbol{\sigma}_k(t)\|_2 + 2M_K\|\boldsymbol{\zeta}_h(t) - \boldsymbol{\zeta}_k(t)\|_2\right] + \|\dot{\boldsymbol{\chi}}_h(t) - \dot{\boldsymbol{\chi}}_k(t)\|_{\mathbb{A}}.$$

Thus $\dot{\boldsymbol{\sigma}}_k$ is a Cauchy sequence in $L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. This implies that $\boldsymbol{\sigma}$ belongs to $H^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and $\boldsymbol{\sigma}_k \to \boldsymbol{\sigma}$ strongly in $H^1([0,T]; L^2(\mathbb{M}^{n\times n}_{sym}))$. Therefore we can pass to the limit in the equations satisfied by $\boldsymbol{\sigma}_k$ and we conclude that $\boldsymbol{\sigma}$ satisfies (3.21) and (3.22).

To prove uniqueness and estimate (3.17) we consider two solutions $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ corresponding to $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ in $C^0([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, respectively. Subtracting (3.21) corresponding to $\boldsymbol{\tau}_1 := \boldsymbol{\sigma}_1 - \boldsymbol{\chi}$ and $\boldsymbol{\tau}_2 := \boldsymbol{\sigma}_2 - \boldsymbol{\chi}$, taking $\hat\sigma = \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)$ as test function, and using Lemma 2.2 we obtain

$$\tfrac{d}{dt}\tfrac{1}{2}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathbb{A}} \le \tfrac{1}{\varepsilon}\left[\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_2 + 2M_K\|\boldsymbol{\zeta}_1(t) - \boldsymbol{\zeta}_2(t)\|_2\right]\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_2.$$

To get (3.17) it is enough to apply Gronwall inequality. The other inequality (3.18) follows from $(\mathrm{ev3})^{\boldsymbol{\zeta}}_{\varepsilon}$ using (2.25) and (3.17). □

The following theorem shows that the modified flow rule $(\mathrm{ev3})_{\varepsilon}$ can be replaced by a suitable stress constraint and an energy-dissipation balance.

**Theorem 3.4.** *Let* $\boldsymbol{\zeta} \in C^0([0,+\infty); L^2(\Omega)^+)$ *and let* $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, $u_0$, $e_0$, $p_0$, $\varepsilon$ *be as in Definition 3.1. Assume that the safe load condition* (2.53)-(2.56) *holds. Let* $(\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon})$ *be a function satisfying* (3.12)*, the initial condition* $(ev0)_{\varepsilon}$*, the kinematic admissibility* $(ev1)^{\boldsymbol{\zeta}}_{\varepsilon}$*, and the equilibrium condition* $(ev2)^{\boldsymbol{\zeta}}_{\varepsilon}$ *of Theorem 3.3, with* $\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) := \mathbb{C}\boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(t)$*.*

*Then* $(\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon})$ *satisfies the regularized flow rule* $(ev3)^{\boldsymbol{\zeta}}_{\varepsilon}$ *of Theorem 3.3 if and only if the following properties are simultaneously satisfied:*

$(\mathrm{ev3}')^{\boldsymbol{\zeta}}_{\varepsilon}$ *modified stress constraint: for* $\mathcal{L}^1$*-a.e.* $t \in [0,+\infty)$

$$\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \varepsilon\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) \in \mathcal{K}(\boldsymbol{\zeta}(t))\,; \tag{3.24}$$

$(\mathrm{ev3}'')^{\boldsymbol{\zeta}}_{\varepsilon}$ *energy-dissipation balance: for every* $T > 0$ *we have*

$$\begin{aligned} &\mathcal{Q}(\boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(T)) - \mathcal{Q}(e_0) + \int_0^T \Big(\mathcal{H}(\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\Big)\,dt + \varepsilon\int_0^T \|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\|_2^2\,dt = \\ &= \int_0^T \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\,dt - \int_0^T \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\,dt + \langle \boldsymbol{\chi}(T), \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(T)\rangle - \langle \boldsymbol{\chi}(0), e_0\rangle\,. \end{aligned} \tag{3.25}$$

*Proof.* Suppose that $(\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon})$ satisfies $(\mathrm{ev3})^{\boldsymbol{\zeta}}_{\varepsilon}$. By (2.37) we have $\partial_p\mathcal{H}(\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) \subset \mathcal{K}(\boldsymbol{\zeta}(t))$. Therefore (3.10) implies $(\mathrm{ev3}')^{\boldsymbol{\zeta}}_{\varepsilon}$.

Since $\mathcal{H}(\cdot,\zeta)$ is convex and positively homogeneous of degree one, the Euler relation gives $\langle\sigma,p\rangle = \mathcal{H}(p,\zeta)$ whenever $\sigma \in \partial_p\mathcal{H}(p,\zeta)$. Therefore, (3.10) implies

$$\mathcal{H}(\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) = \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \varepsilon\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\,, \tag{3.26}$$

which is equivalent to

$$\mathcal{H}(\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) + \varepsilon\|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\|_2^2 = \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\,. \tag{3.27}$$

By (3.13) we have

$$\langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle = \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), E\dot{\boldsymbol{u}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle - \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \dot{\boldsymbol{e}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\,. \tag{3.28}$$

Since $\dot{\boldsymbol{u}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \dot{\boldsymbol{w}}(t) \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ by (3.13), using (2.53) and (3.14) we obtain

$$\langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), E\dot{\boldsymbol{u}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle = \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), E\dot{\boldsymbol{w}}(t)\rangle + \langle \boldsymbol{\chi}(t), E\dot{\boldsymbol{u}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle - \langle \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\,. \tag{3.29}$$

Combining (3.27), (3.28), and (3.29), we deduce that

$$\langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \dot{\boldsymbol{e}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle + \mathcal{H}(\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) + \varepsilon\|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\|_2^2 = \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle + \langle \boldsymbol{\chi}(t), E\dot{\boldsymbol{u}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\,.$$

By (3.13) we have

$$\begin{aligned} &\langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \dot{\boldsymbol{e}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle + \mathcal{H}(\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle + \varepsilon\|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\|_2^2 = \\ &= \langle \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle + \tfrac{d}{dt}\langle \boldsymbol{\chi}(t), \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle - \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\rangle\,. \end{aligned} \tag{3.30}$$

The energy-dissipation balance $(\mathrm{ev3}'')^{\boldsymbol{\zeta}}_{\varepsilon}$ can be obtained from (3.30) by integration.

Conversely, assume that $(\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon})$ satisfies conditions $(\mathrm{ev3}')^{\boldsymbol{\zeta}}_{\varepsilon}$ and $(\mathrm{ev3}'')^{\boldsymbol{\zeta}}_{\varepsilon}$. By differentiating $(\mathrm{ev3}'')^{\boldsymbol{\zeta}}_{\varepsilon}$ we obtain (3.30). Thanks to (3.28) and (3.29), from (3.30) we deduce (3.27), which is equivalent to (3.26). By $(\mathrm{ev3}')^{\boldsymbol{\zeta}}_{\varepsilon}$ for $\mathcal{L}^1$-a.e. $t \in (0,+\infty)$ we have

$$\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \varepsilon\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) \in \mathcal{K}(\boldsymbol{\zeta}(t)) = \partial_p\mathcal{H}(0, \boldsymbol{\zeta}(t))\,. \tag{3.31}$$

Since $\mathcal{H}(\cdot, \boldsymbol{\zeta}(t))$ is convex and $\mathcal{H}(0,\boldsymbol{\zeta}(t)) = 0$, condition (3.10) follows easily from (3.26) and (3.31). □

**Theorem 3.5.** *Let $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, $u_0$, $e_0$, $p_0$, $\varepsilon$ be as in Definition 3.1. Assume that $(u_0, e_0, p_0)$ satisfies the kinematic admissibility condition* (3.11) *at $t = 0$ and that the safe load condition* (2.53)-(2.56) *holds. Then there exists an $\varepsilon$-viscoplastic evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition $(u_0, e_0, p_0, z_0)$.*

*Proof.* Let us fix $T > 0$. We will apply a fixed-point argument in $C^0([0,T]; L^2(\Omega; I_m))$, where $I_m := [\zeta_m, +\infty)$. Given $\boldsymbol{\zeta} \in C^0([0,T]; L^2(\Omega; I_m))$, by Theorem 3.3 we can find a unique function $(\boldsymbol{u}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{p}^{\boldsymbol{\zeta}}_\varepsilon)$, satisfying (3.12) and (ev0)$^{\boldsymbol{\zeta}}_\varepsilon$-(ev3)$^{\boldsymbol{\zeta}}_\varepsilon$. Define

$$\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon(t) := \rho \star \operatorname{tr} \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_\varepsilon(t) . \tag{3.32}$$

As $\operatorname{tr} \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_\varepsilon \in C^0([0,T]; L^2(\Omega))$, we deduce from (2.49) that $\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon \in C^0([0,T]; C^1(\overline{\Omega}))$. By (3.16) we have $\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon \operatorname{tr} \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon \in C^0([0,T]; L^2(\Omega))$, so that (2.49) gives $\rho \star (\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon \operatorname{tr} \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon) \in C^0([0,T]; C^1(\overline{\Omega}))$. Let $\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon \in C^1([0,T]; C^1(\overline{\Omega}))$ be defined by

$$\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon(t) = z_0 + \int_0^t \rho \star (\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon(\tau) \operatorname{tr} \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(\tau))\, d\tau . \tag{3.33}$$

It satisfies

$$\boldsymbol{\zeta}(0) = z_0 \quad \text{and} \quad \dot{\boldsymbol{z}}^{\boldsymbol{\zeta}}_\varepsilon(t) = \rho \star (\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon(t) \operatorname{tr} \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(t) \quad \text{for every } t \in [0,T] .$$

Let us define the operator $\mathcal{G}\colon C^0([0,T]; L^2(\Omega; I_m)) \to C^0([0,T]; L^2(\Omega; I_m))$ by

$$\mathcal{G}(\boldsymbol{\zeta}) := V(\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon) . \tag{3.34}$$

It follows from the definitions that, if $\boldsymbol{\zeta}$ is a fixed point of $\mathcal{G}$, then $(\boldsymbol{u}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{p}^{\boldsymbol{\zeta}}_\varepsilon, \boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon)$ is an $\varepsilon$-viscoplastic evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition $(u_0, e_0, p_0, z_0)$.

To find a fixed point we will apply Schauder's theorem. In the rest of the proof $C$ will denote a positive constant, depending only on $T$, $\varepsilon$, $\Omega$, $e_0$, $\boldsymbol{w}$, $\boldsymbol{\chi}$, $\rho$, $\alpha_Q$, and $\beta_Q$, which may change from line to line. By (2.57) and (3.25) in Theorem 3.4 we have

$$\max_{t\in[0,T]} \|\boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2^2 \le C + C \max_{t\in[0,T]} \|\boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2 ,$$

which implies

$$\max_{t\in[0,T]} \|\boldsymbol{e}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2 \le C . \tag{3.35}$$

Using this inequality in (3.25) and taking into account (2.57), we obtain

$$\int_0^T \|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2^2\, dt \le \tfrac{C}{\varepsilon} . \tag{3.36}$$

From (2.49), (3.32), and (3.35) it follows that

$$\max_{t\in[0,T]} \|\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_\infty \le C . \tag{3.37}$$

Thus, $\|\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon(t) \operatorname{tr} \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2 \le C \|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2$, and hence, by (2.49),

$$\|\dot{\boldsymbol{z}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_\infty \le C \|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2 \quad \text{and} \quad \|\nabla \dot{\boldsymbol{z}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_\infty \le C \|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon(t)\|_2 . \tag{3.38}$$

Inequalities (3.36) and (3.38) imply that the norm of $\dot{\boldsymbol{z}}^{\boldsymbol{\zeta}}_\varepsilon$ in $L^2([0,T]; H^1(\Omega))$ is bounded by a constant independent of $\boldsymbol{\zeta}$. It follows that the norm of $\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon - z_0$ in $H^1([0,T]; H^1(\Omega))$ uniformly bounded. Therefore

$$\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon - z_0 \in C^{0,1/2}([0,T]; H^1(\Omega)) ,$$

and its norm is bounded by a constant independent of $\boldsymbol{\zeta}$ This implies that there exists a closed ball $\mathcal{B}$ in $H^1(\Omega)$ such that

$$\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon - z_0 \in C^{0,1/2}([0,T]; \mathcal{B}) \quad \text{for every } \boldsymbol{\zeta} \in C^0([0,T]; L^2(\Omega; I_m)) .$$

Since $\mathcal{B}$ is compact in $L^2(\Omega)$, the set $\{\boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon : \boldsymbol{\zeta} \in C^0([0,T]; L^2(\Omega; I_m))\}$ is relatively compact in $C^0([0,T]; L^2(\Omega; I_m))$ by the Arzelà-Ascoli Theorem. Therefore the operator $\mathcal{G}$ defined by (3.34) maps $C^0([0,T]; L^2(\Omega; I_m))$ into a compact subset of $C^0([0,T]; L^2(\Omega; I_m))$.

To apply Schauder's Theorem, it is enough to show that the operator $\mathcal{G}$ is continuous from $C^0([0,T];L^2(\Omega;I_m))$ to $C^0([0,T];L^2(\Omega;I_m))$. The continuity of the map $\boldsymbol{\zeta} \mapsto \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_\varepsilon$ follows from (3.17). Then (2.49) and (3.32) imply the continuity of $\boldsymbol{\zeta} \mapsto \boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon$ from $C^0([0,T];L^2(\Omega;I_m))$ to $C^0([0,T];C^1(\overline{\Omega}))$. Using (2.49) and (3.18) we obtain the continuity of $\boldsymbol{\zeta} \mapsto \rho \star (\boldsymbol{a}^{\boldsymbol{\zeta}}_\varepsilon \operatorname{tr} \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_\varepsilon)$ from $C^0([0,T];L^2(\Omega;I_m))$ to $C^0([0,T];C^1(\overline{\Omega}))$. Then (3.33) gives the continuity of $\boldsymbol{\zeta} \mapsto \boldsymbol{z}^{\boldsymbol{\zeta}}_\varepsilon$ from $C^0([0,T];L^2(\Omega;I_m))$ to $C^1([0,T];C^1(\overline{\Omega}))$. The continuity of $\mathcal{G}$ follows now easily from (3.34). □

## 4. Quasistatic evolution

**Definition 4.1.** Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ be as in (2.50), let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_0 \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, and $z_0 \in C^0(\overline{\Omega})$. We say that $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ is a *rescaled viscosity evolution* with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition $(u_0, e_0, p_0, z_0, 0)$ if

$$
\begin{gathered}
\boldsymbol{u}^\circ\colon [0,+\infty) \to BD(\Omega) \quad \text{is weakly}^* \text{ continuous}\,, \\
\boldsymbol{e}^\circ\colon [0,+\infty) \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \quad \text{is weakly continuous}\,, \\
\boldsymbol{p}^\circ\colon [0,+\infty) \to M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}) \quad \text{is locally Lipschitz}\,, \\
\boldsymbol{z}^\circ\colon [0,+\infty) \to C^0(\overline{\Omega})^+ \quad \text{is locally Lipschitz}\,, \\
t^\circ\colon [0,+\infty) \to [0,+\infty) \quad \text{is nondecreasing, surjective, and locally Lipschitz,}
\end{gathered}
\tag{4.1}
$$

and, setting

$$
\boldsymbol{\sigma}^\circ(s) := \mathbb{C}\boldsymbol{e}^\circ(s) \quad \text{and} \quad \boldsymbol{\zeta}^\circ(s) := V(\boldsymbol{z}^\circ(s)) \quad \text{for every } s \in [0,+\infty)\,, \tag{4.2}
$$

$$
\dot{\boldsymbol{p}}^\circ(s) := w^*\text{-}\lim_{h\to 0} \frac{\boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s)}{h} \qquad (w^*\text{-topology of } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}))\,, \tag{4.3}
$$

the following conditions are satisfied:

(ev0)$^\circ$ *Initial condition*: $(\boldsymbol{u}^\circ(0), \boldsymbol{e}^\circ(0), \boldsymbol{p}^\circ(0), \boldsymbol{z}^\circ(0), t^\circ(0)) = (u_0, e_0, p_0, z_0, 0)$.

(ev1)$^\circ$ *Weak kinematic admissibility*: for every $s \in [0,+\infty)$

$$
\begin{gathered}
E\boldsymbol{u}^\circ(s) = \boldsymbol{e}^\circ(s) + \boldsymbol{p}^\circ(s) \quad \text{in } \Omega\,, \\
\boldsymbol{p}^\circ(s) = (\boldsymbol{w}(t^\circ(s)) - \boldsymbol{u}^\circ(s)) \odot \nu\mathcal{H}^{n-1} \quad \text{in } \Gamma_0\,.
\end{gathered}
\tag{4.4}
$$

(ev2)$^\circ$ *Equilibrium condition*: for every $s \in [0,+\infty)$

$$
-\operatorname{div}\boldsymbol{\sigma}^\circ(s) = \boldsymbol{f}(t^\circ(s)) \quad \text{in } \Omega\,, \qquad [\boldsymbol{\sigma}^\circ(s)\nu] = \boldsymbol{g}(t^\circ(s)) \quad \text{on } \Gamma_1. \tag{4.5}
$$

(ev3$'$)$^\circ$ *Partial stress constraint*:

$$
\boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s)) \quad \text{for every } s \in [0,+\infty) \setminus U^\circ, \tag{4.6}
$$

where

$$
U^\circ := \{s \in (0,+\infty) : t^\circ \text{ is constant in a neighbourhood of } s\}. \tag{4.7}
$$

(ev3$''$)$^\circ$ *Energy-dissipation balance*: for every $S \in [0,+\infty)$

$$
\begin{aligned}
&\mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\,ds + \int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))) = \\
&\quad = \int_0^S \Big( \langle \boldsymbol{\sigma}^\circ(s), E\dot{\boldsymbol{w}}(t^\circ(s))\rangle - \langle \boldsymbol{L}(t^\circ(s)), \dot{\boldsymbol{w}}(t^\circ(s))\rangle \Big)\, \dot{t}^\circ(s)\,ds - \\
&\quad - \int_0^S \langle \dot{\boldsymbol{L}}(t^\circ(s)), \boldsymbol{u}^\circ(s)\rangle\, \dot{t}^\circ(s)\,ds + \langle \boldsymbol{L}(t^\circ(S)), \boldsymbol{u}^\circ(S)\rangle - \langle \boldsymbol{L}(0), u_0\rangle\,,
\end{aligned}
\tag{4.8}
$$

where $d_2$ is defined in (2.33).

(ev3$'''$)$^\circ$ *Partial flow-rule*: for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$ with $\boldsymbol{\sigma}^\circ(s) \notin \mathcal{K}(\boldsymbol{\zeta}^\circ(s))$ we have $\dot{\boldsymbol{p}}^\circ(s) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and

$$
\langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle = \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\,. \tag{4.9}
$$

(ev4)$^\circ$ *Evolution law for the internal variable*: for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$ the strong $C^0(\overline{\Omega})$-limit

$$\dot{\boldsymbol{z}}^\circ(s) := s\text{-}\lim_{h\to 0} \frac{\boldsymbol{z}^\circ(s+h) - \boldsymbol{z}^\circ(s)}{h} \tag{4.10}$$

exists, and

$$\dot{\boldsymbol{z}}^\circ(s) = \rho \star \big(\boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s)\big) \quad \text{in } \overline{\Omega} \text{ for } \mathcal{L}^1\text{-a.e. } s \in (0,+\infty)\,, \tag{4.11}$$

where

$$\boldsymbol{a}^\circ(s) := \rho \star \operatorname{tr} \boldsymbol{\sigma}^\circ(s)\,.$$

**Remark 4.2.** For every $\zeta \in C^0(\overline{\Omega})^+$ the function $s \mapsto \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)$ is measurable on $[0,+\infty)$ by [27, Theorem 3.12]. Approximating $s \mapsto \boldsymbol{\zeta}^\circ(s)$ by piecewise constant functions, we find that $s \mapsto \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))$ is measurable on $[0,+\infty)$, so the first integral in (4.8) makes sense.

Let $\varphi_i$ be a dense sequence in the unit ball of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, composed of continuous functions with compact support. Since, taking into account (2.1),

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_2 = \sup_i \langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle,$$

the function $s \mapsto \|\dot{\boldsymbol{p}}^\circ(s)\|_2$ is measurable, so the second integral in (4.8) makes sense.

**Remark 4.3.** Define

$$A^\circ := \{s \in [0,+\infty) : d_2\big(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))\big) > 0\}\,. \tag{4.12}$$

Then (4.9) easily implies that there exists a measurable function $\lambda\colon A^\circ \to [0,+\infty)$ such that

$$\dot{\boldsymbol{p}}^\circ(s) = \lambda(s)\big(\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s))\big) \tag{4.13}$$

for $\mathcal{L}^1$-a.e. $s \in A^\circ$. This justifies the choice of the name *flow-rule* for condition (ev3$'''$)$^\circ$ in Definition 4.1.

**Proposition 4.4.** *Let* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$ *be as in* (2.50). *Assume that* $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, *and* $t^\circ$ *satisfy* (4.1), (4.4), *and* (4.5), *and that the safe load condition* (2.53)-(2.56) *holds. For every* $s \in [0,+\infty)$ *let us define*

$$\boldsymbol{w}^\circ(s) := \boldsymbol{w}(t^\circ(s)) \quad \text{and} \quad \boldsymbol{\chi}^\circ(s) := \boldsymbol{\chi}(t^\circ(s))\,. \tag{4.14}$$

*Then* (4.8) *is equivalent to*

$$\begin{aligned} &\mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\Big)\, ds - \\ &\quad - \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \boldsymbol{\chi}(0), p_0\rangle + \int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds = \\ &= \int_0^S \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{e}^\circ(S)\rangle - \langle \boldsymbol{\chi}(0), e_0\rangle\,, \end{aligned} \tag{4.15}$$

*where* $\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ *and* $\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ *are defined according to* (2.13) *for every* $s \in [0,+\infty)$.

*Proof.* For every $s \in [0,+\infty)$ we define $\boldsymbol{f}^\circ(s) := \boldsymbol{f}(t^\circ(s))$, $\boldsymbol{g}^\circ(s) := \boldsymbol{g}(t^\circ(s))$, and $\boldsymbol{L}^\circ(s) := \boldsymbol{L}(t^\circ(s))$. Since $\boldsymbol{L}^\circ \in H^1_{loc}([0,+\infty); BD(\Omega)')$ and $\boldsymbol{w}^\circ \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^n))$, the scalar function $s \mapsto \langle \boldsymbol{L}^\circ(s), \boldsymbol{w}^\circ(s)\rangle$ belongs to $H^1_{loc}([0,+\infty))$ and its derivative is given by $s \mapsto \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s)\rangle + \langle \boldsymbol{L}^\circ(s), \dot{\boldsymbol{w}}^\circ(s)\rangle$. Therefore we have

$$\begin{aligned} &- \int_0^S \langle \boldsymbol{L}^\circ(s), \dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{u}^\circ(s)\rangle\, ds + \langle \boldsymbol{L}^\circ(S), \boldsymbol{u}^\circ(S)\rangle - \langle \boldsymbol{L}^\circ(0), u_0\rangle = \\ &= \int_0^S \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle\, ds + \langle \boldsymbol{L}^\circ(S), \boldsymbol{u}^\circ(S) - \boldsymbol{w}^\circ(S)\rangle - \langle \boldsymbol{L}^\circ(0), u_0 - \boldsymbol{w}(0)\rangle\,. \end{aligned} \tag{4.16}$$

By (2.52), for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$ we have

$$\langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle = \langle \dot{\boldsymbol{f}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle + \langle \dot{\boldsymbol{g}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle_{\Gamma_1}\,. \tag{4.17}$$

By (2.55) $\dot{\boldsymbol{\chi}}^\circ(s) \in L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$, while (2.53) gives $-\mathrm{div}\,\dot{\boldsymbol{\chi}}^\circ(s) = \dot{\boldsymbol{f}}^\circ(s)$ in $\Omega$ and $[\dot{\boldsymbol{\chi}}^\circ(s)\nu] = \dot{\boldsymbol{g}}^\circ(s)$ on $\Gamma_1$ for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$. Therefore we can apply the integration-by-parts formula (2.16), which together with (4.17) gives

$$\langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle = -\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle + \langle \dot{\boldsymbol{\chi}}^\circ(s), E\boldsymbol{w}^\circ(s)\rangle \tag{4.18}$$

for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$. This proves that $s \mapsto \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ is measurable; by (2.56) and by (2.14), we deduce that $s \mapsto \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ belongs to $L^1_{loc}([0,+\infty))$.

Similarly we prove

$$\langle \boldsymbol{L}^\circ(s), \boldsymbol{u}^\circ(s) - \boldsymbol{w}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s)\rangle - \langle \boldsymbol{\chi}^\circ(s), E\boldsymbol{w}^\circ(s)\rangle \tag{4.19}$$

for every $s \in [0,+\infty)$. By (4.16), (4.18), and (4.19) we have

$$\begin{aligned}
&-\int_0^S \langle \boldsymbol{L}^\circ(s), \dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{u}^\circ(s)\rangle\, ds + \langle \boldsymbol{L}^\circ(S), \boldsymbol{u}^\circ(S)\rangle - \langle \boldsymbol{L}^\circ(0), u_0\rangle = \\
&= -\int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds - \int_0^S \langle \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds + \\
&\quad + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle - \langle \boldsymbol{\chi}(0), p_0\rangle + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{e}^\circ(S)\rangle - \langle \boldsymbol{\chi}(0), e_0\rangle.
\end{aligned}$$

Therefore (4.8) is equivalent to (4.15). □

**Theorem 4.5.** *Assume that the safe load condition* (2.53)-(2.56) *holds. Let* $\boldsymbol{w}$, $u_0$, $e_0$, $p_0$, $z_0$ *be as in Definition 4.1, and let*

$$\sigma_0 := \mathbb{C}e_0 \quad \text{and} \quad \zeta_0 := V(z_0). \tag{4.20}$$

*Assume that the following conditions are satisfied:*

(in1) *Weak kinematic admissibility:*

$$\begin{aligned} Eu_0 &= e_0 + p_0 \quad \text{in } \Omega\,, \\ p_0 &= (\boldsymbol{w}(0) - u_0) \odot \nu \mathcal{H}^{n-1} \quad \text{in } \Gamma_0\,. \end{aligned} \tag{4.21}$$

(in2) *Equilibrium condition:*

$$-\mathrm{div}\,\sigma_0 = \boldsymbol{f}(0) \quad \text{in } \Omega\,, \qquad [\sigma_0\nu] = \boldsymbol{g}(0) \quad \text{on } \Gamma_1. \tag{4.22}$$

(in3) *Stress constraint:*

$$\sigma_0 \in \mathcal{K}(\zeta_0). \tag{4.23}$$

*Then there exists a rescaled viscosity evolution with data* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$, *and initial condition* $(u_0, e_0, p_0, z_0, 0)$.

The proof will be given in Sections 5, 6, and 8.

## 5. Proof of Theorem 4.5: Part one

We start with a technical lemma.

**Lemma 5.1.** *Let* $u \in BD(\Omega)$. *Then there exists a sequence* $u_k$ *of Lipschitz functions from* $\overline{\Omega}$ *into* $\mathbb{R}^n$, *with* $u_k = 0$ *on* $\partial\Omega$, *such that*

$$u_k \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^n), \tag{5.1}$$

$$Eu_k \rightharpoonup (Eu)\llcorner\Omega - u \odot \nu\mathcal{H}^{n-1}\llcorner\Gamma_0 \quad \text{weakly}^* \text{ in } M_b(\Omega\cup\Gamma_0;\mathbb{R}^n). \tag{5.2}$$

*Proof.* It is enough to prove the theorem in a neighbourhood of each point of $\partial\Omega$: the global result can be obtained through a partition of unity. Since $\Omega$ has Lipschitz boundary, we may assume that it is the subgraph of a Lipschitz function, i.e.,

$$\Omega := \{x \in \mathbb{R}^n : \hat{x} \in A\,,\ a < x_n < h(\hat{x})\} \subset R := A\times(a,b)\,, \tag{5.3}$$

where $\hat{x} := (x_1, \dots, x_{n-1})$, $A$ is an open rectangle in $\mathbb{R}^{n-1}$, $a, b \in \mathbb{R}$, $a < b$, and $h \colon \overline{A} \to (a, b)$ is a Lipschitz function. We may also assume that $\operatorname{supp} u \subset\subset R$ and that $\Gamma_0 \subset R \cap \partial\Omega$.

Since $\Omega$ has Lipschitz boundary, by a standard approximation result (see, e.g., [25, Chapter II, Theorem 3.2]) there exists a sequence $v_k \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$, with such that

$$\begin{gathered} v_k \to u \quad \text{in } L^1(\Omega; \mathbb{R}^n), \\ Ev_k \rightharpoonup (Eu) \llcorner \Omega \quad \text{weakly}^* \text{ in } M_b(\bar{\Omega}; \mathbb{R}^n), \\ \|Ev_k\|_1 \to \|Eu\|_1, \end{gathered} \tag{5.4}$$

and therefore (see, e.g., [25, Chapter II, Theorem 3.1])

$$v_k \to u \quad \text{in } L^1(\partial\Omega; \mathbb{R}^n). \tag{5.5}$$

Since $\operatorname{supp} u \subset\subset R$, we may assume that $\operatorname{supp} v_k \subset\subset R$ for every $k$.

Using the special form (5.3) of $\Omega$, we can define a sequence of Lipschitz functions $\psi_j : \overline{\Omega} \to [0, 1]$ by $\psi_j(x) := \min\{j(h(\hat{x}) - x_n), 1\}$. Then $\psi_j = 0$ on the graph of $h$, $\psi_j \to 1$ on $\Omega$, and $\nabla\psi_j \rightharpoonup -\nu\mathcal{H}^{n-1} \llcorner \Gamma_0$ weakly$^*$ in $M_b(\Omega \cup \Gamma_0; \mathbb{R}^n)$. Therefore for every $k$ we have

$$\begin{gathered} \psi_j v_k \to v_k \quad \text{in } L^1(\Omega; \mathbb{R}^n), \\ E(\psi_j v_k) \rightharpoonup Ev_k - v_k \odot \nu\mathcal{H}^{n-1} \llcorner \Gamma_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{R}^n). \end{gathered} \tag{5.6}$$

Since the weak$^*$ convergence is metrisable on bounded sets of $M_b(\Omega \cup \Gamma_0; \mathbb{R}^n)$, it follows from (5.4), (5.5) and (5.6), that for every $k$ we can select $j_k$ so that (5.1) and (5.2) are satisfied by $u_k := \psi_{j_k} v_k$, which vanishes on $\partial\Omega$ by the properties of $\psi_j$ and $v_k$. □

*Proof of Theorem 4.5.* If we apply Lemma 5.1 to $u = u_0 - \boldsymbol{w}(0)$ we find a sequence $u_0^\varepsilon$ in $H^1(\Omega; \mathbb{R}^n)$ such that

$$u_0^\varepsilon = \boldsymbol{w}(0) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0, \tag{5.7}$$

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{weakly}^* \text{ in } BD(\Omega), \tag{5.8}$$

$$Eu_0^\varepsilon \rightharpoonup (Eu_0) \llcorner \Omega + (\boldsymbol{w}(0) - u_0) \odot \nu\mathcal{H}^{n-1} \llcorner \Gamma_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{R}^n). \tag{5.9}$$

We define $p_0^\varepsilon := Eu_0^\varepsilon - e_0$. From the weak kinematic admissibility condition (4.21), and from (5.9), we have

$$p_0^\varepsilon \rightharpoonup p_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{R}^n). \tag{5.10}$$

By Theorem 3.5 there exists an $\varepsilon$-viscoplastic evolution $(\boldsymbol{u}_\varepsilon, \boldsymbol{e}_\varepsilon, \boldsymbol{p}_\varepsilon, \boldsymbol{z}_\varepsilon)$ with boundary datum $\boldsymbol{w}$ and initial condition $(u_0^\varepsilon, e_0, p_0^\varepsilon, z_0)$. The energy equality (3.25), together with (2.46) and (2.57), implies that for every $T > 0$ there exists a constant $C_T$, independent of $\varepsilon$, such that

$$\sup_{t \in [0,T]} \|\boldsymbol{e}_\varepsilon(t)\|_2 \le C_T \quad \text{and} \quad \sup_{t \in [0,T]} \|\boldsymbol{\sigma}_\varepsilon(t)\|_2 \le C_T \tag{5.11}$$

(see the proof of (3.35)). The same equality and the same estimates, together with (5.11), give also that for every $T > 0$ there exists a constant $M_T$, independent of $\varepsilon$, such that

$$\int_0^T \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_1 \, dt \le M_T < +\infty\,. \tag{5.12}$$

Let $s_\varepsilon^\circ \colon [0, +\infty) \to [0, +\infty)$ be the absolutely continuous, increasing, and bijective function defined by

$$s_\varepsilon^\circ(t) := \int_0^t (\|\dot{\boldsymbol{p}}_\varepsilon(\tau)\|_1 + \|\dot{\boldsymbol{\chi}}(\tau)\|_\infty + \|E\dot{\boldsymbol{w}}(\tau)\|_2 + 1) \, d\tau. \tag{5.13}$$

It is easy to see that

$$s_\varepsilon^\circ(t_2) - s_\varepsilon^\circ(t_1) \ge t_2 - t_1 \qquad \text{for every } 0 \le t_1 < t_2 < +\infty. \tag{5.14}$$

Let $t_\varepsilon^\circ : [0, +\infty) \to [0, +\infty)$ be the inverse of $s_\varepsilon^\circ$. By (5.14) $t_\varepsilon^\circ$ satisfies

$$0 < t_\varepsilon^\circ(s_2) - t_\varepsilon^\circ(s_1) \le s_2 - s_1 \qquad \text{for every } 0 \le s_1 < s_2 < +\infty.$$

By the Arzelà-Ascoli Theorem we may assume that $t^\circ_\varepsilon$ converges uniformly on compact sets to a function $t^\circ\colon [0,+\infty)\to[0,+\infty)$ such that

$$0\le t^\circ(s_2)-t^\circ(s_1)\le s_2-s_1 \qquad \text{for every } 0\le s_1<s_2<+\infty.$$

We observe that $t^\circ(0)=0$. Let us prove that

$$t^\circ(s)\to+\infty \text{ when } s\to+\infty \tag{5.15}$$

Indeed, by (2.56), (5.11), (5.12), and (5.13), for every $T>0$ there exists a constant $s_T$, independent of $\varepsilon$, such that $s^\circ_\varepsilon(T)<s_T$. This gives $T\le t^\circ_\varepsilon(s_T)$ for every $\varepsilon$, which implies $T\le t^\circ(s_T)$, and concludes the proof of (5.15).

Define the rescaled functions on $[0,+\infty)$ by

$$\begin{gathered}\boldsymbol{u}^\circ_\varepsilon(s):=\boldsymbol{u}_\varepsilon(t^\circ_\varepsilon(s))\,,\quad \boldsymbol{e}^\circ_\varepsilon(s):=\boldsymbol{e}_\varepsilon(t^\circ_\varepsilon(s))\,,\quad \boldsymbol{p}^\circ_\varepsilon(s):=\boldsymbol{p}_\varepsilon(t^\circ_\varepsilon(s))\,,\quad \boldsymbol{z}^\circ_\varepsilon(s):=\boldsymbol{z}_\varepsilon(t^\circ_\varepsilon(s))\,,\\ \boldsymbol{f}^\circ_\varepsilon(s):=\boldsymbol{f}(t^\circ_\varepsilon(s))\,,\quad \boldsymbol{g}^\circ_\varepsilon(s):=\boldsymbol{g}(t^\circ_\varepsilon(s))\,,\quad \boldsymbol{w}^\circ_\varepsilon(s):=\boldsymbol{w}(t^\circ_\varepsilon(s))\,,\\ \boldsymbol{\sigma}^\circ_\varepsilon(s):=\boldsymbol{\sigma}_\varepsilon(t^\circ_\varepsilon(s))\,,\quad \boldsymbol{\zeta}^\circ_\varepsilon(s):=\boldsymbol{\zeta}_\varepsilon(t^\circ_\varepsilon(s))\,,\qquad \boldsymbol{\chi}^\circ_\varepsilon(s):=\boldsymbol{\chi}(t^\circ_\varepsilon(s))\,.\end{gathered} \tag{5.16}$$

Note that by (3.2)

$$\boldsymbol{\sigma}^\circ_\varepsilon(s):=\mathbb{C}\boldsymbol{e}^\circ_\varepsilon(s) \quad\text{and}\quad \boldsymbol{\zeta}^\circ_\varepsilon(s):=V(\boldsymbol{z}^\circ_\varepsilon(s)) \tag{5.17}$$

for every $s\in[0,+\infty)$. Since $t^\circ_\varepsilon(s)\to t^\circ(s)$ uniformly on compact sets, the continuity properties of $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, and $\boldsymbol{\chi}$ imply that for every $s\in[0,+\infty)$ we have that

$$\begin{aligned}&\boldsymbol{f}^\circ_\varepsilon(s)\to\boldsymbol{f}^\circ(s) \text{ strongly in } L^n(\Omega;\mathbb{R}^n)\,,\quad \boldsymbol{g}^\circ_\varepsilon(s)\to\boldsymbol{g}^\circ(s) \text{ strongly in } L^\infty(\Gamma_1;\mathbb{R}^n)\,,\\ &\boldsymbol{w}^\circ_\varepsilon(s)\to\boldsymbol{w}^\circ(s) \text{ strongly in } H^1(\Omega;\mathbb{R}^n)\,,\quad \boldsymbol{\chi}^\circ_\varepsilon(s)\to\boldsymbol{\chi}^\circ(s) \text{ strongly in } L^2(\Omega;\mathbb{R}^n)\,,\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}\boldsymbol{f}^\circ\in H^1_{loc}([0,+\infty);L^n(\Omega;\mathbb{R}^n))\,,\qquad & \boldsymbol{g}^\circ\in H^1_{loc}([0,+\infty);L^\infty(\Gamma_1;\mathbb{R}^n))\,,\\ \boldsymbol{w}^\circ\in H^1_{loc}([0,+\infty);H^1(\Omega;\mathbb{R}^n))\,,\qquad & \boldsymbol{\chi}^\circ\in H^1_{loc}([0,+\infty);L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\end{aligned}$$

are defined by

$$\boldsymbol{f}^\circ(s):=\boldsymbol{f}(t^\circ(s))\,,\quad \boldsymbol{g}^\circ(s):=\boldsymbol{g}(t^\circ(s))\,,\quad \boldsymbol{w}^\circ(s):=\boldsymbol{w}(t^\circ(s))\,,\quad \boldsymbol{\chi}^\circ(s):=\boldsymbol{\chi}(t^\circ(s)). \tag{5.19}$$

From the definitions of $s^\circ_\varepsilon$ and $t^\circ_\varepsilon$ we obtain easily that

$$\|\boldsymbol{p}^\circ_\varepsilon(s_2)-\boldsymbol{p}^\circ_\varepsilon(s_1)\|_1+\|\boldsymbol{\chi}^\circ_\varepsilon(s_2)-\boldsymbol{\chi}^\circ_\varepsilon(s_1)\|_\infty+\|E\boldsymbol{w}^\circ_\varepsilon(s_2)-E\boldsymbol{w}^\circ_\varepsilon(s_1)\|_2\le s_2-s_1 \tag{5.20}$$

for every $0\le s_1<s_2$, hence

$$\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1+\|\dot{\boldsymbol{\chi}}^\circ_\varepsilon(s)\|_\infty+\|E\dot{\boldsymbol{w}}^\circ_\varepsilon(s)\|_2\le 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } s\in[0,+\infty)\,. \tag{5.21}$$

Let $M$ is an upper bound of $\|p^\varepsilon_0\|_1$ (see (5.10)). From (5.20) we get

$$\|\boldsymbol{p}^\circ_\varepsilon(s)\|_1\le M+s \tag{5.22}$$

for every $s\in[0,+\infty)$. Passing to the limit in (5.20), we obtain

$$\|\boldsymbol{\chi}^\circ(s_2)-\boldsymbol{\chi}^\circ(s_1)\|_\infty+\|E\boldsymbol{w}^\circ(s_2)-E\boldsymbol{w}^\circ(s_1)\|_2\le s_2-s_1 \tag{5.23}$$

for every $0\le s_1<s_2$, hence

$$\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty+\|E\dot{\boldsymbol{w}}^\circ(s)\|_2\le 1 \qquad \text{for } \mathcal{L}^1\text{-a.e. } s\in[0,+\infty)\,. \tag{5.24}$$

For every $S>0$, let

$$\mathcal{B}_S:=\{p\in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}):\|p\|_1\le M+S\}\,.$$

There exists a distance $d_S$ on $\mathcal{B}_S$ inducing the weak$^*$ convergence such that

$$d_S(p,q)\le\|p-q\|_1 \quad \text{for every } p,q\in\mathcal{B}_S. \tag{5.25}$$

By (5.20) we have that $\boldsymbol{p}^\circ_\varepsilon(s)\in\mathcal{B}_S$ for every $s\in[0,S]$ and every $\varepsilon>0$. By (5.20) and (5.25), the sequence $\boldsymbol{p}^\circ_\varepsilon(s)$ is equicontinuous on $[0,S]$ with respect to the distance $d_S$. We then apply the Arzelà-Ascoli Theorem for every $S>0$ and we find that there exists a subsequence, still denoted by $\boldsymbol{p}^\circ_\varepsilon$, and a function $\boldsymbol{p}^\circ\colon [0,+\infty) \to M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ such that

$$\boldsymbol{p}^\circ_\varepsilon(s) \rightharpoonup \boldsymbol{p}^\circ(s) \text{ weakly}^* \text{ in } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}) \tag{5.26}$$

for every $s\in[0,+\infty)$. By lower semicontinuity we obtain from (5.20)

$$\|\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\|_1 \le s_2 - s_1 \tag{5.27}$$

for every $0\le s_1<s_2$, hence

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \le 1 \qquad \text{for } \mathcal{L}^1\text{-a.e. } s\in[0,+\infty)\,. \tag{5.28}$$

where the time derivative $\dot{\boldsymbol{p}}^\circ(s)$ is defined as in (2.3) with $X = M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ and $Y = C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$. Moreover, from (5.20) and (5.26) we obtain that

$$\boldsymbol{p}^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \boldsymbol{p}^\circ(s) \text{ weakly}^* \text{ in } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}) \tag{5.29}$$

for every $s\in[0,+\infty)$ and every $s_\varepsilon\to s$.

We now show that for every $s\in[0,+\infty)$ there exist $\boldsymbol{e}^\circ(s)\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\boldsymbol{u}^\circ(s)\in BD(\Omega)$ such that $(\boldsymbol{u}^\circ(s),\boldsymbol{e}^\circ(s),\boldsymbol{p}^\circ(s),\boldsymbol{w}^\circ(s))$ satisfies the weak kinematic admissibility condition (4.4), $\boldsymbol{\sigma}^\circ(s) := \mathbb{C}\,\boldsymbol{e}^\circ(s)$ satisfies the equilibrium condition (4.5), and

$$\boldsymbol{e}^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \boldsymbol{e}^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{5.30}$$

$$\boldsymbol{u}^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \boldsymbol{u}^\circ(s) \text{ weakly}^* \text{ in } BD(\Omega)\,, \tag{5.31}$$

for every $s_\varepsilon\to s$.

Let us fix $s\in[0,+\infty)$. By (5.11) the sequence $\|\boldsymbol{e}^\circ_\varepsilon(s)\|_2$ is bounded uniformly with respect to $\varepsilon$, thus there exists a subsequence $\boldsymbol{e}^\circ_{\varepsilon_j}(s)$ of $\boldsymbol{e}^\circ_\varepsilon(s)$, possibly depending on $s$, and a function $\boldsymbol{e}^\circ(s)\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ such that

$$\boldsymbol{e}^\circ_{\varepsilon_j}(s) \rightharpoonup \boldsymbol{e}^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,. \tag{5.32}$$

By (5.26) and (5.32), the kinematic admissibility condition (3.3) implies that the sequence $\boldsymbol{u}_{\varepsilon_j}(s)$ is bounded in $BD(\Omega)$. Therefore, up to extracting a further subsequence, it converges weakly$^*$ in $BD(\Omega)$ to a function $\boldsymbol{u}^\circ(s)\in BD(\Omega)$ such that $E\boldsymbol{u}^\circ(s) = \boldsymbol{e}^\circ(s)+\boldsymbol{p}^\circ(s)$ in $\Omega$. By considering suitable extensions and arguing as in [5, Lemma 2.1] we obtain also that $\boldsymbol{p}^\circ(s) = (\boldsymbol{w}^\circ(s)-\boldsymbol{u}^\circ(s))\odot\nu\mathcal{H}^{n-1}$ in $\Gamma_0$. Therefore weak kinematic admissibility condition (4.4) is satisfied.

Passing to the limit in (3.4) we obtain the equilibrium condition (4.5). This implies

$$\mathcal{Q}(\boldsymbol{e}^\circ(s)) \le \mathcal{Q}(\boldsymbol{e}^\circ(s)+E\varphi) - \langle \boldsymbol{f}^\circ(s),\varphi\rangle_\Omega - \langle \boldsymbol{g}^\circ(s),\varphi\rangle_{\Gamma_1} \tag{5.33}$$

for every $\varphi\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. By strict convexity the inequality is strict, unless $E\varphi = 0$ $\mathcal{L}^n$-a.e. in $\Omega$. It remains to prove (5.30) and (5.31) for an arbitrary sequence $s_\varepsilon\to s$. As in the previous step, we see that $\|\boldsymbol{e}^\circ_\varepsilon(s_\varepsilon)\|_2$ is bounded uniformly with respect to $\varepsilon$. Let $\boldsymbol{e}^\circ_{\hat\varepsilon_j}(s_{\hat\varepsilon_j})$ be a subsequence of $\boldsymbol{e}^\circ_\varepsilon(s_\varepsilon)$ which converges to a function $\hat{\boldsymbol{e}}(s)$ weakly in $L^2(\Omega;\mathbb{R}^n)$. The previous arguments, together with (5.29), show that there exists a function $\hat{\boldsymbol{u}}(s)\in BD(\Omega)$ such that $\boldsymbol{u}^\circ_{\hat\varepsilon_j}(s_{\hat\varepsilon_j}) \rightharpoonup \hat{\boldsymbol{u}}(s)$ weakly$^*$ in $BD(\Omega)$, $E\hat{\boldsymbol{u}}(s) = \hat{\boldsymbol{e}}(s)+\boldsymbol{p}^\circ(s)$ in $\Omega$, and $\boldsymbol{p}^\circ(s) = (\boldsymbol{w}^\circ(s)-\hat{\boldsymbol{u}}(s))\odot\nu\mathcal{H}^{n-1}$ in $\Gamma_0$. By difference we obtain that $E(\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s)) = \hat{\boldsymbol{e}}(s)-\boldsymbol{e}^\circ(s)$ in $\Omega$ and $(\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s))\odot\nu = 0$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma_0$. By (2.2) we have $\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s)\in H^1(\Omega;\mathbb{R}^n)$ and $\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s) = 0$ on $\Gamma_0$.

By (5.33) we have $\mathcal{Q}(\boldsymbol{e}^\circ(s)) \le \mathcal{Q}(\hat{\boldsymbol{e}}(s)) - \langle \boldsymbol{f}^\circ(s),\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s)\rangle_\Omega - \langle \boldsymbol{g}^\circ(s),\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s)\rangle_{\Gamma_1}$. Exchanging the roles of $\boldsymbol{e}^\circ(s)$ and $\hat{\boldsymbol{e}}(s)$ we obtain $\mathcal{Q}(\boldsymbol{e}^\circ(s)) = \mathcal{Q}(\hat{\boldsymbol{e}}(s)) - \langle \boldsymbol{f}^\circ(s),\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s)\rangle_\Omega - \langle \boldsymbol{g}^\circ(s),\hat{\boldsymbol{u}}(s)-\boldsymbol{u}^\circ(s)\rangle_{\Gamma_1}$. The strict convexity argument mentioned after (5.33) yields $\boldsymbol{e}^\circ(s) = \hat{\boldsymbol{e}}(s)$ $\mathcal{L}^n$-a.e. in $\Omega$, which in turn gives $\boldsymbol{u}^\circ(s) = \hat{\boldsymbol{u}}(s)$ $\mathcal{L}^n$-a.e. in $\Omega$. This shows that the limit does not depend on the subsequence, and concludes the proof of (5.30) and (5.31).

Let us prove that

$$\boldsymbol{e}^\circ \text{ is weakly continuous in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}). \tag{5.34}$$

Let $s_k$ be a sequence converging to $s$. For every fixed $k$, we can apply (5.30) with $s_\varepsilon = s_k$ for every $\varepsilon$, and we find $\varepsilon_k > 0$ such that $d_w(\boldsymbol{e}^\circ_{\varepsilon_k}(s_k), \boldsymbol{e}^\circ(s_k)) < \frac{1}{k}$, where $d_w$ is a distance which metrises the weak topology on bounded subsets of $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. By (5.30), $\boldsymbol{e}^\circ_{\varepsilon_k}(s_k) \rightharpoonup \boldsymbol{e}^\circ(s)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, so that the previous inequality gives $\boldsymbol{e}^\circ(s_k) \rightharpoonup \boldsymbol{e}^\circ(s)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. This concludes the proof of the weak continuity of $\boldsymbol{e}^\circ$. In a similar way we can prove that $\boldsymbol{u}^\circ\colon [0,+\infty) \to BD(\Omega)$ is weakly$^*$ continuous.

Define now for every $s \in [0, +\infty)$

$$\boldsymbol{a}^\circ_\varepsilon(s) := \boldsymbol{a}_\varepsilon(t^\circ_\varepsilon(s)) = \rho \star \operatorname{tr} \boldsymbol{\sigma}^\circ_\varepsilon(s)\,, \tag{5.35}$$

$$\boldsymbol{a}^\circ(s) := \rho \star \operatorname{tr} \boldsymbol{\sigma}^\circ(s)\,, \tag{5.36}$$

so that, by (3.6) and (5.16),

$$\dot{\boldsymbol{z}}^\circ_\varepsilon(s) = \rho \star \big(\boldsymbol{a}^\circ_\varepsilon(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\big) \tag{5.37}$$

for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$. Using (2.49) and (5.11), we can prove that for every $S > 0$ there exists a constant $C^\circ_S$, independent of $\varepsilon$, such that

$$\sup_{s\in[0,S]} \|\boldsymbol{a}^\circ_\varepsilon(s)\|_\infty \le C^\circ_S\,, \qquad \sup_{s\in[0,S]} \|\nabla \boldsymbol{a}^\circ_\varepsilon(s)\|_\infty \le C^\circ_S\,. \tag{5.38}$$

Therefore for every $s$, the functions $\boldsymbol{a}^\circ_\varepsilon(s)$ are equicontinuous and equibounded on $\overline{\Omega}$. Since $\boldsymbol{\sigma}^\circ_\varepsilon(s) \rightharpoonup \boldsymbol{\sigma}^\circ(s)$ weakly in $L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$, the sequence $\boldsymbol{a}^\circ_\varepsilon(s)$ converges to $\boldsymbol{a}^\circ(s)$ pointwise in $\overline{\Omega}$. It follows that

$$\boldsymbol{a}^\circ_\varepsilon(s) \to \boldsymbol{a}^\circ(s) \quad \text{strongly in } C^0(\overline{\Omega}) \tag{5.39}$$

for every $s \in [0, +\infty)$.

By (5.28) and (5.38) we have $\|\boldsymbol{a}^\circ_\varepsilon(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 \le \sqrt{n} C^\circ_S$ for $\mathcal{L}^1$-a.e. $s \in [0, S]$, and hence by (2.49) and (5.37)

$$\|\dot{\boldsymbol{z}}^\circ_\varepsilon(s)\|_\infty \le \sqrt{n} C^\circ_S \|\rho\|_\infty \quad \text{and} \quad \|\nabla \dot{\boldsymbol{z}}^\circ_\varepsilon(s)\|_\infty \le \sqrt{n} C^\circ_S \|\nabla\rho\|_\infty\,. \tag{5.40}$$

This implies that

$$\|\boldsymbol{z}^\circ_\varepsilon(s_2) - \boldsymbol{z}^\circ_\varepsilon(s_1)\|_\infty + \|\nabla \boldsymbol{z}^\circ_\varepsilon(s_2) - \nabla \boldsymbol{z}^\circ_\varepsilon(s_1)\|_\infty \le M^\circ_S |s_2 - s_1| \tag{5.41}$$

for every $s_1, s_2 \in [0, S]$, where $M^\circ_S := \sqrt{n} C^\circ_S (\|\rho\|_\infty + \|\nabla\rho\|_\infty)$.

We can then apply the Arzelà-Ascoli Theorem as in the proof of (5.26). This gives a subsequence, still denoted $\boldsymbol{z}^\circ_\varepsilon$, such that $\boldsymbol{z}^\circ_\varepsilon(s) \rightharpoonup \boldsymbol{z}^\circ(s)$ weakly$^*$ in $W^{1,\infty}(\Omega)$ for every $s \in [0, +\infty)$, which implies

$$\boldsymbol{z}^\circ_\varepsilon(s) \to \boldsymbol{z}^\circ(s) \text{ strongly in } C^0(\overline{\Omega})\,. \tag{5.42}$$

Using (5.41), we deduce that

$$\boldsymbol{z}^\circ_\varepsilon \to \boldsymbol{z}^\circ \text{ strongly in } C^0([0,S]; C^0(\overline{\Omega}))\,. \tag{5.43}$$

Passing to the limit in (5.41), we get

$$\|\boldsymbol{z}^\circ(s_2) - \boldsymbol{z}^\circ(s_1)\|_\infty + \|\nabla \boldsymbol{z}^\circ(s_2) - \nabla \boldsymbol{z}^\circ(s_1)\|_\infty \le M^\circ_S |s_2 - s_1| \tag{5.44}$$

for every $s_1, s_2 \in [0, S]$.

Let us fix $r > n$. Since $W^{1,r}(\Omega)$ is reflexive, it follows from (5.44) that the strong $W^{1,r}$ limit

$$\dot{\boldsymbol{z}}^\circ(s) := s\text{-}\lim_{h\to 0} \frac{\boldsymbol{z}^\circ(s+h) - \boldsymbol{z}^\circ(s)}{h} \tag{5.45}$$

exists for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$, and that $\dot{\boldsymbol{z}}^\circ \in L^\infty_{loc}([0, +\infty); W^{1,r}(\Omega))$. Since the embedding of $W^{1,r}(\Omega)$ into $C^0(\overline{\Omega})$ is continuous, the limit in (5.45) takes place in $C^0(\overline{\Omega})$ and $\dot{\boldsymbol{z}}^\circ \in L^\infty_{loc}([0, +\infty); C^0(\overline{\Omega}))$. Moreover, from (5.41) and (5.42) we obtain that

$$\boldsymbol{z}^\circ_\varepsilon(s_\varepsilon) \to \boldsymbol{z}^\circ(s) \text{ strongly in } C^0(\overline{\Omega}) \tag{5.46}$$

for every $s \in [0, +\infty)$ and every $s_\varepsilon \to s$.

The initial condition $(\text{ev0})^\circ$ follows easily from the definitions of $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, and $t^\circ$, thanks to (5.8) and (5.10).

For every $s \in [0,+\infty)$ let us define

$$\boldsymbol{\zeta}^\circ(s) := V(\boldsymbol{z}^\circ(s)) . \tag{5.47}$$

To prove $(\text{ev3}')^\circ$ we need the following Lemmas. □

We start with an elementary result about the convergence of inverse functions.

**Lemma 5.2.** *For every* $t \in [0,+\infty)$ *let*

$$s^\circ_-(t) := \sup\{s \in [0,+\infty) : t^\circ(s) < t\} , \tag{5.48}$$

$$s^\circ_+(t) := \inf\{s \in [0,+\infty) : t^\circ(s) > t\} , \tag{5.49}$$

*with the convention* $\sup \emptyset = 0$*, so that* $s^\circ_-(0) = 0$*. Then*

$$s^\circ_-(t) \le s^\circ_+(t) \quad \textit{and} \quad t^\circ(s^\circ_-(t)) = t = t^\circ(s^\circ_+(t)) \tag{5.50}$$

*for every* $t \in [0,+\infty)$*, and*

$$s^\circ_-(t^\circ(s)) \le s \le s^\circ_+(t^\circ(s)) \tag{5.51}$$

*for every* $s \in [0,+\infty)$*. Moreover the set*

$$S^\circ := \{t \in [0,+\infty) : s^\circ_-(t) < s^\circ_+(t)\} \tag{5.52}$$

*is at most countable, and the set* $U^\circ$ *introduced in* (4.7) *satisfies*

$$U^\circ = \bigcup_{t \in S^\circ} (s^\circ_-(t), s^\circ_+(t)) . \tag{5.53}$$

*Finally*

$$s^\circ_-(t) \le \liminf_{\varepsilon \to 0} s^\circ_\varepsilon(t) \le \limsup_{\varepsilon \to 0} s^\circ_\varepsilon(t) \le s^\circ_+(t) \tag{5.54}$$

*for every* $t \in [0,+\infty)$*.*

*Proof.* All assertions are well-known properties of monotone functions, except for the last one. We only prove the first inequality in (5.54). If $s^\circ_-(t) = 0$ the inequality is obvious. If $s^\circ_-(t) > 0$ we fix $0 < s < s^\circ_-(t)$. By the definition of $s^\circ_-$, we have $t^\circ(s) < t$; for $\varepsilon$ small enough, this implies $t^\circ_\varepsilon(s) < t$, hence $s < s^\circ_\varepsilon(t)$. This gives $s \le \liminf_\varepsilon s^\circ_\varepsilon(t)$, and the conclusion follows form the arbitrariness of $s < s^\circ_-(t)$. □

**Lemma 5.3.** *Let* $t \in [0,+\infty) \setminus S^\circ$*, where* $S^\circ$ *is the set defined in* (5.52)*. Then*

$$\boldsymbol{u}_\varepsilon(t) \rightharpoonup \boldsymbol{u}^\circ(s^\circ_-(t)) \ \textit{weakly}^* \ \textit{in } BD(\Omega) , \tag{5.55}$$

$$\boldsymbol{e}_\varepsilon(t) \rightharpoonup \boldsymbol{e}^\circ(s^\circ_-(t)) \ \textit{weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) , \tag{5.56}$$

$$\boldsymbol{p}_\varepsilon(t) \rightharpoonup \boldsymbol{p}^\circ(s^\circ_-(t)) \ \textit{weakly}^* \ \textit{in } M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}) , \tag{5.57}$$

$$\boldsymbol{z}_\varepsilon(t) \to \boldsymbol{z}^\circ(s^\circ_-(t)) \ \textit{strongly in } C^0(\bar\Omega) . \tag{5.58}$$

*Proof.* Since $t \notin S^\circ$, Lemma 5.2 gives $s^\circ_\varepsilon(t) \to s^\circ_-(t)$. By (5.16) we have $\boldsymbol{u}_\varepsilon(t) = \boldsymbol{u}^\circ_\varepsilon(s^\circ_\varepsilon(t))$, $\boldsymbol{e}_\varepsilon(t) = \boldsymbol{e}^\circ_\varepsilon(s^\circ_\varepsilon(t))$, $\boldsymbol{p}_\varepsilon(t) = \boldsymbol{p}^\circ_\varepsilon(s^\circ_\varepsilon(t))$, $\boldsymbol{z}_\varepsilon(t) = \boldsymbol{z}^\circ_\varepsilon(s^\circ_\varepsilon(t))$. Therefore the conclusion follows from (5.29), (5.30), (5.31), and (5.46). □

*Proof of Theorem 4.5 (continuation).* By (2.46), (2.57), (3.25), and (5.11), for every $T > 0$ we have

$$\varepsilon^2 \int_0^T \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_2^2 dt \to 0 .$$

This implies that a subsequence, not relabelled, satisfies

$$\varepsilon \dot{\boldsymbol{p}}_\varepsilon(t) \to 0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$$

for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$. This fact, together with Lemma 5.3, yields that

$$\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \rightharpoonup \boldsymbol{\sigma}^\circ(s^\circ_-(t)) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),$$
$$\boldsymbol{\zeta}_\varepsilon(t) \to \boldsymbol{\zeta}^\circ(s^\circ_-(t)) \text{ strongly in } C^0(\bar{\Omega}),$$

for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$. Since $K$ is convex, the inclusion $\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \in \mathcal{K}(\boldsymbol{\zeta}_\varepsilon(t))$, established in (3.24), passes to the limit and we obtain

$$\boldsymbol{\sigma}^\circ(s^\circ_-(t)) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s^\circ_-(t))). \tag{5.59}$$

By (5.34), (5.44), and the left continuity of $s^\circ$, (5.59) holds for every $t \in [0,+\infty)$. A similar proof shows that

$$\boldsymbol{\sigma}^\circ(s^\circ_+(t)) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s^\circ_+(t))). \tag{5.60}$$

Let $U^\circ$ be the set defined in (4.7) and let $s \in [0,+\infty)\setminus U^\circ$. By (5.51) and (5.53) we have

$$\text{either} \quad s = s^\circ_-(t^\circ(s)) \quad \text{or} \quad s = s^\circ_+(t^\circ(s)). \tag{5.61}$$

The partial stress constraint (ev3$'$)$^\circ$ of Definition 4.1 follows now from (5.59), (5.60), and (5.61).

It remains to prove the energy-dissipation balance (ev3$''$)$^\circ$, the partial flow-rule (ev3$'''$)$^\circ$, and the evolution law for the internal variable (ev4)$^\circ$. In Section 6 we will prove that the left-hand side of (4.8) is less than or equal to the right-hand side. The opposite inequality will be proved in Secton 8, together with the partial flow-rule. The evolution law for the internal variable will be obtained at the end of Section 6. □

## 6. Proof of the energy inequality and of the evolution law

The goal of the first part of this section is to prove that the functions $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, $\boldsymbol{w}^\circ$, $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\boldsymbol{\chi}^\circ$ introduced in the previous section satisfy the energy inequality

$$\begin{aligned}\mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s)) + \langle\dot{\boldsymbol{\chi}}^\circ(s),\boldsymbol{p}^\circ(s)\rangle\Big)\,ds - \\ - \langle\boldsymbol{\chi}^\circ(S),\boldsymbol{p}^\circ(S)\rangle + \langle\chi_0,p_0\rangle + \int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\,ds \le \\ \le \int_0^S \langle\boldsymbol{\sigma}^\circ(s)-\boldsymbol{\chi}^\circ(s),E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds - \int_0^S\langle\dot{\boldsymbol{\chi}}^\circ(s),\boldsymbol{e}^\circ(s)\rangle\,ds + \langle\boldsymbol{\chi}^\circ(S),\boldsymbol{e}^\circ(S)\rangle - \langle\chi_0,e_0\rangle\end{aligned} \tag{6.1}$$

for every $S>0$, where $\chi_0 := \boldsymbol{\chi}(0) = \boldsymbol{\chi}^\circ(0)$. To this aim we prove four lower semicontinuity results concerning the integrals in the left-hand side of (6.1) and the functions $\boldsymbol{p}^\circ_\varepsilon$, $\boldsymbol{\sigma}^\circ_\varepsilon$, $\boldsymbol{\zeta}^\circ_\varepsilon$, and $\boldsymbol{\chi}^\circ_\varepsilon$ defined in (5.16).

**Lemma 6.1.** *For every $S>0$, $\psi \in C^0(\overline{\Omega})^+$, and $\boldsymbol{\zeta} \in C^0([0,+\infty);C^0(\overline{\Omega})^+)$ we have*

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}(s))\,ds \le \liminf_{\varepsilon\to 0}\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}(s))\,ds. \tag{6.2}$$

*Proof.* Since the function $s \mapsto \dot{\boldsymbol{p}}^\circ(s)$ is weakly$^*$ measurable from $[0,+\infty)$ to $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, it is possible to define $\mu_\varepsilon, \mu \in M_b((0,S)\times(\Omega\cup\Gamma_0);\mathbb{M}^{n\times n}_{sym})$ by setting

$$\langle\varphi,\mu_\varepsilon\rangle := \int_0^S \langle\varphi(s,\cdot),\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\,ds \quad \text{and} \quad \langle\varphi,\mu\rangle := \int_0^S\langle\varphi(s,\cdot),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds$$

for every $\varphi \in C^0_0((0,S)\times(\Omega\cup\Gamma_0);\mathbb{M}^{n\times n}_{sym})$. If $\varphi \in C^1_c((0,S)\times(\Omega\cup\Gamma_0);\mathbb{M}^{n\times n}_{sym})$, we have

$$\langle\varphi,\mu_\varepsilon\rangle = -\int_0^S\langle\partial_s\varphi(s,\cdot),\boldsymbol{p}^\circ_\varepsilon(s)\rangle\,ds \to -\int_0^S\langle\partial_s\varphi(s,\cdot),\boldsymbol{p}^\circ(s)\rangle\,ds = \langle\varphi,\mu\rangle,$$

by (5.22) and (5.26). Since $\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 \le 1$ and $\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \le 1$ by (5.21) and (5.28), by uniform approximation we obtain $\langle\varphi,\mu_\varepsilon\rangle \to \langle\varphi,\mu\rangle$ for every $\varphi \in C^0_0((0,S)\times(\Omega\cup\Gamma_0);\mathbb{M}^{n\times n}_{sym}))$, i.e.,

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly}^* \text{ in } M_b((0,S)\times(\Omega\cup\Gamma_0);\mathbb{M}^{n\times n}_{sym}). \tag{6.3}$$

Since $s \mapsto |\dot{\boldsymbol{p}}^\circ(s)|$ is weakly$^*$ measurable from $[0,+\infty)$ to $M_b(\Omega\cup\Gamma_0)$, we define $\lambda_\varepsilon\,,\lambda \in M_b((0,S)\times(\Omega\cup\Gamma_0))$ by setting

$$\langle \phi, \lambda_\varepsilon\rangle := \int_0^S \langle \phi(s,\cdot), |\dot{\boldsymbol{p}}_\varepsilon^\circ(s)|\rangle\, ds \quad \text{and} \quad \langle \phi, \lambda\rangle := \int_0^S \langle \phi(s,\cdot), |\dot{\boldsymbol{p}}^\circ(s)|\rangle\, ds$$

for every $\phi \in C_0^0((0,S)\times(\Omega\cup\Gamma_0))$. It is easy to see that $\mu_\varepsilon \ll \lambda_\varepsilon$ and $\mu \ll \lambda$. Moreover

$$\frac{d\mu_\varepsilon}{d\lambda_\varepsilon}(s,x) = \frac{d\dot{\boldsymbol{p}}_\varepsilon^\circ(s)}{d|\dot{\boldsymbol{p}}_\varepsilon^\circ(s)|}(x) \quad \text{and} \quad \frac{d\mu}{d\lambda}(s,x) = \frac{d\dot{\boldsymbol{p}}^\circ(s)}{d|\dot{\boldsymbol{p}}^\circ(s)|}(x)\,.$$

Using the definition of $\mathcal{H}$, see (2.35), it follows that

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}(s))\, ds = \int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu_\varepsilon}{d\lambda_\varepsilon}(s,x), \boldsymbol{\zeta}(s,x))\, d\lambda_\varepsilon(s,x)\,, \qquad (6.4)$$

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}(s))\, ds = \int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu}{d\lambda}(s,x), \boldsymbol{\zeta}(s,x))\, d\lambda(s,x)\,. \qquad (6.5)$$

By (6.3) we can now apply Reshetnyak's lower semicontinuity Theorem [20, Theorem 2] and we obtain

$$\begin{aligned} &\int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu}{d\lambda}(s,x), \boldsymbol{\zeta}(s,x))\, d\lambda(s,x) \leq \\ &\leq \liminf_{\varepsilon\to 0} \int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu_\varepsilon}{d\lambda_\varepsilon}(s,x), \boldsymbol{\zeta}(s,x))\, d\lambda_\varepsilon(s,x)\,. \end{aligned} \qquad (6.6)$$

Inequality (6.2) follows now from (6.4), (6.5), and (6.6). □

**Lemma 6.2.** *For every* $S>0$*, and every* $\psi \in C^0(\overline{\Omega})^+$*, we have*

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds \leq \liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}_\varepsilon^\circ(s))\, ds\,. \qquad (6.7)$$

*Proof.* As $\boldsymbol{\zeta}^\circ \in C^0([0,+\infty); C^0(\overline{\Omega}))$ by (5.44) and (5.47), we can apply Lemma 6.1 and we obtain

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds \leq \liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds\,,$$

for every $S>0$. Using (2.27), (2.29), (5.21), and the definition of $\mathcal{H}$ we obtain for every $s\in[0,+\infty)$

$$|\mathcal{H}(\psi\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \mathcal{H}(\psi\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}_\varepsilon^\circ(s))| \leq M_K \|\psi\|_\infty \|\boldsymbol{\zeta}_\varepsilon^\circ(s) - \boldsymbol{\zeta}^\circ(s)\|_\infty\,.$$

By (5.17), (5.43), and (5.47) $\|\boldsymbol{\zeta}_\varepsilon^\circ(s) - \boldsymbol{\zeta}^\circ(s)\|_\infty \to 0$ uniformly on compact sets, and inequality (6.7) follows. □

**Lemma 6.3.** *For every* $S>0$*, we have*

$$\begin{aligned} &\int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\Big)\, ds - \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle \leq \\ &\qquad \leq \liminf_{\varepsilon\to 0} \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}_\varepsilon^\circ(s)) - \langle \boldsymbol{\chi}_\varepsilon^\circ(s), \dot{\boldsymbol{p}}_\varepsilon^\circ(s)\rangle\Big)\, ds\,. \end{aligned} \qquad (6.8)$$

*Proof.* We consider a sequence $\psi_k \in C^\infty(\bar{\Omega})$, with $0\le \psi_k \le 1$ in $\bar{\Omega}$ and $\psi_k = 0$ in a neighbourhood of $\bar{\Gamma}_1$, such that $\psi_k(x)\to 1$ for every $x \in \Omega\cup\Gamma_0$. By (2.57) the function $H(\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \boldsymbol{\chi}_\varepsilon^\circ(s){:}\, \dot{\boldsymbol{p}}_\varepsilon^\circ(s)$ is positive $\mathcal{L}^n$-a.e. in $\Omega$ for every $s\in[0,+\infty)$, hence

$$\mathcal{H}(\psi_k\, \dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \langle \psi_k\, \boldsymbol{\chi}_\varepsilon^\circ(s), \dot{\boldsymbol{p}}_\varepsilon^\circ(s)\rangle \leq \mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \langle \boldsymbol{\chi}_\varepsilon^\circ(s), \dot{\boldsymbol{p}}_\varepsilon^\circ(s)\rangle\,. \qquad (6.9)$$

Integrating by parts in time, we have

$$\begin{aligned}\int_0^S \langle \psi_k\, \boldsymbol{\chi}^\circ_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds = -\int_0^S \langle \psi_k\, \dot{\boldsymbol{\chi}}^\circ_\varepsilon(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle\, ds + \\ + \langle \psi_k\, \boldsymbol{\chi}^\circ_\varepsilon(S), \boldsymbol{p}^\circ_\varepsilon(S)\rangle - \langle \psi_k\, \chi_0, p^\varepsilon_0\rangle\,.\end{aligned} \tag{6.10}$$

Performing the change of variables $t = t^\circ_\varepsilon(s)$, we get

$$\int_0^S \langle \psi_k\, \dot{\boldsymbol{\chi}}^\circ_\varepsilon(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle\, ds = \int_0^{T_\varepsilon} \langle \psi_k\, \dot{\boldsymbol{\chi}}(t), \boldsymbol{p}_\varepsilon(t)\rangle\, dt\,, \tag{6.11}$$

where $T_\varepsilon := t^\circ_\varepsilon(S)$. As $\psi_k = 0$ on $\Gamma_1$, integrating by parts in space and using (3.3), we obtain for every $t \in [0, T_\varepsilon]$

$$\begin{aligned}\langle \psi_k\, \dot{\boldsymbol{\chi}}(t), \boldsymbol{p}_\varepsilon(t)\rangle = -\langle \psi_k\, \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t) - E\boldsymbol{w}(t)\rangle - \\ - \langle \dot{\boldsymbol{\chi}}(t), (\boldsymbol{u}_\varepsilon(t) - \boldsymbol{w}(t)) \odot \nabla\psi_k\rangle + \langle \dot{\boldsymbol{f}}(t), \psi_k\, (\boldsymbol{u}_\varepsilon(t) - \boldsymbol{w}(t))\rangle\,.\end{aligned}$$

which, thanks to Lemma 5.3 converges to

$$\begin{aligned}-\langle \psi_k\, \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}^\circ(s^\circ_-(t)) - E\boldsymbol{w}(t)\rangle - \langle \dot{\boldsymbol{\chi}}(t), (\boldsymbol{u}^\circ(s^\circ_-(t)) - \boldsymbol{w}(t)) \odot \nabla\psi_k\rangle + \\ + \langle \dot{\boldsymbol{f}}(t), \psi_k\, (\boldsymbol{u}^\circ(s^\circ_-(t)) - \boldsymbol{w}(t))\rangle\,.\end{aligned}$$

By (2.15), this expression equals to $\langle [\dot{\boldsymbol{\chi}}(t) : \boldsymbol{p}^\circ(s^\circ_-(t))], \psi_k\rangle$; as $\|\boldsymbol{p}_\varepsilon(t)\|_1$ is bounded by (5.10) and (5.12), while $\|\dot{\boldsymbol{\chi}}(t)\|_\infty$ is locally integrable by (2.56), the Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \psi_k\, \dot{\boldsymbol{\chi}}(t), \boldsymbol{p}_\varepsilon(t)\rangle\, dt = \int_0^T \langle [\dot{\boldsymbol{\chi}}(t) : \boldsymbol{p}(s^\circ_-(t))], \psi_k\rangle\, dt\,.$$

Let $\boldsymbol{\omega}(t) := \dot{\boldsymbol{\chi}}(t)$ if the derivative exists at $t$, and $\boldsymbol{\omega}(t) = 0$ otherwise. By (2.4) and (5.19) we get

$$\dot{\boldsymbol{\chi}}^\circ(s) = \boldsymbol{\omega}(t^\circ(s))\, \dot{t}^\circ(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0, S].$$

This equality, together with the change of variables formula (2.5), yields

$$\begin{aligned}\int_0^T \langle [\dot{\boldsymbol{\chi}}(t) : \boldsymbol{p}(s^\circ_-(t))], \psi_k\rangle\, dt = \int_0^T \langle [\boldsymbol{\omega}(t) : \boldsymbol{p}(s^\circ_-(t))], \psi_k\rangle\, dt = \\ = \int_0^S \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s^\circ_-(t^\circ(s)))], \psi_k\rangle\, ds = \int_0^S \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s)], \psi_k\rangle\, ds\,,\end{aligned}$$

where the last equality follows from the fact that $\dot{\boldsymbol{\chi}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in U^\circ$ and that $s^\circ_-(t^\circ(s)) = s$ for $\mathcal{L}^1$-a.e. $s \in [0, S] \setminus U^\circ$ (indeed, by (5.51) and (5.53), the only exceptions are the points of the form $s = s^\circ_+(t)$ for $t \in S^\circ$). We conclude that

$$\lim_{\varepsilon\to 0} \int_0^S \langle \psi_k\, \dot{\boldsymbol{\chi}}^\circ_\varepsilon(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle\, ds = \int_0^S \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s)], \psi_k\rangle\, ds\,. \tag{6.12}$$

Another integration-by-parts argument, using (5.7), (5.8), (5.10), and (5.18), shows that

$$\lim_{\varepsilon\to 0} \Big( \langle \psi_k\, \boldsymbol{\chi}^\circ_\varepsilon(S), \boldsymbol{p}^\circ_\varepsilon(S)\rangle - \langle \psi_k\, \chi_0, p^\varepsilon_0\rangle \Big) = \langle [\boldsymbol{\chi}^\circ(S) : \boldsymbol{p}^\circ(S)], \psi_k\rangle - \langle [\chi_0 : p_0], \psi_k\rangle\,. \tag{6.13}$$

By (6.7), (6.9), (6.12), and (6.13) we finally get

$$\begin{aligned}\int_0^S \Big( \mathcal{H}(\psi_k\, \dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s)], \psi_k\rangle \Big)\, ds - \langle [\boldsymbol{\chi}^\circ(S) : \boldsymbol{p}^\circ(S)], \psi_k\rangle + \\ + \langle [\chi_0 : p_0], \psi_k\rangle \le \liminf_{\varepsilon\to 0} \int_0^S \Big( \mathcal{H}(\psi_k\, \dot{\boldsymbol{p}}^\circ_\varepsilon(s), \boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle \psi_k\, \boldsymbol{\chi}^\circ_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle \Big)\, ds \le \\ \le \liminf_{\varepsilon\to 0} \int_0^S \Big( \mathcal{H}(\dot{\boldsymbol{p}}^\circ_\varepsilon(s), \boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle \boldsymbol{\chi}^\circ_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle \Big)\, ds\,.\end{aligned}$$

Using (2.29), (2.56), (5.27), (5.28), and (5.44) we can pass to the limit as $k \to \infty$, applying the Dominated Convergence Theorem, and we obtain (6.8) . □

We recall that we are adopting convention (2.1) about $L^p$-norms.

**Lemma 6.4.** *Let* $S > 0$*, let* $U^\circ$ *be as in* (4.7)*, and let*

$$A_S^\circ := \{s \in [0, S] : d_2\big(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))\big) > 0\}\,. \tag{6.14}$$

*Then* $A_S^\circ$ *is open,* $A_S^\circ \subset U^\circ$*, and*

$$\int_{A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds \le \liminf_{\varepsilon \to 0+} \int_{A_S^\circ} \|\dot{\boldsymbol{p}}_\varepsilon^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}_\varepsilon^\circ(s), \mathcal{K}(\boldsymbol{\zeta}_\varepsilon^\circ(s)))\, ds\,. \tag{6.15}$$

*Proof.* By convexity, for every $\zeta \in C^0(\bar{\Omega})^+$ the function $\sigma \mapsto d_2(\sigma, \mathcal{K}(\zeta))$ is weakly lower semicontinuous in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. From (2.25), we deduce that

$$|d_2(\sigma, \mathcal{K}(\zeta_1)) - d_2(\sigma, \mathcal{K}(\zeta_2))| \le 2M_K \|\zeta_1 - \zeta_2\|_2$$

for every $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and every $\zeta_1, \zeta_2 \in C^0(\bar{\Omega})^+$. It follows that $(\sigma, \zeta) \mapsto d_2(\sigma, \mathcal{K}(\zeta))$ is lower semicontinuous with respect to the weak topology in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and to the strong topology of $C^0(\overline{\Omega})$. Since $\boldsymbol{e}^\circ$ is continuous for the weak topology of $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ by (5.34), and $\boldsymbol{\zeta}^\circ$ is continuous for the strong topology of $C^0(\overline{\Omega})$ by (5.44), it follows that $s \mapsto d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))$ is lower semicontinuous on $[0, +\infty)$. Therefore the set $A_S^\circ$ is open. The inclusion $A_S^\circ \subset U^\circ$ is a consequence of (4.6).

It only remains to prove (6.15). We fix a compact set $C \subset A_S^\circ$ and a continuous function $\psi\colon C \to [0, +\infty)$ such that

$$d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))) > \psi(s) \qquad \text{for every } s \in C\,. \tag{6.16}$$

We claim that, for $\varepsilon$ sufficiently small, we have

$$d_2(\boldsymbol{\sigma}_\varepsilon^\circ(s), \mathcal{K}(\boldsymbol{\zeta}_\varepsilon^\circ(s))) > \psi(s) \qquad \text{for every } s \in C\,. \tag{6.17}$$

If not, there exist $\varepsilon_k \to 0$ and $s_k \in C$ such that $d_2(\boldsymbol{\sigma}_{\varepsilon_k}^\circ(s_k), \mathcal{K}(\boldsymbol{\zeta}_{\varepsilon_k}^\circ(s_k))) \le \psi(s_k)$. We may assume that $s_k \to s_0 \in C$; now, by (5.30), (5.46), and (5.47), thanks to the lower semicontinuity of $d_2(\sigma, \mathcal{K}(\zeta))$, the previous inequality gives $d_2(\boldsymbol{\sigma}^\circ(s_0), \mathcal{K}(\boldsymbol{\zeta}^\circ(s_0))) \le \psi(s_0)$, which contradicts (6.16). This proves (6.17).

By a standard approximation argument from below, in order to prove (6.15), it suffices to prove

$$\int_C \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, \psi(s)\, ds \le \liminf_{\varepsilon \to 0+} \int_C \|\dot{\boldsymbol{p}}_\varepsilon^\circ(s)\|_2\, \psi(s)\, ds, \tag{6.18}$$

for every compact $C \subset A_S^\circ$ and every continuous function $\psi\colon C \to [0, +\infty)$. To this end, let $\varphi_i$ be a dense sequence in the unit ball of $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, composed of continuous functions with compact support. Since

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_2 = \sup_i \langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle,$$

by the Localisation Lemma (see, e.g., [2, Lemma 2.3.2]) we have

$$\int_C \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, \psi(s)\, ds = \sup \sum_{i=1}^k \int_{C_i} \langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle \psi(s)\, ds, \tag{6.19}$$

where the supremum is taken over all integers $k$ and over all finite Borel partitions $C_1, \dots, C_k$ of $C$. For every $i$ the real-valued functions $s \mapsto \langle \varphi_i, \boldsymbol{p}_\varepsilon^\circ(s)\rangle$ are equi-Lipschitz on $[0, S]$ (by (5.20)) and converge to $s \mapsto \langle \varphi_i, \boldsymbol{p}^\circ(s)\rangle$ for every $s$ (by (5.26)), hence the functions $s \mapsto \langle \varphi_i, \dot{\boldsymbol{p}}_\varepsilon^\circ(s)\rangle$ converge $\langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle$ weakly$^*$ in $L^\infty([0, S])$. It follows that

$$\sum_{i=1}^k \int_{C_i} \langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle \psi(s)\, ds = \lim_{\varepsilon \to 0} \sum_{i=1}^k \int_{C_i} \langle \varphi_i, \dot{\boldsymbol{p}}_\varepsilon^\circ(s)\rangle \psi(s)\, ds \le \liminf_{\varepsilon \to 0} \int_C \|\dot{\boldsymbol{p}}_\varepsilon^\circ(s)\|_2 \psi(s)\, ds.$$

Inequality (6.18) follows now from (6.19). □

We are now in a position to prove the energy inequality (6.1).

*Proof of Theorem 4.5 (continuation).* Let us fix $S>0$ and define $T_\varepsilon := t^\circ_\varepsilon(S)$. By (3.25)

$$\mathcal{Q}(\boldsymbol{e}_\varepsilon(T_\varepsilon)) - \mathcal{Q}(e_0) + \int_0^{T_\varepsilon} \Big(\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}_\varepsilon(t)\rangle\Big)\, dt + \varepsilon \int_0^{T_\varepsilon} \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_2^2\, dt =$$
$$= \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt - \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt + \langle \boldsymbol{\chi}(T_\varepsilon), \boldsymbol{e}_\varepsilon(T_\varepsilon)\rangle - \langle \chi_0, e_0\rangle\,,$$

where $\chi_0 := \boldsymbol{\chi}(0)$. By (3.5) we have

$$\mathcal{Q}(\boldsymbol{e}_\varepsilon(T_\varepsilon)) - \mathcal{Q}(e_0) + \int_0^{T_\varepsilon} \Big(\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}_\varepsilon(t)\rangle\Big)\, dt +$$
$$+ \int_0^{T_\varepsilon} \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_2\, d_2(\boldsymbol{\sigma}_\varepsilon(t), \mathcal{K}(\boldsymbol{\zeta}_\varepsilon(t)))\; dt = \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt -$$
$$- \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt + \langle \boldsymbol{\chi}(T_\varepsilon), \boldsymbol{e}_\varepsilon(T_\varepsilon)\rangle - \langle \chi_0, e_0\rangle\,.$$

Performing the change of variable $t = t^\circ_\varepsilon(s)$ in the left-hand side, we obtain

$$\mathcal{Q}(\boldsymbol{e}^\circ_\varepsilon(S)) - \mathcal{Q}(e_0) + \int_0^{S} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ_\varepsilon(s), \boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle \boldsymbol{\chi}^\circ_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\Big)\, ds +$$
$$+ \int_0^{S} \|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ_\varepsilon(s), \mathcal{K}(\boldsymbol{\zeta}^\circ_\varepsilon(s)))\, ds = \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt - \qquad (6.20)$$
$$- \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt + \langle \boldsymbol{\chi}(T_\varepsilon), \boldsymbol{e}^\circ_\varepsilon(S)\rangle - \langle \chi_0, e_0\rangle\,.$$

By the lower semicontinuity of $\mathcal{Q}$, in view of (5.30) we have

$$\mathcal{Q}(\boldsymbol{e}^\circ(S)) \le \liminf_{\varepsilon\to 0} \mathcal{Q}(\boldsymbol{e}^\circ_\varepsilon(S)). \qquad (6.21)$$

By (5.11) and (5.56) we have

$$\int_0^{T} \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t)) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt = \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt\,, \qquad (6.22)$$

where $T := t^\circ(S)$. Let $\boldsymbol{\omega}(t) := E\dot{\boldsymbol{w}}(t)$ if the derivative exists at $t$, and $\boldsymbol{\omega}(t) = 0$ otherwise. By (2.4) and (5.19) we get

$$E\dot{\boldsymbol{w}}^\circ(s) = \boldsymbol{\omega}(t^\circ(s))\, \dot{t}^\circ(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s\in[0,S].$$

This equality, together with the change of variables formula (2.5), yields

$$\int_0^{T} \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t)) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt = \int_0^{T} \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t)) - \boldsymbol{\chi}(t), \boldsymbol{\omega}(t)\rangle\, dt =$$
$$= \int_0^{S} \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t^\circ(s))) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds = \int_0^{S} \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds,$$

where the last equality follows from the fact that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s\in U^\circ$ and that $s^\circ_-(t^\circ(s)) = s$ for $\mathcal{L}^1$-a.e. $s\in[0,S]\setminus U^\circ$ (see the proof of Lemma 6.3). Therefore, (6.22) gives

$$\int_0^{S} \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds = \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt. \qquad (6.23)$$

Similarly, we prove

$$\int_0^{S} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds = \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt \qquad (6.24)$$

Inequality (6.1) follows now from (5.30), (6.8), (6.15), (6.20), (6.21), (6.23), and (6.24).

To prove the evolution law (4.11) we need two technical results on the convergence of $\dot{\boldsymbol{p}}^\circ_\varepsilon$ to $\dot{\boldsymbol{p}}^\circ$. □

**Lemma 6.5.** *Let $S > 0$ and $\boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$. Then $s \mapsto \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$ is integrable on $[0,S]$ and*

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \quad \text{as } \varepsilon \to 0\,. \tag{6.25}$$

*Proof.* By (5.27) we have $\boldsymbol{p}^\circ \in C^0([0,S], M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}))$. Since

$$\langle \boldsymbol{\varphi}(s), \frac{\boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s)}{h}\rangle \to \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \quad \text{as } h \to 0$$

for $\mathcal{L}^1$-a.e. $s \in [0,S]$, the function $s \mapsto \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable on $[0,S]$. By (5.28) we have $|\langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle| \le \|\boldsymbol{\varphi}(s)\|_\infty$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$. Since $s \mapsto \|\boldsymbol{\varphi}(s)\|_\infty$ is integrable on $[0,S]$, the same property holds for $s \mapsto \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$.

If $\boldsymbol{\varphi} \in C^1_c((0,S); C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$ we can write

$$\int_0^S \langle \boldsymbol{\varphi}(s), \frac{\boldsymbol{p}^\circ_\varepsilon(s+h) - \boldsymbol{p}^\circ_\varepsilon(s)}{h}\rangle\, ds = \int_0^S \langle \frac{\boldsymbol{\varphi}(s-h) - \boldsymbol{\varphi}(s)}{h}, \boldsymbol{p}^\circ_\varepsilon(s)\rangle\, ds\,.$$

Passing to the limit as $h \to 0$ we obtain

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds = -\int_0^S \langle \dot{\boldsymbol{\varphi}}(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle\, ds\,. \tag{6.26}$$

A similar formula holds for $\boldsymbol{p}^\circ$. Thus (6.25) follows from (5.22) and (5.26).

Since $\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 \le 1$ and $\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \le 1$ by (5.21) and (5.28), the same conclusion in the case $\boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$ follows from the density of $C^1_c((0,S); C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$ in $L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$. □

**Lemma 6.6.** *Let $S > 0$ and let $\boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$. Assume that $\boldsymbol{\varphi}_\varepsilon \to \boldsymbol{\varphi}$ strongly in $L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$. Then*

$$\int_0^S \langle \boldsymbol{\varphi}_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \quad \text{as } \varepsilon \to 0\,.$$

*Proof.* Since $\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 \le 1$ by (5.21), we have

$$\Big|\int_0^S \langle \boldsymbol{\varphi}_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds - \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds\Big| \le \int_0^S \|\boldsymbol{\varphi}_\varepsilon(s) - \boldsymbol{\varphi}(s)\|_\infty ds\,.$$

Since the right-hand side tends to $0$ as $\varepsilon \to 0$, the conclusion follows from (6.25). □

We now prove the evolution law (4.11).

*Proof of Theorem 4.5 (continuation).* Let us fix $S > 0$. We first prove that

$$\int_0^S \langle \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ_\varepsilon(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \tag{6.27}$$

for every $\boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}))$. We observe that we can write $\langle \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ_\varepsilon(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle = \langle \boldsymbol{\varphi}(s)\boldsymbol{a}^\circ_\varepsilon(s) I, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle$ and $\langle \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ(s)\rangle = \langle \boldsymbol{\varphi}(s)\boldsymbol{a}^\circ(s) I, \dot{\boldsymbol{p}}^\circ(s)\rangle$. Therefore (6.27) follows from Lemma 6.6, because $\boldsymbol{\varphi}\, \boldsymbol{a}^\circ_\varepsilon I \to \boldsymbol{\varphi}\, \boldsymbol{a}^\circ I$ strongly in $L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$ thanks to (5.38) and (5.39). Using the equalities $\langle \boldsymbol{\varphi}(s), \rho\star\big(\boldsymbol{a}^\circ_\varepsilon(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\big)\rangle = \langle \check{\rho}\star\boldsymbol{\varphi}(s), \boldsymbol{a}^\circ_\varepsilon(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle$ and $\langle \boldsymbol{\varphi}(s), \rho\star\big(\boldsymbol{a}^\circ(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ(s)\big)\rangle = \langle \check{\rho}\star\boldsymbol{\varphi}(s), \boldsymbol{a}^\circ(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ(s)\rangle$, where $\check{\rho}(x) := \rho(-x)$, from (2.49) and (6.27) we obtain

$$\int_0^S \langle \boldsymbol{\varphi}(s), \rho\star\big(\boldsymbol{a}^\circ_\varepsilon(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\big)\rangle\, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \rho\star\big(\boldsymbol{a}^\circ(s)\, \mathrm{tr}\, \dot{\boldsymbol{p}}^\circ(s)\big)\rangle\, ds \tag{6.28}$$

for every $\varphi \in L^1([0,S];L^1(\Omega))$. By (5.37) and (6.28) we have

$$\int_0^S \langle \varphi(s), \dot{z}^\circ_\varepsilon(s)\rangle\, ds \to \int_0^S \langle \varphi(s), \rho \star \big(a^\circ(s) \operatorname{tr} \dot{p}^\circ(s)\big)\rangle\, ds \quad \text{as } \varepsilon \to 0 \tag{6.29}$$

for every $\varphi \in L^1([0,S];L^1(\Omega))$. On the other hand, if $\varphi \in C^1_c((0,S);L^1(\Omega))$, we have

$$\int_0^S \langle \varphi(s), \dot{z}^\circ_\varepsilon(s)\rangle\, ds = -\int_0^S \langle \dot{\varphi}(s), z^\circ_\varepsilon(s)\rangle\, ds\,,$$
$$\int_0^S \langle \varphi(s), \dot{z}^\circ(s)\rangle\, ds = -\int_0^S \langle \dot{\varphi}(s), z^\circ(s)\rangle\, ds\,,$$

so that (5.43) gives

$$\int_0^S \langle \varphi(s), \dot{z}^\circ_\varepsilon(s)\rangle\, ds \to \int_0^S \langle \varphi(s), \dot{z}^\circ(s)\rangle\, ds \quad \text{as } \varepsilon \to 0\,.$$

By (6.29) this implies

$$\int_0^S \langle \varphi(s), \dot{z}^\circ(s)\rangle\, ds = \int_0^S \langle \varphi(s), \rho \star \big(a^\circ(s) \operatorname{tr} \dot{p}^\circ(s)\big)\rangle\, ds$$

for every $\varphi \in C^1_c((0,S);L^1(\Omega))$, and hence

$$\dot{z}^\circ(s) = \rho \star \big(a^\circ(s) \operatorname{tr} \dot{p}^\circ(s)\big) \quad \text{in } \overline{\Omega} \text{ for } \mathcal{L}^1\text{-a.e. } s \in [0,S]\,. \tag{6.30}$$

This concludes the proof of (4.11).

The proof of Theorem 4.5 will be continued in Section 8. □

## 7. Some technical lemmas

In this section we establish some measure theoretic results that are used to prove the opposite of inequality (6.1). The first four lemmas concern a discrete approximation of the integral in the left-hand side of (6.1).

Given $p\colon [0,S] \to M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$ and $\zeta \in C^0(\overline{\Omega})^+$, for every $0 \le a \le b \le S$ we define

$$\operatorname{Var}(p, \zeta; a, b) := \sup \sum_{i=1}^k \mathcal{H}(p(s_i) - p(s_{i-1}), \zeta)\,, \tag{7.1}$$

where the supremum is taken over all finite families $s_0, s_1, \dots, s_k$ such that $a = s_0 \le s_1 \le \dots \le s_k = b$. The following two lemmas provide the properties of $\operatorname{Var}(p,\zeta;0,S)$ that will be used in the proof of Lemma 7.3.

**Lemma 7.1.** *Let $S > 0$, let $p\colon [0,S] \to M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$, let $\zeta \in C^0(\overline{\Omega})^+$, and let $\{s^i_k\}_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$, with*

$$0 = s^0_k \le s^1_k \le \dots \le s^{i_k}_k = S \quad \text{and} \quad \eta_k := \max_{1\le i\le k}(s^i_k - s^{i-1}_k) \to 0\,. \tag{7.2}$$

*Suppose that $p$ is left continuous in $[0,S]$ with respect to the norm topology in $M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$. Then*

$$\operatorname{Var}(p,\zeta;0,S) = \lim_{k\to\infty} \sum_{i=1}^{i_k} \mathcal{H}(p(s^i_k) - p(s^{i-1}_k), \zeta)\,. \tag{7.3}$$

*Proof.* It follows immediately from (7.1) that is enough to prove the inequality

$$\operatorname{Var}(p,\zeta;0,S) \le \liminf_{k\to\infty} \sum_{i=1}^{i_k} \mathcal{H}(p(s^i_k) - p(s^{i-1}_k), \zeta)\,. \tag{7.4}$$

Let us fix $\lambda < \mathrm{Var}(\boldsymbol{p},\zeta;0,S)$. By (7.1) there exist an integer $h$ and a subdivision $0 = s_0 \le s_1 \le \cdots \le s_h = S$ such that

$$\lambda < \sum_{j=1}^{h} \mathcal{H}(\boldsymbol{p}(s_j) - \boldsymbol{p}(s_{j-1}), \zeta) . \tag{7.5}$$

For every $j$ and $k$, let $\iota(j,k)$ be the greatest integer $i$ such that $s_k^i \le s_j$. Since $s_j - \eta_k < s_k^{\iota(j,k)} \le s_j$ and $\eta_k \to 0$, inequality (7.5), together with the left continuity of $\boldsymbol{p}$ and the continuity of $\mathcal{H}$, gives

$$\lambda < \sum_{j=1}^{h} \mathcal{H}(\boldsymbol{p}(s_k^{\iota(j,k)}) - \boldsymbol{p}(s_k^{\iota(j-1,k)}), \zeta)$$

for $k$ large enough. By the triangle inequality (2.28), this implies

$$\lambda < \sum_{i=1}^{i_k} \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \zeta)$$

for $k$ large enough. Inequality (7.4) follows from the arbitrariness of $\lambda < \mathrm{Var}(\boldsymbol{p},\zeta;0,S)$. ☐

**Lemma 7.2.** *Let $S > 0$, let $\boldsymbol{p} \in AC([0,S], M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}))$, let $\zeta \in C^0(\overline{\Omega})^+$, and let $\{s_k^i\}_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2). Then*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \zeta) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \zeta)\, ds \Big| = 0 . \tag{7.6}$$

*Proof.* Arguing as in [5, Theorem 7.1] (see [27, Theorem 3.12]), we can prove that

$$\mathrm{Var}(\boldsymbol{p},\zeta;a,b) = \int_a^b \mathcal{H}(\dot{\boldsymbol{p}}(s), \zeta)\, ds \tag{7.7}$$

for every $0 \le a \le b \le S$. Therefore Lemma 7.1 gives

$$\int_0^S \mathcal{H}(\dot{\boldsymbol{p}}(s), \zeta)\, ds = \lim_{k\to\infty} \sum_{i=1}^{i_k} \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \zeta) . \tag{7.8}$$

Since

$$\mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \zeta) \le \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \zeta)\, ds$$

by (7.7), equality (7.6) is equivalent to (7.8). ☐

**Lemma 7.3.** *Let $S > 0$, let $\boldsymbol{p} \in AC([0,S], M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}))$, let $\boldsymbol{\zeta} \in C^0([0,S], C^0(\overline{\Omega})^+)$, and let $\{s_k^i\}_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2). Then*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_k^{i-1})) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s))\, ds \Big| = 0 , \tag{7.9}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_k^{i})) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s))\, ds \Big| = 0 . \tag{7.10}$$

*Proof.* Since $s \mapsto \boldsymbol{\zeta}(s)$ is continuous, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\|\boldsymbol{\zeta}(s') - \boldsymbol{\zeta}(s)\|_\infty < \varepsilon \quad \text{for every } s',\, s \in [0,S] \text{ with } |s'-s| < \delta(\varepsilon) . \tag{7.11}$$

Let us fix $\varepsilon > 0$ and a subdivision $0 = s_0 < s_1 < \cdots < s_h = S$ such that $s_j - s_{j-1} < \delta(\varepsilon)$ for every $j = 1, \dots, h$. By Lemma 7.2 we have

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_j)) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s_j))\, ds \Big| = 0 \tag{7.12}$$

for every $j = 1, \dots, h$.

If $s_{j-1} < s_k^i \le s_j$, by (7.2) for every $s_k^{i-1} \le s \le s_k^i$ we have $|s - s_{j-1}| < \delta(\varepsilon)$ and $|s - s_j| < \delta(\varepsilon)$ for $k$ sufficiently large. Therefore (2.31), (2.35), and (7.11) give

$$|\mathcal{H}(p, \boldsymbol{\zeta}(s)) - \mathcal{H}(p, \boldsymbol{\zeta}(s_{j-1}))| \le M_K \varepsilon \|p\|_1 \quad \text{and} \quad |\mathcal{H}(p, \boldsymbol{\zeta}(s)) - \mathcal{H}(p, \boldsymbol{\zeta}(s_j))| \le M_K \varepsilon \|p\|_1$$

for every $p \in M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$ and every $s_k^{i-1} \le s \le s_k^i$. Since $\boldsymbol{p}$ is absolutely continuous, this implies, thanks to [5, Theorem 7.1],

$$|\mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_k^{i-1})) - \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_j))| \le$$

$$\le M_K \varepsilon \|\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1})\|_1 \le M_K \varepsilon \int_{s_k^{i-1}}^{s_k^i} \|\dot{\boldsymbol{p}}(s)\|_1 ds\,,$$

$$\Big| \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s))\, ds - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s_j))\, ds \Big| \le M_K \varepsilon \int_{s_k^{i-1}}^{s_k^i} \|\dot{\boldsymbol{p}}(s)\|_1 ds\,,$$

Therefore

$$\sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_k^{i-1})) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s))\, ds \Big| \le$$

$$\le \sum_{j=1}^{h} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_j)) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s_j))\, ds \Big| + 2M_K \varepsilon \int_0^S \|\dot{\boldsymbol{p}}(s)\|_1 ds\,,$$

so (7.12) gives

$$\limsup_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(s_k^i) - \boldsymbol{p}(s_k^{i-1}), \boldsymbol{\zeta}(s_k^{i-1})) - \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}(s))\, ds \Big| \le 2M_K \varepsilon \int_0^S \|\dot{\boldsymbol{p}}(s)\|_1 ds\,.$$

Equality (7.9) follows now from the arbitrariness of $\varepsilon > 0$. The proof of (7.10) is similar. □

**Lemma 7.4.** *Let $S > 0$ and let $\boldsymbol{p} \in C^0([0,S], M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}))$. Suppose that $\boldsymbol{\chi} \in H^1([0,S]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ satisfies (2.53)-(2.56) for suitable $\boldsymbol{f} \in H^1([0,S]; L^n(\Omega; \mathbb{R}^n))$ and $\boldsymbol{g} \in H^1([0,S]; L^\infty(\Gamma_1; \mathbb{R}^n))$, and let $\{s_k^i\}_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2). Assume that there exist a weakly$^*$-continuous function $\boldsymbol{u}\colon [0,S] \to BD(\Omega)$, a weakly continuous function $\boldsymbol{e}\colon [0,S] \to L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $\boldsymbol{w} \in H^1([0,S]; H^1(\Omega; \mathbb{R}^n))$, such that for every $s \in [0,S]$*

$$E\boldsymbol{u}(s) = \boldsymbol{e}(s) + \boldsymbol{p}(s) \quad \text{in } \Omega\,,$$
$$\boldsymbol{p}(s) = (\boldsymbol{w}(s) - \boldsymbol{u}(s)) \odot \nu \mathcal{H}^{n-1} \quad \text{in } \Gamma_0\,.$$

*Then*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \langle \boldsymbol{\chi}(s_k^i) - \boldsymbol{\chi}(s_k^{i-1}), \boldsymbol{p}(s_k^{i-1}) \rangle - \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s) \rangle\, ds \Big| = 0\,, \tag{7.13}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \langle \boldsymbol{\chi}(s_k^i) - \boldsymbol{\chi}(s_k^{i-1}), \boldsymbol{p}(s_k^{i}) \rangle - \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s) \rangle\, ds \Big| = 0\,, \tag{7.14}$$

*where all duality products are defined according to (2.13) for every $s \in [0,S]$.*

*Proof.* Let $\boldsymbol{\psi} := \boldsymbol{\chi}$ or $\boldsymbol{\psi} := \dot{\boldsymbol{\chi}}$. Then the integration-by-parts formula (2.16), together with (2.53)-(2.55), gives

$$\langle \boldsymbol{\psi}(s), \boldsymbol{p}(s') \rangle = \langle \boldsymbol{\psi}(s), E\boldsymbol{w}(s') - \boldsymbol{e}(s') \rangle + \langle \boldsymbol{f}(s), \boldsymbol{u}(s') - \boldsymbol{w}(s') \rangle + \langle \boldsymbol{g}(s), \boldsymbol{u}(s') - \boldsymbol{w}(s') \rangle_{\Gamma_1} \tag{7.15}$$

for every $s, s' \in [0,S]$. This shows that $s \mapsto \langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s)\rangle$ belongs to $L^1([0,S])$ and that for every $s' \in [0,S]$ the function $s \mapsto \langle \boldsymbol{\chi}(s), \boldsymbol{p}(s')\rangle$ is absolutely continuous on $[0,S]$ and its derivative is $\langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s')\rangle$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$. Therefore,

$$\langle \boldsymbol{\chi}(s_k^i) - \boldsymbol{\chi}(s_k^{i-1}), \boldsymbol{p}(s_k^{i-1})\rangle = \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s_k^{i-1})\rangle \, ds \tag{7.16}$$

for every $k$ and every $i$. Let us fix $\delta > 0$. Since $\boldsymbol{p}$ is continuous, by (2.14) and (7.2) for $k$ large enough we have

$$\begin{aligned} &\Big| \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s) - \boldsymbol{p}(s_k^{i-1})\rangle \, ds \Big| \leq \\ &\leq \int_{s_k^{i-1}}^{s_k^i} \|\dot{\boldsymbol{\chi}}(s)\|_\infty \|\boldsymbol{p}(s) - \boldsymbol{p}(s_k^{i-1})\|_1 \, ds \leq \delta \int_{s_k^{i-1}}^{s_k^i} \|\dot{\boldsymbol{\chi}}(s)\|_\infty \, ds \,. \end{aligned} \tag{7.17}$$

It follows from (7.16) and (7.17) that

$$\sum_{i=1}^{i_k} \Big| \langle \boldsymbol{\chi}(s_k^i) - \boldsymbol{\chi}(s_k^{i-1}), \boldsymbol{p}(s_k^{i-1})\rangle - \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}(s), \boldsymbol{p}(s)\rangle \, ds \Big| \leq \delta \int_0^S \|\dot{\boldsymbol{\chi}}(s)\|_\infty \, ds.$$

As the right-hand side is finite by (2.56), the arbitrariness of $\delta$ proves (7.13). The same argument proves (7.14). □

We now prove three lemmas that will be used to obtain a discrete approximation of the integrals in the right-hand side of (6.1). We begin with a lemma concerning the approximation of Lebesgue integrals by Riemann sums.

**Lemma 7.5.** *Let $S > 0$, let $X$ be a Banach space, and let $\boldsymbol{\psi} \colon [0,S] \to X$ be a Bochner integrable function. Then there exists a sequence $(s_k^i)_{0\leq i \leq i_k}$ of subdivisions of the interval $[0,S]$ satisfying (7.2) such that*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \|\boldsymbol{\psi}(s) - \boldsymbol{\psi}(s_k^{i-1})\| \, ds = 0 = \lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \|\boldsymbol{\psi}(s) - \boldsymbol{\psi}(s_k^i)\| \, ds \,. \tag{7.18}$$

*In particular we have*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \boldsymbol{\psi}(s_k^{i-1})(s_k^i - s_k^{i-1}) = \int_0^S \boldsymbol{\psi}(s)\, ds = \lim_{k\to\infty} \sum_{i=1}^{i_k} \boldsymbol{\psi}(s_k^i)(s_k^i - s_k^{i-1}) \,, \tag{7.19}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \|\boldsymbol{\psi}(s_k^i) - \boldsymbol{\psi}(s_k^{i-1})\| (s_k^i - s_k^{i-1}) \,, \tag{7.20}$$

*where the limits in (7.19) are in the strong topology of $X$.*

*Proof.* We omit the proof since (7.19) is well-known (see [12]). For the application we have in mind we need the stronger result (7.18), which is related to the Saks-Henstock lemma (see [24] and [13]) used in the theory of Henstock-Kurzweil integral (see, e.g., [15]). An elementary proof in the framework of Lebesgue integration, based on Fubini's theorem, can be obtained by adapting the arguments developed in [7, page 63]. Equality (7.20) follows from (7.18) by the triangle inequality. □

**Remark 7.6.** If $X_j$ is a sequence of Banach spaces and $\boldsymbol{\psi}_j \colon [0,S] \to X_j$ is a sequence of Bochner integrable function, then there exists a sequence of subdivisions $(s_k^i)_{0\leq i\leq i_k}$, independent of $j$ and satisfying (7.2), such that (7.18) holds simultaneously for each function $\boldsymbol{\psi}_j$. Indeed, we can consider the Banach space $X$ of all sequences $x := (x_j)$ such that $x_j \in X_j$ for every $j$ and $\sum_j \|x_j\|_j < +\infty$, $\|\cdot\|_j$ being the norm in $X_j$, endowed with the norm $\|x\| := \sum_j \|x_j\|_j$. To obtain the result it is enough to apply Lemma 7.5 to the function $\boldsymbol{g}\colon [0,S] \to X$ whose components $\boldsymbol{g}_j$ are given by $\boldsymbol{g}_j(s) := 2^{-j}\boldsymbol{\psi}_j(s)/F_j$, where $F_j := \int_0^S \|\boldsymbol{\psi}_j(s)\|_j\,ds$.

**Lemma 7.7.** *Let $S>0$, let $X$ be a Banach space, let $\boldsymbol{\psi}\colon [0,S]\to X$ be a Bochner integrable function, let $A$ be a measurable set in $[0,S]$ such that $\boldsymbol{\psi}=0$ on $A$, let $B:=[0,S]\setminus A$, and let $(s_k^i)_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2) and (7.18), and hence (7.19). Let us define*

$$I_k^A := \{i : 1\le i\le i_k,\ s_k^{i-1}\in A,\ s_k^i\in A\}\,, \tag{7.21}$$
$$I_k^B := \{i : 1\le i\le i_k,\ s_k^{i-1}\in B,\ s_k^i\in B\}\,, \tag{7.22}$$
$$J_k^{A-} := \{i : 1\le i\le i_k,\ s_k^{i-1}\in A,\ s_k^i\in B\}\,, \tag{7.23}$$
$$J_k^{A+} := \{i : 1\le i\le i_k,\ s_k^{i-1}\in B,\ s_k^i\in A\}\,, \tag{7.24}$$
$$J_k^A := J_k^{A-}\cup J_k^{A+}\,. \tag{7.25}$$

*Then*

$$\lim_{k\to\infty}\sum_{i\in I_k^B}\boldsymbol{\psi}(s_k^{i-1})(s_k^i-s_k^{i-1}) = \int_0^S\boldsymbol{\psi}(s)\,ds = \lim_{k\to\infty}\sum_{i\in I_k^B}\boldsymbol{\psi}(s_k^i)(s_k^i-s_k^{i-1})\,, \tag{7.26}$$
$$\lim_{k\to\infty}\sum_{i\in J_k^A}(\|\boldsymbol{\psi}(s_k^{i-1})\|+\|\boldsymbol{\psi}(s_k^i)\|)(s_k^i-s_k^{i-1}) = 0\,, \tag{7.27}$$
$$\lim_{k\to\infty}\sum_{i\in I_k^A\cup J_k^A}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)\|\,ds = 0\,, \tag{7.28}$$

*where the limits in (7.26) are in the strong topology of $X$.*

*Proof.* By (7.18) we have

$$\lim_{k\to\infty}\sum_{i\in I_k^B}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-1})\|\,ds = 0 = \lim_{k\to\infty}\sum_{i\in I_k^B}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^i)\|\,ds\,, \tag{7.29}$$
$$\lim_{k\to\infty}\sum_{i\in J_k^{A+}}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-1})\|\,ds = 0 = \lim_{k\to\infty}\sum_{i\in J_k^{A-}}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^i)\|\,ds\,, \tag{7.30}$$
$$\lim_{k\to\infty}\sum_{i\in I_k^A\cup J_k^{A-}}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)\|\,ds = 0 = \lim_{k\to\infty}\sum_{i\in I_k^A\cup J_k^{A+}}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)\|\,ds\,. \tag{7.31}$$

Equality (7.28) follows from (7.31). Applying the triangle inequality we obtain (7.27) from (7.28) and (7.30). On the other hand, taking into account (7.21)-(7.24), we have

$$\sum_{i=1}^{i_k}\boldsymbol{\psi}(s_k^{i-1})(s_k^i-s_k^{i-1}) = \sum_{i\in I_k^B}\boldsymbol{\psi}(s_k^{i-1})(s_k^i-s_k^{i-1}) + \sum_{i\in J_k^{A+}}\boldsymbol{\psi}(s_k^{i-1})(s_k^i-s_k^{i-1})\,, \tag{7.32}$$

and the last sum tends to 0 by (7.27). Therefore, the first equality in (7.26) follows from (7.19) and (7.32). The proof of the other equality is similar. ☐

**Remark 7.8.** Let $S$, $A$, and $B$ be as in Lemma 7.7, and let $(s_k^i)_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2) and

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\int_{s_k^{i-1}}^{s_k^i}|1_B(s)-1_B(s_k^{i-1})|\,ds = 0 = \lim_{k\to\infty}\sum_{i=1}^{i_k}\int_{s_k^{i-1}}^{s_k^i}|1_B(s)-1_B(s_k^i)|\,ds\,, \tag{7.33}$$

where $1_B$ denotes the characteristic function of $B$, defined by $1_B(s)=1$ for $s\in B$ and $1_B(s)=0$ for $s\notin B$. It follows from Lemma 7.7, applied to $X=\mathbb{R}$ and $\boldsymbol{\psi}(s)=1_B(s)$, that

$$\lim_{k\to\infty}\sum_{i\in J_k^A}(s_k^i-s_k^{i-1}) = 0 = \lim_{k\to\infty}\sum_{i\in I_k^A\cup J_k^A}\mathcal{L}^1(B\cap[s_k^{i-1},s_k^i])\,. \tag{7.34}$$

**Lemma 7.9.** *Let $S>0$, let $X$ be a Banach space, let $\boldsymbol{\psi}\colon[0,S]\to X$ be a bounded Bochner integrable function, let $A$ be a relatively open set in $[0,S]$, let $B:=[0,S]\setminus A$, let $(s_k^i)_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2), (7.18), and (7.33), and let $I_k^A$, $I_k^B$, $J_k^{A-}$, $J_k^{A+}$, and $J_k^A$ be defined as in (7.21)-(7.25). For every $i\in J_k^{A-}$ let $s_k^{i-\frac12}$ be the supremum of the connected component of $A$ containing $s_k^{i-1}$, and for every $i\in J_k^{A+}$ let $s_k^{i-\frac12}$ be the infimum of the connected component of $A$ containing $s_k^i$. If $1\le i\le i_k$ and $i\notin J_k^A$, we set $s_k^{i-\frac12}:=s_k^i$. Then the subdivision $(\hat s_k^i)_{0\le i\le 2i_k}$ defined by $\hat s_k^i:=s_k^{i/2}$ satisfies (7.2), (7.18), and (7.33). Moreover, if $\hat J_k^{A-}$, $\hat J_k^{A+}$, and $\hat J_k^A$ are defined by (7.23), (7.24), and (7.25), with $\hat s_k^i$ instead of $s_k^i$, then $(\hat s_k^{i-1},\hat s_k^i)\subset A$ for every $i\in\hat J_k^A$.*

*Proof.* Let $M$ is an upper bound of $\|\boldsymbol{\psi}(s)\|$ on $[0,S]$. Since $s_k^{i-\frac12}=s_k^i$ for $i\notin J_k^A$ and $\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-\frac12})\|\le\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-1})\|+2M$ for every $i\in J_k^A$ and every $s\in[0,S]$, we have

$$\sum_{i=1}^{i_k}\Big(\int_{s_k^{i-1}}^{s_k^{i-\frac12}}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-1})\|\,ds+\int_{s_k^{i-\frac12}}^{s_k^i}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-\frac12})\|\,ds\Big)\le$$
$$\le\sum_{i=1}^{i_k}\int_{s_k^{i-1}}^{s_k^i}\|\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s_k^{i-1})\|\,ds+2M\sum_{i\in J_k^A}(s_k^i-s_k^{i-1})\,.$$

Since the right-hand side tends to 0 by (7.18) and (7.34), we obtain the first equality in (7.18) for $\hat s_k^i$. A similar argument proves the other equality, as well as (7.33). The final statement of the lemma follows immediately from the definition of $s_k^{i-\frac12}$. □

## 8. Proof of Theorem 4.5: Conclusion

In this section $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, $u_0$, $e_0$, $p_0$, and $z_0$ are as in Definition 4.1 and satisfy the uniform safe-load condition (2.53)-(2.56). We assume that $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, $t^\circ$, $\boldsymbol{\sigma}^\circ$, and $\boldsymbol{\zeta}^\circ$ satisfy (4.1) and (4.2), together with conditions (ev0)°, (ev1)°, (ev2)°, and (ev3′)° of Definition 4.1. We define $\boldsymbol{\chi}^\circ(s):=\boldsymbol{\chi}(t^\circ(s))$ and $\boldsymbol{w}^\circ(s):=\boldsymbol{w}(t^\circ(s))$, and we assume that (5.24) is satisfied. Let us fix $S>0$ and let $A_S^\circ$ be the open set defined in (6.14). We assume also that

$$\int_{A_S^\circ}\|\dot{\boldsymbol{p}}^\circ(s)\|_2\,d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\,ds<+\infty\,,\tag{8.1}$$

so that $\dot{\boldsymbol{p}}^\circ(s)$, defined by (4.3), belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for $\mathcal{L}^1$-a.e. $s\in A_S^\circ$.

The goal of this section is to prove that the functions $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, $\boldsymbol{w}^\circ$, $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\boldsymbol{\chi}^\circ$ satisfy the energy inequality

$$\begin{aligned}&\mathcal{Q}(\boldsymbol{e}^\circ(S))-\mathcal{Q}(e_0)+\int_0^S\Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))+\langle\dot{\boldsymbol{\chi}}^\circ(s),\boldsymbol{p}^\circ(s)\rangle\Big)\,ds-\\&-\langle\boldsymbol{\chi}^\circ(S),\boldsymbol{p}^\circ(S)\rangle+\langle\chi_0,p_0\rangle+\int_{A_S^\circ}\langle\boldsymbol{\sigma}^\circ(s)-\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\ge\\&\ge\int_0^S\langle\boldsymbol{\tau}^\circ(s),E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds-\int_0^S\langle\dot{\boldsymbol{\chi}}^\circ(s),\boldsymbol{e}^\circ(s)\rangle\,ds+\langle\boldsymbol{\chi}^\circ(S),\boldsymbol{e}^\circ(S)\rangle-\langle\chi_0,e_0\rangle\,,\end{aligned}\tag{8.2}$$

where $\chi_0:=\boldsymbol{\chi}(0)=\boldsymbol{\chi}^\circ(0)$ and $\boldsymbol{\tau}^\circ:=\boldsymbol{\sigma}^\circ-\boldsymbol{\chi}^\circ$. The first five lemmas concern the properties of the functions $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, and $\boldsymbol{p}^\circ$ on $A_S^\circ$.

**Lemma 8.1.** *Let $(a,b)$ be a connected component of $A_S^\circ$, and let $c\in(a,b)$. Then $\boldsymbol{p}^\circ-\boldsymbol{p}^\circ(c)\in AC_{loc}((a,b);L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. In particular, for $\mathcal{L}^1$-a.e. $s\in(a,b)$, $\dot{\boldsymbol{p}}^\circ(s)$ is the strong limit in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, as $h\to0$, of the difference quotient $\frac1h(\boldsymbol{p}^\circ(s+h)-\boldsymbol{p}^\circ(s))$, and $\dot{\boldsymbol{p}}^\circ\in L^1_{loc}((a,b);L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. Moreover, for every $s_1,s_2\in(a,b)$, we have*

$$\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\quad\text{and}\quad\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)=\int_{s_1}^{s_2}\dot{\boldsymbol{p}}^\circ(s)\,ds\,,\tag{8.3}$$

*where the last term is a Bochner integral in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.

*Proof.* By the lower semicontinuity of $d_2\big(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s))\big)$, obtained in the proof of Lemma 6.4, for every $[a_1,b_1]\subset(a,b)$, there exists a constant $\eta_1>0$ such that $d_2\big(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s))\big)\geq\eta_1$ for every $s\in[a_1,b_1]$. By (8.1) this gives

$$\int_{a_1}^{b_1}||\dot{\boldsymbol{p}}^\circ(s)||_2\,ds<+\infty\,. \tag{8.4}$$

This inequality and the measurability of $s\mapsto\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle$ for every $\varphi\in C^0_0(\Omega;\mathbb{M}^{n\times n}_{sym})$ imply that $s\mapsto\langle\psi,\dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable for every $\psi\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, hence $\dot{\boldsymbol{p}}^\circ\colon[a_1,b_1]\to L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is weakly measurable. By Pettis Theorem it is also strongly measurable, so that (8.4) implies that $\dot{\boldsymbol{p}}^\circ\in L^1_{loc}((a,b);L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. For every $\varphi\in C^0_0(\Omega;\mathbb{M}^{n\times n}_{sym})$, the function $s\mapsto\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable and bounded, hence, for every $s_1,s_2\in(a,b)$, we have

$$\langle\varphi,\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle=\int_{s_1}^{s_2}\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds=\Big\langle\varphi,\int_{s_1}^{s_2}\dot{\boldsymbol{p}}^\circ(s)\,ds\Big\rangle\,,$$

where the last equality follows from the fact that the Bochner integral of $\dot{\boldsymbol{p}}^\circ$ in the last term is well defined in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. By the arbitrariness of $\varphi$, this proves (8.3). The inclusion $\boldsymbol{p}^\circ-\boldsymbol{p}^\circ(c)\in AC_{loc}((a,b);L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ follows now from (8.3), as well as the statement about the difference quotients, thanks to the Differentiation Theorem for Bochner integrals. □

**Lemma 8.2.** *Let* $(a,b)$ *be a connected component of* $A^\circ_S$. *Then, for every* $a<s_1<s_2<b$,

$$\boldsymbol{u}^\circ(s_2)-\boldsymbol{u}^\circ(s_1)\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)\,, \tag{8.5}$$

*where* $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ *is defined by* (2.6).

*Proof.* Let us fix $a<s_1<s_2<b$. From the weak kinematic admissibility (4.4), we have

$$E\boldsymbol{u}^\circ(s_2)-E\boldsymbol{u}^\circ(s_1)=\boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)+\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\quad\text{ in }\Omega\,, \tag{8.6}$$

$$\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)=((\boldsymbol{w}^\circ(s_2)-\boldsymbol{w}^\circ(s_1))-(\boldsymbol{u}^\circ(s_2)-\boldsymbol{u}^\circ(s_1)))\odot\nu\mathcal{H}^{n-1}\quad\text{ in }\Gamma_0\,. \tag{8.7}$$

As the measure $\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)$ belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, it does not charge $\Gamma_0$, so that the left-hand side of (8.7) is 0; since $\boldsymbol{w}^\circ(s)$ is constant in $(a,b)$ by the inclusion $A^\circ_S\subset U^\circ$ proved in Lemma 6.4, we get $\boldsymbol{u}^\circ(s_2)-\boldsymbol{u}^\circ(s_1)=0$ $\mathcal{H}^{n-1}$-a.e. on $\Gamma_0$. Moreover, the right-hand side of (8.6) belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. By (2.2) we have $\boldsymbol{u}^\circ(s_2)-\boldsymbol{u}^\circ(s_1)\in H^1(\Omega;\mathbb{R}^n)$. □

**Lemma 8.3.** *The function* $\boldsymbol{e}^\circ$ *belongs to* $AC_{loc}(A^\circ_S;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ *and*

$$\alpha_Q\|\dot{\boldsymbol{e}}^\circ(s)\|_2\leq\beta_Q\|\dot{\boldsymbol{p}}^\circ(s)\|_2 \tag{8.8}$$

*for* $\mathcal{L}^1$*-a.e.* $s\in A^\circ_S$.

*Proof.* Let $(a,b)$ be a connected component of $A^\circ_S$ and let $a<s_1<s_2<b$. By the inclusion $A^\circ_S\subset U^\circ$ proved in Lemma 6.4 and by (4.5) we have that $\boldsymbol{\sigma}^\circ(s_2)-\boldsymbol{\sigma}^\circ(s_1)$ belongs to the set $\Sigma_0(\Omega)$ defined by (2.17), so that from (2.18), (8.5), and (8.6), we get

$$\langle\boldsymbol{\sigma}^\circ(s_2)-\boldsymbol{\sigma}^\circ(s_1),\boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\rangle=\langle\boldsymbol{\sigma}^\circ(s_2)-\boldsymbol{\sigma}^\circ(s_1),\boldsymbol{p}^\circ(s_1)-\boldsymbol{p}^\circ(s_2)\rangle\,; \tag{8.9}$$

by (2.9) this yields $2\alpha_Q\|\boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\|_2\leq2\beta_Q\|\boldsymbol{p}^\circ(s_1)-\boldsymbol{p}^\circ(s_2)\|_2$, and the conclusion follows from Lemma 8.1 □

**Lemma 8.4.** *Let* $(a,b)$ *be a connected component of* $A^\circ_S$. *Then there exists an increasing sequence* $s_k\to b$ *such that* $d_2(\boldsymbol{\sigma}^\circ(s_k),\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k)))\to0$.

*Proof.* We argue by contradiction. If the conclusion does not hold, there exist $c \in (a,b)$ and $\eta > 0$ such that $d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))) \geq \eta$ for every $s \in [c,b)$. Then (4.2), (8.1), and (8.8) imply that

$$\int_c^b ||\dot{\boldsymbol{\sigma}}^\circ(s)||_2\, ds < +\infty.$$

It follows that $\boldsymbol{\sigma}^\circ(s)$ has a strong limit in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ as $s \to b^-$. Since $\boldsymbol{\sigma}^\circ(s) \rightharpoonup \boldsymbol{\sigma}^\circ(b)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ as $s \to b^-$, we deduce that $\boldsymbol{\sigma}^\circ(s) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ as $s \to b^-$. Since $\boldsymbol{\sigma}^\circ(b) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(b))$, we conclude that $d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))) \to 0$ as $s \to b^-$, which contradicts our assumption on $\eta$. □

In the following lemma we use a weak $L^1$-estimate for gradients of solutions of the elliptic system of linearized elasticity. For every measurable set $B$ and for every measurable function $f$ defined on $B$ with values in a finite dimensional Hilbert space, we define

$$\|f\|_{1,w,B} := \sup_{t>0} t\,\mathcal{L}^n(\{|f| > t\} \cap B)\,. \tag{8.10}$$

It is well-known that $\|f\|_{1,w,B} \leq \|f\|_{1,B}$ (Chebychev Inequality) and that $\|f_1 + f_2\|_{1,w,B} \leq 2\|f_1\|_{1,w,B} + 2\|f_2\|_{1,w,B}$ for every pair of functions $f_1, f_2$.

**Lemma 8.5.** *Let $(a,b)$ be a connected component of $A^\circ_S$. Then there exists an increasing sequence $s_k \to b$ such that $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.*

*Proof.* Let $s_k$ be the sequence given by Lemma 8.4. Let us fix $h < k$. By Lemma 8.2 we have $\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k) \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$, while $\boldsymbol{\sigma}^\circ(s_h) - \boldsymbol{\sigma}^\circ(s_k) \in \Sigma_0(\Omega)$ by (4.5), thanks to the inclusion $A^\circ_S \subset U^\circ$ proved in Lemma 6.4. Then (8.6) implies that

$$-\mathrm{div}\,(\mathbb{C}E(\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k))) = -\mathrm{div}\,(\mathbb{C}(\boldsymbol{p}^\circ(s_h) - \boldsymbol{p}^\circ(s_k))).$$

Let us fix an open set $\Omega' \subset\subset \Omega$. We can apply the regularity result proved in the Appendix (Theorem 9.1), and we find that there exists a constant $C$ such that

$$\|E(\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k))\|_{1,w,\Omega'} \leq C\|\boldsymbol{p}^\circ(s_h) - \boldsymbol{p}^\circ(s_k)\|_1 + C\|\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k)\|_1\,;$$

then (4.2), (8.6), the Lipschitz continuity of $\boldsymbol{p}^\circ$, and the strong continuity of $\boldsymbol{u}\colon [0,S] \to L^1(\Omega;\mathbb{R}^n)$ entail that $\boldsymbol{\sigma}^\circ(s_k)$ is a Cauchy sequence with respect to convergence in measure in $\Omega$. As $\boldsymbol{\sigma}^\circ(s_k) \rightharpoonup \boldsymbol{\sigma}^\circ(b)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, it follows that $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ in measure. We now consider the decomposition

$$\boldsymbol{\sigma}^\circ(s_k) = \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k)) + (\boldsymbol{\sigma}^\circ(s_k) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k))\,. \tag{8.11}$$

The sequence $\boldsymbol{\sigma}^\circ(s_k) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k))$ converges to 0 strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ by Lemma 8.4. As $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ in measure, this implies that $\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k)) \to \boldsymbol{\sigma}^\circ(b)$ in measure. Since $\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k))$ is uniformly bounded in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$, by the Dominated Convergence Theorem we have $\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k)) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, therefore (8.11) gives $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, as required. □

The next four lemmas provide a discrete approximation of the integrals in (8.2). Let

$$B^\circ_S := \{s \in [0,S] : \boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s))\} = [0,S] \setminus A^\circ_S\,. \tag{8.12}$$

Since $A^\circ_S$ is open, $B^\circ_S$ is compact. We recall that $\boldsymbol{\tau}^\circ := \boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ$.

**Lemma 8.6.** *For every $s_1, s_2 \in B^\circ_S$ with $s_1 < s_2$ we have*

$$\begin{aligned}\tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2) - E\boldsymbol{w}^\circ(s_1)\rangle + \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_1) + \boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1)\rangle \leq \\ \leq \mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) + \tfrac{1}{2}\mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1)) + \\ + \tfrac{1}{2}\mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_2)) - \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_1) + \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle\,,\end{aligned} \tag{8.13}$$

*or equivalently*

$$\begin{aligned}&\tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1)+\boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2)-E\boldsymbol{w}^\circ(s_1)\rangle - \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_2)-\boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_2)+\boldsymbol{e}^\circ(s_1)\rangle -\\ &\quad - \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_2)-\boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_2)+\boldsymbol{p}^\circ(s_1)\rangle \le \mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) +\\ &\quad + \tfrac{1}{2}\mathcal{H}(\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1)) + \tfrac{1}{2}\mathcal{H}(\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_2)) -\\ &\quad - \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2)\rangle + \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1)\rangle - \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2)\rangle + \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_1)\rangle\,.\end{aligned} \tag{8.14}$$

*Proof.* The equivalence between (8.13) and (8.14) is trivial, hence we limit ourselves to proving (8.13). Let $s_1$ and $s_2$ be as in the statement of the lemma. By (2.16) and (7.15) we have

$$\begin{aligned}&\mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) = \tfrac{1}{2}\langle \boldsymbol{\sigma}^\circ(s_2)+\boldsymbol{\sigma}^\circ(s_1), \boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\rangle =\\ &= \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1)+\boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2)-E\boldsymbol{w}^\circ(s_1)\rangle - \tfrac{1}{2}\langle \boldsymbol{\sigma}^\circ(s_1)+\boldsymbol{\sigma}^\circ(s_2), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle +\\ &\quad + \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_1)+\boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle + \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_1)+\boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\rangle.\end{aligned}$$

Since $\boldsymbol{\sigma}^\circ(s_i) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s_i))$ and $\boldsymbol{\zeta}^\circ(s_i) \in C^0(\overline{\Omega})$ for $i=1,2$, we can adapt [3, Proposition 3.3], following the lines of [27, Proposition 3.2], and we obtain $\langle \boldsymbol{\sigma}^\circ(s_i), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle \le \mathcal{H}(\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_i))$. With this, (8.13) easily follows from the previous equalities. □

**Lemma 8.7.** *Let $(a,b)$ be a connected component of $A^\circ_S$ and let $a \le s_1 < s_2 \le b$. Then*

$$\begin{aligned}&\langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2)\rangle - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}(s_1)\rangle \le \mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) +\\ &+ \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\,ds - \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2)\rangle + \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}(s_1)\rangle +\\ &\quad + \int_{s_1}^{s_2} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\,.\end{aligned} \tag{8.15}$$

*Proof.* We first observe that $\boldsymbol{\chi}^\circ$ is constant on $(a,b)$ by the inclusion $A^\circ_S \subset U^\circ$ proved in Lemma 6.4. By Lemma 8.3 the function $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s)\rangle$ is locally absolutely continuous on $(a,b)$ and

$$\tfrac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle \tag{8.16}$$

for a.e. $s \in (a,b)$. Similarly, by Lemma 8.1 and (2.14) the function $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ is locally absolutely continuous on $(a,b)$ and

$$\tfrac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \tag{8.17}$$

for a.e. $s \in (a,b)$, where the right-hand side is the usual scalar product of $L^2$. These continuity results, together with the weak lower semicontinuity of $\mathcal{Q}$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, imply that it is enough to prove the inequality in (8.15) when $a < s_1$. By Lemma 8.5 we may also assume $s_2 < b$.

Since $s \mapsto \mathcal{Q}(\boldsymbol{e}^\circ(s))$ is locally absolutely continuous in $(a,b)$ by Lemma 8.3, taking into account (8.16) and (8.17), inequality (8.15) easily follows from the inequality $\langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle \le \frac{d}{ds}\mathcal{Q}(\boldsymbol{e}^\circ(s)) + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle$, which is equivalent to

$$\begin{aligned}&\langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle \le \langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) -\\ &\quad - \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,.\end{aligned} \tag{8.18}$$

As $(\boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ)(s) \in \Sigma_0(\Omega)$ by (2.53) and (4.5), from (2.18), (8.5), and (8.6) we get

$$\langle (\boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ)(s), \boldsymbol{e}^\circ(s+h) - \boldsymbol{e}^\circ(s)\rangle = -\langle (\boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ)(s), \boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s)\rangle\,;$$

by Lemmas 8.1 and 8.3, we conclude that $\langle (\boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ)(s), \dot{\boldsymbol{e}}^\circ(s)\rangle = -\langle (\boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ)(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$, therefore (8.18) is equivalent to

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \le \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,;$$

this inequality can be proved by observing that

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle = \langle \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle \leq$$
$$\leq \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,,$$

where the inequality follows from the definition of $\mathcal{H}$. This concludes the proof. □

**Lemma 8.8.** *Let $(s^i_k)_{0\leq i\leq i_k}$ be a sequence of subdivisions of $[0,S]$ satisfying (7.2) and (7.18), with $\boldsymbol{\psi}$ given by the following functions: $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\sigma}^\circ 1_{B^\circ_S}$, $\boldsymbol{\chi}^\circ 1_{B^\circ_S}$, and $1_{B^\circ_S}$, the first three with $X = L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Let $I^A_k$, $I^B_k$, and $J^A_k$ be defined by (7.21), (7.22), and (7.25), with $A = A^\circ_S$ and $B = B^\circ_S$. Then*

$$\lim_{k\to\infty} \sum_{i\in I^B_k} \langle \boldsymbol{\tau}^\circ(s^{i-1}_k), E\boldsymbol{w}^\circ(s^i_k) - E\boldsymbol{w}^\circ(s^{i-1}_k)\rangle = \int_0^S \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds\,, \tag{8.19}$$

$$\lim_{k\to\infty} \sum_{i\in I^A_k\cup J^A_k} \int_{s^{i-1}_k}^{s^i_k} (\|\boldsymbol{\sigma}^\circ(s)\|_2 + \|\boldsymbol{\chi}^\circ(s)\|_2 + 1)\, 1_{B^\circ_S}(s)\,ds = 0\,, \tag{8.20}$$

$$\lim_{k\to\infty} \sum_{i\in I^B_k} \langle \boldsymbol{\chi}^\circ(s^i_k) - \boldsymbol{\chi}^\circ(s^{i-1}_k), \boldsymbol{e}^\circ(s^{i-1}_k)\rangle = \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\,ds\,, \tag{8.21}$$

$$\lim_{k\to\infty} \sum_{i\in I^B_k} \langle \boldsymbol{\chi}^\circ(s^i_k) - \boldsymbol{\chi}^\circ(s^{i-1}_k), \boldsymbol{p}^\circ(s^{i-1}_k)\rangle = \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\,ds\,. \tag{8.22}$$

*These equalities continue to hold if $\boldsymbol{\tau}^\circ(s^{i-1}_k)$, $\boldsymbol{e}^\circ(s^{i-1}_k)$, and $\boldsymbol{p}^\circ(s^{i-1}_k)$ are replaced by $\boldsymbol{\tau}^\circ(s^i_k)$, $\boldsymbol{e}^\circ(s^i_k)$, and $\boldsymbol{p}^\circ(s^i_k)$, respectively.*

*Proof.* Equality (8.20) follows from (7.28), with $\boldsymbol{\psi}$ given by $\boldsymbol{\sigma}^\circ 1_{B^\circ_S}$, $\boldsymbol{\chi}^\circ 1_{B^\circ_S}$, and $1_{B^\circ_S}$. Now, recalling that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s\in A^\circ_S$ by the inclusion $A^\circ_S \subset U^\circ$ proved in Lemma 6.4, and that $\|E\dot{\boldsymbol{w}}^\circ(s)\|_2 \leq 1$ for $\mathcal{L}^1$-a.e. $s\in[0,S]$ by (5.24), we get

$$\Big|\sum_{i\in I^B_k} \langle \boldsymbol{\tau}^\circ(s^{i-1}_k), E\boldsymbol{w}^\circ(s^i_k) - E\boldsymbol{w}^\circ(s^{i-1}_k)\rangle - \int_0^S \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds\Big| \leq$$
$$\leq \sum_{i\in I^B_k} \int_{s^{i-1}_k}^{s^i_k} \big|\langle \boldsymbol{\tau}^\circ(s^{i-1}_k) - \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\big|\,ds + \sum_{i\in I^A_k\cup J^A_k} \int_{s^{i-1}_k}^{s^i_k} |\langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle|\,ds \leq$$
$$\leq \sum_{i\in I^B_k} \int_{s^{i-1}_k}^{s^i_k} \|\boldsymbol{\tau}^\circ(s^{i-1}_k) - \boldsymbol{\tau}^\circ(s)\|_2\,ds + \sum_{i\in I^A_k\cup J^A_k} \int_{s^{i-1}_k}^{s^i_k} \|\boldsymbol{\tau}^\circ(s)\|_2 1_{B^\circ_S}(s)\,ds\,.$$

The first term in the right-hand side vanishes in the limit since $\boldsymbol{\tau}^\circ = \boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ$ , $\boldsymbol{\sigma}^\circ$ satisfies (7.18), and $\boldsymbol{\chi}^\circ$ is continuous. As the second one tends to 0 by (8.20), equality (8.19) is proved.

Since $\dot{\boldsymbol{\chi}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s\in A^\circ_S \subset U^\circ$, and $\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty \leq 1$ for $\mathcal{L}^1$-a.e. $s\in[0,S]$ by (5.24), by adapting the previous argument we can prove (8.21). We finally observe that, by (2.14) and (5.24),

$$\sum_{i\in I^A_k\cup J^A_k} \Big|\int_{s^{i-1}_k}^{s^i_k} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\,ds\Big| \leq M \sum_{i\in I^A_k\cup J^A_k} \int_{s^{i-1}_k}^{s^i_k} 1_{B^\circ_S}(s)\,ds\,,$$

where $M$ is an upper bound of $\|\boldsymbol{p}^\circ(s)\|_1$ on $[0,S]$, and the right-hand side vanishes in the limit as $k\to\infty$ by (8.20). Together with (7.13) and (7.14), this proves (8.22). The last assertion of the lemma can be proved in a similar way. □

**Lemma 8.9.** *Let $(s_k^i)_{0\le i\le i_k}$, $I_k^A$, $I_k^B$, and $J_k^A$ be as in Lemma 8.8. Assume that $(s_k^{i-1}, s_k^i)$ is contained in $A_S^\circ$ for every $i \in J_k^A$. Then there exists a sequence $R_k \to 0$ such that*

$$\begin{aligned}\sum_{i\in I_k^A\cup J_k^A}\Big(\mathcal{Q}(\boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i}\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds - \langle\boldsymbol{\chi}^\circ(s_k^i),\boldsymbol{p}^\circ(s_k^i)\rangle +\\ + \langle\boldsymbol{\chi}^\circ(s_k^{i-1}),\boldsymbol{p}^\circ(s_k^{i-1})\rangle + \int_{A_k^{i-1,i}}\langle\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\Big) \ge\\ \ge \sum_{i\in I_k^A\cup J_k^A}\Big(\langle\boldsymbol{\chi}^\circ(s_k^i),\boldsymbol{e}^\circ(s_k^i)\rangle - \langle\boldsymbol{\chi}^\circ(s_k^{i-1}),\boldsymbol{e}^\circ(s_k^{i-1})\rangle\Big) - R_k\,,\end{aligned} \tag{8.23}$$

*where $A_k^{i-1,i} := A_S^\circ \cap (s_k^{i-1}, s_k^i)$.*

*Proof.* Define

$$\hat{I}_k^A := \{i\in I_k^A\cup J_k^A : (s_k^{i-1},s_k^i)\subset A_S^\circ\}\quad\text{and}\quad \check{I}_k^A := \{i\in I_k^A : (s_k^{i-1},s_k^i)\cap B_S^\circ \ne \emptyset\};$$

our assumption on $J_k^A$ implies that $\hat{I}_k^A \cup \check{I}_k^A = I_k^A \cup J_k^A$. By Lemma 8.7, we have

$$\begin{aligned}\sum_{i\in \hat{I}_k^A}\Big(\mathcal{Q}(\boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i}\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds - \langle\boldsymbol{\chi}^\circ(s_k^i),\boldsymbol{p}^\circ(s_k^i)\rangle +\\ + \langle\boldsymbol{\chi}^\circ(s_k^{i-1}),\boldsymbol{p}^\circ(s_k^{i-1})\rangle + \int_{A_k^{i-1,i}}\langle\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\Big) \ge\\ \ge \sum_{i\in \hat{I}_k^A}\Big(\langle\boldsymbol{\chi}^\circ(s_k^i),\boldsymbol{e}^\circ(s_k^i)\rangle - \langle\boldsymbol{\chi}^\circ(s_k^{i-1}),\boldsymbol{e}^\circ(s_k^{i-1})\rangle\Big)\,.\end{aligned}$$

For every $i\in\check{I}_k^A$, we define $s_k^{i-\frac{2}{3}}$ (respectively $s_k^{i-\frac{1}{3}}$) as the supremum (respectively the infimum) of the connected component of $A_S^\circ$ containing $s_k^{i-1}$ (respectively $s_k^i$). Notice that both $s_k^{i-\frac{1}{3}}$ and $s_k^{i-\frac{2}{3}}$ belong to the set $B_S^\circ$. By Lemma 8.7, we have

$$\begin{aligned}\sum_{i\in \check{I}_k^A}\Big(\mathcal{Q}(\boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})) + \int_{s_k^{i-\frac{1}{3}}}^{s_k^i}\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds - \langle\boldsymbol{\chi}^\circ(s_k^i),\boldsymbol{p}^\circ(s_k^i)\rangle +\\ + \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}),\boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}})\rangle + \int_{s_k^{i-\frac{1}{3}}}^{s_k^i}\langle\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\Big) \ge\\ \ge \sum_{i\in \check{I}_k^A}\Big(\langle\boldsymbol{\chi}^\circ(s_k^i),\boldsymbol{e}^\circ(s_k^i)\rangle - \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}),\boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})\rangle\Big)\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in \check{I}_k^A}\Big(\mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^{i-\frac{2}{3}}}\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds - \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}),\boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}})\rangle +\\ + \langle\boldsymbol{\chi}^\circ(s_k^{i-1}),\boldsymbol{p}^\circ(s_k^{i-1})\rangle + \int_{s_k^{i-1}}^{s_k^{i-\frac{2}{3}}}\langle\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\Big) \ge\\ \ge \sum_{i\in \check{I}_k^A}\Big(\langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}),\boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})\rangle - \langle\boldsymbol{\chi}^\circ(s_k^{i-1}),\boldsymbol{e}^\circ(s_k^{i-1})\rangle\Big)\,.\end{aligned}$$

Therefore to prove (8.23) it is enough to show that there exists $R_k \to 0$ such that

$$\begin{aligned}\sum_{i\in \check{I}_k^A}\Big(\mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})) + \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}}\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds - \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}),\boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}})\rangle +\\ + \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}),\boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}})\rangle + \int_{A_k^{i-\frac{2}{3},i-\frac{1}{3}}}\langle\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)),\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds -\\ - \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}),\boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})\rangle + \langle\boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}),\boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})\rangle\Big) \ge -R_k\,,\end{aligned} \tag{8.24}$$

where $A_k^{i-\frac{2}{3},i-\frac{1}{3}} := A_S^\circ \cap (s_k^{i-\frac{2}{3}}, s_k^{i-\frac{1}{3}})$.

Let $\check{B}_k$ be the union of the intervals $(s_k^{i-1}, s_k^i)$ for $i \in \check{I}_k^A$. By the definition of $\check{I}_k^A$ each point of $\check{B}_k$ has distance from $B_S^\circ$ less than the constant $\eta_k$ introduced in (7.2). Since $B_S^\circ$ is compact, we have $\mathcal{L}^1(\check{B}_k \cap A_S^\circ) \to 0$. By (8.1) this implies that

$$\int_{\check{B}_k \cap A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds \to 0\,. \tag{8.25}$$

By Lemma 8.6 we have

$$\begin{aligned}
&\mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})) + \tfrac{1}{2}\mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}}) - \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac{2}{3}})) + \\
&+ \tfrac{1}{2}\mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}}) - \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac{1}{3}})) - \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}), \boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}})\rangle + \\
&+ \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}})\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}), \boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})\rangle + \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})\rangle \geq \\
&\geq \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_k^{i-\frac{2}{3}}) + \boldsymbol{\tau}^\circ(s_k^{i-\frac{1}{3}}), E\boldsymbol{w}^\circ(s_k^{i-\frac{1}{3}}) - E\boldsymbol{w}^\circ(s_k^{i-\frac{2}{3}})\rangle - \\
&- \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s^{i-\frac{1}{3}}) - \boldsymbol{\chi}^\circ(s^{i-\frac{2}{3}}), \boldsymbol{e}^\circ(s^{i-\frac{1}{3}}) + \boldsymbol{e}^\circ(s^{i-\frac{2}{3}})\rangle - \\
&- \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s^{i-\frac{1}{3}}) - \boldsymbol{\chi}^\circ(s^{i-\frac{2}{3}}), \boldsymbol{p}^\circ(s^{i-\frac{1}{3}}) + \boldsymbol{p}^\circ(s^{i-\frac{2}{3}})\rangle\,.
\end{aligned}$$

Now, recalling that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in A_S^\circ \subset U^\circ$, and that $\|E\dot{\boldsymbol{w}}^\circ(s)\|_2 \leq 1$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$ by (5.24), we get

$$\begin{aligned}
&\big|\tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_k^{i-\frac{2}{3}}) + \boldsymbol{\tau}^\circ(s_k^{i-\frac{1}{3}}), E\boldsymbol{w}^\circ(s_k^{i-\frac{1}{3}}) - E\boldsymbol{w}^\circ(s_k^{i-\frac{2}{3}})\rangle\big| \leq \\
&\leq C_1 \|E\boldsymbol{w}^\circ(s_k^{i-\frac{1}{3}}) - E\boldsymbol{w}^\circ(s_k^{i-\frac{2}{3}})\|_2 \leq C_1 \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} 1_{B_S^\circ}(s)\, ds \leq C_1 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,,
\end{aligned}$$

where $C_1$ is an upper bound of $\|\boldsymbol{\tau}^\circ(s)\|_2$ on $[0,S]$. Similarly, as $\dot{\boldsymbol{\chi}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in A_S^\circ \subset U^\circ$ and $\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty \leq 1$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$ by (5.24), we get

$$\big|\tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s^{i-\frac{1}{3}}) - \boldsymbol{\chi}^\circ(s^{i-\frac{2}{3}}), \boldsymbol{e}^\circ(s^{i-\frac{1}{3}}) + \boldsymbol{e}^\circ(s^{i-\frac{2}{3}})\rangle\big| \leq C_2 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,,$$

where $C_2$ is an upper bound of $\|\boldsymbol{e}^\circ(s)\|_1$ on $[0,S]$. Arguing as before, by (2.14) and (5.24), we can also prove that

$$\big|\tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s^{i-\frac{1}{3}}) - \boldsymbol{\chi}^\circ(s^{i-\frac{2}{3}}), \boldsymbol{p}^\circ(s^{i-\frac{1}{3}}) + \boldsymbol{p}^\circ(s^{i-\frac{2}{3}})\rangle\big| \leq C_3 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,,$$

where $C_3$ is an upper bound of $\|\boldsymbol{p}^\circ(s)\|_1$ on $[0,S]$. Setting $C := C_1 + C_2 + C_3$, from the previous inequalities we obtain that (8.24) holds with

$$\begin{aligned}
R_k := C \sum_{i \in \check{I}_k^A} &\int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds + \int_{\check{B}_k \cap A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds + \\
&+ \tfrac{1}{2} \sum_{i \in \check{I}_k^A} \Big(\mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}}) - \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac{2}{3}})) - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) ds\Big) + \\
&+ \tfrac{1}{2} \sum_{i \in \check{I}_k^A} \Big(\mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}}) - \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac{1}{3}})) - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds\Big)\,.
\end{aligned}$$

From Lemma 7.3 and from (7.28) and (8.25) we obtain $R_k \to 0$, concluding the proof. □

*Proof of Theorem 4.5 (conclusion).* Let us fix $S > 0$ and let $A_S^\circ$ and $B_S^\circ$ be the sets defined in (6.14) and (8.12). Let $(s_k^i)_{0 \leq i \leq i_k}$, $I_k^A$, $I_k^B$, and $J_k^A$ be as in Lemma 8.8. By Lemma

7.9 we may assume that $(s_k^{i-1}, s_k^i) \subset A_S^\circ$ for every $i \in J_k^A$. By Lemma 8.8 there exists a sequence $\rho_k^1 \to 0$ such that

$$\int_0^S \Big( \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle \Big) ds + \sum_{i\in I_k^B} \Big( \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{e}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{e}^\circ(s_k^{i-1})\rangle \Big) \le$$
$$\le \tfrac12 \sum_{i\in I_k^B} \langle \boldsymbol{\tau}^\circ(s_k^{i-1}) + \boldsymbol{\tau}^\circ(s_k^i), E\boldsymbol{w}^\circ(s_k^i) - E\boldsymbol{w}^\circ(s_k^{i-1})\rangle -$$
$$- \tfrac12 \sum_{i\in I_k^B} \langle \boldsymbol{\chi}^\circ(s_k^i) - \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{e}^\circ(s_k^{i-1}) + \boldsymbol{e}^\circ(s_k^i)\rangle +$$
$$+ \sum_{i\in I_k^B} \Big( \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{e}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{e}^\circ(s_k^{i-1})\rangle \Big) + \rho_k^1 \,.$$

By Lemma 8.6 we then deduce that

$$\int_0^S \Big( \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle \Big) ds + \sum_{i\in I_k^B} \Big( \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{e}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{e}^\circ(s_k^{i-1})\rangle \Big) \le$$
$$\le \sum_{i\in I_k^B} (\mathcal{Q}(\boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-1}))) + \tfrac12 \sum_{i\in I_k^B} \mathcal{H}(\boldsymbol{p}^\circ(s_k^i) - \boldsymbol{p}^\circ(s_k^{i-1}), \boldsymbol{\zeta}^\circ(s_k^{i-1})) +$$
$$+ \tfrac12 \sum_{i\in I_k^B} \mathcal{H}(\boldsymbol{p}^\circ(s_k^i) - \boldsymbol{p}^\circ(s_k^{i-1}), \boldsymbol{\zeta}^\circ(s_k^i)) + \tfrac12 \sum_{i\in I_k^B} \langle \boldsymbol{\chi}^\circ(s_k^{i-1}) - \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i) + \boldsymbol{p}^\circ(s_k^{i-1})\rangle -$$
$$- \sum_{i\in I_k^B} \Big( \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle \Big) + \rho_k^1 \,.$$

By (8.22), Lemma 7.3 provides a sequence $\varrho_k^2 \to 0$ such that

$$\int_0^S \Big( \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle \Big) ds + \sum_{i\in I_k^B} \Big( \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{e}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{e}^\circ(s_k^{i-1})\rangle \Big) \le$$
$$\le \sum_{i\in I_k^B} (\mathcal{Q}(\boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-1}))) + \sum_{i\in I_k^B} \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s)), \boldsymbol{\zeta}^\circ(s))\, ds +$$
$$+ \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds - \sum_{i\in I_k^B} \Big( \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle \Big) + \rho_k^1 + \rho_k^2 \,.$$

Adding (8.23), where $A_k^{i-1,i} := A_S^\circ \cap (s_k^{i-1}, s_k^i)$, we get

$$\int_0^S \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{e}^\circ(S)\rangle - \langle \chi_0, e_0\rangle \le$$
$$\le \sum_{i=1}^{i_k} (\mathcal{Q}(\boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_k^{i-1}))) + \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s)), \boldsymbol{\zeta}^\circ(s))\, ds + \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds -$$
$$- \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle + \sum_{i\in I_k^A \cup J_k^A} \int_{A_k^{i-1,i}} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds + \rho_k^3 \le$$
$$\le \mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}^\circ(s))\, ds + \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds -$$
$$- \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle + \int_{A_S^\circ} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds + \rho_k^3 \,,$$

with

$$\varrho_k^3 := \varrho_k^1 + \varrho_k^2 + R_k + \int_{B_k \cap A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2 \, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds\,, \tag{8.26}$$

where $B_k$ is the union of the intervals $(s_k^{i-1}, s_k^i)$ for $i \in I_k^B$. By the definition of $I_k^B$ each point of $B_k$ has distance from $B_S^\circ$ less than the constant $\eta_k$ introduced in (7.2). Since $B_S^\circ$ is compact, we have $\mathcal{L}^1(B_k \cap A_S^\circ) \to 0$. By (8.1) this implies that the integral in (8.26) tends to 0 as $k \to \infty$. Therefore $\varrho_k^3 \to 0$, and the last chain of inequalities yields (8.2). Together with inequality (6.1), proved in Section 6, this gives (4.15) and (4.9). By Proposition 4.4 this proves (4.8) and concludes the proof of Theorem 4.5. □

## 9. Appendix

We now prove the regularity result used in Lemma 8.5.

**Theorem 9.1.** *For every open set $\Omega' \subset\subset \Omega$ there exists a constant $C$ depending only on $\Omega'$, $\Omega$, and $\mathbb{C}$ such that, if $p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $u \in H^1_{loc}(\Omega; \mathbb{R}^n)$ satisfies the equation*

$$-\mathrm{div}\,(\mathbb{C}Eu) = -\mathrm{div}\,(\mathbb{C}p) \quad \text{in } \Omega\,, \tag{9.1}$$

*then we have the estimate*

$$\|\nabla u\|_{1,w,\Omega'} \le C(\|p\|_{1,\Omega} + \|u\|_{1,\Omega})\,, \tag{9.2}$$

*where $\|\cdot\|_{1,w,\Omega'}$ is defined in* (8.10).

*Proof.* Let $\Omega''$ be an open set such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and let $\varphi \in C_c^\infty(\Omega)$ be a cutoff function with $\varphi = 1$ on $\Omega''$ and $0 \le \varphi \le 1$. Let $p$ and $u$ be as in the statement, and let $q := \mathbb{C}p$, and $v := \varphi u$. It turns out that $v$ has compact support and satisfies the equation

$$-\mathrm{div}\,(\mathbb{C}Ev) = -\mathrm{div}\,(\varphi q) + q\nabla\varphi - (\mathbb{C}Eu)\nabla\varphi - \mathrm{div}\,(\mathbb{C}(u \odot \nabla\varphi) \quad \text{in } \mathbb{R}^n\,. \tag{9.3}$$

The fundamental solution of the operator $-\mathrm{div}\,(\mathbb{C}Eu)$ is given by

$$G(x) := a\, g(x)\, I + b\, \nabla g(x) \otimes x\,, \tag{9.4}$$

where $g$ is the fundamental solution of the Laplace operator, $a = \frac{1}{2(\lambda+2\mu)} + \frac{1}{2\mu}$, and $b = \frac{1}{2(\lambda+2\mu)} - \frac{1}{2\mu}$ (see [19, Section 2.5.2] and [25, Chapter II, formula (1.46)]). Since $v$ has compact support, equation (9.3) gives the representation

$$v_i(x) = \sum_{h,k=1}^n \int_{\mathbb{R}^n} D_k G_{ih}(x-y)(\varphi q)_{hk}(y)\, dy + \sum_{h=1}^n \int_{\mathbb{R}^n} G_{ih}(x-y)(q\nabla\varphi)_h(y)\, dy -$$

$$- \sum_{h=1}^n \int_{\mathbb{R}^n} G_{ih}(x-y)(\mathbb{C}Eu\nabla\varphi)_h(y)\, dy + \sum_{h,k=1}^n \int_{\mathbb{R}^n} D_k G_{ih}(x-y)(\mathbb{C}(u\odot\nabla\varphi))_{hk}(y)\, dy\,.$$

For a.e. $x \in \Omega'$ it follows that

$$D_j v_i(x) = \alpha(x) + \beta(x) - \gamma(x) + \delta(x)\,,$$

where

$$\alpha(x) := \sum_{h,k=1}^n \int_{\mathbb{R}^n} D_j D_k G_{ih}(x-y)(\varphi q)_{hk}(y)\, dy\,,$$

$$\beta(x) := \sum_{h=1}^n \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y)(q\nabla\varphi)_h(y)\, dy\,,$$

$$\gamma(x) := \sum_{h=1}^n \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y)(\mathbb{C}Eu\nabla\varphi)_h(y)\, dy\,,$$

$$\delta(x) := \sum_{h,k=1}^n \int_{\Omega\setminus\Omega''} D_j D_k G_{ih}(x-y)(\mathbb{C}(u\odot\nabla\varphi))_{hk}(y)\, dy\,.$$

The function $D_j D_k G_{ih}$ is homogeneous of degree $-n$. Using the explicit expression of $G_{ih}$ given by (9.4) we can check that $D_j D_k G_{ih}$ has mean value 0 on the boundary of each ball around the origin. Therefore we can apply the Calderon-Zygmund estimate contained in [28, Chapter II, Theorem 4], obtaining

$$\|\alpha\|_{1,w,\Omega} \leq C_1 \|p\|_{1,\Omega}\,, \tag{9.5}$$

where the constant $C_1$ only depends on the function $G$, and the elasticity tensor $\mathbb{C}$.

To estimate the term $\gamma(x)$ we introduce the cartesian components $c_{hk}^{lm}$ of the tensor $\mathbb{C}$, defined by

$$(\mathbb{C}Eu)_{hk} = \sum_{l,m=1}^{n} c_{hk}^{lm} D_l u_m\,.$$

It follows that

$$\gamma(x) = \sum_{h,k,l,m=1}^{n} c_{hk}^{lm} \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y) D_l u_m(y) D_k \varphi(y)\, dy\,.$$

For $x \in \Omega'$, the function $y \mapsto G_{ih}(x-y)$ is of class $C^\infty$ in $\Omega \setminus \Omega''$. Integrating by parts, we obtain

$$\gamma(x) = -\sum_{h,k,l,m=1}^{n} c_{hk}^{lm} \int_{\Omega\setminus\Omega''} D_j D_l G_{ih}(x-y) u_m(y) D_k \varphi(y)\, dy - \\ -\sum_{h,k,l,m=1}^{n} c_{hk}^{lm} \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y) u_m(y) D_l D_k \varphi(y)\, dy\,.$$

As $D_j D_l G_{ih}(x-y)$ and $D_j G_{ih}(x-y)$ are uniformly bounded when $x \in \Omega'$ and $y \in \Omega \setminus \Omega''$, we obtain the estimate

$$\|\gamma\|_{\infty,\Omega'} \leq C_3 \|u\|_{1,\Omega}\,, \tag{9.6}$$

where the constant $C_3$ depends on the function $G$, on the elasticity tensor $\mathbb{C}$, on the pair $\Omega'$, $\Omega''$, and on the function $\varphi$.

In a similar, and easier, way we prove the estimates

$$\|\beta\|_{\infty,\Omega'} \leq C_2 \|p\|_{1,\Omega} \quad \text{ and } \|\delta\|_{\infty,\Omega'} \leq C_4 \|u\|_{1,\Omega}\,, \tag{9.7}$$

where the constants $C_2$ and $C_4$ depend on the function $G$, on the elasticity tensor $\mathbb{C}$, on the pair $\Omega'$, $\Omega''$, and on the function $\varphi$. Inequality (9.2) follows now from (9.5), (9.6), and (9.7). □

## Acknowledgments

We thank Alexei Demyanov, who worked with us in the early stages of the preparation of this paper, and gave many contributions to Section 3. We also thank Riccarda Rossi and Giuseppe Savaré for a very useful discussion on the additional term accounting for the instantaneous viscous dissipation at jump times, which suggested to us the precise form of (1.1). Finally, we thank Giovanni Alberti for a useful remark leading to a simplified proof of the result in the Appendix.

The lecture [30] by Lev Truskinovsky has been a useful source of inspiration.

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