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QUASI-SYMMETRIC EMBEDDINGS IN EUCLIDEAN SPACES

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ABSTRACT. We consider quasi-symmetric embeddings $f\colon G \to R^n$, G open in R^p, $p \leqslant n$. If $p = n$, quasi-symmetry implies quasi-conformality. The converse is true if G has a sufficiently smooth boundary. If $p < n$, the Hausdorff dimension of fG is less than n. If fG has a finite p-measure, f preserves the property of being of p-measure zero. If $p < n$ and $n \geqslant 3$, R^n contains a quasi-symmetric p-cell which is topologically wild.

We also prove auxiliary results on the relations between Hausdorff measure and Čech cohomology.

1. Introduction.

1.1. The quasi-symmetric (QS) embeddings were introduced in [TV] as a natural generalization of quasi-conformal maps $f\colon R^n \to R^n$. Let us recall the definition. Suppose that X and Y are metric spaces. The distance between two points a, b in either space is written as $|a - b|$. An embedding $f\colon X \to Y$ is QS or η-QS if there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that $|f(a) - f(x)| \leqslant \eta(\rho)|f(b) - f(x)|$ whenever $\rho \geqslant 0$ and $|a - x| \leqslant \rho|b - x|$. If this is only supposed to be true for $\rho = 1$, we say that f is weakly QS or weakly H-QS, $H = \rho(1)$.

In this paper we shall mainly consider the case $X \subset R^p$ and $Y = R^n$, $p \leqslant n$. Moreover, X is usually assumed to be an open set $G \subset R^p$. In the case $p = n$, there is a close connection between quasi-symmetry and quasi-conformality. This will be studied in §2. In §3 we present auxiliary results on Čech cohomology and Hausdorff measure, which may be of independent interest. In §4 we prove that if $p < n$, the Hausdorff dimension of fG is always less than n. §5 deals with the case where the p-measure of fG is finite. This case has been considered by F. W. Gehring [Ge_3] for maps which are restrictions of n-dimensional quasi-conformal maps. We shall extend all his results to QS embeddings. In §6 we show that a QS p-cell can be topologically wild in every R^n, $n \geqslant \max(p + 1, 3)$. In this respect, they differ from Lipschitz p-cells, which are topologically flat in R^n if $n \geqslant 3p + 1$.

1.2. *Terminology and notation*. We shall use the same terminology as in [TV]. All spaces are assumed to be metric. If $p < n$, we identify R^p with the subspace $R^p \times 0$ of R^n. The open ball $\{y\colon |y - x| < r\}$ in a metric space X is written as $B(x, r)$. If $X = R^n$, we may use the notation $B^n(x, r)$ and the abbreviation $B^n = B^n(0, 1)$. Similarly, $S(x, r) = \{y\colon |y - x| = r\}$, $S^{n-1}(x, r)$, and S^{n-1}. Furthermore, I^n will denote the closed unit n-cube $[-1, 1]^n$ in R^n, and Σ^{n-1} its boundary ∂I^n. The

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normalized Hausdorff p-measure, $p \geqslant 0$, of a set A in a metric space X is written as $m_p(A)$. For the definition, see [Fe, 2.10.2]. In particular, if $X = R^n$, m_n is the ordinary Lebesgue measure. The Hausdorff dimension $\dim_H A$ of A is $\inf\{p: m_p(A) = 0\} = \sup\{p: m_p(A) = \infty\}$. We set $\Omega_n = m_n(B^n)$ and $\omega_n = m_n(S^n)$.

2. Quasi-symmetry and quasi-conformality.

2.1. In this section we shall consider embeddings $f: G \to R^n$, G open in R^n, $n \geqslant 2$. For such maps, quasi-symmetry implies quasi-conformality (Theorem 2.3). The converse is not true. For example, a Möbius transformation of a ball onto a half-space is quasi-conformal (even conformal) but not QS, since a QS image of a bounded set is bounded [TV, 2.6]. In the spherical metric, Möbius transformations are QS, but there is no η such that every Möbius transformation of S^n is η-QS. We shall show, however, that in some particular cases quasi-conformality implies quasi-symmetry. Moreover, the concepts locally QS and locally quasi-conformal are equivalent for embeddings $f: G \to R^n$, G open in R^n.

We use the abbreviation QC for quasi-conformal. For the basic theory of QC maps, see [Vä$_1$].

2.2. Definitions. Let $f: X \to Y$ be an embedding (or more generally, an immersion). If every point x of X has a neighborhood U such that $f|U$ is QS, f is said to be locally QS. The concepts locally η-QS, locally weakly QS, locally weakly H-QS, locally QC, and locally K-QC are defined similarly. Observe, however, that a locally K-QC embedding is K-QC.

2.3. Theorem. *Let G be open in R^n, $n \geqslant 2$, and let $f: G \to R^n$ be a locally weakly H-QS embedding. Then f is H^{n-1}-QC.*

Proof. This is well known, see [Vä$_1$, Remark 34.2]. □

2.4. Theorem. *Suppose that G is open in R^n, $n \geqslant 2$, and that $f: G \to R^n$ is K-QC. Suppose also that $x_0 \in G$, $\alpha > 1$, $r > 0$ such that $B(x_0, \alpha r) \subset G$. Then $f|B(x_0, r)$ is η-QS where η depends only on n, K and α.*

Proof. The following proof is due to J. Sarvas. It is published with his permission, and it simplifies the author's original proof.

By [TV, 2.16], it suffices to show that f is weakly H-QS with $H = H(n, K, \alpha)$. We may normalize $r = 1$. Let $h: B(x_0, 4) \to B(x_0, \alpha)$ be a K_1-QC radial homeomorphism with $h|B(x_0, 1) = \mathrm{id}$ and $K_1 = K_1(n, \alpha)$. Then $g = fh: B(x_0, 4) \to R^n$ is a KK_1-QC embedding, and $g = f$ in $B(x_0, 1)$. Let $a, b, x \in B(x_0, 1)$ with $0 < |a - x| \leqslant |b - x|$. We want to estimate the ratio $\rho' = |f(a) - f(x)|/|f(b) - f(x)|$.

Using the standard notation (see for example [Vä$_1$, p. 78]), we set

$$L = L(x, f, |a - x|) \quad \text{and} \quad l = l(x, f, |a - x|).$$

Observe that $|a - x| \leqslant 2$ and thus $\overline{B}(x, |a - x|) \subset B(x_0, 3)$. Clearly $\rho' \leqslant L/l$. Choose $z \in S(x, |x - a|)$ such that $|g(z) - g(x)| = L$. Let F be the ray from $g(x)$ through $g(z)$, and let $J \subset F$ be a segment joining $g(z)$ to $gS(x, 3)$ in $gB(x, 3)$. Then $g^{-1}J$ is a continuum joining the spheres $S(x, |x - a|)$ and $S(x, 3)$. Set $A = \overline{B}(f(x), l)$. Then $g^{-1}A$ is a continuum joining x and $S(x, |x - a|)$. Let Γ be the family of all paths joining $g^{-1}A$ and $g^{-1}J$ in $B(x, 3)$. Then $M(\Gamma) \geqslant c_n > 0$. This can be proved, for example, as [**Vä**$_1$, Theorem 11.7(4)]. On the other hand, $M(g\Gamma) < \omega_{n-1}(\log(L/l))^{1-n}$. Since $M(\Gamma) < KK_1 M(g\Gamma)$, we obtain $L/l < e^t$ with $t = (KK_1\omega_{n-1}/c_n)^{1/(n-1)}$. □

2.5. Corollary. *An embedding $f\colon R^n \to R^n$ is QS if and only if it is QC. In this case $fR^n = R^n$.*

Proof. The last statement follows from [**Vä**$_1$, 17.4]. □

2.6. Corollary. *Let G be open in R^n and let $f\colon G \to R^n$ be an embedding. Then the following statements are equivalent*:
(1) *f is locally QC*,
(2) *f is locally QS*,
(3) *f is locally weakly QS*. □

2.7. Theorem. *Suppose that G is open in R^n, that F is compact in G and that f: $G \to R^n$ is a K-QC embedding. Then $f|F$ is QS. If G is connected, $f|F$ is η-QS, where η depends only on G, F, and K.*

Proof. Since a locally QS embedding of a compact space is QS [**TV**, 2.23], the first part of the theorem follows from Theorem 2.4.

Suppose that G is connected. Choose a connected compact polyhedron P such that $F \subset P \subset G$. By [**LV**, 2.34], P is quasi-convex and hence of bounded turning. It follows from [**TV**, 2.16] that in the second part of the theorem, it suffices to show that $f|P$ is weakly H-QS with H depending only on G, F, and K. Suppose that this is not true. Then there is a sequence of K-QC embeddings $f_j\colon G \to R^n$ and points a_j, b_j, x_j in P such that $|a_j - x_j| < |b_j - x_j|$ and such that

$$\rho_j = |f_j(a_j) - f_j(x_j)|/|f_j(b_j) - f_j(x_j)| \to \infty.$$

Replacing f_j by $g_j f_j$, where g_j is a suitable similarity map, we may assume that $0 \in f_j P$ and that $d(f_j P) = 1$. Passing to a subsequence, we may assume that $a_j \to a_0$, $b_j \to b_0$, $x_j \to x_0$. By 2.4, x_0 has a neighborhood in which each f_j is η-QS with η depending only on n and K. Hence $b_0 \neq x_0$.

Since every f_j omits ∞ and since $f_j P \subset \overline{B}^n$, it follows from [**Vä**$_1$, 19.4(1)] that the family of all f_j is equicontinuous. Hence we may assume that they converge to a limit $f\colon G \to R^n$ uniformly in compact sets. Since $d(fF) = 1$, f is not constant. By [**Vä**$_1$, 21.1], f is a K-QC embedding. Thus $\rho_j \to |f(a_0) - f(x_0)|/|f(b_0) - f(x_0)|$, which gives a contradiction. □

2.8. Remarks. 1. The quantitative version of 2.6 is also true: The conditions (1) f is K-QC, (2) f is locally η-QS, (3) f is locally weakly H-QS, are equivalent in the sense that, for example, (1) implies (2) with η depending only on K and n.

2. The conditions (2) and (3) in 2.6 and in 2.8.1 are equivalent in the more general case $f\colon G \to R^n$, G open in R^p, $p < n$. This follows from [**TV**, 2.16].

3. The quasi-conformality of an embedding $f\colon G \to R^n$, G open in R^n, $n > 2$, can also be characterized by means of the linear dilatation

$$H_L(x) = \limsup_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)}.$$

Indeed, f is QC if and only if H_L is bounded in G [**Vä**$_1$, 34.1]. Hence the local boundedness of H_L implies that f is locally QS. This result is not true for general metric spaces, not even for homeomorphisms $f\colon R^1 \to R^1$. In fact, it is not difficult to construct a homeomorphism $f\colon R^1 \to R^1$ which is not locally QS at some point but which has a positive derivative at every point. Thus $H_L(x) = 1$ for all x. Another example will be given in 5.11.

4. If D and D' are bounded domains in R^n such that ∂D and $\partial D'$ are QC flat $(n-1)$-spheres, then every QC homeomorphism $f\colon D \to D'$ is QS. Indeed, the QC Schoenflies theorem provides an extension of f to a QC homeomorphism $R^n \to R^n$, and the result follows from 2.5.

We give a more general result whose proof does not involve the Schoenflies theorem:

2.9. THEOREM. *Let D be a bounded domain in R^n with the following property: For every point $b \in \partial D$ there exist a neighborhood U of b and a QC homeomorphism of U onto B^n such that $U \cap D$ is mapped onto an open half-ball. Let D' be another bounded domain with the same property. Then every QC homeomorphism $f\colon D \to D'$ is QS.*

PROOF. By [**Vä**$_1$, 17.18], f can be extended to a homeomorphism $f^*\colon \overline{D} \to \overline{D}'$. By [**Vä**$_1$, 35.5], every boundary point of D has a neighborhood U such that $f|U \cap D$ can be extended to a QC embedding of U. By 2.6, f^* is locally QS. Since $\overline{D}$ is compact, f^* is QS [**TV**, 2.23]. □

3. Cohomology and measure.

3.1. The purpose of this section is to present auxiliary results on Čech cohomology and on its relations with the Hausdorff measure. Lemma 3.2 will be needed in §§4 and 5, and Theorem 3.6 in §5. It is possible that these results have independent interest.

The Čech cohomology groups with integral coefficients of a space X are written as $H^j(X)$. If $\mathcal{Q}$ is a family of sets in X, we let $N(\mathcal{Q})$ and $H^j(\mathcal{Q})$ denote the nerve of $\mathcal{Q}$ and the j-dimensional cohomology group of $N(\mathcal{Q})$, respectively. If $\mathcal{Q}$ is an open covering of X, there are canonical homomorphisms $\gamma\colon H^j(\mathcal{Q}) \to H^j(X)$. If $i\colon E \to X$ is an inclusion, we set $\mathcal{Q}|E = \{U \cap E\colon U \in \mathcal{Q}\}$. There is a natural simplicial map $\varphi\colon N(\mathcal{Q}|E) \to N(\mathcal{Q})$ and a commutative diagram:

$$\begin{array}{ccc} H^j(\mathcal{Q}) & \xrightarrow{\varphi^*} & H^j(\mathcal{Q}|E) \\ \downarrow\gamma_1 & & \downarrow\gamma_2 \\ H^j(X) & \xrightarrow[i^*]{} & H^j(E) \end{array}$$

The homomorphism $\gamma_2\varphi^* = i^*\gamma_1\colon H^j(\mathcal{Q}) \to H^j(E)$ will also be called canonical.

Let $\mathcal{W}_p$ be the covering of $R^p \setminus 0$ consisting of open half-spaces $W_j = \{x \in R^p\colon x_j > 0\}$ and $W_{p+j} = \{x \in R^p\colon x_j < 0\}$, $1 \leq j \leq p$. Observe that a subfamily $\mathcal{B}$ of $\mathcal{W}_p$ has an empty intersection if and only if there is j such that $\{W_j, W_{p+j}\} \subset \mathcal{B}$. We shall later also make use of the sets $W_j^* = \{x \in R^p\colon x_j \geq |x_i| \text{ for all } i \neq j\}$ and $W_{p+j}^* = \{x \in R^p\colon x_j \leq -|x_i| \text{ for all } i \neq j\}$. Observe that $\overline{W}_\nu^* \setminus 0 \subset W_\nu$ and that $\{\overline{W}_1^*, \ldots, \overline{W}_{2p}^*\}$ is a covering of R^p.

3.2. LEMMA. *Let E be a compact set in R^p separating* 0 *from* ∞. *Then the canonical homomorphism* $\gamma\colon H^{p-1}(\mathfrak{V}_p) \to H^{p-1}(E)$ *is not the zero map.*

PROOF. Choose $r_2 > r_1 > 0$ such that E is contained in the annulus $A = \{x\colon r_1 < |x| < r_2\}$. Since all intersections of the elements of $\mathfrak{V}_p|A$ are cohomologically trivial and since $\varphi\colon N(\mathfrak{V}_p|A) \to N(\mathfrak{V}_p)$ is a simplicial isomorphism, the canonical homomorphisms $\gamma_1\colon H^j(\mathfrak{V}_p) \to H^j(A)$ are isomorphisms. See **[Go**, p. 213]; an elementary proof can in this case also be given. Let $i\colon E \to A$ be the inclusion. Since $\gamma = i^*\gamma_1$, it suffices to show that $i^*\colon H^{p-1}(A) \to H^{p-1}(E)$ is not the zero map. By Alexander duality, i^* can be identified with the map $j_*\colon \tilde{H}_0(R^p \setminus A) \to \tilde{H}_0(R^p \setminus E)$ induced by the inclusion. Since the two components of $R^p \setminus A$ are contained in different components of $R^p \setminus E, j_* \neq 0$. □

3.3. The rest of this section will be devoted to the study of the relations between the Hausdorff measure of a set and the cohomological properties of its neighborhoods. In view of the applications in §5, we shall work in Σ^n, but it is clear that similar results hold in R^n. The results and the methods were inspired by **[Ge$_2$]**.

We introduce some notation. If $b \in R^n$ and $A \subset R^n$, we let bA denote the rectilinear join of b and A, consisting of all segments ba, $a \in A$. If $A \subset \Sigma^n$ and $t > 0$, $A(t)$ is the open t-neighborhood of A in Σ^n, $A(t) = \{x \in \Sigma^n\colon d(x, A) < t\}$. Given $z \in R^{n+1}$ and $0 < s < 2$, we shall consider the decomposition K of R^{n+1} to congruent closed cubes by the planes $x_j = z_j + ks$, $1 \leqslant j \leqslant n+1$, $k \in Z$. The base point z will be chosen in such a way that none of these planes is of the form $x_j = \pm 1$. Then K induces a decomposition L of Σ^n. Let K^k and L^k be the k-skeletons of K and L, respectively. Then $L^k = K^{k+1} \cap \Sigma^n$. Each L^k is a union of PL k-cells Q_j^k, where j runs through a finite index set J_k. These cells are ordinary k-cubes of side s except for those intersecting the $(n-1)$-skeleton of Σ^{n-1}. Anyway, every Q_j^k can be written as a join $x_j^k E_j^k$ where E_j^k is the manifold boundary of Q_j^k and $E_j^k \subset L^{k-1}$. Moreover, $d(Q_j^k) < s(n+1)^{1/2}$ for all k and j.

3.4. LEMMA. *Let A be a compact set in Σ^n, let $p \in [0, n]$ be an integer, and let $A \cap L^{n-p} = \varnothing$. Then there is a compact set $C \subset \Sigma^n$ such that $H^q(C) = 0$ for all $q \geqslant p$, and $A \subset C \subset A(t)$, where $t = s(n+1)^{1/2}$.*

PROOF. This is a modification of the proof of **[Ge$_2$**, Lemma 4]. We set $C^{n-p} = \varnothing$ and construct inductively compact sets $C^k \subset L^k$ for $n - p \leqslant k \leqslant n$ as follows: Suppose that C^{k-1} has been constructed. Let $B_j^k = (A \cup C^{k-1}) \cap Q_j^k$. If $B_j^k = \varnothing$, we set $C_j^k = \varnothing$; otherwise $C_j^k = x_j^k B_j^k$. Then let $C^k = \bigcup \{C_j^k\colon j \in J_k\}$. We claim that $C = C^n$ has the desired properties. Observe that $C \cap L^k = C^k$.

It follows from the construction that $A \subset C$ and that $C \cap Q_j^n \neq \varnothing$ implies $A \cap Q_j^n \neq \varnothing$. Hence $C \subset A(t)$.

Since $C \cap L^0 = \varnothing$, C is a proper subset of Σ^n, and therefore $H^q(C) = 0$ for $q \geqslant n$. It remains to show that $H^q(C) = 0$ whenever $p \leqslant q \leqslant n-1$.

We say that $P \subset \Sigma^n$ is a k-set, $0 \leqslant k \leqslant n$, if P is a union of some of the PL cells Q_j^i, $i \leqslant k$. We shall prove by induction on k that $H^q(C \cap P) = 0$ for every k-set P and for every $q \in [p + k - n, n - 1]$. Since Σ^n is an n-set, this will prove the theorem.

If $k < n - p$, $C \cap P = \emptyset$, and thus $H^q(C \cap P) = 0$ for all q. Suppose that $H^q(C \cap P) = 0$ for all k-sets P with $k < l - 1$ and for all $q \in [p + k - n, n - 1]$. Let P be an l-set and let $p + l - n \leqslant q < n - 1$. Let a be the number of l-cells Q_j^l in P. If $a = 0$, then P is an $(l - 1)$-set and thus $H^q(C \cap P) = 0$. Proceeding by a subinduction on a, we assume that $H^q(C \cap P) = 0$ whenever $a < b - 1$, and let $a = b$. Choose an l-cell Q_j^l in P. Then $P' = \mathrm{cl}(P \setminus Q_j^l)$ is also an l-set. Setting $E_1 = C \cap P'$ and $E_2 = C \cap C_j^l$ we consider the Mayer-Vietoris sequence

$$H^{q-1}(E_1 \cap E_2) \to H^q(E_1 \cup E_2) \to H^q(E_1) \oplus H^q(E_2).$$

Here $H^q(E_1) = 0$ by the subinductive hypothesis. Since E_2 is contractible, $H^q(E_2) = 0$. Since $E_1 \cap E_2$ is an $(l - 1)$-set, the main induction implies $H^{q-1}(E_1 \cap E_2) = 0$. Hence $H^q(C \cap P) = H^q(E_1 \cup E_2) = 0$. □

3.5. THEOREM. *Let $0 < r < 1/(n + 1)$, let $p \in [1, n]$ be an integer, and let A be a compact set in Σ^n such that $m_p(A) < r^p$. Then there is a compact set $C \subset \Sigma^n$ such that $H^q(C) = 0$ for all $q \geqslant p$ and $A \subset C \subset A(t)$, where $t = 2(n + 1)^{3/2}r$.*

PROOF. Set $s = 2(n + 1)r$. Then $0 < s < 2$, and we can introduce the corresponding cube decomposition K of R^{n+1} as in 3.3. By [**Ge$_2$**, Lemma 1], we can choose the base point z of K in such a way that $m_{p-k}(A \cap K^{n+1-k}) < s^{p-k}$ for all $k \in [0, p]$. For $k = p$ this yields $A \cap K^{n+1-p} = \emptyset$. A slight change of z ensures that no n-face of I^{n+1} is in K^n. Since $L^j = \Sigma^n \cap K^{j+1}$, $A \cap L^{n-p} = \emptyset$, and the theorem follows from 3.4. □

3.6. THEOREM. *Suppose that A is a compact set in Σ^n, that $p \in [1, n]$ is an integer and that $0 < t < 2(n + 1)^{1/2}$. Suppose also that the homomorphism $H^p(A(t)) \to H^p(A)$ is not the zero map. Then $m_p(A) \geqslant (t/2(n + 1)^{3/2})^p$.*

PROOF. Suppose that the theorem is false. Then $m_p(A) < r^p$ for some $r < t/2(n + 1)^{3/2}$, which implies $r < 1/(n + 1)$. Let C be the set given by 3.5. Then the map $H^p(A(t)) \to H^p(A)$ factorizes through the zero group $H^p(C)$, which gives the contradiction. □

4. The Hausdorff dimension of fG.

4.1. THEOREM. *Let G be open in R^p, let $p < n$, and let $f\colon G \to R^n$ be a locally weakly H-QS embedding. Then $\dim_H fG \leqslant \beta < n$ where β depends only on n and H.*

PROOF. Let $x_0 \in G$. Choose a ball $B = B^p(x_0, r_0)$ such that $\bar{B} \subset G$ and such that $f|\bar{B}$ is weakly H-QS. Let Q be a closed n-cube in $R^n \setminus f[G \setminus B]$. Since fB can be covered with a countable number of such cubes, it suffices to find an estimate $\dim_H(Q \cap fB) \leqslant \beta(n, H) < n$. We shall prove that there is an integer $k = k(n, H)$ such that if Q is divided into k^n congruent closed n-cubes, at least one of these does not meet fB. It is well known that this yields the assertion. See, for example, [**Sa**, 3.3].

First suppose that $p = 1$. This case could be included in the proof of the general case, but the proof in this case is considerably simpler. Choose an integer $k \geqslant 6$ and divide Q into k^n closed cubes Q_j. Assume that every Q_j meets fB. Choose a cube Q_ν which contains the center of Q. Next choose a point $t \in B$ with $f(t) \in Q_\nu$. Then $f(t)$ divides $f\bar{B}$ into two subarcs A_1, A_2. Since both of these meet ∂Q and since ∂Q is connected, we can find Q_i, Q_j such that the sets $A_1 \cap Q_i$, $A_2 \cap Q_j$ and $Q_i \cap Q_j \cap \partial Q$ are nonempty. Choose $t_1, t_2 \in \bar{B}$ such that $f(t_1) \in A_1 \cap Q_i$ and $f(t_2) \in A_2 \cap Q_j$. Then $|t - t_1| < |t_2 - t_1|$, and hence $|f(t) - f(t_1)| < H|f(t_2) - f(t_1)|$. On the other hand, let r be the side of Q. Then $|f(t) - f(t_1)| > r/2 - 2r/k$ and $|f(t_1) - f(t_2)| \leqslant 2rn^{1/2}/k$. This yields the desired estimate $k < 4 + 4Hn^{1/2}$.

We now turn to the case $p \geqslant 2$. Instead of the connectedness of ∂Q, we shall make use of the fact that $H^{p-1}(\partial Q) = 0$. Since B is convex, it follows from **[TV**, 2.16] that f is η-QS with η depending only on n and H. We claim that if $k > 8n^{1/2}\eta(n^{1/2})$, then fB does not meet all cubes Q_j. Assume that fB meets every Q_j. Then there is $x_1 \in B$ such that $f(x_1)$ belongs to a cube of this subdivision containing the center of Q. We may assume that $x_1 = 0$. Set

$$P = \cup \{Q_j \colon Q_j \cap \partial Q \neq \varnothing\} \quad \text{and} \quad E = f^{-1}P.$$

Then P separates $f(0)$ from $f\partial B$, and therefore E separates 0 from ∂B and thus from ∞.

Consider the covering $\mathcal{W}_p = \{W_1, \dots, W_{2p}\}$ of $R^p \setminus 0$ and the sets W_ν^*, defined in 3.1. If $a, b \in E$ and $a \in \overline{W_\nu^*}$, $b \notin W_\nu$, then $|a| < p^{1/2}|a - b| < n^{1/2}|a - b|$. Since $k > 8$, this implies $\eta(n^{1/2})|f(a) - f(b)| > |f(a) - f(0)| > r/2 - 2r/k > r/4$. On the other hand, if $Q_i \cap Q_j \neq \varnothing$ and $Q_i \cup Q_j \subset P$, then $d(Q_i \cup Q_j) < 2rn^{1/2}/k < r/4\eta(n^{1/2})$. Consequently, if the set $f^{-1}Q_i \cup f^{-1}Q_j$ meets $\overline{W_\nu^*}$, it is contained in W_ν. Replace each Q_i by a slightly larger concentric open cube A_i, all of the same size, such that these still have this property. Let $\mathcal{Q}$ be the family of all A_i, and let $f^{-1}\mathcal{Q} = \{f^{-1}A \colon A \in \mathcal{Q}\}$. Observe that $f^{-1}A \neq \varnothing$ for all $A \in \mathcal{Q}$.

We define simplicial maps φ, ψ, τ in the diagram

$$\begin{array}{ccc} N(\mathcal{W}_p) & & \\ \tau\uparrow & \nwarrow \psi & \\ N(f^{-1}\mathcal{Q}) & \underset{\varphi}{\rightarrow} & N(\mathcal{Q}) \end{array}$$

as follows: Firstly, φ is the map induced by f. Thus φ maps a vertex $f^{-1}A$ of $N(f^{-1}\mathcal{Q})$ to A. Secondly, for every $A \in \mathcal{Q}$ we choose a member W_ν of $\mathcal{W}_p$ such that $f^{-1}A$ meets $\overline{W_\nu^*}$, and set $\psi(A) = W_\nu$. By what was proved above, ψ is simplicial. Thirdly, we define $\tau = \psi\varphi$. Then τ is a projection map, that is, $\tau(U) \supset U$ for every $U \in f^{-1}\mathcal{Q}$. Passing to cohomology we obtain the commutative diagram

$$\begin{array}{ccccc} & & H^{p-1}(\mathcal{W}_p) & & \\ & \gamma_2 \swarrow & \downarrow\tau^* & \searrow \psi^* & \\ H^{p-1}(E) & \underset{\gamma_1}{\leftarrow} & H^{p-1}(f^{-1}\mathcal{Q}) & \underset{\varphi^*}{\leftarrow} & H^{p-1}(\mathcal{Q}) \end{array}$$

where γ_1 and γ_2 are canonical homomorphisms. Since the elements of $\mathcal{Q}$ are convex, the cohomology of $\mathcal{Q}$ is isomorphic to the Čech cohomology of $\cup\mathcal{Q}$ **[Go**, p. 213]. This space is homotopy equivalent to S^{n-1}. Since $p < n$, $H^{p-1}(\mathcal{Q}) = 0$.

Thus $\gamma_2 = \gamma_1\varphi^*\psi^*$ is the zero map. This contradicts Lemma 3.2 and proves the theorem. $\square$

4.2. REMARK. P. Tukia has pointed out to the author that an alternative proof of 4.1 can be obtained using equicontinuity and Ascoli's theorem. This proof does not give any explicit estimate for $\beta(n, H)$.

5. The case $m_p(fG) < \infty$.

5.1. In this section we consider QS embeddings f: $G \to R^n$, G open in R^p, $2 \leqslant p \leqslant n$, such that $m_p(fG) < \infty$. These maps have been studied by Gehring [**Ge$_3$**] under the additional assumption that f is QC flat, that is, f is a restriction of a QC map of an open set in R^n. Those maps do not include all QS embeddings, not even all Lipschitz embeddings. For example, there is a Lipschitz embedding f: $B^2 \to R^3$ such that fB^2 is not topologically flat [**Ge$_1$**, p. 316]. Since f is Lipschitz, $m_2(fB^2) < \infty$.

We shall extend all results of [**Ge$_3$**] for QS embeddings. In particular, we show that f maps every set of p-measure zero onto a set with the same property. Moreover, we show that f is ACL^p, differentiable a.e., and satisfies a path family inequality. We shall closely follow the proofs of Gehring, but the linking arguments will be replaced by cohomological considerations.

5.2. THEOREM. *Let G be a bounded open set in R^p, let $1 \leqslant p \leqslant n$, and let f: $\overline{G} \to R^n$ be an η-QS embedding. If $x \in G$, then $m_p(fG) \geqslant cd(f(x), f\partial G)^p$ where c is a positive constant depending only on n and η.*

PROOF. We remark that the result is trivial if $p = 1$ or if $p = n$. The theorem was proved by Gehring [**Ge$_3$**, Theorem 2] for QC flat embeddings.

Suppose that $p < n$. We normalize the situation by $x = 0$, $f(x) = 0$. Let $r_1 = d(0, f\partial G)n^{-1/2}$, and let Q_1 be the cube $r_1 I^n$. It suffices to find a constant $c > 0$ such that

$$m_{p-1}(fG \cap r\Sigma^{n-1}) \geqslant cr^{p-1} \tag{5.3}$$

for all $r \in (0, r_1)$. Indeed, setting $g(x) = \max_i |x_i|$ we obtain a 1-Lipschitz map g: $Q_1 \to R^1$, and then [**Fe**, 2.10.25] implies

$$m_p(fG \cap Q_1) \geqslant \frac{\Omega_p}{\Omega_{p-1}\Omega_1}\int_0^{r_1} m_{p-1}(fG \cap r\Sigma^{n-1})\,dr \geqslant \frac{c\Omega_p d(0, f\partial G)^p}{pn^{p/2}\Omega_{p-1}\Omega_1}.$$

So let $0 < r < r_1$. Set $Q = rI^n$, $E = f^{-1}\partial Q$, and $E' = fE = \partial Q \cap fG$. Since ∂Q separates 0 from $f\partial G$, E separates 0 from ∂G and hence from ∞. Consider the family $\mathfrak{D}_p$ of half-spaces $W_1, \ldots, W_{2p}$ and the subsets $W_\nu^* \subset W_\nu$ defined in 3.1. If $a, b \in E$ and if $a \in \overline{W_\nu^*}$, $b \notin W_\nu$, then $|a| \leqslant p^{1/2}|a - b| \leqslant n^{1/2}|a - b|$, and therefore $\eta(n^{1/2})|f(a) - f(b)| \geqslant |f(a)| \geqslant r$. Consequently, if A is a subset of ∂Q with $d(A) < r/\eta(n^{1/2})$ and if $f^{-1}A$ meets $\overline{W_\nu^*}$, then $f^{-1}A \subset W_\nu$.

Set $t = r/5\eta(n^{1/2})$ and $V_x = B(f(x), t) \cap \partial Q$ for $x \in E$. Let $\mathcal{Q}$ be the family of all V_x, $x \in E$, and let $f^{-1}\mathcal{Q} = \{f^{-1}V_x\colon x \in E\}$. Since $f(x) \in V_x$, the elements of $\mathcal{Q}$ and $f^{-1}\mathcal{Q}$ are nonempty. As in the proof of 4.1, we define simplicial maps φ: $N(f^{-1}\mathcal{Q}) \to N(\mathcal{Q})$ and ψ: $N(\mathcal{Q}) \to \mathfrak{D}_p$. Thus φ is induced by f, and ψ maps a vertex

V_x of $N(\mathcal{A})$ to an element $W_\nu \in \mathcal{D}_p$ such that $\overline{W_\nu^*} \cap f^{-1}V_x \neq \varnothing$. The choice of t ensures that if $V_x \cap V_y \neq \varnothing$, then $f^{-1}V_x \cup f^{-1}V_y$ cannot meet both $\overline{W_\nu^*}$ and $R^p \setminus W_\nu$. Hence ψ is simplicial. The map $\tau = \psi\varphi$: $N(f^{-1}\mathcal{A}) \to N(\mathcal{D}_p)$ is a projection, that is, $\tau(U) \supset U$ for every $U \in f^{-1}\mathcal{A}$. Observe that the union of all V_x is $E'(t) = \{y \in \partial Q: d(y, E') < t\}$. Let i: $E' \to E'(t)$ be the inclusion. Then we have the following commutative diagram:

$$\begin{tikzcd}
H^{p-1}(\mathcal{D}_p) \arrow[d, "\tau^*"'] \arrow[dr, "\psi^*"] & & \\
H^{p-1}(f^{-1}\mathcal{A}) \arrow[d, "\gamma_1"'] & H^{p-1}(\mathcal{A}) \arrow[l, "\varphi^*"'] \arrow[d, "\gamma_2"] \arrow[dr, "\gamma_3"] & \\
H^{p-1}(E) & H^{p-1}(E') \arrow[l, "f^*"] & H^{p-1}(E'(t)) \arrow[l, "i^*"]
\end{tikzcd}$$

Here each γ_j is a canonical homomorphism. The map $\gamma = \gamma_1\tau^*$ is also a canonical homomorphism. By Lemma 3.2, it is not the zero map. Since $\gamma = f^*i^*\gamma_3\psi^*$, $i^* \neq 0$. Using Theorem 3.6 and an auxiliary similarity map $\partial Q \to \Sigma^{n-1}$ we conclude that $m_{p-1}(E') \geqslant (t/2(n+1)^{3/2})^{p-1}$. This proves (5.3) with

$$c = \left(10(n+1)^{3/2}\eta(n^{1/2})\right)^{1-n}. \quad \square$$

5.4. Notation. Suppose that G is open in R^p and that f: $G \to R^n$ is continuous. For $x \in G$ and $r \in (0, d(x, \partial G))$ we use the following (fairly standard) notation:

$$L(x, f, r) = \sup\{|f(y) - f(x)|: |y - x| \leqslant r\}, \qquad L(x, f) = \limsup_{r\to 0} \frac{L(x, f, r)}{r},$$

$$H_O(x, f, r) = \frac{\Omega_p L(x, f, r)^p}{m_p(fB^p(x, r))}, \qquad H_O(x, f) = \limsup_{r\to 0} H_O(x, f, r),$$

$$\mu_f'(x) = \limsup_{r\to 0} \frac{m_p(fB^p(x, r))}{\Omega_p r^p}.$$

If f is differentiable at $x \in G$, then $J_pf(x)$ will denote the p-jacobian $\|\bigwedge_p Df(x)\|$ of f at x [**Fe**, 3.2.1].

5.5. We shall consider maps f: $G \to R^n$ which are locally weakly H-QS for some $H \geqslant 1$. It follows from [**TV**, 2.16] that if $B = B(x_0, r)$ is a ball such that $\overline{B} \subset G$ and $f|\overline{B}$ is weakly H-QS, then $f|\overline{B}$ is η-QS where η depends only on n and H. Moreover, $H_O(x, f, r) \leqslant \Omega_p H^p/c$ where $c = c(n, H)$ is the constant given by 5.2. Hence $H_O(x, f) \leqslant \Omega_p H^p/c$ for all $x \in G$. From [**Ge**$_3$, Lemma 3] we obtain the first part of the following theorem.

5.6. Theorem. *Suppose that G is open in R^p, that $2 \leqslant p \leqslant n$, that f: $G \to R^n$ is locally weakly H-QS, and that $m_p(fF) < \infty$ for every compact $F \subset G$. Then f is ACL^p, differentiable a.e., and $L(x, f)^p \leqslant C\mu_f'(x)$ a.e. in G, where the constant C depends only on n and H. Furthermore, $M_p^p(\Gamma) \leqslant CM_p^p(f\Gamma)$ for every path family Γ in G.*

PROOF. It remains to prove the path family inequality. The modulus M_p^p is defined in [**Ge**$_3$, p. 91]. Observe that $M_p^p(\Gamma) = M_p(\Gamma)$. Our proof is essentially the same as in the case $p = n$ [**Vä**$_1$, 32.3]. It somewhat simplifies the proof of [**Ge**$_3$, Theorem 4].

Let Γ_0 be the family of all locally rectifiable paths $\alpha \in \Gamma$ such that f is absolutely continuous on every closed subpath of α. Since f is ACL^p, it follows from Fuglede's theorem [**Vä**$_1$, 28.2] that $M_p(\Gamma_0) = M_p(\Gamma)$. Hence it suffices to show that $M_p(\Gamma_0) \leqslant CM_p^p(f\Gamma)$.

Let $\rho'\colon R^n \to R^1 \cup \infty$ be a function in $F(f\Gamma)$. This means that $\rho' \geqslant 0$ is a Borel function and that the line integral of ρ' along every path in $f\Gamma$ is at least one. Define $\rho\colon G \to R^1 \cup \infty$ by $\rho(x) = \rho'(f(x))L(x, f)$. If $\alpha \in \Gamma_0$, then a transformation formula for line integrals [**Vä**$_1$, 5.3] yields $\int_\alpha \rho\, ds \geqslant \int_{f\circ\alpha} \rho'\, ds \geqslant 1$. Hence $\rho \in F(\Gamma)$, which implies

$$\begin{aligned} M_p(\Gamma_0) &\leqslant \int_G \rho^p\, dm_p = \int_G \rho'(f(x))^p L(x, f)^p\, dm_p(x) \\ &\leqslant C\int_G \rho'(f(x))^p \mu_f'(x)\, dm_p(x). \end{aligned}$$

By a standard transformation inequality, for example, by a slightly generalized version of [**Vä**$_1$, 24.5], this implies $M_p(\Gamma_0) \leqslant C\int_{fG} \rho'^p\, dm_p \leqslant C\int_{R^n} \rho'^p\, dm_p$. Taking the infimum over all $\rho' \in F(f\Gamma)$ we obtain $M_p(\Gamma_0) \leqslant CM_p^p(f\Gamma)$. □

5.7. LEMMA. *Suppose that G is a bounded open set in R^p, that $2 \leqslant p \leqslant n$, that $f\colon \overline{G} \to R^n$ is an η-QS embedding and that $m_p(f\overline{G}) < \infty$. Let $x_0 \in G$ and let $0 < r_0 < n^{-1/2}d(f(x_0), f\partial G)$. Then*

$$\int_E J_p f\, dm_p \geqslant c r_0^p$$

where $E = f^{-1}[f(x_0) + r_0 I^n]$ and the constant c depends only on n and η.

PROOF. Our strategy is, as in [**Ge**$_3$, Lemma 4], to approximate f by smooth maps g_j, and then apply the argument of the proof of 5.2 to the maps g_j. To avoid repetition we shall omit some details.

We normalize the situation by $x_0 = 0$, $f(x_0) = 0$. Replacing G by a slightly smaller open set, we may assume that f is defined and η-QS is a neighborhood U of $\overline{G}$. By 5.6, f is ACL^p. Hence there is a sequence of C^1-maps g_j, defined in a neighborhood of $\overline{G}$, such that $g_j \to f$ uniformly in $\overline{G}$ and such that the partial derivatives $\partial_i g_j$ converge to $\partial_i f$ in $L^p(\overline{G})$. It follows that $J_p g_j \to J_p f$ in $L^p(\overline{G})$. Set $A_j = E \cap g_j^{-1}[r\Sigma^n]$. Arguing as in the proof of [**Ge**$_3$, Lemma 4] we see that it suffices to find an estimate

$$\liminf_{j\to\infty} m_{p-1}(g_j A_j) \geqslant c_1 r^{p-1} \tag{5.8}$$

for $0 < r < r_0$, where $c_1 > 0$ depends only on n and η.

Fix $r \in (0, r_0)$, and set $\varepsilon_j = \sup\{|g_j(x) - f(x)|\colon x \in \overline{G}\}$. Then $\varepsilon_j \to 0$ as $j \to \infty$. For large j, $r\Sigma^n$ separates $g_j(0)$ from $g_j\partial E$, and therefore A_j separates 0 from ∞.

Consider again the sets W_ν and W_ν^*, defined in 3.1. If $a, b \in A_j$ and $a \in \overline{W}_\nu^*$, $b \notin W_\nu$, then $|a| \leq n^{1/2}|a - b|$ and hence $|f(a) - f(b)| \geq |f(a)|/\eta(n^{1/2}) \geq (r - \varepsilon_j)/\eta(n^{1/2})$, which implies $|g_j(a) - g_j(b)| \geq (r - \varepsilon_j)/\eta(n^{1/2}) - 2\varepsilon_j > r/2\eta(n^{1/2})$ for large j, say for $j \geq j_0$. Set $t = r/8\eta(n^{1/2})$. If $j \geq j_0$, $C \subset A_j$, $d(g_j C) \leq 4t$, and C meets $\overline{W}_\nu^*$, then $C \subset W_\nu$.

We can now proceed as in the proof of Theorem 5.2 and obtain $m_{p-1}(g_j A_j) \geq (t/2(n+1)^{3/2})^{p-1}$ for large j. This proves (5.8) with $c_1 = (16(n+1)^{3/2}\eta(n^{1/2}))^{1-n}$. □

5.9. THEOREM. *Suppose that G is open in R^p, that $2 \leq p \leq n$, that $f\colon G \to R^n$ is locally QS, and that $m_p(fF) < \infty$ for every compact $F \subset G$. Then f maps every set of p-measure zero onto a set of p-measure zero. Furthermore, $\mu_f'(x) = J_p f(x)$ a.e. in G, and*

$$m_p(fE) = \int_E J_p f\, dm_p$$

for every Borel set $E \subset G$.

PROOF. Repeating the proof of [**Ge$_3$**, Lemma 5] with balls replaced by cubes, we see that the first part of the theorem follows from Lemma 5.7. The rest of the theorem can be proved as [**Ge$_3$**, Lemma 6]. □

5.10. QUESTION. Let f be as in 5.9. Does $m_p(fE) = 0$ imply $m_p(E) = 0$?

5.11. EXAMPLE. If $p = n$, there is a converse of the first part of Theorem 5.6: If $f\colon G \to R^n$ is ACL and if $L(x, f)^n \leq K\mu_f'(x)$ a.e., then f is QC with $K_O(f) \leq K$ [**Vä$_1$**, 32.3 and 32.5.1]. If $p < n$, the corresponding statement is no longer true. We show this by giving an example, for all p and n with $1 \leq p < n$, of an embedding $f\colon B^p \to R^n$ which is Lipschitz and satisfies the condition $L(x, f)^p \leq C\mu_f'(x)$ a.e. but which is not locally QS at the origin.

For every positive integer j, let $A_j \subset R^2$ be the broken line segment with consecutive vertices $2^{-j}e_1$, $2^{-j}e_1 + 2^{-j-2}e_2$, $(2^{-j} - 2^{-2j-2})e_1 + 2^{-j-2}e_2$, $(2^{-j} - 2^{-2j-2})e_1$, $2^{-j-1}e_1$, and let A be the union of all A_j and the origin. Then A is an arc of length one. Let $\alpha = (\alpha_1, \alpha_2)\colon [0, 1] \to R^2$ be the parametrization of A by the arc-length with $\alpha(0) = 0$. Define $f\colon \overline{B}^p \to R^n$ by rotating α, that is, if $e \in \partial B^p$ and $0 \leq r \leq 1$, $f(re) = \alpha_1(r)e + \alpha_2(r)e_{p+1}$.

It is not difficult to verify that f is L_1-Lipschitz for some $L_1 > 0$. Moreover, there is $L_2 > 0$ such that f^{-1} is locally L_2-Lipschitz at every point of $f\overline{B}^p \setminus 0$. Consequently, $L(x, f)^p \leq L_1^p L_2^p \mu_f'(x)$ for every $x \in B^p \setminus 0$. However, f is not locally QS at the origin, since $f[rB^p]$ is not of bounded turning for any $r \in (0, 1)$, cf. [**TV**, 2.11].

Observe that the linear dilatation H_L of f, defined in 2.8.3, is bounded. In fact, $H_L(0) = 1$ and $H_L(x) \leq L_1 L_2$ for $x \neq 0$. Hence, for $p < n$, it is not possible to characterize the quasi-symmetry of an embedding in terms of its linear dilatation.

6. Wild quasi-symmetric cells.

6.1. DEFINITIONS. A *p-cell* is a space X homeomorphic to I^p. If the homeomorphism can be chosen to be QS, X is a *quasi-symmetric* p-cell. A p-cell X in R^n is (topologically) *wild* if there is no homeomorphism $f\colon R^n \to R^n$ such that $fX = I^p$.

6.2. REMARKS. We shall show that R^n contains a wild QS p-cell whenever $p < n$ and $n \geqslant 3$. In this respect, the quasi-symmetric embeddings differ from Lipschitz embeddings. In fact, if $f\colon I^p \to R^n$ is an embedding and Lipschitz (f^{-1} need not be) and if $n \geqslant 3p + 1$, then fI^p is (topologically) flat [**Vä$_2$**, 3.8]. In particular, a Lipschitz arc is flat in R^n for $n \geqslant 4$. On the other hand, the wild arc of Fox and Artin in R^3 [**Ru**, Example 2.4.2, p. 65] can be made Lipschitz (and hence QS).

6.3. THEOREM. *Let $n \geqslant 3$ and $1 \leqslant p < n$. Then R^n contains a wild QS p-cell.*

PROOF. Let $E \subset R^n$ be the necklace of Antoine-Blankinship [**Bl**]. Thus E is a Cantor set such that no homeomorphism of R^n carries E into a line. We shall construct a QS p-cell X such that $E \subset X \subset R^n$. Then X is wild in R^n. This follows from [**Bl**, Theorem 3E] or from the Klee trick ([**Ru**, Theorem 2.5.1, p. 74] applied to Cantor sets). The construction is classical, see [**Ru**, Figure 2.4.16, p. 73]. However, we must take care to make the homeomorphism $f\colon I^p \to X$ QS. To do this, we make all auxiliary constructions, including the construction of E, in the category PL. Then f will be a limit of PL embeddings and in fact PL outside $f^{-1}E$.

We first give a PL version of the construction of E. It is the intersection of a descending sequence $E_0 \supset E_1 \supset \cdots$ of compact sets in R^n. The set E_0 is a solid n-torus, that is, a set PL homeomorphic to $I^2 \times \Sigma^1 \times \cdots \times \Sigma^1$ with $n-2$ Σ^1-factors. The set E_1 is the union of disjoint solid n-tori $T_1, \dots, T_m$ in int E_0, suitably linked with each other. We may assume that $d(E_0) = 1$ and that the sets T_j are similar to E_0 and mutually congruent. Thus there are $\alpha \in (0, 1)$ and sense-preserving similarity maps $g_j\colon R^n \to R^n$ such that $g_jE_0 = T_j$ and $|g_j(x) - g_j(y)| = \alpha|x - y|$ for all $j \geqslant 0$ and for all $x, y \in R^n$. The integer m depends only on n.

To describe the general step we use the free monoid W generated by the set $\{1, \dots, m\}$. Thus W is the set of all words $w = w_1 \dots w_q$, $1 \leqslant w_j \leqslant m$; also the empty word $0 \in W$. The length $l(w)$ of a word $w = w_1 \dots w_q$ is the number $q \geqslant 0$.

For every $w = w_1 \dots w_q \in W$ we set $g_w = g_{w_1} \dots g_{w_q}$, $g_0 = \mathrm{id}$, and $T_w = g_wE_0$. Then $E_q = \cup \{T_w\colon l(w) = q\}$. Observe that $T_{vw} = g_vT_w \subset T_v$ for all words v, w and that $d(T_w) = \alpha^{l(w)}$.

We next choose disjoint congruent oriented closed p-cubes $Q_1, \dots, Q_m$ in int I^p such that $d(Q_i) < d(Q_i, \Sigma^{p-1})$ for all i and such that $d(Q_i) < d(Q_i, Q_j)$ for $i \neq j$. In each Q_j we choose a smaller concentric cube D_j, all of equal size, such that $d(D_j) < d(D_j, \partial Q_j)$. Let A_j be the annulus $Q_j \setminus \mathrm{int}\, D_j$, and let C_0 be the cheese (disc-with-holes) $\mathrm{cl}(I^p \setminus (Q_1 \cup \cdots \cup Q_m))$. Let $h_j\colon R^n \to R^n$ be the similarity map $h_j(x) = \beta x + b_j$ where 2β is the side and b_j the center of D_j. Then $h_jI^p = D_j$. For every word $w = w_1 \dots w_q$ we set $h_w = h_{w_1} \dots h_{w_q}$, $h_0 = \mathrm{id}$, and $D_w = h_wI^p$, $C_w = h_wC_0$. Setting $Q_0 = h_j^{-1}Q_j$ and $A_0 = h_j^{-1}A_j = Q_0 \setminus \mathrm{int}\, I^p$ (both independent of j) we also define $Q_w = h_wQ_0$ and $A_w = h_wA_0$. Then each annulus A_w, $l(w) \geqslant 1$, meets exactly two cheeses, namely C_w (inside A_w) and $C_{w_1 \dots w_{q-1}}$ (outside A_w). Moreover, $I^p = D_0$ is the union of a Cantor set F and all C_w, $w \in W$, and A_w, $l(w) \geqslant 1$.

We now construct the embedding $f\colon I^p \to R^n$. It will be a limit of PL embeddings $f_q\colon I^p \to E_0$. Let $\varphi\colon I^p \to \partial E_0$ be a similarity embedding. Choose a PL embedding $f_0\colon I^p \to E_0$ such that

(1) $f_0 = \varphi$ in C_0,
(2) $f_0 = g_j \varphi h_j^{-1}$ in D_j, $1 \leqslant j \leqslant m$,
(3) $f_0[\operatorname{int} A_j] \subset \operatorname{int} E_0 \setminus E_1$, $1 \leqslant j \leqslant m$.

The existence of f_0 is rather obvious. A rigorous proof can be based on the regular neighborhood theory [**RS**, §3]; it is only necessary to consider the case $p = n - 1$ in view of Remark 6.4.1. Then $f_0 A_j$ is a tube joining ∂E_0 and ∂T_j. Suppose that we have constructed $f_0, \ldots, f_{q-1}$. Then we set $f_q = f_{q-1}$ outside the cubes D_w, $l(w) = q$, and $f_q = g_w f_0 h_w^{-1}$ in D_w. The maps f_q are PL embeddings which clearly converge to an embedding $f\colon I^p \to E_0$. Outside the Cantor set F the convergence is trivial: $f = f_q$ outside the cubes Q_w, $l(w) = q + 2$. Moreover, $fD_w = T_w \cap fI^p$ and $fF = E$.

It remains to show that f is QS. Since I^p is convex, it suffices to prove that f is weakly QS [**TV**, 2.16]. So let a, b, x be distinct points in I^p with $|a - x| \leqslant |b - x|$. We must find an upper bound for the ratio $\rho' = |f(a) - f(x)|/|f(b) - f(x)|$. The proof is elementary but lengthy in view of the many cases and subcases that have to be considered.

We choose $\lambda > 0$ such that the following conditions are satisfied for $i, j \in [1, m]$:
(1) $\lambda \leqslant d(T_i, \partial E_0)$,
(2) $\lambda \leqslant d(fA_i, fQ_{ij})$,
(3) $\lambda \leqslant d(fQ_i, fQ_j)$, $i \neq j$.

We also choose $L \geqslant 1$ such that f_1 and f_1^{-1} are L-Lipschitz.

Case 1. For no $w \in W$ and $i \in [1, m]$, $\{b, x\}$ is contained in $A_w \cup C_w$ or in $A_{wi} \cup C_w$. Then b and x are separated by some A_w or C_w. Let w be the shortest word for which this happens. Set $q = l(w)$.

Subcase 1. A_w separates b and x. Write $v = w_1 \ldots w_{q-1}$. Then C_v does not separate b and x. It follows that $\{b, x\} \subset D_v$, and therefore $d(D_v) \geqslant |b - x| \geqslant |a - x|$, which implies $a \in D_u$, $u = w_{q-2}$. Hence $|f(a) - f(x)| \leqslant d(T_u) = \alpha^{q-2}$. On the other hand, $|f(b) - f(x)| \geqslant d(fD_w, fC_v) = \alpha^{q-1} d(T_j, \partial E_0) \geqslant \alpha^{q-1}\lambda$. Hence $\rho' \leqslant 1/\alpha\lambda$.

Subcase 2. A_w does not separate b and x. Then C_w separates, and therefore $\{b, x\} \subset Q_w$. Hence $d(Q_w) \geqslant |b - x| \geqslant |a - x|$, which implies $a \in D_v$, $v = w_1 \ldots w_{q-1}$. Thus $|f(a) - f(x)| \leqslant d(T_v) = \alpha^{q-1}$. On the other hand, $|f(b) - f(x)| \geqslant d(fA_w, fQ_{wj})$ for some $j \in [1, m]$ or $|f(b) - f(x)| \geqslant d(fQ_{wi}, fQ_{wj})$ for some $i \neq j$. In both cases $|f(b) - f(x)| \geqslant \alpha^q \lambda$. Hence $\rho' \leqslant 1/\alpha\lambda$.

Case 2. There is $w \in W$ such that $\{b, x\} \subset A_w \cup C_w$. Set $q = l(w)$ and $v = w_1 \ldots w_{q-1}$. Let G be the union of C_v, A_w, C_w, and all A_{wi}, $i \in [1, m]$. Then $f|G$ is the composition of similarity maps and f_1. Consequently, $\rho' \leqslant L^2$ if $a \in G$. Suppose that $a \notin G$. Since $|a - x| \leqslant |b - x| \leqslant d(Q_w)$, $a \in D_v$. Since $a \notin G$, $a \in D_{wi}$ for some $i \in [1, m]$. Thus $|b - x| \geqslant |a - x| \geqslant d(C_w, D_{wi}) \geqslant d(D_{wi}) \geqslant \beta^{q+1}$, which implies $|f(b) - f(x)| \geqslant \alpha^{q-1} L^{-1} \beta^{1-q} |b - x| \geqslant \alpha^{q-1} L^{-1} \beta^2$. On the other hand, $|f(a) - f(x)| \leqslant d(T_v) = \alpha^{q-1}$. Hence $\rho' \leqslant L/\beta^2$.

Case 3. There are $w \in W$ and $i \in [1, m]$ such that $\{b, x\} \subset A_{wi} \cup C_w$. Set $q = l(w)$, and let G' be the union of A_w, C_w, A_{wj}, and C_{wj} over all $j \in [1, m]$. If $a \in G'$, we obtain $\rho' \leqslant L^2$ as in Case 2. Suppose that $a \notin G'$. Then $|a - x| \leqslant |b - x| \leqslant d(D_w)$, which implies $a \in Q_w$. Hence $a \in Q_{wjk}$ for some $j, k \in [1, m]$. Then

$|b - x| \geqslant |a - x| \geqslant d(Q_{wjk}, A_{wj}) \geqslant d(Q_{wjk}) \geqslant \beta^{q+2}$, which yields $|f(b) - f(x)| \geqslant \alpha^q L^{-1}\beta^{-q}|b - x| \geqslant \alpha^q L^{-1}\beta^2$. On the other hand, $|f(a) - f(x)| \leqslant d(T_w) = \alpha^q$, and we obtain $\rho' \leqslant L/\beta^2$. $\square$

6.4. REMARKS. 1. Suppose that $p = n - 1$. We may choose the cubes Q_j so that their centers lie on I^1. Then also $F \subset I^1$. We thus obtain a QS embedding f: $I^{n-1} \to R^n$ such that fI^p is a wild p-cell for every $p \in [1, n - 1]$.

2. Let J^p be a face of Σ^p, let α: $J^p \to I^p$ be a similarity map, and let f: $I^p \to R^n$ be the QS embedding constructed in 6.3. Then it is easy to extend $f\alpha$ piecewise linearly to a QS embedding g: $\Sigma^p \to R^n$. Then $g\Sigma^p$ is a QS *p-sphere* which is wild in R^n.

3. P. Tukia has pointed out to the author that using the idea in **[Tu**, §14] it is possible to find a QS p-cell which is TOP flat but not QS flat in R^n.

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