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Quasisymmetric parametrizations of two-dimensional metric spheres

Mario Bonk⋆, **Bruce Kleiner**⋆⋆

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

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1. Introduction

According to the classical uniformization theorem, every smooth Riemannian surface Z homeomorphic to the 2-sphere is conformally diffeomorphic to $\mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$ equipped with the Riemannian metric induced by the ambient Euclidean metric). The availability of a similar uniformization procedure for spheres with a "generalized conformal structure" is highly desirable, in particular in connection with Thurston's hyperbolization conjecture. This was addressed by Cannon in his combinatorial Riemann mapping theorem [7]. He considers topological surfaces equipped with a sequence of "shinglings"—a combinatorial structure that leads to a notion of approximate conformal moduli of rings. He then finds conditions that imply the existence of coordinate systems on the surface that relate these combinatorial moduli to classical analytic moduli in the plane.

In this paper we develop a uniformization theory for a different type of generalized conformal structure. We start with a metric space Z homeomorphic to $\mathbb{S}^2$ and ask for conditions under which Z can be mapped onto $\mathbb{S}^2$ by a quasisymmetric homeomorphism. The class of quasisymmetries is an appropriate analog of conformal[1] mappings in a metric space context. Quasisymmetric homeomorphisms also arise in the theory of Gromov hyperbolic metric spaces—quasi-isometries between Gromov hyperbolic spaces induce quasisymmetric boundary homeomorphisms. Our setup has the advantage that we can exploit recent notions and methods from Analysis

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[1] A homeomorphism between compact Riemannian manifolds is quasisymmetric iff it is quasiconformal. There seems to be no hope of a general existence theory for *conformal* mappings beyond the Riemannian setting: by any reasonable definition, two norms on $\mathbb{R}^2$ define locally conformally equivalent metrics iff the corresponding normed spaces are isometric.

on metric spaces. Our main result, Theorem 11.1, gives a necessary and sufficient condition for Z to be quasisymmetrically equivalent to $\mathbb{S}^2$. Since the formulation of this theorem requires some preparation, we postpone stating it until Sect. 11 (see Corollary 11.4 for a more accessible special case). In this introduction we formulate two consequences of our methods that are easier to state. The first result answers a question of Heinonen and Semmes affirmatively (cf. [16], Question 3, and the discussion in [28], Sect. 8) and was the original motivation for this paper.

Theorem 1.1. *Let Z be an Ahlfors 2-regular metric space homeomorphic to $\mathbb{S}^2$. Then Z is quasisymmetric to $\mathbb{S}^2$ if and only if Z is linearly locally contractible.*

We recall that a metric space Z is Ahlfors Q-regular if there is a constant $C > 0$ such that the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ of every open r-ball $B(a, r)$ satisfies

$$C^{-1}r^Q \leq \mathcal{H}^Q(B(a, r)) \leq Cr^Q,$$

when $0 < r \leq \operatorname{diam}(Z)$. A metric space is linearly locally contractible if there is a constant C such that every small ball is contractible inside a ball whose radius is C times larger; for closed surfaces linear local contractibility is equivalent to linear local connectedness, see Sect. 2.

The statement of Theorem 1.1 is quantitative in a sense that will be explained below (see the comment after the proof of Theorem 1.1 in Sect. 10).

The problem considered here is just a special case of the general problem of characterizing a metric space Z up to quasisymmetry. Particularly interesting are the cases when Z is $\mathbb{R}^n$ or the standard sphere $\mathbb{S}^n$. Quasisymmetric characterizations of $\mathbb{R}$ and $\mathbb{S}^1$ have been given by Tukia and Väisälä [33]. If $n \geq 3$ then results by Semmes [27] show that natural conditions which one might expect to imply that a metric space is quasisymmetric to $\mathbb{S}^n$ (or $\mathbb{R}^n$), are in fact insufficient; at present these cases look intractable.

A result similar to Theorem 1.1 has been proved by Semmes [24] under the additional assumption that Z is a smooth Riemannian surface. The hypothesis of 2-regularity in the theorem is essential. A metric 2-sphere containing an open set bilipschitz equivalent to the unit disk $B(0, 1) \subset \mathbb{R}^2$ with the metric

$$d_\alpha((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|^\alpha,$$

where $0 < \alpha < 1$, will never be quasisymmetrically homeomorphic to $\mathbb{S}^2$, see [31, 36]. We also mention that the construction of Laakso [17] provides examples of Ahlfors 2-regular, linearly locally contractible 2-spheres which are not bilipschitz homeomorphic to $\mathbb{S}^2$; this shows that one cannot replace the word "quasisymmetric" with "bilipschitz" in the statement of the theorem. Finally we point out that the n-dimensional analog of Theorem 1.1 is false for $n > 2$ according to the results by Semmes [27]: for $n > 2$ there are linearly locally contractible and n-regular metric n-spheres which are not quasisymmetric to the standard n-sphere. However, if an n-regular n-sphere admits an appropriately large group of symmetries, then it must be quasisymmetrically homeomorphic to the standard n-sphere, see [2].

Theorem 1.1 is closely related to a theorem of Semmes [26] which shows that an Ahlfors n-regular metric space that is a linearly locally contractible topological n-manifold satisfies a $(1,1)$-Poincaré inequality (see Sect. 7) and hence has nice analytic properties. His result shows in particular that a 2-sphere as in our theorem satisfies a Poincaré inequality. We will not use this result, since it does not substantially simplify our arguments, and in fact our theorem together with a result by Tyson [34] gives a different way to establish a Poincaré inequality in our case. Our methods could also easily be adapted to show this directly.

From an analytic perspective it is interesting to consider metric spaces that satisfy Poincaré inequalities by assumption (cf. [15,26,12,3,4,18]). For an Ahlfors Q-regular metric space a $(1,Q)$-Poincaré inequality is equivalent to the Q-Loewner property as introduced by Heinonen and Koskela [15], see Sect. 7. It turns out that in dimension 2, this is a very restrictive condition:

Theorem 1.2. *Let $Q \geq 2$ and Z be an Ahlfors Q-regular metric space homeomorphic to $\mathbb{S}^2$. If Z is Q-Loewner, then $Q = 2$ and Z is quasisymmetric to $\mathbb{S}^2$.*

By the result of Semmes [26] the space Z will actually satisfy a $(1,1)$-Poincaré inequality.

The analog of Theorem 1.2 in higher dimensions is false—one has the examples of Semmes cited above. Also, the standard Carnot metric on the 3-sphere is Ahlfors 4-regular and 4-Loewner. In view of these examples one can summarize Theorem 1.2 by saying that there are no exotic geometric structures on $\mathbb{S}^2$ that are analytically nice.

Another source of examples of Ahlfors regular, linearly locally contractible metric spheres is the theory of Gromov hyperbolic groups. The boundary $\partial_\infty G$ of a hyperbolic group G has a natural family of Ahlfors regular metrics which are quasisymmetric to one another by the identity homeomorphism. When $\partial_\infty G$ is homeomorphic to a sphere, then these metrics are all linearly locally contractible. Cannon [7] has conjectured that when $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$, then G admits a discrete, cocompact, and isometric action on hyperbolic 3-space $\mathbb{H}^3$. This conjecture is a major piece of Thurston's hyperbolization conjecture for 3-manifolds[2]. By a theorem of Sullivan [30] Cannon's conjecture is equivalent to the following conjecture:

Conjecture 1.3. If G is a hyperbolic group and $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$, then $\partial_\infty G$ (equipped with one of the metrics mentioned above) is quasisymmetric to $\mathbb{S}^2$.

[2] The Hyperbolization Conjecture is part of the full Geometrization Conjecture. It says that a closed, irreducible, aspherical 3-manifold admits a hyperbolic structure provided its fundamental group does not contain a copy of $\mathbb{Z} \times \mathbb{Z}$.

It is an interesting problem (especially in view of Theorem 1.2) to find additional assumptions on the hyperbolic group G which imply that $\partial_\infty G$ is quasisymmetric to a space with "nice" analytic properties, i.e., to a Q-regular metric space with a $(1, Q)$-Poincaré inequality. A natural question is whether this is always true if $\partial_\infty G$ is connected and has no local cut points. By work of Bestvina-Mess, Bowditch, and Swarup, this last property of $\partial_\infty G$ is equivalent to the property that the Gromov hyperbolic group G is non-elementary and none of its finite index subgroups (including itself) virtually splits over a virtually cyclic group.

Recently, M. Bourdon and H. Pajot answered this question in the negative [5]: they found examples of infinite hyperbolic groups G such that $\partial_\infty G$ is connected and has no local cut points, but such that $\partial_\infty G$ is not quasisymmetric to any Q-regular metric space satisfying a $(1, Q)$-Poincaré inequality.

We now turn to the problem of finding necessary and sufficient conditions for a metric space to be quasisymmetric to $\mathbb{S}^2$. It follows easily from the definitions that a compact metric space Z which is quasisymmetric to a doubling (respectively linearly locally contractible) metric space is itself doubling (respectively linearly locally contractible). Therefore any metric space quasisymmetric to a standard sphere is doubling and linearly locally contractible. In Sect. 10 we give two different necessary and sufficient conditions for a metric 2-sphere to be quasisymmetric to $\mathbb{S}^2$, Theorems 10.1 and 10.4. Roughly speaking, Theorem 10.4 says that a doubling, linearly locally contractible metric 2-sphere Z is quasisymmetric to $\mathbb{S}^2$ if and only if the following condition is true. If one considers a sequence of finer and finer "graph approximations" of Z, then the corresponding combinatorial moduli of any pair of continua (E, F) are small provided the relative distance $\Delta(E, F)$ as defined in (2.9) is big. Theorem 10.1 is similar, except that one assumes instead that if the moduli of the pair (E, F) are small then the relative distance $\Delta(E, F)$ is big. We refer the reader to Sect. 10 for the precise statements of these two theorems.

The problem of finding necessary and sufficient conditions for a metric sphere to be quasisymmetric to $\mathbb{S}^2$ has some features in common with Cannon's work [7] on the combinatorial Riemann mapping theorem. We will discuss this in Sect. 11. In this section we prove Theorem 11.1 which is an improvement of Theorem 10.4. One can use Theorem 11.1 to verify that certain self-similar examples are quasisymmetric to $\mathbb{S}^2$. We also formulate another necessary and sufficient condition in Corollary 11.4; readers may find the statement of Corollary 11.4 more accessible than Theorems 10.1, 10.4, and 11.1, as it does not rely on the language of K-approximations.

We now outline the proofs of Theorems 1.1 and 1.2.

The first step is to use the linear local contractibility to produce an embedded graph with controlled geometry which approximates our space Z on a given scale. This can actually be done for any doubling, linearly locally connected metric space. If Z is a topological 2-sphere, then we can obtain a graph approximation which is, in addition, the 1-skeleton of a triangulation. In the second step we apply a uniformization procedure. We invoke the circle packing theorem of Andreev-Koebe-Thurston, which ensures that every triangulation of the 2-sphere is combinatorially equivalent to the triangulation dual to some circle packing, and then map each vertex of the graph to the center of the associated circle. In this way we get a mapping f from the vertex set of our approximating graph to the sphere[3]. The way to think about the map is that it provides a coarse conformal change of the metric: the scale attached to a given vertex of the graph approximation is changed to the scale given by the radius of the corresponding disk in the circle packing. The third step is to show that (after suitably normalizing the circle packing) the mapping f has controlled quasisymmetric distortion. Since in some sense f changes the metric conformally, we control its quasisymmetric distortion (in fact it is the quasi-Möbius distortion which enters more naturally) via modulus estimates. There are two main ingredients in our implementation of this idea—the Ferrand cross-ratio (cf. [19,4]), which mediates between the quasisymmetric distortion and the "conformal" distortion, and a modulus comparison proposition which allows one to relate (under suitable conditions) the 2-modulus of a pair of continua $E, F \subset Z$ with the combinatorial 2-modulus of their discrete approximations in the approximating graph. In the final step we take a sequence of graph approximations at finer and finer scales, and apply Arzelà-Ascoli to see that the corresponding mappings subconverge to a quasisymmetric homeomorphism from Z to $\mathbb{S}^2$.

We suggest that readers who are unfamiliar with modulus arguments read the basic definitions in Sects. 2, 3, 7, and Proposition 9.1. The proposition is a simplified version of later arguments which bound quasi-Möbius distortion.

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[3] Alternatively, one can use the classical uniformization theorem to produce such a map. To do this, one endows the sphere with a piecewise flat metric so that each 2-simplex of the topological triangulation is isometric to an equilateral Euclidean triangle with side length 1. Such a piecewise flat metric defines a flat Riemannian surface with isolated conical singularities, and one can then apply the classical uniformization theorem to get a map from this Riemann surface to $\mathbb{S}^2$.

2. Cross-ratios

We use the notation $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\mathbb{R}^+ = (0, \infty)$, and $\mathbb{R}_0^+ = [0, \infty)$.

Let (Z, d) be a metric space. We denote by $B_Z(a, r)$ and by $\bar{B}_Z(a, r)$ the open and closed ball in Z centered at $a \in Z$ of radius $r > 0$, respectively. We drop the subscript Z if the space Z is understood.

The *cross-ratio*, $[z_1, z_2, z_3, z_4]$, of a four-tuple of distinct points (z_1, z_2, z_3, z_4) in Z is the quantity

$$[z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}.$$

Note that

$$[z_1, z_2, z_3, z_4] = [z_2, z_1, z_3, z_4]^{-1} = [z_1, z_2, z_4, z_3]^{-1} = [z_3, z_4, z_1, z_2]. \tag{2.1}$$

It is convenient to have a quantity that is quantitatively equivalent to the cross-ratio and has a geometrically more transparent meaning. Let $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for $a, b \in \mathbb{R}$. If (z_1, z_2, z_3, z_4) is a four-tuple of distinct points in Z define

$$\langle z_1, z_2, z_3, z_4 \rangle := \frac{d(z_1, z_3) \wedge d(z_2, z_4)}{d(z_1, z_4) \wedge d(z_2, z_3)}. \tag{2.2}$$

Then the following is true.

Lemma 2.3. *Let (Z, d) be a metric space and $\eta_0(t) = 3(t \vee \sqrt{t})$ for $t > 0$. Then for every four-tuple (z_1, z_2, z_3, z_4) of distinct points in Z we have*

$$\langle z_1, z_2, z_3, z_4 \rangle \le \eta_0([z_1, z_2, z_3, z_4]). \tag{2.4}$$

Proof. Suppose there is a four-tuple (z_1, z_2, z_3, z_4) for which the left hand side in (2.4) exceeds the right hand side. Let $t_0 = [z_1, z_2, z_3, z_4]$. We may assume $d(z_1, z_3) \le d(z_2, z_4)$. Then

$$\begin{aligned} d(z_1, z_4) &\le d(z_1, z_3) + d(z_3, z_2) + d(z_2, z_4) \\ &\le 2d(z_2, z_4) + d(z_2, z_3). \end{aligned}$$

Similarly, $d(z_2, z_3) \le 2d(z_2, z_4) + d(z_1, z_4)$, and so by our assumption we have

$$\begin{aligned} d(z_1, z_4) \vee d(z_2, z_3) &\le 2d(z_2, z_4) + d(z_1, z_4) \wedge d(z_2, z_3) \\ &\le \left(2 + \frac{1}{\eta_0(t_0)}\right) d(z_2, z_4). \end{aligned}$$

Hence,

$$t_0 = [z_1, z_2, z_3, z_4] = \frac{d(z_1, z_3)d(z_2, z_4)}{(d(z_1, z_4) \wedge d(z_2, z_3))(d(z_1, z_4) \vee d(z_2, z_3))}$$
$$\geq \frac{d(z_1, z_3)\eta_0(t_0)}{(d(z_1, z_4) \wedge d(z_2, z_3))(1 + 2\eta_0(t_0))}$$
$$\geq \frac{\eta_0(t_0)^2}{1 + 2\eta_0(t_0)} > t_0.$$

This is a contradiction. □

Using the symmetry properties (2.1) for the cross-ratio which are also true for the modified cross-ratio defined in (2.2), we obtain an inequality as in (2.4) with the roles of the cross-ratios reversed and the function η_0 replaced by the function $t \mapsto 1/\eta_0^{-1}(1/t)$. In particular, we conclude that $[z_1, z_2, z_3, z_4]$ is small if and only if $\langle z_1, z_2, z_3, z_4 \rangle$ is small, where the quantitative dependence is given by universal functions.

A metric space (Z, d) is called λ*-linearly locally contractible* where $\lambda \geq 1$, if every ball $B(a, r)$ in Z with $0 < r \leq \operatorname{diam}(Z)/\lambda$ is contractible inside $B(a, \lambda r)$, i.e., there exists a continuous map $H\colon B(a, r) \times [0, 1] \to B(a, \lambda r)$ such that $H(\cdot, 0)$ is the identity on $B(a, r)$ and $H(\cdot, 1)$ is a constant map. The space is called linearly locally contractible, if it is λ-linearly locally contractible for some $\lambda \geq 1$. Similar language will be employed for other notions that depend on numerical parameters.

A metric space (Z, d) is called λ-LLC for $\lambda \geq 1$ (LLC stands for linearly locally connected) if the following two conditions are satisfied:

(λ-LLC_1) If $B(a, r)$ is a ball in Z and $x, y \in B(a, r)$, then there exists a continuum $E \subset B(a, \lambda r)$ containing x and y.

(λ-LLC_2) If $B(a, r)$ is a ball in Z and $x, y \in Z \setminus B(a, r)$, then there exists a continuum $E \subset Z \setminus B(a, r/\lambda)$ containing x and y.

We remind the reader that a continuum is a compact connected set consisting of more than one point.

Linear local contractibility implies the LLC condition for compact connected topological n-manifolds, and is equivalent to it when $n = 2$:

Lemma 2.5. *Suppose Z a metric space which is a compact connected topological n-manifold. Then:*

(i) *If Z is λ-linearly locally contractible, then Z is λ'-LLC for each $\lambda' > \lambda$.*
(ii) *If $n = 2$ and Z is LLC, then Z is linearly locally contractible. The linear local contractibility constant depends on Z and not just on the LLC constant.*

Proof. (i) We first verify the LLC_1 condition. If $a \in Z$, and $r > \operatorname{diam}(Z)/\lambda$, then $B(a, \lambda r) = Z$, so in this case the λ-LLC_1 condition follows from the connectedness of Z. If $r \leq \operatorname{diam}(Z)/\lambda$, then the inclusion $i\colon B(a, r) \to$

$B(a, \lambda r)$ is homotopic to a constant map. Hence it induces the zero homomorphism on reduced 0-dimensional homology, which means that λ-LLC_1 holds.

Let $\lambda' > \lambda$. To see that λ'-LLC_2 holds, we have to show that if $B(a, r') \subset Z$ is a ball with $Z \setminus B(a, r') \neq \emptyset$, then the inclusion $i\colon Z \setminus B(a, r') \to Z \setminus B(a, r'/\lambda')$ induces the zero homomorphism

$$\tilde{H}_0(Z \setminus B(a, r')) \xrightarrow{0} \tilde{H}_0(Z \setminus B(a, r'/\lambda')) \tag{2.6}$$

for reduced singular homology with coefficients in $\mathbb{Z}_2$. Note that $Z \setminus B(a, r') \neq \emptyset$ implies $r' \leq \operatorname{diam}(Z)$. Moreover, we can find $0 < r < r'$ close enough to r' such that $\bar{B}(a, r'/\lambda') \subset B(a, r/\lambda)$. Let $K_1 := \bar{B}(a, r'/\lambda')$ and $K_2 := \bar{B}(a, r)$. Then K_1 and K_2 are compact, and we have $B(a, r'/\lambda') \subset K_1 \subset K_2 \subset B(a, r')$. So in order to show (2.6), it is enough to show that the inclusion $i\colon Z \setminus K_2 \to Z \setminus K_1$ induces the zero homomorphism

$$\tilde{H}_0(Z \setminus K_2) \xrightarrow{0} \tilde{H}_0(Z \setminus K_1). \tag{2.7}$$

It follows from the path connectedness of Z and the long exact sequence for singular homology that the natural map $\partial\colon H_1(Z, Z \setminus K_i) \to \tilde{H}_0(Z \setminus K_i)$ is surjective for $i \in \{1, 2\}$. Hence (2.7) is true, if the inclusion $i\colon (Z, Z \setminus K_2) \to (Z, Z \setminus K_1)$ induces the zero homomorphism

$$H_1(Z, Z \setminus K_2) \xrightarrow{0} H_1(Z, Z \setminus K_1). \tag{2.8}$$

Now duality [29, Theorem 17, p. 296] shows that for each compact subset $K \subset Z$ we have an isomorphism $H_1(Z, Z \setminus K) \simeq \check{H}^{n-1}(K)$, where $\check{H}^*$ denotes Čech cohomology with coefficients in $\mathbb{Z}_2$. This isomorphism is natural, and hence compatible with inclusions. Since $K_1 \subset B(a, r/\lambda) \subset B(a, r) \subset K_2$ and $r < r' \leq \operatorname{diam}(Z)$, it follows from our assumptions that K_1 contracts to a point inside K_2. Hence the inclusion $i : K_1 \to K_2$ induces the zero homomorphism $\check{H}^{n-1}(K_2) \xrightarrow{0} \check{H}^{n-1}(K_1)$. Therefore, (2.8) holds which implies (2.6) as we have seen.

(ii) Suppose Z is λ-LLC. It is enough to show that the inclusion $i\colon B(a, r) \to B(a, \lambda r)$ is homotopic to a constant map, if $r > 0$ is sufficiently small independent of $a \in Z$. Since Z is a compact 2-manifold, every sufficiently small ball lies precompactly in an open subset of Z homeomorphic to $\mathbb{R}^2$. So without loss of generality we may assume that the sets $U := B(a, r)$ and $V := B(a, \lambda r)$ are bounded and open subsets of $\mathbb{R}^2$ with $U \subset V$. Now λ-LLC_1 implies that U lies in a single component of V. So in order to show that U is contractible inside V, it is enough to show that each component Ω of U is contained in a simply connected (and hence contractible) subregion of V.

The condition λ-LLC_2 implies that $\mathbb{R}^2 \setminus V$ lies in one, namely the unbounded component of $\mathbb{R}^2 \setminus U$. It follows in particular that if γ is a Jordan curve in U, then the interior region $I(\gamma)$ of γ is contained in V.

A well-known fact from plane topology is that every bounded region Ω can be written as an nondecreasing union $\Omega = \bigcup_{i=0}^{\infty} \Omega_i$, where Ω_i is a region with $\bar{\Omega}_i \subset \Omega$ whose boundary consists of finitely many disjoint Jordan curves. One of the boundary components γ_i of Ω_i is a Jordan curve whose interior $I(\gamma_i)$ contains Ω_i. Now if Ω is a component of U, then $\gamma_i \subset \Omega \subset U$, and so $I(\gamma_i) \subset V$ as we have seen. Hence $\Omega \subset \bigcup_{i=0}^{\infty} I(\gamma_i) \subset V$ lies in the union of a nondecreasing sequence of Jordan subregions of V. This union is a simply connected subregion of V containing Ω. □

In view of the lemma we prefer to work with the weaker *LLC* condition instead of linear local contractibility in the following.

If E and F are continua in Z we denote by

$$\Delta(E, F) := \frac{\operatorname{dist}(E, F)}{\operatorname{diam}(E) \wedge \operatorname{diam}(F)} \tag{2.9}$$

the *relative distance* of E and F.

Lemma 2.10. *Suppose (Z, d) is λ-LLC. Then there exist functions $\delta_1, \delta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on λ with the following properties. Suppose $\epsilon > 0$ and (z_1, z_2, z_3, z_4) is a four-tuple of distinct points in Z.*

(i) *If $[z_1, z_2, z_3, z_4] < \delta_1(\epsilon)$, then there exist continua $E, F \subset Z$ with $z_1, z_3 \in E$, $z_2, z_4 \in F$ and $\Delta(E, F) \geq 1/\epsilon$.*
(ii) *If there exist continua $E, F \subset Z$ with $z_1, z_3 \in E$, $z_2, z_4 \in F$ and $\Delta(E, F) \geq 1/\delta_2(\epsilon)$, then $[z_1, z_2, z_3, z_4] < \epsilon$.*

As the proof will show, the function δ_2 can actually be chosen as a numerical function independent of λ.

Proof. We have to show that $[z_1, z_2, z_3, z_4]$ is small if and only if there exist two continua with large relative distance containing $\{z_1, z_3\}$ and $\{z_2, z_4\}$, respectively.

Suppose $s = [z_1, z_2, z_3, z_4]$ is small. Then by Lemma 2.3 the quantity

$$t := \langle z_1, z_2, z_3, z_4 \rangle = \frac{d(z_1, z_3) \wedge d(z_2, z_4)}{d(z_1, z_4) \wedge d(z_2, z_3)}. \tag{2.11}$$

is small, quantitatively. We may assume $t < 1$ and $r := d(z_1, z_3) \leq d(z_2, z_4)$. Since Z is λ-*LLC* and $z_1, z_3 \in B(z_1, 2r)$, there exists a continuum E connecting z_1 and z_3 in $B(z_1, 2\lambda r)$. Let $R := r(1/t - 1) > 0$. Then $d(z_1, z_4) \geq r/t > R$ and $d(z_1, z_2) \geq d(z_2, z_3) - d(z_1, z_3) \geq r(1/t - 1) = R$. Thus z_2, z_4 are in the complement of $B(z_1, R)$, and so there exists a continuum F connecting z_2 and z_4 in $Z \setminus B(z_1, R/\lambda)$. For the relative distance of E and F we get

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\operatorname{diam}(E) \wedge \operatorname{diam}(F)} \geq \frac{R/\lambda - 2\lambda r}{4\lambda r} > 1/(4\lambda^2 t) - 1,$$

which is uniformly large if s and so t are small.

Now suppose that there exist continua $E, F \subset Z$ with with $z_1, z_3 \in E$ and $z_2, z_4 \in F$ for which $\Delta(E, F)$ is large. Since

$$\langle z_1, z_2, z_3, z_4 \rangle = \frac{d(z_1, z_3) \wedge d(z_2, z_4)}{d(z_1, z_4) \wedge d(z_2, z_3)} \leq \frac{\operatorname{diam}(E) \wedge \operatorname{diam}(F)}{\operatorname{dist}(E, F)} = 1/\Delta(E, F),$$

we conclude from Lemma 2.3 that $[z_1, z_2, z_3, z_4]$ is uniformly small. □

In the proof of this lemma we used for the first time the expression “If A is small, then B is small, quantitatively.” This and similar language will be very convenient in the following, but it requires some explanation. By this expression we mean that an inequality $B \leq \Psi(A)$ for the quantities A and B holds, where Ψ is a positive function with $\Psi(t) \to 0$ if $t \to 0$ that depends only on the *data*. The data are some ambient parameters associated to the given space, function, etc. In the proof above the data consisted just of the parameter λ in the LLC-condition for Z.

3. Quasi-Möbius maps

Let $\eta \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a homeomorphism, i.e., a strictly increasing nonnegative function with $\eta(0) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$, and let $f \colon X \to Y$ be an injective map between metric spaces (X, d_X) and (Y, d_Y). The map f is an *η-quasi-Möbius map* if for every four-tuple (x_1, x_2, x_3, x_4) of distinct points in X, we have

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]).$$

Note that by exchanging the roles of x_1 and x_2, one gets the lower bound

$$\eta([x_1, x_2, x_3, x_4]^{-1})^{-1} \leq [f(x_1), f(x_2), f(x_3), f(x_4)].$$

Hence the inverse $f^{-1} \colon f(X) \to X$ is also quasi-Möbius.

Another way to express the condition that f is quasi-Möbius is to say that the cross-ratio $[x_1, x_2, x_3, x_4]$ of a four-tuple of distinct points is small if and only if the cross-ratio $[f(x_1), f(x_2), f(x_3), f(x_4)]$ is small, quantitatively. This is easy to see using the symmetry properties (2.1) of cross-ratios.

The map f is *η-quasisymmetric* if

$$\frac{d_Y(f(x_1), f(x_2))}{d_Y(f(x_1), f(x_3))} \leq \eta\left(\frac{d_X(x_1, x_2)}{d_X(x_1, x_3)}\right)$$

for every triple (x_1, x_2, x_3) of distinct points in X. Again it is easy to see that the inverse map $f^{-1} \colon f(X) \to X$ is also quasisymmetric. Two metric spaces X and Y are called *quasisymmetric*, if there exists a homeomorphism $f\colon X \to Y$ that is quasisymmetric.

Intuitively, a quasisymmetry is a map between metric spaces that maps balls to roundish objects that can be trapped between two balls whose radius ratio is bounded by a fixed constant. Based on this it is easy to verify the quasisymmetric invariance of properties like linear local contractibility or linear local connectivity.

We list some properties of quasi-Möbius and quasisymmetric maps (cf. [35]):

(1) Quasi-Möbius and quasisymmetric maps are homeomorphisms onto their images.
(2) The post-composition of an η_1-quasi-Möbius map with an η_2-quasi-Möbius map is an $\eta_2 \circ \eta_1$-quasi-Möbius map.
(3) An η-quasisymmetric map is $\tilde{\eta}$-quasi-Möbius with $\tilde{\eta}$ depending only on η.

    Conversely, every quasi-Möbius map between bounded spaces is quasisymmetric. This statement is not quantitative in general, but we have:

(4) Suppose (X, d_X) and (Y, d_Y) are bounded metric spaces, $f\colon X \to Y$ is η-quasi-Möbius, and $\lambda \geq 1$. Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are triples of distinct points in X and Y, respectively, such that $f(x_i) = y_i$ for $i \in \{1, 2, 3\}$, $d_X(x_i, x_j) \geq \operatorname{diam}(X)/\lambda$ and $d_Y(y_i, y_j) \geq \operatorname{diam}(Y)/\lambda$ for $i, j \in \{1, 2, 3\}, i \neq j$. Then f is $\tilde{\eta}$-quasisymmetric with $\tilde{\eta}$ depending only on η and λ.
(5) An η-quasisymmetric map from a dense subset A of a metric space X into a complete metric space Y has a unique extension to an η-quasisymmetric map on X.

We will need the following convergence property of quasi-Möbius maps which we state as a separate lemma.

Lemma 3.1. *Suppose (X, d_X) and (Y, d_Y) are compact metric spaces, and $f_k\colon D_k \to Y$ for $k \in \mathbb{N}$ is an η-quasi-Möbius map defined on a subset D_k of X. Suppose*

$$\lim_{k\to\infty} \sup_{x\in X} \operatorname{dist}(x, D_k) = 0$$

and that for $k \in \mathbb{N}$ there exist triples (x_1^k, x_2^k, x_3^k) and (y_1^k, y_2^k, y_3^k) of points in $D_k \subset X$ and Y, respectively, such that $f(x_i^k) = y_i^k$, $k \in \mathbb{N}$, $i \in \{1, 2, 3\}$,

$$\inf \left\{ d_X\left(x_i^k, x_j^k\right) : k \in \mathbb{N},\ i, j \in \{1, 2, 3\},\ i \neq j \right\} > 0$$

and

$$\inf \left\{ d_Y\left(y_i^k, y_j^k\right) : k \in \mathbb{N},\ i, j \in \{1, 2, 3\},\ i \neq j \right\} > 0.$$

Then the sequence (f_k) subconverges uniformly to an η-quasi-Möbius map $f\colon X \to Y$, i.e. there exists an increasing sequence (k_n) in $\mathbb{N}$ such that

$$\lim_{n\to\infty} \sup_{x\in D_{k_n}} d_Y(f(x), f_{k_n}(x)) = 0.$$

The assumptions imply that the functions f_k are equicontinuous (cf. [35, Thm. 2.1]). The proof of the lemma then follows from standard arguments, and we leave the details to the reader.

Lemma 3.2. *Suppose (X, d_X) and (Y, d_Y) are metric spaces, and $f: X \to Y$ is an η-quasi-Möbius map. Then there exists a function $\Phi: \mathbb{R}^+ \to \mathbb{R}^+$ with* $\lim_{t\to\infty} \Phi(t) = \infty$ *depending only on η such that the following statement holds.*

If $E, F \subset X$ are disjoint continua, then

$$\Delta(f(E), f(F)) \geq \Phi(\Delta(E, F)).$$

If f is surjective, and we apply the lemma to the inverse map f^{-1}, we get a similar inequality with the roles of sets and images sets reversed. These inequalities say that the relative distance of two continua is large if and only if the relative distance of the image sets under a quasi-Möbius map is large, quantitatively.

Since every quasisymmetric map is also quasi-Möbius, this last statement is also true for quasisymmetric maps.

Proof. Let $E' := f(E)$ and $F' := f(F)$. Then E' and F' are continua. Hence there exist points $y_1 \in E'$ and $y_3 \in F'$ with $d_Y(y_1, y_3) = \operatorname{dist}(E', F')$. Moreover, we can find points $y_4 \in E'$ and $y_2 \in F'$ with $d_Y(y_1, y_4) \geq \operatorname{diam}(E')/2$ and $d_Y(y_2, y_3) \geq \operatorname{diam}(F')/2$. Then

$$\Delta(E', F') \geq 2\langle y_1, y_2, y_3, y_4\rangle.$$

On the other hand, if $x_i := f^{-1}(y_i)$, then

$$\Delta(E, F) \leq \langle x_1, x_2, x_3, x_4\rangle$$

by the very definition of these quantities.

Now if $\Delta(E, F)$ is large, then $\langle x_1, x_2, x_3, x_4\rangle$ is at least as large. Since f is η-quasi-Möbius it follows from Lemma 2.3 that $\langle y_1, y_2, y_3, y_4\rangle$ and hence $\Delta(E', F')$ are large, quantitatively. □

A metric space (Z, d) is called *weakly λ-uniformly perfect*, $\lambda > 1$, if for every $a \in Z$ and $0 < r \leq \operatorname{diam}(Z)$ the following is true: if the ball $\bar{B}(a, r/\lambda)$ contains a point distinct from a, then $B(a, r) \setminus \bar{B}(a, r/\lambda) \neq \emptyset$.

This condition essentially says that at each point $a \in Z$ the space is uniformly perfect in the usual sense above the scale at which there exist points different from a. Note that every connected metric space, or more generally, every dense set in a connected metric space is weakly λ-uniformly perfect for $\lambda > 2$.

A metric space (Z, d) is called *C_0-doubling*, $C_0 \geq 1$, if every ball of radius $r > 0$ can be covered by at most C_0 balls of radius $r/2$. A set $A \subset Z$ is called *ϵ-separated*, $\epsilon > 0$, if $d(x, y) \geq \epsilon$ for $x, y \in A$, $x \neq y$. Later we will use the fact that for every $\epsilon > 0$ there exists an ϵ-separated set $A \subset Z$ that is maximal (with respect to inclusion). This follows from Zorn's lemma.

If Z is C_0-doubling, and $A \subset Z$ is an ϵ-separated set in a ball of radius $r > 0$, then the cardinality of A is bounded by a number only depending on C_0 and the ratio r/ϵ.

Lemma 3.3. *Suppose (X, d_X) and (Y, d_Y) are metric spaces, and $f: X \to Y$ is a bijection. Suppose that X is weakly λ-uniformly perfect, Y is C_0-doubling, and there exists a function $\delta_0: \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$[f(x_1), f(x_2), f(x_3), f(x_4)] < \delta_0(\epsilon) \Rightarrow [x_1, x_2, x_3, x_4] < \epsilon, \tag{3.1}$$

whenever $\epsilon > 0$ and (x_1, x_2, x_3, x_4) is a four-tuple of distinct points in X. Then f is η-quasi-Möbius with η depending only on λ, C_0, and δ_0.

As we remarked above, a bijection is quasi-Möbius if it has the property that a cross-ratio of four points is small if and only if the cross-ratio of the image points is small, quantitatively. The lemma says that for suitable spaces this equivalence, which consists of implications in two directions, can be replaced by one of these implications.

Proof. We have to show that for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, \lambda, C_0, \delta_0) > 0$ such that

$$[x_1, x_2, x_3, x_4] < \delta \Rightarrow [f(x_1), f(x_2), f(x_3), f(x_4)] < \epsilon, \tag{3.5}$$

whenever (x_1, x_2, x_3, x_4) is a four-tuple of distinct points in X. By Lemma 2.3, for this purpose it is enough to show the following: if $\epsilon \in (0, 1]$ and (x_1, x_2, x_3, x_4) is a four-tuple of distinct points in X with $\langle x_1, x_2, x_3, x_4\rangle < \delta$ and $\langle y_1, y_2, y_3, y_4\rangle \geq \epsilon$, where $y_i = f(x_i)$, $i \in \{1, 2, 3, 4\}$, then we obtain a contradiction if δ is smaller than a positive number depending on ϵ, λ, C_0, and δ_0.

We may assume that $s := d_X(x_1, x_3) \leq d_X(x_2, x_4)$. Let

$$t := \min\{d_Y(y_i, y_j) : i \in \{1, 3\},\ j \in \{2, 4\}\}. \tag{3.6}$$

Then

$$d_Y(y_i, y_j) \geq \epsilon t \quad \text{for} \quad i, j \in \{1, 2, 3, 4\},\ i \neq j. \tag{3.7}$$

We have that

$$\begin{aligned} \operatorname{diam}(X) &\geq \min\{d_X(x_i, x_j) : i \in \{1, 3\},\ j \in \{2, 4\}\} \\ &\geq d_X(x_1, x_4) \wedge d_X(x_2, x_3) - d_X(x_1, x_3) \ \geq\ (1/\delta - 1)s. \end{aligned}$$

Since we may assume that $(1/\delta - 1) > \lambda^2$, we can choose $N \in \mathbb{N}$ such that

$$\lambda^{2N} < (1/\delta - 1) \leq \lambda^{2N+2}. \tag{3.8}$$

Since X is weakly λ-uniformly perfect, $x_3 \in \bar{B}(x_1, s)$ and $\lambda^{2N}s < \operatorname{diam}(X)$, there exist points $z_i \in X$ for $i \in \{1, \ldots, N\}$ such that

$$z_i \in B(x_1, \lambda^{2i}s) \setminus \bar{B}(x_1, \lambda^{2i-1}s).$$

Then

$$d_X(z_i, x_1) \vee d_X(z_i, x_3) \le (\lambda^{2i} + 1)s \quad \text{for} \quad i \in \{1, \dots, N\}$$

and

$$d_X(z_i, z_j) \ge \lambda^{2j-2}(\lambda - 1)s \quad \text{for} \quad i, j \in \{1, \dots, N\},\ i < j.$$

It follows that

$$\langle z_i, p, z_j, q \rangle \ge c(\lambda) > 0,$$

whenever $i, j \in \{1, \dots, N\}$, $i \neq j$, $p \in \{x_1, x_3\}$ and $q \in \{x_2, x_4\}$. By our hypotheses and Lemma 2.3 there exists $c_1 \in (0, 1]$ depending only on δ_0 and λ such that

$$\langle f(z_i), u, f(z_j), v \rangle \ge c_1 > 0, \tag{3.9}$$

whenever $i, j \in \{1, \dots, N\}$, $i \neq j$, $u \in \{y_1, y_3\}$, and $v \in \{y_2, y_4\}$.

We claim that

$$d_Y(f(z_i), f(z_j)) \ge c_1 \epsilon t/3 =: c_2 t \tag{3.10}$$

for $i, j \in \{1, \dots, N\}, i \neq j$. For otherwise, by (3.7) we can pick $u \in \{y_1, y_3\}$ and $v \in \{y_2, y_4\}$ such that

$$\operatorname{dist}(\{f(z_i), f(z_j)\}, \{u, v\}) \ge t\epsilon/3$$

and we get a contradiction to (3.9).

Choose $u_0 \in \{y_1, y_3\}$ and $v_0 \in \{y_2, y_4\}$ such that $d_Y(u_0, v_0) = t$. Then at most one of the points $f(z_i)$ can lie outside $\bar{B}(u_0, c_3 t)$ where $c_3 = 1 + 1/c_1$. For if this were true for $f(z_i)$ and $f(z_j)$, $i \neq j$, then again we get a contradiction to (3.9) with $u = u_0$ and $v = v_0$.

The doubling property of Y now shows that the number of points in $\bar{B}(u_0, c_3 t)$ which are $(c_2 t)$-separated is bounded by a constant C depending only on C_0, $c_2 = c_2(\epsilon, \lambda, \delta_0)$ and $c_3 = c_3(\epsilon, \lambda, \delta_0)$. Hence $N - 1 \le C$. By (3.8) this leads to a contradiction if δ is smaller than a constant depending on ϵ, λ, C_0, and δ_0. □

4. Approximations of metric spaces

Suppose G is a graph with vertex set V. We assume that there are no loops in G, i.e., no vertex is connected to itself by an edge, and that two arbitrary distinct vertices are not connected by more than one edge. If $v_1, v_2 \in V$ are connected by an edge or are identical we write $v_1 \sim v_2$. The combinatorial structure of the graph is completely determined by the vertex set V and this reflexive and symmetric relation $\sim$. Hence we will write $G = (V, \sim)$.

A *chain* is a sequence $x_1, \dots, x_n$ of vertices with $x_1 \sim x_2 \sim \cdots \sim x_n$. It *connects* two subsets $A \subset V$ and $B \subset V$ if $x_1 \in A$ and $x_n \in B$.

If $x, y \in V$ we let $k_G(x, y) \in \mathbb{N}_0 \cup \{\infty\}$ be the *combinatorial distance* of x and y, i.e., $k_G(x, y) + 1$ is the smallest cardinality $\#M$ of a chain M connecting x and y. If G is connected, then (V, k_G) is a metric space, and we define $B_G(v, r) := \{u \in V : k_G(u, v) < r\}$ and $\bar{B}_G(v, r) := \{u \in V : k_G(u, v) \le r\}$ for $v \in V$ and $r > 0$. We drop the subscript G if the graph under consideration is understood. The cardinality of the set $\{u \in V : k_G(u, v) = 1\}$ is the *valence* of $v \in V$. The *valence* of G is the supremum of the valences over all vertices in G.

Now let (Z, d) be a metric space. We consider quadruples $\mathcal{A} = (G, p, r, \mathcal{U})$, where $G = (V, \sim)$ is a graph with vertex set V, $p\colon V \to Z$, $r\colon V \to \mathbb{R}^+$ and $\mathcal{U} = \{U_v : v \in V\}$ is an open cover of Z indexed by the set V. We let $p_v := p(v)$ and $r_v := r(v)$ for $v \in V$. Let

$$N_\epsilon(U) := \{z \in Z : \operatorname{dist}(z, U) < \epsilon\}$$

for $U \subset Z$ and $\epsilon > 0$, and define the L*-star* of $v \in V$ with respect to $\mathcal{A}$ for $L > 0$ as

$$\mathcal{A}\text{-St}_L(v) := \bigcup \{U_u : u \in V, k(u, v) < L\}.$$

We simply write $\text{St}_L(v)$, if no confusion can arise. We call $\mathcal{A}$ a K*-approximation* of Z, $K \ge 1$, if the following conditions are satisfied:

(1) Every vertex of G has valence at most K.
(2) $B(p_v, r_v) \subset U_v \subset B(p_v, Kr_v)$ for $v \in V$.
(3) If $u \sim v$ for $u, v \in V$, then $U_u \cap U_v \ne \emptyset$, and $K^{-1}r_u \le r_v \le Kr_v$. If $U_u \cap U_v \ne \emptyset$ for $u, v \in V$, then $k(u, v) < K$.
(4) $N_{r_v/K}(U_v) \subset \text{St}_K(v)$ for $v \in V$.
(5) If $v \in V$, $z_1, z_2 \in U_v$, then there is a path γ in Z connecting z_1 to z_2 so that $\gamma \subset \text{St}_K(v)$.

The point p_v should be thought of as a basepoint of U_v. By condition (2) we can think of the number r_v as the "local scale" associated with v. Condition (3) says that the local scale only changes by a bounded factor if we move to a neighbor of a vertex, and that the incidence pattern of the cover $\mathcal{U}$ resembles the incidence pattern of the vertices in G, quantitatively. Condition (4) means that we can thicken up a set U_v by a fixed amount comparable to the local scale by passing to the K-star of v. Finally, condition (5) allows us to connect any two points in U_v by a path contained in the K-star of v.

We point out some immediate consequences of the conditions (1)–(5):

(6) If Z is connected, then G is connected; this follows from (3).
(7) The multiplicity of $\mathcal{U}$ is bounded by a constant $C = C(K)$: if $U_{v_1} \cap \ldots \cap U_{v_n} \ne \emptyset$ then $\{v_1, \ldots, v_n\} \subset B(v_1, K)$ by (3), and $\#B(v_1, K) \le C = C(K)$ by (1). Similarly, it can be shown that for fixed $L > 0$, the multiplicity of the cover $\{\text{St}_L(v) : v \in V\}$ is bounded by a number $C = C(K, L)$.

(8) For the curve γ in (5) we have $\operatorname{diam}(\gamma) \le Cr_v$ with $C = C(K)$; this follows from (2) and (3).

The *mesh size* of the K-approximation $\mathcal{A}$ is defined to be

$$\operatorname{mesh}(\mathcal{A}) := \sup_{v \in V} r_v.$$

The next lemma shows that K-approximations behave well under quasisymmetric maps.

Lemma 4.1. *Suppose (X, d_X) and (Y, d_Y) are connected metric spaces, and $f\colon X \to Y$ is an η-quasisymmetric homeomorphism. Suppose $K \ge 1$ and $\mathcal{A} = ((V, \sim), p, r, \mathcal{U})$ is a K-approximation of X. Assume that*

$$\operatorname{mesh}(\mathcal{A}) < \operatorname{diam}(X)/2. \tag{4.2}$$

For $v \in V$ define $p'_v := f(p_v)$, $U'_v := f(U_v)$ and

$$r'_v := \inf\{d_Y(f(x), p'_v) : x \in X, d_X(x, p_v) \ge r_v\}. \tag{4.3}$$

Let $\mathcal{U}' = \{U'_v : v \in V\}$. Then $\mathcal{A}' = ((V, \sim), p', r', \mathcal{U}')$ is a K'-approximation of Y with K' depending only on K and η.

We emphasize that the underlying graphs of $\mathcal{A}$ and $\mathcal{A}'$ are the same.

Note that by condition (4.2) the set in (4.3) over which the infimum is taken is nonempty. The continuity of f^{-1} implies that r'_v is positive. The number r'_v is roughly the diameter of U'_v. Up to multiplicative constants, it is essentially the only possible choice for r'_v. Our particular definition guarantees $B_Y(p'_v, r'_v) \subset f(B_X(p_v, r_v)) \subset f(U_v) = U'_v$.

Up to this ambiguity in the choice of r'_v, the K'-approximation $\mathcal{A}'$ is canonically determined by $\mathcal{A}$ and the map f. In this sense we can say that $\mathcal{A}'$ is the "image" of $\mathcal{A}$ under f.

Proof. We denote image points under f by a prime, i.e., $x' = f(x)$ for $x \in X$. We also denote by $K_1, K_2, \dots$ positive constants that can be chosen only to depend on η and K.

Since X is connected and the complement of $B_X(p_v, r_v)$ is nonempty, for every $v \in V$ we can choose a point $x_v \in X$ with with $d_X(x_v, p_v) = r_v$. The quasisymmetry of f implies

$$r'_v \le d_Y(x'_v, p'_v) \le K_1 r'_v.$$

If $x \in X$ and $d_X(x, p_v) < Kr_v$ then

$$d_Y(x', p'_v) < d_Y(x'_v, p'_v)\eta(K) \le K_2 r'_v.$$

This and the definition of r'_v show

$$B_Y(p'_v, r'_v) \subset f(B_X(p_v, r_v)) \subset f(U_v) = U'_v \subset f(B_X(p_v, Kr_v)) \subset B_Y(p'_v, K_2 r'_v). \tag{4.4}$$

If $u \sim v$, then $U_u \cap U_v \neq \emptyset$ and $r_u \leq K r_v$. In particular,

$$d_X(p_u, p_v) \leq K(r_u + r_v) \leq K_3 r_v$$

and

$$d_X(x_u, p_v) \leq d_X(x_u, p_u) + d_X(p_u, p_v) \leq r_u + K_3 r_v \leq K_4 r_v.$$

Hence

$$\begin{aligned} r'_u &\leq d_Y(x'_u, p'_u) \leq d_Y(p'_u, p'_v) + d_Y(x'_u, p'_v) \\ &\leq d_Y(x'_v, p'_v)(\eta(K_3) + \eta(K_4)) \leq K_5 r'_v. \end{aligned} \tag{4.5}$$

Suppose $z \in U_v$. Since $d_Y(x'_v, p'_v) \geq r'_v$, there exists $y \in \{p_v, x_v\}$ such that $d_Y(y', z') \geq r'_v/2$. Then $d_X(y, z) \leq 2K r_v$. If now $x \in X$ is an arbitrary point with $d_X(x, z) \geq r_v/K$, then

$$r'_v \leq 2 d_Y(y', z') \leq 2 d_Y(x', z') \eta(2K^2) \leq K_6 d_Y(x', z').$$

This implies that

$$B_Y(z', r'_v/K_6) \subset f(B_X(z, r_v/K)) \subset f(\mathcal{A}\text{-St}_K(v)) = \mathcal{A}'\text{-St}_K(v) \ \text{ for } z \in U_v. \tag{4.6}$$

The assertion now follows from the fact that $\mathcal{A}$ is a K-approximation and (4.4)–(4.6). $\square$

Lemma 4.7. *Suppose (Z, d) is a connected metric space and $((V, \sim), p, r, \mathcal{U})$ is a K-approximation of Z. Suppose $L \geq K$ and $W \subset V$ is a maximal set of combinatorially L-separated vertices. Then $M = p(W) \subset Z$ is weakly λ-uniformly perfect with λ depending only on L and K.*

Proof. Note that property (3) of a K-approximation implies

$$K^{-k(u,v)} \leq \frac{r(u)}{r(v)} \leq K^{k(u,v)} \quad \text{for} \quad u, v \in V.$$

Since $d(p(u), p(v)) \leq K(r(u) + r(v))$ whenever $u, v \in V$ with $u \sim v$, we obtain

$$d(p(u), p(v)) \leq 2r(u) k(u, v) K^{1+k(u,v)} \quad \text{for} \quad u, v \in V.$$

Let $\lambda = 16 L^2 K^{4+4L}$. Suppose $w_0, w_1 \in W$ such that for $z_0 = p(w_0)$ and $z_1 = p(w_1)$ we have that $z_0 \neq z_1$ and $z_1 \in \bar{B}(z_0, r/\lambda)$, where $0 < r \leq \operatorname{diam}(M) \leq \operatorname{diam}(Z)$. We claim that $B(z_0, r) \setminus \bar{B}(z_0, r/\lambda)$ contains a point in M. Since $w_0 \neq w_1$ we have $k(w_0, w_1) \geq L \geq K$ and so $U_{w_0} \cap U_{w_1} = \emptyset$ by property (3) of a K-approximation. This implies

$$r(w_0) \leq d(z_0, z_1) \leq r/\lambda. \tag{4.8}$$

Since $\lambda > 4$ there exist points in Z outside $B(z_0, r/\sqrt{\lambda})$. The connectedness of Z then implies that there actually exists $z \in Z$ with $d(z_0, z) = r/\sqrt{\lambda}$. Since $\mathcal{U}$ is a cover of Z, we have $z \in U_v$ for some $v \in V$. Then

$$r(v) \leq Kr/\sqrt{\lambda}. \tag{4.9}$$

For otherwise,

$$\operatorname{dist}(z_0, U_v) \leq d(z_0, z) = r/\sqrt{\lambda} < r(v)/K,$$

and so $z_0 \in N_{r(v)/K}(U_v) \subset \mathrm{St}_K(v)$. This implies $k(w_0, v) \leq 2K$ which leads to

$$r(w_0) \geq K^{-2K} r(v) \geq K^{1-2K} r/\sqrt{\lambda} > r/\lambda,$$

contradicting (4.8).

Since W is a maximal L-separated set in V, there exists $w_2 \in W$ such that $k(w_2, v) < L$. Let $z_2 = p(w_2) \in M$. We claim that $d(z_2, z_0) > r/\lambda$. Otherwise, $d(z_2, z_0) \leq r/\lambda$. If $w_2 \neq w_0$, then similarly as above we conclude $r(w_2) \leq r/\lambda$. But by (4.8) this is also true if $w_2 = w_0$. Hence we get in this case

$$\begin{aligned} r/\sqrt{\lambda} = d(z_0, z) &\leq d(z_0, z_2) + d(z_2, p(v)) + d(p(v), z) \\ &\leq r/\lambda + r(w_2) 2LK^{L+1} + Kr(v) \\ &\leq r/\lambda + (2LK^{L+1} + K^{L+1}) r(w_2) \\ &\leq (1 + 2LK^{L+1} + K^{L+1}) r/\lambda < r\sqrt{\lambda}, \end{aligned}$$

which is a contradiction.

Moreover, by (4.9)

$$\begin{aligned} d(z_0, z_2) &\leq d(z_0, z) + d(z, p(v)) + d(p(v), z_2) \\ &\leq r/\sqrt{\lambda} + Kr(v) + 2LK^{L+1} r(v) \\ &\leq (1 + K^2 + 2LK^{L+2}) r/\sqrt{\lambda} \; < \; r. \end{aligned}$$

This shows that the point $z_2 \in M$ is contained in $B(z_0, r) \setminus \bar{B}(z_0, r/\lambda)$. $\square$

5. Circle packings

In Sects. 5 and 6 we will consider embeddings of a graph G in a metric space Z. In this context we will regard $G = (V, \sim)$ as a topological space by choosing a unit interval $I := [0, 1]$ for each two-element set $\{u, v\} \subset V$ with $u \sim v$, where we let the endpoints of I correspond to u and v. We then glue these intervals together whenever endpoints of intervals correspond to the same vertex in V. An *embedding* of G into Z is then just a map of this topological space into Z which is a homeomorphism onto its image.

If the graph G is embedded in Z we will identify G with its image under the embedding. This image is viewed as a subset of Z with certain points and arcs distinguished as vertices and edges, respectively, so that their incidence pattern is the same as the incidence pattern of the graph. In this case we will write $G = (V, E)$, where V is the set of vertices and E is the set of edges of G.

Suppose the graph G is combinatorially equivalent to the 1-skeleton of a triangulation T of a topological 2-sphere. By the Andreev-Koebe-Thurston circle packing theorem (cf. for example [20]) the graph G can be realized as the incidence graph of a circle packing. This means the following. Let $G = (V, \sim)$. Then there is a family $\mathcal{C}$ of pairwise disjoint open spherical disks C_v, $v \in V$, in $\mathbb{S}^2$ such that $\bar{C}_u \cap \bar{C}_v \neq \emptyset$ for $u, v \in V$ if and only if $u \sim v$.

We can always assume that the circle packing is *normalized*. By this we mean that among the centers of the disks of the circle packing, there are three normalizing points which lie on a great circle of $\mathbb{S}^2$ and are equally spaced. A normalization of a circle packing can always be achieved by replacing the original circles by their images under a suitably chosen Möbius transformation. To see this note that the boundary circles of three distinct disks D_1, D_2, D_3 determine distinct hyperbolic planes H_1, H_2, H_3 in hyperbolic three-space $\mathbb{H}^3$ (as viewed in the unit ball model). It is easy to see that there exists a point $z_0 \in \mathbb{H}^3$ that minimizes the sum of the (signed) hyperbolic distances to the planes H_i. The unit vectors in the tangent space $T_{z_0}\mathbb{H}^3$ of $\mathbb{H}^3$ at z_0 determined by the directions from z_0 to the planes H_i will then lie in a two-dimensional subspace of $T_{z_0}\mathbb{H}^3$ and form an equilateral triangle. If we move the point z_0 to the center of the unit ball by a Möbius transformation g, the centers of the image disks $g(D_1), g(D_2), g(D_3)$ will then be equally spaced points on a great circle.

In a normalized circle packing all disks are smaller than hemispheres. In particular, if two different disks in the packing have a common boundary point, then there is a unique geodesic joining the centers. If we join the centers of adjacent disks in the circle packing in this way, then we get an embedding of G on the sphere. The closures of the complementary regions of this embedded graph are closed spherical triangles Δ forming a triangulation T' of $\mathbb{S}^2$ combinatorially equivalent to T. If $v \in V$ let $p(v)$ be the center of the disk C_v corresponding to v, and let $r(v)$ be the spherical radius of C_v. Let U_v be the interior of the union of all triangles $\Delta \in T'$ having $p(v)$ as a vertex. Then U_v is open, starlike with respect to $p(v)$ and contains C_v. Moreover, the sets U_v, $v \in V$, form a cover $\mathcal{U}$ of $\mathbb{S}^2$.

Given these definitions we claim:

Lemma 5.1. *Suppose G is combinatorially equivalent to a 1-skeleton of a triangulation of $\mathbb{S}^2$, and $\mathcal{C}$ is a normalized circle packing realizing G. Then $(G, p, r, \mathcal{U})$ is a K-approximation of $\mathbb{S}^2$ with K depending only on the valence of G.*

Proof. It is a well-known fact that for a circle packing of Euclidean circles the ratio of the radii of two adjacent disks in the packing is bounded by a constant depending only on the number of neighbors of (one of) these disks (this is called the "Ring Lemma"; cf. [23]). For a packing of spherical circles a similar statement is true if no disk in the packing is larger than a hemisphere, in particular if the packing is normalized. In other words, if $u, v \in V$ and $u \sim v$, then $C^{-1} \leq r_u/r_v \leq C$ with C depending only on the valence of G. Choosing K suitably depending on the valence of G, it is easy to see that the conditions (1)–(5) of a K-approximation are true for $(G, p, r, \mathcal{U})$. We omit the details. $\square$

6. Construction of good graphs

In this section we will work with a modification of the LLC_1-condition for a metric space (Z, d):

(λ-$\widetilde{LLC}_1$) If $x, y \in Z$, $x \neq y$, then there exists an arc γ with endpoints x and y such that

$$\operatorname{diam}(\gamma) \leq \lambda d(x, y).$$

Here $\lambda \geq 1$. Obviously, λ-$\widetilde{LLC}_1$ implies $(1 + 2\lambda)$-LLC_1. A similar quantitative implication in the other direction will not be true in general, unless Z is locally "nice". For example, if Z is locally Euclidean, then a simple covering argument shows that λ-LLC_1 implies 3λ-$\widetilde{LLC}_1$. So for topological manifolds LLC_1 and $\widetilde{LLC}_1$ are quantitatively equivalent.

Lemma 6.1. *Suppose (Z, d) is a metric space which is C_0-doubling and λ-$\widetilde{LLC}_1$. Let $0 < r \leq \operatorname{diam}(Z)$ and suppose $A \subset Z$ is a maximal r-separated set. Then there exists a connected graph $\Gamma = (V, E)$ which is embedded in Z and has the following properties:*

(i) *The valence of Γ is bounded by K.*
(ii) *The vertex set V contains A.*
(iii) *If $u, v \in A$ with $d(u, v) < 2r$, then Γ contains an edge path γ joining u and v with $\operatorname{diam}(\gamma) \leq Kr$. Each edge in Γ belongs to one of these paths γ.*
(iv) *For all balls $B(a, r) \subset Z$ we have $\#(B(a, r) \cap V) \leq K$.*

Here the constant $K \geq 1$ depends only on C_0 and λ.

Implicit in this statement is that Γ satisfies our standing assumptions on graphs; namely, every edge in Γ has two distinct vertices as endpoints, and two distinct vertices are connected by at most one edge.

Note that (iii) implies $\operatorname{diam}(e) \leq Kr$ for $e \in E$. It follows from (iv) and the doubling property of Z that a ball of radius R in Z meets at most C vertices or edges of Γ, where C is a number depending only on C_0, K and R/r.

Proof. For all two-element subsets $\{u, v\} \subset A$ with $d(u, v) < 2r$ choose an arc α with endpoints u, v and $\operatorname{diam}(\alpha) \le 2\lambda r$. Let $\mathcal{A}$ be the family of arcs thus obtained.

We claim that there exists $N = N(C_0, \lambda) \in \mathbb{N}$ such that $\mathcal{A}$ can be written as a disjoint union $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_N$, where each of the subfamilies $\mathcal{A}_i$ has the property that if $\alpha, \alpha' \in \mathcal{A}_i$ are two distinct arcs, then

$$\operatorname{dist}(\alpha, \alpha') > 8\lambda r. \tag{6.2}$$

To see that this can be done, note first that since Z is C_0-doubling there exists $N_1 = N_1(C_0, \lambda) \in \mathbb{N}$ such that

$$\#(\bar{B}(a, 12\lambda r) \cap A) < N_1 \quad \text{for} \quad a \in Z.$$

Hence if $\alpha \in \mathcal{A}$, then

$$\#\{\alpha' \in \mathcal{A} : \operatorname{dist}(\alpha, \alpha') \le 8\lambda r\} < N_1(N_1 - 1)/2. \tag{6.3}$$

Let $N = N_1(N_1 - 1)/2$. An argument using Zorn's lemma and (6.3) shows that there exists a labeling of the arcs in $\mathcal{A}$ by the numbers $1, \dots, N$ such that no two distinct arcs $\alpha, \alpha' \in \mathcal{A}$ with $\operatorname{dist}(\alpha, \alpha') \le 8\lambda r$ have the same label. If we define $\mathcal{A}_i$ to be the set of all arcs with label i, we get the desired decomposition $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_N$.

Since Z is doubling, there exists $N_2 = N_2(C_0, \lambda) \in \mathbb{N}$ such that each arc in $\mathcal{A}$ can be covered by at most N_2 open balls of radius r. Now define graphs $\Gamma_i = (V_i, E_i)$ for $i = 1, \dots, N$ inductively as follows. The graphs Γ_i will be embedded in Z, their edges will have diameter bounded by $2\lambda r$ and we will have

$$M_i := \max_{a \in Z} \#\{e \in E_i : e \cap B(a, r) \neq \emptyset\} \le (2N_2 + 4)^i. \tag{6.4}$$

Let Γ_1 be the union of the arcs in $\mathcal{A}_1$, where we consider these arcs as the edges of Γ_1 and the set of their endpoints as the set of vertices. Note that by (6.2) the graph Γ_1 is embedded in Z and by the choice of the arcs in $\mathcal{A}$ the diameter of each edge will be bounded by $2\lambda r$. Moreover, each ball $B(a, r)$ can only meet at most one arc in $\mathcal{A}_1$, so (6.4) is true for $i = 1$.

Suppose Γ_{i-1} has been constructed. We consider an arbitrary arc $\alpha \in \mathcal{A}_i$ and will modify it to obtain an arc with the same endpoints such that for each edge $e \in E_{i-1}$ the set $\alpha \cap e$ is connected. Note first that the number of edges in E_{i-1} that α meets is bounded by $N_2 M_{i-1}$, and in particular finite. This follows from the definition of N_2 and M_{i-1}.

Let $e_1, \dots, e_k \in E_{i-1}$ be the edges that meet α. Assume inductively that we have modified α into an arc (also called α by abuse of notation) such that

$$\text{the sets} \quad \alpha \cap e_1, \ \dots, \ \alpha \cap e_{j-1} \quad \text{are connected.} \tag{6.5}$$

Let γ be the smallest (possibly degenerate) subarc of α which contains $\alpha \cap e_j$. Then the endpoints of γ are contained in e_j, and $\alpha \setminus \gamma$ is disjoint from e_j.

Replace $\gamma \subset \alpha$ by the subarc of e_j which has the same endpoints as γ. This new curve α is an arc and the set $\alpha \cap e_j$ is connected. Since the edges in E_{i-1} are nonoverlapping (i.e., they have disjoint interiors), the statement (6.5) is still true for the new arc α (some of the intersections in (6.5) may have become empty) and there are no new edges that α meets. After at most k modifications, the arc α will have the same endpoints as before, and will have a subdivision into nonoverlapping subarcs which consists of the sets $\alpha \cap e$ for $e \in E_{i-1}$ and their complementary subarcs. Hence α is subdivided into at most $2k+1 \leq 2N_2M_{i-1}+1$ subarcs which all have diameter bounded by $2\lambda r$. Note that the endpoints of these subarcs will always belong to the original arc α. Hence the diameter of the new arc α will be bounded by $2\lambda r + \sup_{e\in E_{i-1}} \operatorname{diam}(e) \leq 4\lambda r$. Let $\tilde{\mathcal{A}}_i$ be the set of the new arcs α. Then for any two distinct arcs in $\tilde{\mathcal{A}}_i$ we have

$$\operatorname{dist}(\alpha, \alpha') > 2\lambda r. \tag{6.6}$$

The graph $\Gamma_i = (V_i, E_i)$ is now obtained from Γ_{i-1} and the set of modified arcs $\tilde{\mathcal{A}}_i$ as follows. If for $e \in E_{i-1}$ there exists $\alpha \in \tilde{\mathcal{A}}_i$ which meets e, subdivide e by introducing new vertices into at most three new edges such that $e \cap \alpha$ becomes a vertex or an edge. Every edge $e \in E_{i-1}$ is subdivided at most once, since it cannot meet two distinct arcs in $\tilde{\mathcal{A}}_i$ by (6.6). To this graph obtained by subdividing some of the edges of Γ_{i-1}, we add the edges and vertices from the subdivision of the arcs $\alpha \in \tilde{\mathcal{A}}_i$. Obviously, Γ_i is embedded in Z and all its edges have diameter bounded by $2\lambda r$. It can be shown inductively that Γ_i has the property that every edge in Γ_i has two distinct vertices as endpoints, and that two distinct vertices are connected by at most one edge.

If $B(a, r)$ is an arbitrary ball, then an edge $e \in E_i$ meeting $B(a, r)$ is either a subset of an edge in E_{i-1} meeting $B(a, r)$ or it is an edge obtained from the subdivision of some arc $\alpha \in \tilde{\mathcal{A}}_i$. By (6.6) all these latter edges lie on the same arc α. Hence

$$M_i \leq 3M_{i-1} + 2N_2M_{i-1} + 1 \leq (2N_2 + 4)^i.$$

Now let $\Gamma = \Gamma_N$. Then the underlying set of Γ is equal to the union of the arcs in $\mathcal{A}_1 \cup \tilde{\mathcal{A}}_2 \cup \cdots \cup \tilde{\mathcal{A}}_N$. This shows (ii) and (iii). These conditions imply that Γ is connected. Suppose v is a vertex of Γ. If an edge e has a vertex v as an endpoint, then $e \cap B(v, r) \neq \emptyset$. From (6.4) it follows that the number of edges with endpoint v is bounded by M_N which gives (i). Finally, (iv) follows from (6.4) and

$$\#(B(a, r) \cap V) \leq 2\#\{e \in E : e \cap B(a, r) \neq \emptyset\}.$$

□

Proposition 6.7. *Suppose (Z, d) is a metric space homeomorphic to $\mathbb{S}^2$. If (Z, d) is C_0-doubling and λ-LLC, then for given $0 < r \leq \operatorname{diam}(Z)$ and any maximal r-separated set $A \subset Z$ there exists an embedded graph $G = (V, E)$ which is the 1-skeleton of a triangulation T of Z such that:*

(i) *The valence of G is bounded by K.*
(ii) *The vertex set V of G contains A.*
(iii) *If $e \in E$, then* $\operatorname{diam}(e) < Kr$. *If $u, v \in V$ and $d(u, v) < 2r$, then $k_G(u, v) < K$.*
(iv) *For all balls $B(a, r) \subset Z$ we have $\#(B(a, r) \cap V) \leq K$.*

Here the constant $K \geq 1$ depends only on C_0 and λ.

Note that (iii) implies: If $u, v \in V$ and $d(u, v) \leq Lr$, then we have $k_G(u, v) \leq C(L, K, C_0, \lambda)$.

Since G is embedded in Z, the vertices and edges of G are subsets of Z. For $v \in V$ let $p(v) := v$, $r(v) := r$ and $U_v := B_Z(v, Kr)$. Then $\mathcal{U} := \{U_v : v \in V\}$ is a cover of Z. Hence under the above assumptions we immediately have:

Corollary 6.8. $(G, p, r, \mathcal{U})$ *is a K'-approximation of Z, where K' depends only on λ and C_0.*

Corollary 6.9. *Suppose Z is a metric space homeomorphic to $\mathbb{S}^2$. If Z is C_0-doubling and λ-LLC, then there exist $K \geq 1$ only depending on C_0 and λ and a sequence $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ of K-approximations of Z, whose graphs $G_k = (V_k, E_k)$ are 1-skeletons of triangulations T_k of Z and for which*

$$\lim_{k \to \infty} \operatorname{mesh}(\mathcal{A}_k) = 0.$$

Proof. This follows immediately from Corollary 6.8 if we apply Proposition 6.7 for a maximal $(1/k)$-separated set A_k. □

Proof of Proposition 6.7. First we claim that every (continuous) loop $\phi \colon \mathbb{S}^1 \to Z$ such that $\phi(\mathbb{S}^1) \subset B(p, R)$ for some $p \in Z$ and $R > 0$ is null-homotopic in $B(p, \lambda R)$. For this note that since Z is λ-LLC, the compact set $A = Z \setminus B(p, \lambda R)$ is contained in a component of $Z \setminus \phi(\mathbb{S}^1)$. Since Z is homeomorphic to $\mathbb{S}^2$ it follows that ϕ is null-homotopic in $Z \setminus A = B(p, \lambda R)$.

Since Z is a topological manifold and λ-LLC, it is λ'-$\widetilde{LLC}$ with $\lambda' = 3\lambda$. Let $\Gamma_1 = (V_1, E_1)$ be a graph embedded in Z that satisfies the conditions (i)–(iv) of Lemma 6.1 with some constant K' depending on the data of Z. The idea for constructing G is to subdivide the components of $Z \setminus \Gamma_1$ into triangles. For this to result in a graph as desired, we have to bound the diameter of such a component. We need two lemmas.

Lemma 6.10. *Given a continuous map $f_0 \colon \mathbb{S}^1 \to Z$, there is a continuous map $f_1 \colon \mathbb{S}^1 \to \Gamma_1 \subset Z$ and a homotopy $f_0 \sim f_1$ so that the tracks of the homotopy have diameter bounded by $C_1 r$ where C_1 depends only on C_0 and λ.*

Proof. Since $A \subset V$ is a maximal r-separated set, we have $\operatorname{dist}(z, A) < r$ for all $z \in Z$. Since $f_0(\mathbb{S}^1)$ is compact, for some $r' \in (0, r)$ we have $\operatorname{dist}(f_0(\zeta), A) < r'$ for all $\zeta \in \mathbb{S}^1$. Since f_0 is uniformly continuous, we can find a finite set $S \subset \mathbb{S}^1$ containing at least two points such that if $J \subset \mathbb{S}^1 - S$ is a maximal complementary arc, then $\operatorname{diam}(f_0(J)) < r - r'$. For each $\zeta \in S$ we can find a point $f_1(\zeta) \in A$ such that $d(f_0(\zeta), f_1(\zeta)) < r'$. Let $J \subset \mathbb{S}^1 - S$ be a maximal complementary arc and suppose its endpoints are $\zeta, \zeta' \in S$. Then $\operatorname{dist}(f_1(\zeta), f_1(\zeta')) < 2r$ and so by property (iii) of Γ_1 we can extend f_1 continuously to $\bar{J}$ such that $f_1(\bar{J})$ is a path in Γ_1 of diameter at most $K'r$. If we extend f_1 in this way to all such arcs $\bar{J}$, then we get a continuous map $f_1 \colon \mathbb{S}^1 \to \Gamma_1$.

We build a homotopy $H \colon \mathbb{S}^1 \times I \to Z$ (where $I = [0, 1]$) from f_0 to f_1 as follows. We set $H(\zeta, 0) = f_0(\zeta)$ and $H(\zeta, 1) = f_1(\zeta)$ for all $\zeta \in \mathbb{S}^1$. For each $\zeta \in S$, define $H|_{\{\zeta\}\times I}$ to be a path connecting $f_0(\zeta)$ to $f_1(\zeta)$ of diameter bounded by $\lambda' r = 3\lambda r$. We have defined H on $(\mathbb{S}^1 \times \{0, 1\}) \cup (S \times I)$. If $J \subset \mathbb{S}^1 - S$ is a maximal complementary arc, then we can extend H to $\bar{J} \times I$ so that the image of this set is contained in a ball of radius Cr where $C = C(C_0, \lambda)$. Here we use the fact that the boundary of the "square" $\bar{J} \times I$ is mapped into a ball of radius $R = (3\lambda + K' + 1)r$ and this loop is null-homotopic in a ball with the same center and radius λR. It follows that the tracks $t \mapsto H(\zeta, t)$ of the homotopy have diameter bounded by $C_1 r$ with $C_1 = C_1(C_0, \lambda)$. □

Lemma 6.11. *The diameter of each connected component of $Z \setminus \Gamma_1$ is bounded by $C_2 r$ where C_2 depends only on C_0 and λ.*

Proof. We have to show that if C_2 is large enough depending on the data, then the set Γ_1 separates every point $p \in Z \setminus \Gamma_1$ from every point $q \in Z \setminus \Gamma_1$ outside $B(p, C_2 r)$. Indeed, with the notation of the last lemma we can choose $C_2 = 4 + 2C_1$. To see this note first that

$$M := \bar{B}\big(p, \tfrac{1}{2}(C_2 + 1)r\big) \setminus B\big(p, \tfrac{1}{2}(C_2 - 1)r\big)$$

separates p from q. Using the fact that Z is homeomorphic to $\mathbb{S}^2$, it is easy to see that there is a Jordan curve in an arbitrarily small neighborhood of M separating p from q. In particular, there exists a loop $f_0 \colon \mathbb{S}^1 \to Z$ such that

$$f_0(\mathbb{S}^1) \subset B\big(p, \tfrac{1}{2}(C_2 + 2)r\big) \setminus \bar{B}\big(p, \tfrac{1}{2}(C_2 - 2)r\big)$$

and the winding number of f_0 with respect to p differs from the winding number of f_0 with respect to q. By the previous lemma we can find a loop $f_1 \colon \mathbb{S}^1 \to \Gamma_1$ homotopic to f_0 such that the tracks of the homotopy stay inside

$$B\left(p, \tfrac{1}{2}(C_2 + 2 + 2C_1)r\right) \setminus \bar{B}\left(p, \tfrac{1}{2}(C_2 - 2 - 2C_1)r\right) \subset B(p, C_2 r) \setminus \{p\}.$$

In particular, the winding number of f_1 with respect to p will still be different from the winding number of f_1 with respect to q. Hence $f_1(\mathbb{S}^1)$ also separates p from q, and so does $\Gamma_1 \supset f_1(\mathbb{S}^1)$. □

Since Γ_1 is connected, a component Ω of $Z \setminus \Gamma_1$ is a simply connected region whose boundary $\partial\Omega$ is a finite union of edges in Γ_1. Note that by the previous lemma, the number of these edges is bounded by a number depending only on the data of Z.

Now define a new graph $\Gamma_2 = (V_2, E_2)$ as follows: Subdivide the edges of Γ_1 by choosing for each edge a point in its interior. Moreover for each component Ω of $Z \setminus \Gamma_1$ choose a point in its interior. These points together with the set V_1 form the vertex set V_2 of Γ_2. The edges of Γ_2 consist of the arcs obtained by the subdivision of the edges in Γ_1 and new edges obtained as follows for each component Ω of $Z \setminus \Gamma_1$. The vertices in V_2 on the boundary of Ω can be brought into a natural cyclic order $v_1, \dots, v_N, v_{N+1} = v_1$, possibly with repetitions, such that successive vertices are adjacent, i.e., endpoints of an arc obtained from the subdivision of the edges in Γ_1. Note that each vertex can occur at most twice in this given cyclic order. Hence N is bounded by a number depending only on the data. Since Ω is simply connected, we can connect the vertex v chosen in the interior of Ω with each of the vertices v_i by an arc e_i such that $e_i \setminus \{v_i\} \subset \Omega$ and such that two of these arcs have only the point v in common.

The graph Γ_2 is embedded in Z, and has complementary regions whose closures are topological triangles, i.e., there are exactly three different vertices and edges in successive order on the boundary of such a region. One of these vertices is a vertex contained in $Z \setminus \Gamma_1$, one will be in the interior of an edge $e \in E_1$ and one vertex will be also a vertex of Γ_1. In particular, the components of $Z \setminus \Gamma_2$ are Jordan regions. In general, the set of these triangles which are the closures of components of $Z \setminus \Gamma_2$ will not be a triangulation of Z, because it may happen that two such triangles have the same vertex set without being identical. This situation arises from components of $Z \setminus \Gamma_1$ which are not Jordan regions.

Define a graph $G = (V, E)$ obtained from Γ_2 in the same way as Γ_2 was obtained from Γ_1. Then the closures of the complementary components of $Z \setminus G$ are topological triangles which triangulate Z so that the 1-skeleton of this triangulation is G. The other desired properties of G follow immediately from the previous lemma and the properties of Γ_1. $\square$

7. Modulus

Suppose (Z, d, μ) is a metric measure space, i.e., d is a complete metric and μ a Borel measure on Z. Moreover, we assume that μ is locally finite and has dense support. The space (Z, d, μ) is called (*Ahlfors*) *Q-regular*, $Q > 0$, if the measure μ satisfies

$$C^{-1} R^Q \leq \mu(B(a, R)) \leq C R^Q \tag{7.1}$$

for each open ball $B(a, R)$ of radius $0 < R \leq \operatorname{diam}(Z)$ and for some constant $C \geq 1$ independent of the ball. The numbers Q and C are called the *data* of Z. If (7.1) is true for some measure μ, then a similar inequality holds for Q-dimensional Hausdorff measure $\mathcal{H}^Q$. Hence, if in a Q-regular space the measure is not specified, then we assume that the underlying measure μ is the Hausdorff measure $\mathcal{H}^Q$.

Let $U \subset Z$ be an open set. We call a Borel function $\rho\colon U \to [0, \infty]$ an *upper gradient* of a function $u\colon U \to \mathbb{R}$ if

$$|u(x) - u(y)| \leq \int_\gamma \rho\, ds,$$

whenever $x, y \in U$ and γ is a rectifiable path joining x and y in U. Here integration is with respect to arclength on γ.

Suppose $B = B(a, r)$ is an open ball in Z. If $\lambda > 0$ we let $\lambda B := B(a, \lambda r)$. Moreover, if $u\colon B \to \mathbb{R}$ is a locally integrable function on B, we denote by u_B the average of u over B, i.e.,

$$u_B = \frac{1}{\mu(B)} \int_B u\, d\mu.$$

The metric measure space is said to satisfy a $(1, Q)$-*Poincaré inequality*, where $Q \geq 1$, if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |u - u_B|\, d\mu \leq C(\operatorname{diam}(B)) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} \rho^Q\, d\mu \right)^{1/Q},$$

whenever B is an open ball in Z, the function u is locally integrable on λB, and ρ is an upper gradient of u on λB.

A *density* (on Z) is a Borel function $\rho\colon Z \to [0, \infty]$. A density ρ is called *admissible* for a path family Γ in Z, if

$$\int_\gamma \rho\, ds \geq 1$$

for each rectifiable path $\gamma \in \Gamma$. Here integration is with respect to arclength on γ. If $Q \geq 1$, the Q-*modulus* of a family Γ of paths in Z is the number

$$\operatorname{Mod}_Q(\Gamma) = \inf \int \rho^Q\, d\mu, \tag{7.2}$$

where the infimum is taken over all densities $\rho\colon Z \to [0, \infty]$ that are admissible for Γ. If E and F are (nondegenerate) continua in Z, we let $\operatorname{Mod}_Q(E, F)$ denote the Q-modulus of the family of paths in Z connecting E and F.

Suppose Z is a rectifiably connected metric measure space. Then Z is called a Q-*Loewner space*, $Q \geq 1$, if there exists a positive decreasing function $\Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\operatorname{Mod}_Q(E, F) \geq \Psi(\Delta(E, F)) \tag{7.3}$$

whenever E and F are disjoint continua in Z. Recall that $\Delta(E, F)$ is the relative distance of E and F as defined in (2.9). The number Q and the function Ψ are the data of the Loewner space Z.

The Loewner condition was introduced in [15] and quantifies the idea that a space has many rectifiable curves. According to Thm. 5.7 and Thm. 5.12 in [15], a proper Q-regular metric space Z satisfies a $(1, Q)$-Poincaré inequality if and only if Z is Q-Loewner (note that the assumption of φ-convexity in [15, Thm. 5.7] is unnecessary, since a proper Q-regular metric space satisfying a $(1, Q)$-Poincaré inequality is quasiconvex [12, Appendix]).

We will use the following fact about Loewner spaces.

Proposition 7.4. *Suppose (Z, d, μ) is a Q-regular Q-Loewner space, $Q > 1$. Then there exist constants $\lambda \geq 1$ and $C > 0$ depending only on the data of Z with the following property.*

If $z \in Z$, $s > 0$, and $Y_1, Y_2 \subset Z$ are continua with $Y_i \cap B(z, s) \neq \emptyset$ and $\operatorname{diam}(Y_i) \geq s/4$ *for $i \in \{1, 2\}$, then for every Borel function $\rho : Z \to [0, \infty]$ there exists a rectifiable path η in Z joining Y_1 and Y_2 such that*

$$\int_\eta \rho \, ds \leq C \left(\int_{B(z,\lambda s)} \rho^Q \, d\mu \right)^{1/Q}.$$

We will skip the proof of this proposition which is very similar to the proof of Lem. 3.17 in [15]. Essentially, the result is true because the relative distance of Y_1 and Y_2 is bounded by a fixed constant. Hence the regularity and the Loewner condition imply that if λ is large enough depending on the data, then the modulus of the family of paths inside $B(z, \lambda s)$ joining Y_1 and Y_2 is bigger than a constant.

Suppose $G = (V, \sim)$ is a graph, and A, B are subsets of V. We will define the *combinatorial Q-modulus* $\operatorname{mod}_Q^G(A, B)$ of the pair A and B as follows. Call a weight function $w \colon V \to [0, \infty]$ *admissible* for the pair A and B, if

$$\sum_{i=1}^{n} w(x_i) \geq 1,$$

whenever $x_1, \ldots, x_n$ is chain connecting A and B.

Now let

$$\operatorname{mod}_Q^G(A, B) = \inf \sum_{v \in V} w(v)^Q,$$

where the infimum is taken over all weights w that are admissible for A and B. Note that $\operatorname{mod}_Q^G(A, B) \geq 1$ if $A \cap B \neq \emptyset$. We drop the superscript G in $\operatorname{mod}_Q^G(A, B)$ if the graph G is understood.

If $A \subset V$ and $s > 0$ we denote by $N_s(A)$ the s-neighborhood of A, i.e., the set of all $u \in V$ for which there exists $a \in A$ with $k_G(a, u) < s$.

If we want to estimate the Q-modulus of the pair (A, B), then the following lemma will allow us to change the sets A and B with quantitative control.

Lemma 7.5. *Suppose $G = (V, \sim)$ is a graph with valence bounded by $d_0 \geq 1$. For every $Q \geq 1$ and $s > 0$ there exists a number $C = C(d_0, s, Q)$ with the following property: If $A, B, A', B' \subset V$, $A' \subset N_s(A)$, and $B' \subset N_s(B)$, then*

$$\operatorname{mod}_Q(A', B') \leq C \operatorname{mod}_Q(A, B).$$

Proof. Note that if w is admissible for A and B, then $\tilde{w}\colon V \to [0, \infty]$ defined by

$$\tilde{w}(v) = \sum_{u \in B(v,s)} w(u) \quad \text{for} \quad v \in V$$

is admissible for (A', B'). Moreover, since the valence of G is bounded by d_0, it follows that each ball $B(v, s)$ has a cardinality bounded by a constant depending only on s and d_0. It follows that

$$\sum_{v \in V} \tilde{w}(v)^Q \leq C \sum_{v \in V} w(v)^Q,$$

with $C = C(s, d_0, Q)$. The lemma follows. □

8. K-approximations and modulus comparison

In this section we relate the Q-modulus on a metric space to the Q-modulus on the graph of a K-approximation. Results of this general nature are well-known. The (minor) novelty here is that the local scales may vary from point to point.

Let (Z, d) be a metric space. Throughout this section $\mathcal{A} = (G, p, r, \mathcal{U})$ will be a K-approximation of Z with graph $G = (V, \sim)$. For each subset $E \subset Z$ we define $V_E := \{v \in V \mid U_v \cap E \neq \emptyset\}$. Note that $V_E \subset V$ depends on $\mathcal{A}$, but we suppress this dependence in our notation. If $\gamma\colon J \to Z$ is a path, we will denote the image set $\gamma(J)$ also by γ for simplicity.

Proposition 8.1. *Let (Z, d, μ) be a Q-regular metric measure space, $Q \geq 1$, and let $\mathcal{A}$ be a K-approximation of Z. Then there exists a constant $C \geq 1$ depending only on K and the data of Z with the following property:*

If $E, F \subset Z$ are continua and if $\operatorname{dist}(V_E, V_F) \geq 4K$, then

$$\operatorname{Mod}_Q(E, F) \leq C \operatorname{mod}_Q(V_E, V_F). \tag{8.2}$$

Proof. Let $w\colon V \to [0, \infty]$ be an admissible function for the pair (V_E, V_F): if $v_1 \sim \cdots \sim v_k$ is a chain in V with $v_1 \in V_E$ and $v_k \in V_F$, then $\sum_{i=1}^k w(v_i) \geq 1$. Define $\tilde{w}\colon V \to [0, \infty]$ by the formula

$$\tilde{w}(v) = \sum_{u \in B(v,K)} w(u),$$

and

$$\rho := \sum_{v \in V} \left(\frac{\tilde{w}(v)}{r_v} \right) \chi_{\mathrm{St}_K(v)},$$

where χ_Y denotes the characteristic function of $Y \subset Z$.

Mass bound for ρ. The cover $\{\mathrm{St}_K(v) : v \in V\}$ has controlled overlap depending on K and there exists a constant $C = C(K)$ such that $\mathrm{St}_K(v) \subset B(p_v, Cr_v)$ for $v \in V$. Moreover, Z is Q-regular and every K-ball in V has cardinality controlled by $C(K)$. So we have that

$$\begin{aligned} \int_Z \rho^Q \, d\mu &\lesssim \sum_{v \in V} \int_Z \left(\frac{\tilde{w}(v)}{r_v} \chi_{\mathrm{St}_K(v)} \right)^Q d\mu \\ &\lesssim \sum_{v \in V} \tilde{w}(v)^Q \lesssim \sum_{v \in V} w(v)^Q. \end{aligned} \tag{8.3}$$

Admissibility of ρ. Now let $\gamma \colon J \to Z$ be a rectifiable path connecting E to F. Since $\mathcal{U}$ is a cover of the path γ, there exists a set $W = \{v_1, \dots, v_k\}$ in V such that $\gamma \cap U_{v_i} \neq \emptyset$ for $i \in \{1, \dots, k\}$, $U_{v_i} \cap U_{v_{i+1}} \neq \emptyset$ for $i \in \{1, \dots, k-1\}$, and $v_1 \in V_E$ and $v_k \in V_F$. The combinatorial distance of v_i and v_{i+1} is less than K. Hence there exists a chain A in V connecting V_E and V_F satisfying $W \subset A \subset N_K(W)$.

For each $v \in W$, let $J_v := \gamma^{-1}(\mathrm{St}_K(v))$ and $\gamma_v := \gamma|_{J_v}$. Then the definition of ρ gives

$$\rho(\gamma(t)) \geq \tilde{w}(v)/r_v \quad \text{for} \quad t \in J_v.$$

By our assumption that $\mathrm{dist}(V_E, V_F) \geq 4K$ the path γ is not contained in any K-star of a vertex. For if $\gamma \subset \mathrm{St}_K(u)$, then there exist $u_1, u_2 \in V$ with $k(u_1, u) < K$, $k(u_2, u) < K$, $U_{v_1} \cap U_{u_1} \neq \emptyset$, and $U_{v_k} \cap U_{u_2} \neq \emptyset$. Then $k(v_1, u_1) < K$ and $k(v_k, u_2) < K$ which implies $\mathrm{dist}(V_E, V_F) \leq k(v_1, v_k) < 4K$.

Since γ is not contained in any K-star of a vertex, we have that if a set U_v meets γ, then $\mathrm{length}(\gamma \cap \mathrm{St}_K(v)) \geq r_v/K$ by condition (4) of a K-approximation. In particular, for each $v \in W$ we have $\mathrm{length}(\gamma_v) \geq r_v/K$, and so

$$\int_{\gamma_v} \rho \, ds \gtrsim \left(\frac{\tilde{w}(v)}{r_v} \right) \mathrm{length}(\gamma_v) \gtrsim \tilde{w}(v).$$

Hence

$$\sum_{v \in W} \int_{\gamma_v} \rho \, ds \gtrsim \sum_{v \in W} \tilde{w}(v) \gtrsim \sum_{v \in N_K(W)} w(v) \geq 1,$$

since $N_K(W)$ contains the chain A connecting V_E and V_F and w is admissible. The sets $\mathrm{St}_K(v)$ and hence the sets $J_v \subset J$ for $v \in W$ have controlled overlap depending on K, giving

$$\int_\gamma \rho\, ds \gtrsim \sum_{v\in W} \int_{\gamma_v} \rho\, ds \gtrsim 1. \tag{8.4}$$

Combining (8.3) with (8.4) we get

$$\mathrm{Mod}_Q(E, F) \lesssim \mathrm{mod}_Q(V_E, V_F).$$

□

It is an interesting question when an inequality like (8.2) holds in the opposite direction. We will not need such a result for the proof of our theorems, but we will nevertheless explore this question, because it illuminates the general picture. In order to get the desired inequality, we have to add an analytic assumption on Z to our hypotheses. It suffices to assume that Z is a Q-regular Q-Loewner space, but as the next proposition shows it is enough that a Loewner type condition holds locally on the scale of our K-approximation $\mathcal{A}$.

Proposition 8.5. *Let (Z, d, μ) be a Q-regular metric measure space, $Q \geq 1$, and let $\mathcal{A}$ be a K-approximation of Z.*

Suppose that there exist constants $c_1, C_1 > 0$ with the following property: Let $v \in V$, $z \in U_v$, and $0 < s \leq c_1 r_v$. If $Y_1, Y_2 \subset Z$ are continua with $Y_i \cap B(z, s) \neq \emptyset$ and $\mathrm{diam}(Y_i) \geq s/4$ for $i \in \{1, 2\}$, then for every Borel function $\rho\colon Z \to [0, \infty]$ there exists a rectifiable path η connecting Y_1 and Y_2 such that

$$\int_\eta \rho\, ds \leq C_1 \left(\int_{\mathrm{St}_K(v)} \rho^Q\, d\mu \right)^{1/Q}. \tag{8.6}$$

Then there exists a constant $C \geq 1$ depending only on K, the data of Z, and the constants associated to the analytic condition (8.6) with the following property:

If $E, F \subset Z$ are continua not contained in any set $\mathrm{St}_{2K}(v)$ for $v \in V$, then

$$\mathrm{mod}_Q(V_E, V_F) \leq C\mathrm{Mod}_Q(E, F). \tag{8.7}$$

Note that by Proposition 7.4 and by the properties of a K-approximation every Q-regular Q-Loewner space Z with $Q > 1$ satisfies the analytic condition (8.6) with appropriate constants depending only on K and the data of Z. So Proposition 8.1 and Proposition 8.5 together imply the following corollary.

Corollary 8.8. *Let Z be a Q-regular Q-Loewner space, $Q > 1$, and let $\mathcal{A}$ be a K-approximation of Z. Then there exists a constant $C \geq 1$ depending only on K and the data of Z with the following property:*

If $E, F \subset Z$ are continua not contained in any $(2K)$-star and if $\operatorname{dist}(V_E, V_F) \geq 4K$, *then*

$$C^{-1}\mathrm{Mod}_Q(E, F) \leq \mathrm{mod}_Q(V_E, V_F) \leq C\mathrm{Mod}_Q(E, F). \tag{8.9}$$

Proof of Proposition 8.5. Let $\rho\colon Z \to [0, \infty]$ be an admissible Borel function for the pair (E, F), i.e.

$$\int_\gamma \rho\, ds \geq 1$$

for any rectifiable path γ joining E with F. Define $w\colon V \to [0, \infty]$ by

$$w(v) := \left(\int_{\mathrm{St}_{3K}(v)} \rho^Q\, d\mu\right)^{1/Q}.$$

Mass bound for w. Since the numbers $\#B(v, 3K)$ for $v \in V$ and the multiplicity of the cover $\mathcal{U}$ are bounded by a constant depending only on K, we have

$$\begin{aligned}\sum_{v\in V} w(v)^Q &\leq \sum_{v\in V}\sum_{u\in B(v,3K)} \int_{U_u} \rho^Q\, d\mu \\ &\lesssim \sum_{v\in V}\int_{U_v} \rho^Q\, d\mu \\ &\lesssim \int_Z \rho^Q\, d\mu.\end{aligned} \tag{8.10}$$

Admissibility of w. This step in the proof is modelled on arguments from [15], and is based on repeated application of our analytic condition. We use this near a single set U_v to prove that under our assumptions we have:

Lemma 8.11. *Suppose $v \in V$, and $Y_1, Y_2 \subset Z$ are continua with $Y_i \cap$* $\mathrm{St}_K(v) \neq \emptyset$, *and* $\operatorname{diam}(Y_i) \geq c_0 r_v$, *where $c_0 > 0$. Then there is a rectifiable path η connecting Y_1 and Y_2 such that*

$$\int_\eta \rho\, ds \leq Cw(v), \tag{8.12}$$

where $C > 0$ depends only on c_0, K, and the data of Z.

Proof. Pick $z_1, z_2 \in \mathrm{St}_K(v)$ so that $z_i \in Y_i \cap \mathrm{St}_K(v)$. Applying condition (5) of a K-approximation repeatedly, we find a path γ joining z_1 to z_2 so that $\gamma \subset \mathrm{St}_{2K}(v)$. Let

$$s := (c_0 \wedge c_1) \min_{u\in B(v,2K)} r(u) \simeq r(v),$$

where c_1 is the constant in the hypothesis of Proposition 8.5. Since Z is Q-regular, it is doubling. Moreover, $s \simeq r(v)$ and $\operatorname{diam}(\gamma) \lesssim r(v)$. Hence the cardinality of a maximal $(s/2)$-separated set on γ is bounded by a number depending only on the data. Since γ is connected, we can find an appropriate subset $x_1, \ldots, x_N$ of such a maximal set such that $d(z_1, x_1) < s$, $d(z_2, x_N) < s$, and $d(x_{i-1}, x_i) < s$ for $i \in \{2, \ldots, N\}$, where $N \in \mathbb{N}$ is bounded by a number depending only on the data.

Now let $\lambda_1 := Y_1$ and $\lambda_{N+1} := Y_2$. Then $\operatorname{diam}(\lambda_1) \wedge \operatorname{diam}(\lambda_{N+1}) \geq s/4$ by our assumptions. If $N \geq 2$, we have $\operatorname{diam}(\gamma) \geq s/2$ and so in addition we can find continua $\lambda_i \subset \gamma$ with $x_i \in \lambda_i \subset B(x_i, s)$ and $\operatorname{diam}(\lambda_i) \geq s/4$ for $i \in \{2, \ldots, N\}$.

Now $x_i \in \gamma \subset \mathrm{St}_{2K}(v)$ and so $x_i \in U_{u_i}$ for some $u_i \in V$ with $k(u_i, v) < 2K$. Then by definition of s we have $s \leq c_1 r_{u_i}$. Hence we can inductively find rectifiable paths $\eta_1, \ldots, \eta_N$ such that η_i joins $\lambda_1 \cup \eta_1 \cup \cdots \cup \eta_{i-1}$ and λ_{i+1}, and

$$\int_{\eta_i} \rho\, ds \lesssim \left(\int_{\mathrm{St}_K(u_i)} \rho^Q \, d\mu \right)^{1/Q} \leq \left(\int_{\mathrm{St}_{3K}(v)} \rho^Q \, d\mu \right)^{1/Q} = w(v). \quad (8.13)$$

This follows from an application of our analytic assumption to the ball $B(x_i, s)$ and the pair $\lambda_1 \cup \eta_1 \cup \cdots \cup \eta_{i-1}$ and λ_{i+1}. Note that λ_{i+1} meets $B(x_i, s)$. The same is true for the set $\lambda_1 \cup \eta_1 \cup \cdots \cup \eta_{i-1}$, since it meets λ_i by induction hypothesis. The union $\eta_1 \cup \ldots \cup \eta_N$ contains a rectifiable path η connecting Y_1 and Y_2 with

$$\int_{\eta} \rho\, ds \lesssim N w(v) \simeq w(v). \qquad \square$$

Now suppose $v_1, \ldots, v_k$ are the vertices of a chain in G joining V_E to V_F. Then $U_{v_1} \cap E \neq \emptyset$, $U_{v_k} \cap F \neq \emptyset$, and $U_{v_{i-1}} \cap U_{v_i} \neq \emptyset$ for $i \in \{2, \ldots, k\}$. Set $\lambda_1 := E$, $\lambda_{k+1} := F$, and for $i \in \{2, \ldots, k\}$ let λ_i be a continuum with $\lambda_i \subset \mathrm{St}_K(v_{i-1}) \cap \mathrm{St}_K(v_i)$ and

$$\operatorname{diam}(\lambda_i) \geq (r_{v_{i-1}} \wedge r_{v_i})/(2K) \geq (r_{v_{i-1}} \vee r_{v_i})/(2K^2).$$

These sets exist by condition (4) of a K-approximation and the fact that the complement of any K-star contains elements of E and F and is thus nonempty.

We can inductively find rectifiable paths $\eta_1, \ldots, \eta_k$ with

$$\int_{\eta_i} \rho\, ds \leq C_1 w(v_i)$$

so that η_i joins $\lambda_1 \cup \eta_1 \cup \ldots \cup \eta_{i-1}$ to λ_{i+1}. Here C_1 depends only on K and the data of Z. This follows from an application of Lemma 8.11 with $v = v_i$, $Y_1 := \lambda_1 \cup \eta_1 \cup \ldots \cup \eta_{i-1}$, $Y_2 := \lambda_{i+1}$, and a constant c_0 only depending on K. Indeed, note that Y_2 meets $\mathrm{St}_K(v_i)$, and $\operatorname{diam}(Y_2) \geq r_{v_i}/(2K^2)$. The set $Y_1 = \lambda_1 \cup \eta_1 \cup \ldots \cup \eta_{i-1}$ also meets $\mathrm{St}_K(v_i)$, since it meets λ_i by induction hypothesis. Moreover, since $E = \lambda_1 \subset Y_1$ and E is not contained in any $(2K)$-star, condition (4) of a K-approximation shows that we have $\operatorname{diam}(Y_1) \geq c(K) r_{v_i}$, where $c(K) > 0$ depends on K only.

The union $\eta_1 \cup \ldots \cup \eta_k$ will contain a rectifiable path η joining E to F with

$$1 \leq \int_\eta \rho\, ds \leq C_1 \sum_{i=1}^k w(v_i).$$

Therefore $C_1 w$ is an admissible test function for (V_E, V_F). Hence by (8.10)

$$\operatorname{mod}_Q(V_E, V_F) \lesssim \operatorname{Mod}_Q(E, F).$$

This completes the proof of Proposition 8.5. □

9. The Ferrand cross-ratio

If a map quantitatively distorts the modulus of path families, then in some situations it follows that the map is quasi-Möbius. A result of this type is the following proposition, which illustrates the importance of the concept of a Loewner space (cf. Remark 4.25 in [15], where a related result is mentioned without proof.)

Proposition 9.1. *Let X and Y be metric spaces, $f\colon X \to Y$ a homeomorphism, and $Q > 1$. Suppose X is a Q-regular Q-Loewner space, Y is Q-regular and LLC, and that there exists a constant $K > 0$ such that*

$$\operatorname{Mod}_Q(\Gamma) \leq K \operatorname{Mod}_Q(f(\Gamma)) \tag{9.2}$$

for every family Γ of paths in X.

Then f is η-quasi-Möbius with η depending only on K and the data of X and Y.

Here $f(\Gamma)$ is the family of all paths $f \circ \gamma$ with $\gamma \in \Gamma$.

Proof. Being a Loewner space, X is λ-LLC with λ depending on the data of X, and in particular connected. Moreover, Y is C_0-doubling with C_0 depending only on the data of Y. So by Lemma 3.3 it is enough to show that if (x_1, x_2, x_3, x_4) is a four-tuple of distinct points in X with $[y_1, y_2, y_3, y_4]$ small, where $y_i = f(x_i)$, then $[x_1, x_2, x_3, x_4]$ is small, quantitatively.

Now if $[y_1, y_2, y_3, y_4]$ is small, then by Lemma 2.10 we can find continua $E', F' \subset Y$ with $y_1, y_3 \in E'$, and $y_2, y_4 \in F'$ such that $\Delta(E', F')$ is large, quantitatively. Let Γ' be the family of all paths in Y joining E' and F', and let Γ be the family of all paths in X joining $E := f^{-1}(E')$ and $F := f^{-1}(F')$. Then $\Gamma' = f(\Gamma)$ and so by our hypotheses we have

$$\operatorname{Mod}_Q(E, F) = \operatorname{Mod}_Q(\Gamma) \leq K \operatorname{Mod}_Q(\Gamma') = K \operatorname{Mod}_Q(E', F').$$

Since Y is Q-regular, we have that

$$\operatorname{Mod}_Q(E', F') \lesssim \frac{1}{(\log(1 + \Delta(E', F')))^{Q-1}}.$$

This is a standard fact following from the upper mass bound for the Hausdorff measure in Y. It can be be established similarly as Proposition 9.9 below. Hence if $\Delta(E', F')$ is large, then $\operatorname{Mod}_Q(E', F')$ and $\operatorname{Mod}_Q(E, F)$ are small, quantitatively. But in a Loewner space, we have

$$\Phi(\Delta(E, F)) \leq \operatorname{Mod}_Q(E, F),$$

where $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a positive decreasing function. It follows that $\Delta(E, F)$ is large, quantitatively. Finally, by Lemma 2.10 again, this means that for the points $x_1, x_3 \in E$ and $x_2, x_4 \in F$ we have that $[x_1, x_2, x_3, x_4]$ is small, quantitatively. □

We will actually not use this proposition, but rather corresponding discrete versions of this result (the closest discrete analog is Proposition 9.8). We included Proposition 9.1 to clarify the basic idea.

The relevant point in the preceding proof was that the cross-ratio of four points can be quantitatively controlled by an appropriate modulus. So suppose X is a metric measure space and let (x_1, x_2, x_3, x_4) be a four-tuple of distinct points. For $Q \geq 1$ define the *Ferrand cross-ratio* of the four points to be

$$[x_1, x_2, x_3, x_4]_Q = \inf \operatorname{Mod}_Q(E, F), \tag{9.3}$$

where the infimum is taken over all continua $E, F \subset X$ with $x_1, x_3 \in E$ and $x_2, x_4 \in F$.

Using Lemma 2.10, it is not hard to see that if X is a Q-regular Q-Loewner space, then the cross-ratio $[x_1, x_2, x_3, x_4]$ is small if and only if the Ferrand cross-ratio $[x_1, x_2, x_3, x_4]_Q$ is small. Moreover, if X is only LLC and Q-regular, then at least one of these implication holds. Namely, if $[x_1, x_2, x_3, x_4]$ is small, then $[x_1, x_2, x_3, x_4]_Q$ is small. The purpose of this section is to establish similar results for vertices in a graph coming from a K-approximation.

Assume $Q \geq 1$ is fixed and let $G = (V, \sim)$ be a connected graph. Imitating the definition of the Ferrand cross-ratio in a metric measure space Z, we define the Ferrand cross-ratio of a four-tuple (v_1, v_2, v_3, v_4) of distinct points in V by

$$[v_1, v_2, v_3, v_4]_Q^G = \inf \operatorname{mod}_Q^G(A, B),$$

where the infimum is taken over all chains $A, B \subset V$ with $v_1, v_3 \in A$ and $v_2, v_4 \in B$. The superscript G will be dropped, if no confusion can arise.

Proposition 9.4. *Let Z be a metric space which is LLC, let $\mathcal{A} = (G, p, r, \mathcal{U})$ be a K-approximation of Z, and $Q \geq 1$. Suppose that there exist a number $L > 0$ and a function $\Psi\colon \mathbb{R}^+ \to (0, \infty]$ with $\lim_{t\to\infty} \Psi(t) = 0$ such that*

$$\mathrm{mod}_Q(V_E, V_F) \leq \Psi(\Delta(E, F)), \tag{9.5}$$

whenever $E, F \subset Z$ are continua not contained in any L-star.

Then there exists a function $\delta\colon \mathbb{R}^+ \to \mathbb{R}^+$ depending only on K, L, Q, Ψ and the data of Z with the following property:

If $\epsilon > 0$ and (v_1, v_2, v_3, v_4) is an arbitrary four-tuple of vertices in G such that $k(v_i, v_j) \geq 2(K + L)$ for $i \neq j$, then we have:

$$[p(v_1), p(v_2), p(v_3), p(v_4)] < \delta(\epsilon) \Rightarrow [v_1, v_2, v_3, v_4]_Q < \epsilon.$$

We will see below (cf. Proposition 9.9) that if Z is LLC and Q'-regular with $Q' \leq Q$, then condition (9.5) is satisfied with $L = K$ and some function Ψ only depending on K and the data of Z (and not on $\mathcal{A}$).

Proof. Let $p_i = p(v_i)$ for $i \in \{1, \dots, 4\}$. Our assumption on the combinatorial separation of the vertices v_i and properties (2) and (3) of a K-approximation imply that the points p_i are distinct. Hence $[p_1, p_2, p_3, p_4]$ is well-defined.

We have to show that if $k(v_i, v_j) \geq 2(K+L)$ for $i \neq j$ and $[p_1, p_2, p_3, p_4]$ is small, then $[v_1, v_2, v_3, v_4]_Q$ is small, quantitatively. If $[p_1, p_2, p_3, p_4]$ is small, then by Lemma 2.10 there exist continua E and F with $p_1, p_3 \in E$, $p_2, p_4 \in F$ and $\Delta(E, F)$ large, quantitatively. Since E is a continuum, we can find a chain $A \subset N_K(V_E)$ connecting $v_1, v_3 \in V_E$. Similarly, we can find a chain $B \subset N_K(V_F)$ connecting $v_2, v_4 \in V_F$. Lemma 7.5 implies that there exists a constant $C = C(K)$ such that

$$\mathrm{mod}_Q(A, B) \leq C \mathrm{mod}_Q(V_E, V_F).$$

The set $E \supset \{p_1, p_3\}$ is not contained in the L-star of any $v \in V$. For if $E \subset \mathrm{St}_L(v)$, then there exist $u_1, u_2 \in V$ with $k(v, u_1) < L$, $k(v, u_2) < L$, $p_1 \in U_{u_1}$, and $p_3 \in U_{u_2}$. But then $p_1 \in U_{u_1} \cap U_{v_1}$ which implies $k(v_1, u_1) < K$ by property (3) of a K-approximation. Similarly, $k(v_3, u_2) < K$. Putting these inequalities together we get $k(v_1, v_3) < 2(K + L)$ which contradicts our assumption on the combinatorial separation of the vertices v_i. In the same way we see that F cannot be contained in any L-star either. Now from our assumption we obtain

$$[v_1, v_2, v_3, v_4]_Q \leq \mathrm{mod}_Q(A, B) \lesssim \mathrm{mod}_Q(V_E, V_F) \leq \Psi(\Delta(E, F)).$$

Since $\Delta(E, F)$ is large and $\Psi(t) \to 0$ as $t \to \infty$, this implies that $[v_1, v_2, v_3, v_4]_Q$ is small, quantitatively. $\square$

Proposition 9.6. *Let Z be a metric space, let $\mathcal{A} = (G, p, r, \mathcal{U})$ be a K-approximation of Z, and $Q \geq 1$. Suppose that there exist a number $M > 0$ and a decreasing positive function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\Phi(\Delta(E, F)) \leq \operatorname{mod}_Q(V_E, V_F), \tag{9.7}$$

whenever $E, F \subset Z$ are continua with $\operatorname{dist}(V_E, V_F) \geq M$.

Then there exists a function $\delta\colon \mathbb{R}^+ \to \mathbb{R}^+$ depending only on K, M, Q, and Φ with the following property:

If $\epsilon > 0$ and (v_1, v_2, v_3, v_4) is an arbitrary four-tuple of vertices in G such that $k(v_i, v_j) \geq K$ for $i \neq j$, then we have:

$$[v_1, v_2, v_3, v_4]_Q < \delta(\epsilon) \Rightarrow [p(v_1), p(v_2), p(v_3), p(v_4)] < \epsilon.$$

It follows from Proposition 8.1 that if Z is a Q-regular Q-Loewner space, then condition (9.7) is satisfied with $M = 4K$ and some function Φ depending only on K and the data of Z (and not on $\mathcal{A}$).

Proof. Let $p_i = p(v_i)$ for $i \in \{1, \dots, 4\}$. Our assumption on the combinatorial separation of the vertices v_i implies that the points p_i are distinct and $[p_1, p_2, p_3, p_4]$ is well-defined.

If $[v_1, v_2, v_3, v_4]_Q$ is small, then there exist chains A, B in G with $v_1, v_3 \in A$ and $v_2, v_4 \in B$ for which $\operatorname{mod}_Q(A, B)$ is small, quantitatively.

We may assume $\operatorname{dist}(A, B) \geq M+4K$. Otherwise, $A' = N_{M+4K}(A)$ and $B' = N_{M+4K}(B)$ have nonempty intersection which by Lemma 7.5 leads to

$$1 \leq \operatorname{mod}_Q(A', B') \leq C(K, M, Q)\operatorname{mod}_Q(A, B).$$

Since A is a chain connecting v_1 and v_3, there are elements u_i in A with $u_1 = v_1 \sim \cdots \sim u_n = v_3$. Then $U_{u_i} \cap U_{u_{i+1}} \neq \emptyset$ and we can find a path $\gamma_i \subset \operatorname{St}_K(u_i) \cup \operatorname{St}_K(u_{i+1})$ connecting $p(u_i)$ and $p(u_{i+1})$ for $i \in \{1, \dots, n-1\}$. The union $E = \gamma_1 \cup \cdots \cup \gamma_{n-1}$ is a continuum joining p_1 and p_3 with

$$E \subset \bigcup_{i=1}^{n} \operatorname{St}_K(u_i).$$

If $u \in V_E$, then $U_u \cap U_w \neq \emptyset$ for some $w \in N_K(A)$. Hence $V_E \subset N_{2K}(A)$. A continuum F in Z connecting p_2 and p_4 with $V_F \subset N_{2K}(B)$ can be constructed in the same way. Then $\operatorname{dist}(V_E, V_F) \geq \operatorname{dist}(A, B) - 4K \geq M$ and so from our hypotheses and Lemma 7.5 we conclude

$$\Phi(\Delta(E, F)) \leq \operatorname{mod}_Q(V_E, V_F) \lesssim \operatorname{mod}_Q(A, B).$$

Since $\operatorname{mod}_Q(A, B)$ is small, we see that $\Delta(E, F)$ is large, quantitatively. Lemma 2.10 implies that $[p_1, p_2, p_3, p_4]$ is small, quantitatively. $\square$

Now we can prove a discrete version of Proposition 9.1.

Proposition 9.8. *Let $Q \geq 1$, and let X and Y be metric spaces with K-approximations $\mathcal{A} = (G, p, r, \mathcal{U})$ and $\mathcal{A}' = (G, p', r', \mathcal{U}')$, respectively, whose underlying graph $G = (V, \sim)$ is the same. Suppose X is connected, and X and $\mathcal{A}$ satisfy condition (9.7) for some $M > 0$ and some function Φ. Suppose Y is LLC and doubling, and Y and $\mathcal{A}'$ satisfy condition (9.5) for some $L > 0$ and some function Ψ. Assume $W \subset V$ is a maximal set of vertices with mutual combinatorial distance at least s, where $s \geq 2(K+L)$. Let $A = p(W)$, $B = p'(W)$ and define*

$$f\colon A \to B, x \mapsto p'(p^{-1}(x)).$$

Then f is η-quasi-Möbius with η depending only on K, Q, L, M, s, Φ, Ψ, and the data of Y (i.e., the parameters in the LLC and doubling conditions).

Since the concept of modulus on a graph is independent of the concept of a K-approximation, the analog of (9.2) in this proposition is the assumption that the underlying graphs of $\mathcal{A}$ and $\mathcal{A}'$ are equal.

By the remarks following Propositions 9.4 and 9.6, this proposition can be applied if $\mathcal{A}$ and $\mathcal{A}'$ are K-approximations of a Q-regular Q-Loewner space X with $Q > 1$ and of a Q'-regular space Y with $Q' \leq Q$, respectively. This special case corresponds to the situation in Proposition 9.1.

Proof. By properties (2) and (3) of a K-approximation, the restrictions $p'|W$ and $p|W$ are injective. Hence f is well-defined and a bijection.

By Lemma 4.7 the set A is weakly λ-uniformly perfect with λ depending only on s and K. Since Y is doubling, the subset B is also doubling, quantitatively. Hence by Lemma 3.3, in order to establish that f is uniformly quasi-Möbius it is enough to show that if (x_1, x_2, x_3, x_4) is a four-tuple of distinct points in A, and $[f(x_1), f(x_2), f(x_3), f(x_4)]$ is small, then $[x_1, x_2, x_3, x_4]$ is small, quantitatively. To see this let $v_i = p^{-1}(x_i) = {p'}^{-1}(f(x_i))$. Then Proposition 9.4 shows that if $[f(x_1), f(x_2), f(x_3), f(x_4)]$ is small, then $[v_1, v_2, v_3, v_4]_Q$ is also small quantitatively. This in turn implies by Proposition 9.6 that $[x_1, x_2, x_3, x_4]$ is small, quantitatively. □

As already mentioned, condition (9.5) is true if $Q > 1$ and Z is Q'-regular with $Q' \leq Q$. This is proved in the following proposition.

Proposition 9.9. *Suppose $Q > 1$ and let (Z, d, μ) be a metric measure space which is LLC and Q'-regular for some $Q' \leq Q$. Let $\mathcal{A} = (G, p, r, \mathcal{U})$ be a K-approximation of Z. Then there exists a function $\Psi\colon \mathbb{R}^+ \to (0, \infty]$ with $\lim_{t\to\infty} \Psi(t) = 0$ depending only on K, Q and the data of Z such that*

$$\operatorname{mod}_Q(V_E, V_F) \leq \Psi(\Delta(E, F)), \tag{9.10}$$

whenever $E, F \subset Z$ are continua not contained in any K-star.

Proof. We may assume $\Delta(E, F) \geq 2$ and $R := \operatorname{diam}(E) \leq \operatorname{diam}(F)$. Fix $z_0 \in E$. Since $\mathcal{A}$ is a K-approximation, we have that

$$|d(z_0, p(u)) - d(z_0, p(v))| \leq C_1 r(u) \quad \text{for} \quad u, v \in V,\ u \sim v, \tag{9.11}$$

where $C_1 = C_1(K)$. If $d(z_0, p(v)) < r(v)$ for some $v \in V$, then $U_v \cap E \neq \emptyset$. Hence $r(v) \leq C_2 \operatorname{diam}(E)$, where $C_2 = C_2(K) > 0$, because E is not contained in $\mathrm{St}_K(v)$. Therefore, there exists $C_3 = C_3(K) > 0$ such that

$$r(v) \leq C_3(R + d(z_0, p(v))) \quad \text{for} \quad v \in V. \tag{9.12}$$

Together with (9.11) this shows that there exists $C_4 = C_4(K) \geq 1$ such that

$$C_4^{-1} \leq \frac{R + d(z_0, p(v))}{R + d(z_0, p(u))} \leq C_4 \quad \text{for} \quad u, v \in V,\ u \sim v. \tag{9.13}$$

Now define $w : V \to \mathbb{R}^+$ as follows. Let

$$w(v) = \frac{r(v)}{\log(\Delta(E, F))(R + d(z_0, p(v)))}$$

if $0 \leq d(z_0, p(v)) \leq R\Delta(E, F)$ and let $w(v) = 0$ otherwise. There exists $N \in \mathbb{N}$ such that

$$2^{N-1} \leq \Delta(E, F) < 2^N. \tag{9.14}$$

Let $B_i := B(z_0, 2^i R)$ for $i \in \{0, \ldots, N\}$ and let $B_{-1} := \emptyset$. By property (2) of a K-approximation and by (9.12) there exist $C_5 > 0$ depending only on the data such that $U_v \subset B(z_0, C_5 2^i R)$ whenever $v \in V$ and $p(v) \in B_i$. Using (9.12) and the Q'-regularity of μ we obtain for the total mass of w

$$\begin{aligned}
\sum_{v \in V} w(v)^Q &\leq \sum_{i=0}^{N} \sum_{p(v) \in B_i \setminus B_{i-1}} w(v)^Q \\
&\lesssim \frac{1}{(\log \Delta(E, F))^Q} \sum_{i=0}^{N} \sum_{p(v) \in B_i \setminus B_{i-1}} \frac{r(v)^{Q'}}{(R + d(z_0, p(v)))^{Q'}} \\
&\lesssim \frac{1}{(\log \Delta(E, F))^Q} \sum_{i=0}^{N} \sum_{p(v) \in B_i} \frac{\mu(U_v)}{2^{iQ'} R^{Q'}} \\
&\lesssim \frac{1}{(\log \Delta(E, F))^Q} \sum_{i=0}^{N} \frac{\mu(B(z_0, C_5 2^i R))}{2^{iQ'} R^{Q'}} \\
&\lesssim \frac{N+1}{(\log \Delta(E, F))^Q} \lesssim \frac{1}{(\log \Delta(E, F))^{Q-1}}.
\end{aligned}$$

In the last inequality we used (9.14) and the fact $\Delta(E, F) \geq 2$.

On the other hand, let $u_1 \sim \cdots \sim u_n$ be an arbitrary chain with $u_1 \in V_E$ and $u_n \in V_F$. Let $d_i := R + d(z_0, p(u_i))$, $i \in \{1, \ldots, n\}$. Then there is a largest number $k \in \mathbb{N}$, $k \le n$, such that $d(z_0, p(u_i)) \le R\Delta(E, F) = \operatorname{dist}(E, F)$ for $i \in \{1, \ldots, k\}$.

We claim $d_k \gtrsim R\Delta(E, F)$. For otherwise, $d(z_0, p(u_k)) < d_k << R\Delta(E, F)$. If $k = n$ this implies $r(u_k) \simeq \operatorname{diam}(U_{u_k}) \gtrsim R\Delta(E, F)$, because U_{u_k} then meets F and contains $p(u_k)$, which is close to E. But $r(u_k) \gtrsim R\Delta(E, F)$ is also true if $k < n$, because then by (9.11) we have $r(u_k) \gtrsim |d_{k+1} - d_k| \simeq d(z_0, p(u_{k+1})) > R\Delta(E, F)$.

Now the inequalities $d(z_0, p(u_k)) << R\Delta(E, F)$ and $r(u_k) \gtrsim R\Delta(E, F)$ are incompatible if $\Delta(E, F)$ is larger than a constant depending on the data, which we may assume. For in this case $E \subset N_{r(u_k)/K}(U_{u_k}) \subset \operatorname{St}_K(u_k)$ which is a contradiction.

Note that since $r(v) \lesssim \operatorname{diam}(E)$ for $v \in V_E$, we have $d_1 \lesssim R$. Hence $\log(d_k/d_1) \gtrsim \log \Delta(E, F)$, and by using (9.11) and (9.13) we arrive at

$$\begin{aligned}\sum_{i=1}^{n} w(v_i) &\ge \frac{1}{\log \Delta(E, F)} \sum_{i=1}^{k} \frac{r(u_i)}{d_i} \\ &\gtrsim \frac{1}{\log \Delta(E, F)} \sum_{i=1}^{k-1} \frac{|d_{i+1} - d_i|}{d_i \wedge d_{i+1}} \\ &\ge \frac{1}{\log \Delta(E, F)} \sum_{i=1}^{k-1} \int_{d_i}^{d_{i+1}} \frac{ds}{s} \\ &= \frac{\log(d_k/d_1)}{\log \Delta(E, F)} \gtrsim 1.\end{aligned}$$

This and the mass bound for w show

$$\operatorname{mod}_Q(V_E, V_F) \lesssim \frac{1}{(\log \Delta(E, F))^{Q-1}}.$$

The assertion follows from this and $Q > 1$. □

In the previous proof we used (9.12) in the second of the inequalities used to derive the mass bound for w. If we do not use (9.12), then the proof actually shows

$$\operatorname{mod}_Q(V_E, V_F) \le \left(\frac{\operatorname{mesh}(\mathcal{A})}{\operatorname{diam}(E) \wedge \operatorname{diam}(F)} \right)^{Q-Q'} \frac{C}{(\log \Delta(E, F))^{Q-1}}, \tag{9.15}$$

where C is a constant depending only on K, Q and the data of Z. This inequality will be useful in the proof of Theorem 1.2.

The goal in the proofs of Theorems 1.1 and 1.2 is the construction of a quasisymmetric map between two spaces. Based on Proposition 9.8 one can prove a general result in this direction if one considers K-approximations of the spaces with mesh size tending to zero.

Proposition 9.16. *Let $Q, K, K' \geq 1$, and let (X, d_X) and (Y, d_Y) be compact metric spaces. Assume that $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ and $\mathcal{A}'_k = (G_k, p'_k, r'_k, \mathcal{U}'_k)$ for $k \in \mathbb{N}$ are K-approximations of X and K'-approximations of Y, respectively, whose underlying graphs $G_k = (V^k, \sim)$ are the same.*

Suppose that X is connected, and that there exist $M > 0$ and some function Φ such that X and $\mathcal{A}_k$ for $k \in \mathbb{N}$ satisfy condition (9.7). Suppose Y is LLC and doubling, and that there exist $L > 0$ and some function Ψ such that Y and $\mathcal{A}'_k$ for $k \in \mathbb{N}$ satisfy condition (9.5).

Finally, suppose that there exist $\lambda > 0$ and vertices $v_1^k, v_2^k, v_3^k \in V^k$ for $k \in \mathbb{N}$ such that

$$d_X\left(p_k\left(v_i^k\right), p_k\left(v_j^k\right)\right) \geq \lambda \operatorname{diam}(X) \text{ and } d_Y\left(p'_k\left(v_i^k\right), p'_k\left(v_j^k\right)\right) \geq \lambda \operatorname{diam}(Y)$$

for $k \in \mathbb{N}$, $i, j \in \{1, 2, 3\}$, $i \neq j$.

If $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{A}_k) = 0$, *then there exists an η_1-quasisymmetric map $f\colon X \to Y$, where η_1 depends only on the data.*

If $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{A}'_k) = 0$, *then there exists an η_2-quasisymmetric map $g\colon Y \to X$, where η_2 depends only on the data.*

The data here consist of K, K', L, M, Q, λ, the functions Φ and Ψ, and the LLC and the doubling constants of Y. Note that we do not claim that f or g are surjective. If both $\operatorname{mesh}(\mathcal{A}'_k) \to 0$ and $\operatorname{mesh}(\mathcal{A}'_k) \to 0$, then the maps f and g can be constructed so that they are inverse to each other. In this case the spaces X and Y are quasisymmetrically equivalent.

The natural question arises what the relation of the conditions $\operatorname{mesh}(\mathcal{A}_k) \to 0$ and $\operatorname{mesh}(\mathcal{A}'_k) \to 0$ is. We will later see (cf. Proposition 11.7) that even under slightly weaker assumptions $\operatorname{mesh}(\mathcal{A}'_k) \to 0$ actually implies $\operatorname{mesh}(\mathcal{A}_k) \to 0$. The other direction is less clear.

We will apply this proposition in the case that X and Y are topological 2-spheres. In this case f and g are forced to be surjective, since a sphere can not be embedded into a proper subset of an another sphere of the same dimension (this fact easily follows from invariance of domain).

Proof. Increasing K or K' to $K \vee K'$, we may assume $K = K'$.

If $\operatorname{mesh}(\mathcal{A}_k) \to 0$ or $\operatorname{mesh}(\mathcal{A}'_k) \to 0$, then the mutual combinatorial distance of the vertices v_1^k, v_2^k, v_3^k becomes arbitrarily large as $k \to \infty$. So if k is sufficiently large, $k \geq k_0$ say, then there exists a maximal $(2K + 2L)$-separated set $W_k \subset V^k$ containing v_1^k, v_2^k, v_3^k. Assume $k \geq k_0$ for the rest of the proof.

Let $A_k := p_k(W_k)$, $B_k := p'_k(W_k)$ and $f_k\colon A_k \to B_k$, $x \mapsto p'_k(p_k^{-1}(x))$. Then by Proposition 9.8, the maps f_k are $\tilde{\eta}_1$-quasi-Möbius with $\tilde{\eta}_1$ depending on the data (and not on k). Hence the inverse maps $g_k = f_k^{-1}\colon B_k \to A_k$ are $\tilde{\eta}_2$-quasi-Möbius with $\tilde{\eta}_2$ depending on the data. Moreover, let $x_i^k := p(v_i^k)$ and $y_i^k := p'_k(v_i^k)$ for $i \in \{1, 2, 3\}$. Then $d_X(x_i^k, x_j^k) \geq \lambda \operatorname{diam}(X)$ and

$d_Y(y_i^k, y_j^k) \geq \lambda \mathrm{diam}(Y)$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, and we have $f_k(x_i^k) = y_i^k$ and $g_k(y_i^k) = x_i^k$.

Every vertex $v \in V^k$ has combinatorial distance at most $2K + 2L$ to the set W_k. Moreover, the sets U_v, $v \in V^k$, form a cover of X. It follows from the properties of a K-approximation that every point in X lies within distance $C(K, L)\mathrm{mesh}(\mathcal{A}_k)$ of the set A_k. So if $\mathrm{mesh}(\mathcal{A}_k) \to 0$, then $\sup_{x \in X} \mathrm{dist}(x, A_k) \to 0$ as $k \to \infty$. In this case the maps f_k subconverge to an $\tilde{\eta}_1$-quasi-Möbius map $f\colon X \to Y$ by Lemma 3.1.

Passing to appropriate subsequences we may assume that $x_i^k \to x_i \in X$ and $y_i^k \to y_i \in Y$ as $k \to \infty$, and $f(x_i) = y_i$ for $i \in \{1, 2, 3\}$. Then $d_X(x_i, x_j) \geq \lambda \mathrm{diam}(X)$ and $d_X(y_i, y_j) \geq \lambda \mathrm{diam}(Y)$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. It follows from remark (4) in Sect. 3 that f is a η_1-quasisymmetric with η_1 depending on λ and $\tilde{\eta}_1$, and hence only on the data.

If $\mathrm{mesh}(\mathcal{A}'_k) \to 0$, then by considering the maps g_k one can construct an η_2-quasisymmetric map $g\colon Y \to X$ with η_2 depending on the data in a similar way.

If both $\mathrm{mesh}(\mathcal{A}_k) \to 0$ and $\mathrm{mesh}(\mathcal{A}'_k) \to 0$, then we first find a subsequence $(f_{k_l})_{l \in \mathbb{N}}$ of the sequence f_k converging to a map f. Then a subsequence of the sequence $(g_{k_l})_{l \in \mathbb{N}}$ will converge to a map g. Then f and g will be quasisymmetries as desired, and we have in addition that f and g are inverse to each other. $\square$

10. The proofs of Theorems 1.1 and 1.2

We will derive our Theorems 1.1 and 1.2 from more general theorems that give necessary and sufficient conditions for a metric 2-sphere to be quasisymmetric to $\mathbb{S}^2$. In Theorems 10.1 and 10.4 we will assume that Z is linearly locally connected and doubling. These conditions are necessary for Z to be quasisymmetric to $\mathbb{S}^2$. Moreover, by Corollary 6.9, a sequence of K-approximations as specified always exists under these necessary a priori assumptions.

Theorem 10.1. *Let Z be metric space homeomorphic to $\mathbb{S}^2$ which is linearly locally connected and doubling. Suppose $K \geq 1$ and $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ for $k \in \mathbb{N}$ are K-approximations of Z whose graphs $G_k = (V^k, \sim)$ are combinatorially equivalent to* 1*-skeletons of triangulations T_k of $\mathbb{S}^2$ and for which*

$$\lim_{k \to \infty} \mathrm{mesh}(\mathcal{A}_k) = 0. \tag{10.2}$$

Suppose there exist numbers $Q \geq 2$, $k_0 \in \mathbb{N}$, $M > 0$, and a positive decreasing function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following property:

If $k \geq k_0$ and $E, F \subset Z$ are continua with $\mathrm{dist}(V_E^k, V_F^k) \geq M$*, then*

$$\Phi(\Delta(E, F)) \leq \mathrm{mod}_Q^{G_k}\left(V_E^k, V_F^k\right). \tag{10.3}$$

Then there exists an η-quasisymmetric homeomorphism $f\colon Z \to \mathbb{S}^2$ with η depending only on the data.

Conversely, if Z is quasisymmetric to $\mathbb{S}^2$, then condition (10.3) for the given sequence $\mathcal{A}_k$ is satisfied for $Q = 2$, some numbers $k_0 \in \mathbb{N}$, $M > 0$, and an appropriate function Φ.

The data in the first part of the theorem are Q, K, M, Φ, and the LLC and doubling constants of Z.

Proof. Fix a triple (z_1, z_2, z_3) of distinct points in Z such that $d(z_i, z_j) \geq \operatorname{diam}(Z)/2$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Since $\operatorname{mesh}(\mathcal{A}_k) \to 0$, for sufficiently large k, say $k \geq k_1 \geq k_0$, we can find $v_i^k \in V^k$ such that for $x_i^k := p_k(v_i^k)$ we have $d(z_i, x_i^k) < \operatorname{diam}(Z)/8$ for $i \in \{1, 2, 3\}$. Then $d(x_i^k, x_j^k) \geq \operatorname{diam}(Z)/4$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Assume $k \geq k_1$ for the rest of the proof.

The triangulation T_k can be realized as a circle packing on $\mathbb{S}^2$ (Sect. 5). We normalize the circle packing so that the vertices v_1^k, v_2^k, v_3^k correspond to points y_1, y_2, y_3 in $\mathbb{S}^2$ equally spaced on some great circle. The circle packings induce canonical K'-approximations $\mathcal{A}'_k = (G_k, p'_k, r'_k, \mathcal{U}'_k)$ of $\mathbb{S}^2$, where K' depends only on the valence of G_k and hence only on K. Then $p'_k(v_i^k) = y_i$ and so the vertices v_i^k satisfy the condition in Proposition 9.16, where λ is a numerical constant.

Since $\mathbb{S}^2$ is LLC and 2-regular, and $Q \geq 2$, we see by Proposition 9.9 that condition (9.5) is true for the space $\mathbb{S}^2$ and the K'-approximations $\mathcal{A}'_k$ with $L = K'$ and a uniform function Ψ independent of k. Therefore, the hypotheses of Proposition 9.16 are satisfied for $X = Z$, $Y = \mathbb{S}^2$ and our sequence of approximations. We conclude that there exists an η-quasisymmetry $f\colon Z \to \mathbb{S}^2$ where η depends only on the data. Since Z is a topological sphere, this embedding has to be surjective and is hence a homeomorphism.

Conversely, assume that there exists an η-quasisymmetry $f\colon Z \to \mathbb{S}^2$. Since (10.2) implies the condition (4.2) in Lemma 4.1 for sufficiently large k, say for $k \geq k_0$, we can use the quasisymmetric images of the K-approximations $\mathcal{A}_k$ as in Lemma 4.1 to obtain K'-approximations $\mathcal{A}'_k = (G_k, p'_k, r'_k, \mathcal{U}'_k)$ of $\mathbb{S}^2$. Here K' depends only on K and η.

Since $\mathbb{S}^2$ is a 2-regular 2-Loewner space, by Proposition 8.1 condition (9.7) is true for the space $\mathbb{S}^2$ and the K'-approximations $\mathcal{A}'_k$ with $Q = 2$, the constant $M = 4K'$ and a function Φ' independent of k.

Now let $k \geq k_0$, and suppose that $E, F \subset Z$ are continua such that $\operatorname{dist}(V_E^k, V_E^k) \geq M$. The underlying graphs of $\mathcal{A}_k$ and $\mathcal{A}'_k$ are the same. Moreover, the combinatorics of the covers $\mathcal{U}_k$ and $\mathcal{U}'_k$ correspond under the mapping f. This shows that for $E' = f(E)$ and $F' = f(F)$ we have $V_E^k = V_{E'}^k$, $V_F^k = V_{F'}^k$, and $\operatorname{dist}(V_E^k, V_F^k) = \operatorname{dist}(V_{E'}^k, V_{F'}^k) \geq M$, where the sets V_E^k, et cetera, are interpreted with respect to the appropriate approximations. Hence we get

$$\Phi'(\Delta(E', F')) \leq \operatorname{mod}_2^{G_k}\left(V_{E'}^k, V_{F'}^k\right) = \operatorname{mod}_2^{G_k}\left(V_E^k, V_F^k\right).$$

Condition (10.3) for an appropriate function Φ independent of k will follow from this, if we can show that $\Delta(E, F)$ is large if and only if $\Delta(E', F')$ is large, quantitatively. But this last statement follows from the quasisymmetry of f and the discussion after Lemma 3.2. □

As an immediate application of this theorem we get a proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose Z is Q-regular and Q-Loewner for $Q \geq 2$. Then Z is LLC and doubling. Corollary 6.9 shows that there exist $K \geq 1$ and a sequence of K-approximations $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ whose graphs $G_k = (V^k, \sim)$ are combinatorially equivalent to 1-skeletons of triangulations T_k of Z and for which (10.2) is true. Now the Q-regularity of Z, Proposition 8.1, and the Q-Loewner property of Z show that condition (9.7) is true for the K-approximations $\mathcal{A}_k$ with $M = 4K$ and a function Φ independent of k. Theorem 10.1 implies that there exists a quasisymmetric homeomorphism $f : Z \to \mathbb{S}^2$. A result by Tyson [34] shows that if a Q-regular Q-Loewner space is quasisymmetrically mapped onto a Q'-regular space, then $Q' \geq Q$. But $\mathbb{S}^2$ is 2-regular, and so we can apply this for $Q' = 2$ and get $2 \geq Q$. Since also $Q \geq Q' = 2$ by assumption, we must have $Q = 2$. The proof of Theorem 1.2 is complete. □

It may be worthwhile to point out that in the previous proof an argument can be given that avoids invoking Tyson's theorem.

Suppose Z is Q-regular Q-Loewner space and $f : Z \to \mathbb{S}^2$ a quasisymmetric homeomorphism. Let $\mathcal{A}_k$ be a sequence of K-approximations of Z with underlying graphs $G_k = (V^k, \sim)$ such that $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{A}_k) = 0$. Let $\mathcal{A}'_k$ be the K'-approximation of $\mathbb{S}^2$ obtained as the image of $\mathcal{A}_k$ under f. Then $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{A}'_k) = 0$. Let $E, F \subset Z$ be two disjoint continua and $E' := f(E)$, $F' := f(F)$. Then by Proposition 8.1 and by the remark following the proof of Proposition 9.9 we have for sufficiently large k

$$\begin{aligned} \Phi(\Delta(E, F)) \leq \operatorname{Mod}_Q(E, F) &\lesssim \operatorname{mod}_Q^{G_k}\left(V_E^k, V_F^k\right) = \operatorname{mod}_Q^{G_k}\left(V_{E'}^k, V_{F'}^k\right) \\ &\lesssim \left(\frac{\operatorname{mesh}(\mathcal{A}'_k)}{\operatorname{diam}(E') \wedge \operatorname{diam}(F')}\right)^{Q-2} \frac{1}{(\log \Delta(E', F'))^{Q-1}}. \end{aligned}$$

Here Φ is a positive function provided by the Q-Loewner property of Z. Moreover, the multiplicative constants implicit in this inequality are independent of E, F and k. Note that the additional assumptions on the combinatorial separation in Propositions 8.1 and 9.9 are true for our continua if k is sufficiently large. If $Q > 2$ then the last term in the inequality tends to zero, since the mesh size tends to zero. But this is impossible, since the first term is independent of k and positive. Hence $Q = 2$.

Theorem 10.4. *Let Z be metric space homeomorphic to $\mathbb{S}^2$ which is linearly locally connected and doubling. Suppose $K \geq 1$ and $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$*

for $k \in \mathbb{N}$ are K-approximations of Z whose graphs $G_k = (V^k, \sim)$ are combinatorially equivalent to 1-skeletons of triangulations T_k of $\mathbb{S}^2$ and for which

$$\lim_{k\to\infty} \mathrm{mesh}(\mathcal{A}_k) = 0. \tag{10.5}$$

Suppose that there exist numbers $k_0 \in \mathbb{N}$, $L > 0$, and a function $\Psi\colon \mathbb{R}^+ \to (0, \infty]$ with $\lim_{t\to\infty} \Psi(t) = 0$ satisfying the following property:

If $k \geq k_0$ and $E, F \subset Z$ are continua not contained in any L-star of $\mathcal{A}_k$, then

$$\mathrm{mod}_2^{G_k}\left(V_E^k, V_F^k\right) \leq \Psi(\Delta(E, F)). \tag{10.6}$$

Then there exists an η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$ with η depending only on the data.

Conversely, if Z is quasisymmetric to $\mathbb{S}^2$, then condition (10.6) for the given sequence $\mathcal{A}_k$ is satisfied for some numbers $k_0 \in \mathbb{N}$, $L > 0$, and an appropriate function Ψ.

The data in the first part of the theorem are K, L, Ψ, and the LLC and doubling constants of Z.

Proof. The proof of this theorem is very similar to the proof of Theorem 10.1. For the sufficiency part note again that the triangulation T_k can be realized as a normalized circle packing on $\mathbb{S}^2$. The circle packings induce canonical K'-approximations $\mathcal{A}'_k = (G_k, p'_k, r'_k, \mathcal{U}'_k)$ of $\mathbb{S}^2$, where K' depends only on K.

As in the proof of Theorem 10.1, for sufficiently large k we can find vertices $v_1^k, v_2^k, v_3^k \in V^k$ satisfying the condition in Proposition 9.16 where $\lambda > 0$ is a numerical constant. Since $\mathbb{S}^2$ is 2-regular and 2-Loewner, Proposition 8.1 implies that condition (9.7) is true for the space $\mathbb{S}^2$ and the K'-approximations $\mathcal{A}'_k$ with $M = 4K'$ and a function Φ independent of k.

It follows that the hypotheses of Proposition 9.16 are satisfied for $X = \mathbb{S}^2$ and the K'-approximations $\mathcal{A}'_k$ and $Y = Z$ and the K-approximations $\mathcal{A}_k$. (Note that the roles of $\mathcal{A}_k$ and $\mathcal{A}'_k$ in this proof and in Proposition 9.16 are reversed). Since $\mathrm{mesh}(\mathcal{A}_k) \to 0$ it follows that there exists an η-quasisymmetry $g\colon Z \to \mathbb{S}^2$ where η depends only on the data. Again g has to be a homeomorphism.

For the converse assume that there exists an η-quasisymmetry $g\colon Z \to \mathbb{S}^2$. Again for sufficiently large k we obtain K'-approximations $\mathcal{A}'_k$ of $\mathbb{S}^2$ with $K' = K'(\eta, K)$ as the quasisymmetric images under g of the K-approximations $\mathcal{A}_k$. The sphere $\mathbb{S}^2$ is 2-regular, so by Proposition 9.9 we have condition (9.5) for $Q = 2$, $L := K'$ and an appropriate function Ψ' independent of k. Now suppose E, F are continua not contained in any L-star with respect to $\mathcal{A}_k$. We have $\mathcal{A}'_k\text{-}\mathrm{St}_L(v) = g(\mathcal{A}_k\text{-}\mathrm{St}_L(v))$. This implies that

$E' = g(E)$ and $F' = g(F)$ are not contained in any L-star with respect to $\mathcal{A}'_k$. Hence

$$\operatorname{mod}_2^{G_k}\left(V_E^k, V_F^k\right) = \operatorname{mod}_2^{G_k}\left(V_{E'}^k, V_{F'}^k\right) \leq \Psi'(\Delta(E', F')).$$

Now $\Delta(E', F')$ is large if and only if $\Delta(E, F)$ is large, quantitatively. Hence condition (10.6) follows with $L = K'$, and an appropriate function Ψ independent of k. □

Proof of Theorem 1.1. As we remarked in the introduction, only the sufficiency part of Theorem 1.1 demands a proof. Since linear local contractibility implies linear local connectivity quantitatively for topological 2-spheres, we can assume that Z is LLC. We will show that there exists an η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$, where η depends only on the data. Here we call the LLC constant, and the constant that enters the condition for 2-regularity (where $\mu = \mathcal{H}^2$) the data of Z.

Note that Z is doubling with a constant only depending on the data. Corollary 6.9 shows that there exist $K \geq 1$ depending on the data and a sequence of K-approximations $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ whose graphs $G_k = (V^k, \sim)$ are combinatorially equivalent to 1-skeletons of triangulations T_k of Z and for which condition (10.5) is true. Since Z is LLC and 2-regular, Proposition 9.9 shows that the condition (10.6) is true for $L = K$ and an appropriate function Ψ depending on the data. Now Theorem 10.4 shows that there exists a η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$, where η depends only on the data. □

Theorem 1.1 is quantitative as the proof above shows. Namely, if Z is a metric space homeomorphic to $\mathbb{S}^2$ that is Ahlfors 2-regular and LLC, then there exists an η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$, where η depends only on the data, i.e., the constants in the Ahlfors 2-regularity and the LLC conditions. Conversely, if Z is a metric space for which there exists an η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$, then Z is λ-LLC with λ only depending on η.

11. Asymptotic conditions

Cannon's paper [7] provides a framework that allows one to speak of modulus for subsets of a topological space. A shingling $\mathcal{S}$ of a topological space Z is a locally finite cover consisting of compact connected subsets of Z. When Z is homeomorphic to $\mathbb{S}^2$ and $R \subset Z$ is an annulus, Cannon defines invariants $M(\mathcal{S}, R)$ and $m(\mathcal{S}, R)$ which are combinatorial analogs for the classical moduli of annuli. He then studies a sequence of shinglings $\mathcal{S}_j$ of Z with mesh size tending to zero. His main theorem—the combinatorial Riemann mapping theorem—is a necessary and sufficient condition for the existence of a homeomorphism $f : Z \to \mathbb{S}^2$ such that for every annulus $R \subset Z$, the moduli $M(\mathcal{S}_j, R)$ and $m(\mathcal{S}_j, R)$ agree with the standard 2-modulus of $f(R)$ to within a fixed multiplicative factor, for sufficiently large j.

The combinatorial Riemann mapping theorem is similar in spirit to Theorems 10.1 and 10.4: all three results give necessary and sufficient conditions for a "conformally flavored" structure on the 2-sphere to be equivalent to the standard structure modulo a homeomorphism.

Any of these theorems can be used to give necessary and sufficient conditions for a Gromov hyperbolic group to admit a discrete, cocompact, and isometric action on hyperbolic space $\mathbb{H}^3$. The paper [11] uses [7] and [30, Corollary, p. 468] to give such conditions; the conditions in [11] are in turn applied in [10]. Our Theorems 10.1 or 10.4 can be combined directly with Sullivan's theorem. The point here is that the action $G \curvearrowright \partial_\infty G$ of a non-elementary hyperbolic group on its boundary is by uniformly quasi-Möbius homeomorphisms, and if one conjugates this action by a quasisymmetric homeomorphism $\partial_\infty G \to \mathbb{S}^2$, the resulting action $G \curvearrowright \mathbb{S}^2$ is also uniformly quasi-Möbius, in particular uniformly quasiconformal, so that [30] may be applied.

On the other hand, there are significant differences between our approach and Cannon's approach. Cannon's hypotheses and conclusions do not involve metric information, and only relate to the limiting behavior of the combinatorial moduli. In contrast, Theorems 10.1 and 10.4 hypothesize inequalities between relative distance (which is metric based) and combinatorial modulus which hold uniformly for every K-approximation in the given sequence; and they assert that the metric space is quasisymmetric to $\mathbb{S}^2$, which is a metric conclusion.

The interesting parts of Theorems 10.1 and 10.4 are the sufficient conditions. An upper bound for a modulus is easier to establish than a lower bound, because for a lower bound an inequality for the total mass of *all* admissible test functions has to be shown whereas an upper bound already follows from a mass bound for *one* test function. In this respect, Theorem 10.4 seems to be more useful, because its hypotheses require upper modulus bounds. In view of Cannon's work it seems worthwhile to find a sufficient condition in the spirit of Theorem 10.4 that works with an asymptotic condition for the graph modulus as in (10.6). The following theorem provides such a result where we further weaken the requirements for which sets E and F an asymptotic modulus inequality has to hold.

Theorem 11.1. *Let Z be a metric space homeomorphic to $\mathbb{S}^2$ which is linearly locally connected and doubling. Suppose $K \geq 1$, and $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ for $k \in \mathbb{N}$ are K-approximations of Z whose graphs $G_k = (V^k, \sim)$ are combinatorially equivalent to 1-skeletons of triangulations T_k of $\mathbb{S}^2$ and for which*

$$\lim_{k\to\infty} \operatorname{mesh}(\mathcal{A}_k) = 0. \tag{11.2}$$

Suppose there exist numbers $C > 0$ and $\lambda > 1$ with the following property: If $B = B(a, r)$ and $\lambda B = B(a, \lambda r)$ are balls in Z, then we have

$$\limsup_{k\to\infty} \operatorname{mod}_2^{G_k}\left(V^k_B, V^k_{Z\setminus\lambda B}\right) < C. \tag{11.3}$$

Then there exists an η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$ with η depending only on the data.

Conversely, if Z is quasisymmetric to $\mathbb{S}^2$, then there exist $C > 0$ and $\lambda > 1$ such that condition (11.3) is satisfied for the given sequence $\mathcal{A}_k$.

The data are K, C, λ, the LLC constant, and the doubling constant.

If B is a ball in Z and $\lambda > 1$, let A be the "annulus" $A = \lambda B \setminus B$. The 2-modulus of A can be defined as the 2-modulus of the path family Γ joining the disjoint sets B and $Z \setminus \lambda B$. The appropriate combinatorial version of this modulus with respect to the K-approximation $\mathcal{A}_k$ is $\mathrm{mod}_2^{G_k}(V_B^k, V_{Z\setminus\lambda B}^k)$ which appears in (11.3). So this inequality essentially says that the combinatorial analog of the 2-modulus of A is asymptotically bounded above by a fixed constant.

We now formulate a version of Theorem 11.1 which does depend on the language of K-approximations.

Corollary 11.4. *Let Z be a doubling, linearly locally connected metric space homeomorphic to $\mathbb{S}^2$. Suppose $r_k > 0$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} r_k = 0$, and for each $k \in \mathbb{N}$, $\hat{V}_k \subset Z$ is a maximal r_k-separated set. We let $\hat{G}_k$ be the incidence graph of the cover $\{B(v, r_k)\}_{v\in\hat{V}_k}$, and for each subset $A \subset Z$ we set $\hat{V}_A^k := \{v \in \hat{V}_k : A \cap B(v, r_k) \neq \emptyset\}$. Then Z is quasisymmetric to $\mathbb{S}^2$ if and only if there exist constants $C > 0$ and $\lambda > 1$ with the following property: if $B = B(a, r)$ and $\lambda B = B(a, \lambda r)$ are balls in Z, then we have*

$$\limsup_{k\to\infty} \mathrm{mod}_2^{\hat{G}_k}\left(\hat{V}_B^k, \hat{V}_{\lambda B}^k\right) < C.$$

Proof. We give a proof, omitting some technical details.

By applying Proposition 6.7 to the r_k-separated subset $\hat{V}_k \subset Z$, one obtains a K-approximation $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$, where $G_k = (V_k, \sim)$ and $\hat{V}_k \subset V_k$. It follows readily from properties (iii)–(iv) of Proposition 6.7 that there are constants $C_1, C_2 > 0$ independent of k such that for all k the inclusion $\hat{V}_k \to V_k$ is C_1-bilipschitz onto its image (with respect to the combinatorial distances in the graphs $\hat{G}_k$ and G_k respectively), and every $v \in V_k$ is within combinatorial distance at most C_2 from a vertex in $\hat{V}_k$. Using this and the fact that the graphs $\hat{G}_k$ and G_k have uniformly bounded valence, one easily checks that for all pairs of subsets $E, F \subset Z$, the quantities $\limsup_{k\to\infty} \mathrm{mod}_2^{G_k}(V_E^k, V_F^k)$ and $\limsup_{k\to\infty} \mathrm{mod}_2^{\hat{G}_k}(\hat{V}_E^k, \hat{V}_F^k)$ are quantitatively equivalent. Hence the corollary reduces to Theorem 11.1. $\square$

In order to prove Theorem 11.1 we have to revisit some of the material in Sect. 9 and prove asymptotic versions. The next proposition should be compared with Proposition 9.4.

Proposition 11.5. *Let Z be a locally compact metric space which is λ-LLC, $\lambda \geq 1$. Suppose $K \geq 1$, and $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ for $k \in \mathbb{N}$*

are K-approximations of Z with graphs $G_k = (V^k, \sim)$. Assume that $\mathrm{mesh}(\mathcal{A}_k) \to 0$ *as* $k \to \infty$.

Let $Q \geq 1$, and suppose that there exists a function $\Psi\colon \mathbb{R}^+ \to (0, \infty]$ *with* $\lim_{t\to\infty} \Psi(t) = 0$ *such that*

$$\limsup_{k\to\infty} \mathrm{mod}_Q^{G_k}\left(V_E^k, V_F^k\right) \leq \Psi(\Delta(E, F)), \tag{11.6}$$

whenever $E, F \subset Z$ are disjoint continua.

Then there exists a function $\phi\colon \mathbb{R}_0^+ \to [0, \infty]$ *with* $\lim_{t\to 0}\phi(t) = \phi(0) = 0$ *depending only on K, Q, Ψ and the data of Z with the following property:*

Suppose (z_1, z_2, z_3, z_4) is a four-tuple of points in Z with $\{z_1, z_3\} \cap \{z_2, z_4\} = \emptyset$, *and assume that for $k \in \mathbb{N}$ and $i \in \{1, 2, 3, 4\}$ we have vertices $v_i^k \in V^k$ such that $p_k(v_i^k) \to z_i$ for $k \to \infty$, $i \in \{1, 2, 3, 4\}$. Then*

$$\limsup_{k\to\infty} \left[v_1^k, v_2^k, v_3^k, v_4^k\right]_Q^{G_k} \leq \phi([z_1, z_2, z_3, z_4]).$$

We want to allow the possibility $z_1 = z_3$ or $z_2 = z_4$ here. In this case we set $[z_1, z_2, z_3, z_4] = 0$, which is a consistent extension of the definition of the cross-ratio. Note that $[v_1^k, v_2^k, v_3^k, v_4^k]_Q^{G_k}$ is a cross-ratio with respect to G_k. The proposition says that if $[z_1, z_2, z_3, z_4]$ is small, then $[v_1^k, v_2^k, v_3^k, v_4^k]_Q^{G_k}$ is asymptotically small, quantitatively.

Proof. If $[z_1, z_2, z_3, z_4]$ is small, then by Lemma 2.10 there exist continua E' and F' with $z_1, z_3 \in E$, $z_2, z_4 \in F$ and $\Delta(E', F')$ large, quantitatively. If $z_1 = z_3$ or $z_2 = z_4$ then $\Delta(E', F')$ can be made arbitrarily large. Since Z is locally compact and LLC and hence locally connected, we can find compact connected neighborhoods E and F of E' and F', respectively, such that $\Delta(E, F)$ is large, quantitatively. Since $\mathrm{mesh}(\mathcal{A}_k) \to 0$ we will have $p_k(v_1^k) \in U_{v_1^k} \cap E$ and $p_k(v_3^k) \in U_{v_3^k} \cap E$ for large k. In particular, $v_1^k, v_3^k \in V_E^k$. Similarly, $v_2^k, v_4^k \in V_F^k$ for large k. The rest of the proof now proceeds as the proof of Proposition 9.4. For large k we can find chains $A_k \subset N_K(V_E^k)$ connecting v_1^k, v_3^k and chains $B_k \subset N_K(V_F^k)$ connecting v_2^k, v_4^k. Then by Lemma 7.5 we have

$$\left[v_1^k, v_2^k, v_3^k, v_4^k\right]_Q^{G_k} \leq \mathrm{mod}_Q^{G_k}(A_k, B_k) \leq C(K)\mathrm{mod}_Q^{G_k}\left(V_E^k, V_F^k\right).$$

So our assumptions imply

$$\limsup_{k\to\infty} \left[v_1^k, v_2^k, v_2^k, v_3^k\right]_Q^{G_k} \leq C(K)\Psi(\Delta(E, F)).$$

Since $\Delta(E, F)$ is large and $\Psi(t) \to 0$ as $t \to \infty$ we get the desired quantitative conclusion. ☐

The following proposition corresponds to one of the parts of Proposition 9.16. We have replaced condition (9.5) by the asymptotic condition (11.6).

Proposition 11.7. *Let $Q, K, K' \geq 1$, and let (X, d_Y) and (Y, d_Y) be compact metric spaces. Assume that $\mathcal{A}_k = (G_k, p_k, r_k, \mathcal{U}_k)$ and $\mathcal{A}'_k = (G_k, p'_k, r'_k, \mathcal{U}'_k)$ for $k \in \mathbb{N}$ are K-approximations of X and K'-approximations of Y, respectively, whose underlying graphs $G_k = (V^k, \sim)$ are the same.*

Suppose X is connected, and there exist $M > 0$ and some function Φ such that X and $\mathcal{A}_k$ for $k \in \mathbb{N}$ satisfy condition (9.7). Suppose Y is LLC and doubling, and Y and $\mathcal{A}'_k$ satisfy condition (11.6) for some function Ψ.

Suppose that there are vertices $v_1^k, v_2^k, v_3^k \in V^k$ for $k \in \mathbb{N}$ such that for some constant $\lambda > 0$ we have

$$d_X\left(p_k\left(v_i^k\right), p_k\left(v_j^k\right)\right) \geq \lambda \operatorname{diam}(X) \text{ and } d_Y\left(p'_k\left(v_i^k\right), p'_k\left(v_j^k\right)\right) \geq \lambda \operatorname{diam}(Y)$$

for $k \in \mathbb{N}$, $i, j \in \{1, 2, 3\}$, $i \neq j$.

If $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{A}'_k) = 0$, then there exists an η-quasisymmetric map $f\colon X \to Y$, where η depends only on the data.

The data here consist of K, K', Q, M, λ, the functions Φ and Ψ, and the LLC and the doubling constants of Y.

In the proof we will show that $\operatorname{mesh}(\mathcal{A}_k) \to 0$. Since condition (9.5) is stronger than condition (11.6), this justifies the remark after Proposition 9.16. Namely, that that under the assumptions of this proposition we have that $\operatorname{mesh}(\mathcal{A}'_k) \to 0$ implies $\operatorname{mesh}(\mathcal{A}_k) \to 0$.

Proof. 1. In this proof we will call distortion functions those functions $\phi\colon \mathbb{R}_0^+ \to [0, \infty]$ for which $\phi(t) \to \phi(0) = 0$ as $t \to 0$. We will first establish the existence of a distortion function ϕ_1 depending on the data with the following property. If $z_1, z_3 \in X$, $w_1, w_3 \in Y$, $u_1^k, u_3^k \in V^k$ for $k \in \mathbb{N}$, and $p_k(u_i^k) \to z_i$ and $p'_k(u_i^k) \to w_i$ as $k \to \infty$ for $i \in \{1, 3\}$, then

$$\frac{d_X(z_1, z_3)}{\operatorname{diam}(X)} \leq \phi_1\left(\frac{d_Y(w_1, w_3)}{\operatorname{diam}(Y)}\right). \tag{11.8}$$

To prove this we may assume $d_Y(w_1, w_3) < (\lambda/3)\operatorname{diam}(Y)$. Hence if $w_i^k := p'_k(u_i^k)$ for $i \in \{1, 3\}$ we have $d_Y(w_1^k, w_3^k) < (\lambda/3)\operatorname{diam}(Y)$ for large k. For such k there will be at least two among the vertices v_1^k, v_2^k, v_3^k, call them u_2^k and u_4^k, such that we have $\operatorname{dist}(\{w_1^k, w_3^k\}, \{w_2^k, w_4^k\}) \geq (\lambda/3)\operatorname{diam}(Y)$, where we set $w_i^k = p'_k(u_i^k)$ also for $i \in \{2, 4\}$. Then for large k we obtain

$$\left[w_1^k, w_2^k, w_3^k, w_4^k\right] \leq C(\lambda)\frac{d_Y\left(w_1^k, w_3^k\right)}{\operatorname{diam}(Y)}.$$

We may assume that we have limits $w_2^k \to w_2$ and $w_4^k \to w_4$ for $k \to \infty$. Then $\{w_1, w_3\} \cap \{w_2, w_4\} = \emptyset$, and so Proposition 11.5 and the previous inequality show that there exist distortion functions ϕ_2 and ϕ_3 depending on the data such that

$$\limsup_{k\to\infty} \left[u_1^k, u_2^k, u_3^k, u_4^k\right]_Q^{G_k} \leq \phi_2([w_1, w_2, w_3, w_4]) \leq \phi_3\left(\frac{d_Y(w_1, w_3)}{\operatorname{diam}(Y)}\right).$$

Since $\operatorname{mesh}(\mathcal{A}'_k) \to 0$ as $k \to \infty$ and among the points w_1, w_2, w_3, w_4 only w_1 and w_3 can be identical, the combinatorial separation of any two of the vertices $u_1^k, u_2^k, u_3^k, u_4^k$ becomes arbitrarily large as $k \to \infty$ with the possible exception of u_1^k and u_3^k. We make the momentary extra assumption that the combinatorial separation of u_1^k and u_3^k is at least K for large k. Let $z_i^k = p_k(u_i^k)$. Note that $d_X(z_2^k, z_4^k) \geq (\lambda/2)\operatorname{diam}(X)$ for large k by choice of u_2^k and u_4^k. Then from Proposition 9.6 we infer that for sufficiently large k

$$\frac{d_X\left(z_1^k, z_3^k\right)}{\operatorname{diam}(X)} \leq C(\lambda)\left[z_1^k, z_2^k, z_3^k, z_4^k\right] \leq \phi_4\left(\left[u_1^k, u_2^k, u_3^k, u_4^k\right]_Q^{G_k}\right),$$

where ϕ_4 is a distortion function depending on the data. Letting k tend to infinity, the claim (11.8) follows under the additional assumption on the combinatorial separation of u_1^k and u_3^k.

2. In order to establish the general case of (11.8), we first show that $\operatorname{mesh}(\mathcal{A}_k) \to 0$ as $k \to \infty$. Arguing by contradiction and passing to a subsequence if necessary, we may assume there there exists $\delta > 0$ and $a_1^k \in V^k$ with $r_k(a_1^k) \geq \delta > 0$ for $k \in \mathbb{N}$. Since the mesh size of $\mathcal{A}'_k$ tends to 0, the cardinality of G_k tends to infinity. Moreover, G_k is connected and its valence is uniformly bounded. Thus, for sufficiently large k we can find a vertex $a_3^k \in V^k$ with $K \leq k_{G_k}(a_1^k, a_3^k) \leq 2K$. Then $U_{a_1^k} \cap U_{a_3^k} = \emptyset$ and it follows $d_X(p_k(a_1^k), p_k(a_3^k)) \geq r_k(a_1^k) \geq \delta$. Letting $x_i^k := p_k(a_i^k)$ and $y_i^k := p'_k(a_i^k)$ and passing to subsequences, we may assume that $x_i^k \to x_i$ and $y_i^k \to y_i$ for $k \to \infty$, $i \in \{1, 3\}$. Then $d_X(x_1, x_3) \geq \delta > 0$. On the other hand, $y_1 = y_3$, since the combinatorial distance of a_1^k and a_3^k is uniformly bounded by choice of a_3^k, and the mesh size of $\mathcal{A}'_k$ tends to zero. But the combinatorial distance of a_1^k and a_3^k was at least K for large k, so we can apply (11.8) and get a contradiction.

3. Once we know that the mesh size of $\mathcal{A}_k$ tends to zero, we can verify (11.8) without the additional assumption on the combinatorial separation of u_1^k and u_3^k. For if $z_1 = z_3$, then there is nothing to prove. If $z_1 \neq z_3$, then the combinatorial distance of u_1^k and u_3^k becomes arbitrarily large, since $\operatorname{mesh}(\mathcal{A}_k) \to 0$ as $k \to \infty$.

4. Let A be a countable dense subset of X. For $z \in A$ and $k \in \mathbb{N}$ we can find $u_k(z) \in V^k$ with $z \in U_{u_k(z)}$. Since $\operatorname{mesh}(\mathcal{A}_k) \to 0$, we have $p_k(u_k(z)) \to z$ as $k \to \infty$, $z \in A$. Define $f_k(z) := p'_k(u_k(z))$. By passing to successive subsequences and taking a final "diagonal subsequence" we may assume that the countably many sequences $(f_k(z))_{k \in \mathbb{N}}$, $z \in A$, converge, $f_k(z) \to f(z)$ say, as $k \to \infty$. From (11.8) and this definition of f, we get (11.8) for arbitrary $z_1, z_3 \in A$ and $w_1 = f(z_1)$ and $w_3 = f(z_3)$. In particular, $f\colon A \to Y$ is injective.

5. We claim that the map f is $\tilde{\eta}$-quasi-Möbius with $\tilde{\eta}$ only depending on the data. To see this note that as a dense subset of connected metric space, the set A is weakly λ'-uniformly perfect with a fixed constant, $\lambda' = 3$ say.

Since Y is doubling, the subset $f(A)$ is also doubling, quantitatively. Hence by Lemma 3.3, in order to establish that f is uniformly quasi-Möbius it is enough to show that if (x_1, x_2, x_3, x_4) is a four-tuple of distinct points in A, and $[f(x_1), f(x_2), f(x_3), f(x_4)]$ is small, then $[x_1, x_2, x_3, x_4]$ is small, quantitatively. By definition of f, we can find $u_i^k \in V^k$ such $x_i \in U_{u_i^k}$ and $p'_k(u_i^k) \to y_i := f(x_i)$ for $k \to \infty$, $i \in \{1, \ldots, 4\}$. Then Proposition 11.5 shows that if $[y_1, y_2, y_3, y_4]$ is small, then $\limsup_{k\to\infty}[u_1^k, u_2^k, u_3^k, u_4^k]_Q^{G_k}$ is also small, quantitatively. Since the points y_i are distinct, the combinatorial separation of the vertices u_i^k is arbitrarily large for $k \to \infty$. This implies by Proposition 9.6 that $[p_k(u_1^k), p_k(u_2^k), p_k(u_3^k), p_k(u_4^k)]$ for large k is small, quantitatively. Passing to the limit we conclude that

$$[x_1, x_2, x_3, x_4] = \lim_{k\to\infty} \left[p_k\left(u_1^k\right), p_k\left(u_2^k\right), p_k\left(u_3^k\right), p_k\left(u_4^k\right)\right]$$

is small, quantitatively.

6. There are points z_1, z_2, z_3 in A whose mutual distance is at least $\operatorname{diam}(X)/4$. The estimate (11.8) and the definition of f show that the mutual distance of the points $f(z_1)$, $f(z_2)$, $f(z_3)$ is bounded below by $c\operatorname{diam}(Y)$, where $c > 0$ is a constant depending on the data. Hence $f\colon A \to Y$ is η-quasisymmetric with η depending on the data. Since A is dense and Y is compact, there is a unique extension of f to an η-quasisymmetric map on X (cf. (5) in Sect. 3). Calling this map also f, we get the desired quasisymmetry. □

Proof of Theorem 11.1. To prove sufficiency, we want to apply Proposition 11.7 for $Q = 2$, $X = \mathbb{S}^2$ and $Y = Z$. As in the proof of Theorem 10.1 one can realize the triangulations T_k as normalized circle packings. The circle packings induce canonical K'-approximations $\mathcal{A}'_k = (G_k, p'_k, r'_k, \mathcal{U}'_k)$ of $\mathbb{S}^2$, where K' depends only on K. Again as in the proof of Theorem 10.1 we can use suitable normalizations so that for sufficiently large k we can find vertices $v_1^k, v_2^k, v_3^k \in V^k$ satisfying the condition in Proposition 11.7 where $\lambda > 0$ is a numerical constant. Since $\mathbb{S}^2$ is 2-regular and 2-Loewner, Proposition 8.1 implies that condition (9.7) is true for the space $X = \mathbb{S}^2$ and the K'-approximations $\mathcal{A}'_k$ with $M = 4K'$ and a function Φ independent of k.

Since $\operatorname{mesh}(\mathcal{A}_k) \to 0$ the only thing that remains to be verified is that with $Y = Z$, the K-approximations $\mathcal{A}_k$ satisfy the asymptotic condition (11.6) for some function Ψ depending on the data.

To see that this is true, let E and F be arbitrary disjoint continua. We have to show that the combinatorial modulus $\operatorname{mod}_2^{G_k}(V_E^k, V_E^k)$ for large k is small if the relative distance of E and F is large, quantitatively.

We may assume $\operatorname{diam}(E) \leq \operatorname{diam}(F)$. Pick $a \in E$, let $r = 2\operatorname{diam}(E)$ and $B_i := B(a, \lambda^{2i-2}r)$ for $i \in \mathbb{N}$. Then $E \subset B_1$ and $B_i \subset \lambda B_i \subset \lambda^2 B_i = B_{i+1}$

for $i \in \mathbb{N}$. Let N be the largest integer such that $r\lambda^{2N-1} < \operatorname{dist}(E, F)$. Note that N is large if and only if $\Delta(E, F)$ is large, quantitatively. Then

$$E \subset B_1 \subset \lambda B_1 \subset \lambda^2 B_1 = B_2 \subset \lambda B_2 \subset \ldots \subset B_N \subset \lambda B_N \subset Z \setminus F.$$

Since $\operatorname{mesh}(\mathcal{A}_k) \to 0$, there exists $k_1 \in \mathbb{N}$ such that if $k \geq k_1$ and $v \in V^k_{\lambda B_i}$ for some $i \in \{1, \ldots, N-1\}$, then $v \notin V^k_{Z \setminus B_{i+1}}$. For suppose $v \in V^k_{\lambda B_i} \cap V^k_{Z \setminus B_{i+1}}$. Then $U_v \cap \lambda B_i \neq \emptyset$ and $U_v \cap (Z \setminus B_{i+1}) \neq \emptyset$. Hence $2Kr_v \geq \operatorname{diam}(U_v) \geq \lambda^{2i}(1 - 1/\lambda)r \geq (1 - 1/\lambda)r$. This is impossible if $\operatorname{mesh}(\mathcal{A}_k)$ is small enough.

By our hypothesis we can find $k_2 \in \mathbb{N}$ such that for $k \geq k_2$ and $i \in \{1, \ldots, N\}$ we have $\operatorname{mod}_2^{G_k}(V^k_{B_i}, V^k_{Z \setminus \lambda B_i}) < C$. Consider a fixed K-approximation $\mathcal{A}_k$ for $k \geq k_3 := k_1 \vee k_2$. To simplify notation we drop the sub- or superscript k.

By our assumption on k, there exists a weight $w_i \colon V \to [0, \infty)$ which is admissible for the pair $(V_{B_i}, V_{Z \setminus \lambda B_i})$ and satisfies

$$\sum_{v \in V} w_i(v)^2 < C.$$

Define $w(v) := \sup_{i \in \{1,\ldots,N\}} w_i(v)$ for $v \in V$. Then

$$\sum_{v \in V} w(v)^2 \leq \sum_{i=1}^{N} \sum_{v \in V} w_i(v)^2 \leq NC. \tag{11.9}$$

Now let $v_1 \sim \cdots \sim v_l$ be a chain connecting V_E and V_F. For $i \in \{1, \ldots, N\}$ let m_i be the largest index with $v_{m_i} \in V_{B_i}$. Since $v_1 \in V_E \subset V_{B_i}$ the number m_i is well defined. Moreover, $m_i \leq m_{i+1}$. Let m'_i be the smallest index $\geq m_i$ with $v_{m'_i} \in V_{Z \setminus \lambda B_i}$. Note that m'_i is well defined since $v_l \in V_F \subset V_{Z \setminus \lambda B_i}$. Then $v_{m_i} \sim \cdots \sim v_{m'_i}$ is a chain connecting V_{B_i} and $V_{Z \setminus \lambda B_i}$ and we obtain from the admissibility of w_i

$$\sum_{\nu = m_i}^{m'_i} w_i(v_\nu) \geq 1.$$

We claim that the index sets $\{m_i, \ldots, m'_i\}$ for $i \in \{1, \ldots, N\}$ are pairwise disjoint. To see this let $i \in \{1, \ldots, N-1\}$ and $j := m'_i$. Assume $m_i < m'_i$. Then $v_{j-1} \notin V_{Z \setminus \lambda B_i}$ by definition of m'_i. This means $U_{v_{j-1}} \subset \lambda B_i$. Then $\emptyset \neq U_{v_{j-1}} \cap U_{v_j} \subset \lambda B_i \cap U_{v_j}$, and so $v_j \in V_{\lambda B_i}$. This is also true if $m'_i = m_i$. By our assumption on k and the choice of k_1, we have $v_j \notin V_{Z \setminus B_{i+1}}$ which implies $j < l$ and $U_{v_j} \subset B_{i+1}$. Therefore, we have that $\emptyset \neq U_{v_j} \cap U_{v_{j+1}} \subset B_{i+1} \cap U_{v_{j+1}}$. Thus $v_{j+1} \in V_{B_{i+1}}$ and we conclude $m_{i+1} \geq j + 1 > m'_i$. The claim follows from this and we get

$$\sum_{\nu=1}^{l} w(v_\nu) \geq \sum_{i=1}^{N} \sum_{\nu = m_i}^{m'_i} w_i(v_\nu) \geq N.$$

We conclude that w/N is admissible for the pair (V_E, V_F), and so by (11.9) we have

$$\operatorname{mod}_2(V_E, V_F) \leq C/N.$$

Returning to the usual notation, this means that $\operatorname{mod}_2^{G_k}(V_E^k, V_F^k)$ is small for $k \geq k_3$, if $\Delta(E, F)$ is large, quantitatively.

Proposition 11.7 now shows that there exists an $\tilde{\eta}$-quasisymmetric map $f\colon \mathbb{S}^2 \to Z$, where $\tilde{\eta}$ depends only on the data. This map has to be a homeomorphism. Its inverse map will be an η-quasisymmetric homeomorphism $g\colon Z \to \mathbb{S}^2$, where η depends only on the data.

Conversely, suppose that Z is quasisymmetric to $\mathbb{S}^2$. Assume that Z is λ_0-LLC, where $\lambda_0 > 1$. By Theorem 10.4 condition (10.6) will be satisfied for $L > 0, k_0 \in \mathbb{N}$, and a suitable function Ψ. We can find $t_0 > 0$ and $C > 0$ such that $\Psi(t) < C$ for $t \geq t_0$. Let $\lambda := (2t_0 + 1)\lambda_0^2 > 1$. Suppose $B = B(a, r)$ is a ball in Z. From λ_0-LLC_1 it follows that there exists a continuum E with $B \subset E \subset \bar{B}(a, \lambda_0 r)$. Moreover, assume that $Z \setminus \lambda B \neq \emptyset$. Then λ_0-LLC_2 implies that there exists a continuum F with $Z \setminus \lambda B \subset F \subset Z \setminus B(a, \lambda r/\lambda_0)$. We have $\Delta(E, F) \geq (\lambda - \lambda_0^2)/(2\lambda_0^2) = t_0$. Since $\operatorname{mesh}(\mathcal{A}_k) \to 0$, we have that E and F are not contained in any L-star of $\mathcal{A}_k$ for sufficiently large k. It follows that for large k we have

$$\operatorname{mod}_2^{G_k}\left(V_B^k, V_{Z\setminus\lambda B}^k\right) \leq \operatorname{mod}_2^{G_k}\left(V_E^k, V_F^k\right) < C.$$

If $Z \setminus \lambda B = \emptyset$, then $\operatorname{mod}_2^{G_k}(V_B^k, V_{Z\setminus\lambda B}^k) = 0$ by definition of the modulus. In any case we see that condition (11.3) is satisfied. $\square$

12. Concluding remarks

(1) Theorems similar to Theorem 1.1 are true for more general surfaces. In the case when Z is homeomorphic to $\mathbb{R}^2$ the following statement holds:

Let Z be an Ahlfors 2-regular complete metric space homeomorphic to $\mathbb{R}^2$. Then Z is quasisymmetric to $\mathbb{R}^2$ (equipped with the standard Euclidean metric) if and only if Z is linearly locally connected.

(2) Theorem 1.1 can be used to give a canonical model for 2-regular 2-spheres that are linearly locally contractible. To make this precise we remind the reader of the concept of a deformation of a metric space (Z, d) by a metric doubling measure. Suppose μ is a Borel measure on Z. The measure is called *doubling* if there exists a constant $C \geq 1$ such that

$$\mu(B(a, 2r)) \leq C\mu(B(a, r)),$$

whenever $a \in Z$ and $r > 0$. If $x, y \in Z$ let $B_{xy} := B(x, d(x, y)) \cup B(y, d(x, y))$. Suppose $Q \geq 1$ is fixed. Then we introduce a function $\delta_\mu(x, y) := \mu(B_{x,y})^{1/Q}$. The measure μ is called a *metric doubling measure* (with exponent Q) if δ_μ is a metric up to a bounded multiplicative constant, i.e., there exists a metric δ on Z and a constant $C \geq 1$ such that

$$(1/C)\delta(x, y) \leq \delta_\mu(x, y) \leq C\delta(x, y) \quad \text{for} \quad x, y \in Z.$$

Suppose μ is a metric doubling measure. As long as an ambiguity caused by a multiplicative constant is harmless, the distance function δ_μ is as good as a metric and we can talk about the metric space (Z, δ_μ) and quasisymmetric maps of this space etc. It is easy to see that the "metric space" (Z, δ_μ) is Ahlfors Q-regular and quasisymmetric to (Z, d) by the identity map.

If $Z = \mathbb{S}^n$ and $Q = n \geq 2$, then every metric doubling measure μ is absolutely continuous with respect to spherical measure σ_n, i.e., there exists a measurable weight $w\colon \mathbb{S}^n \to [0, \infty]$ such that $d\mu = w\, d\sigma_n$. The weight is an A_∞-weight. Weights that arise from metric doubling measures in this way are called strong A_∞-weights.

Theorem 1.1 now implies the following statement:

A metric 2-sphere (Z, d) is Ahlfors 2-regular and linearly locally contractible if and only if (Z, d) is bilipschitz to a space $(\mathbb{S}^2, \delta_\mu)$, where μ is a metric doubling measure on $\mathbb{S}^2$ with exponent $Q = 2$.

Indeed, if (Z, d) is Ahlfors 2-regular and linearly locally contractible, then there exists a quasisymmetric homeomorphism $f\colon \mathbb{S}^2 \to Z$ by Theorem 1.1. Define the measure μ on $\mathbb{S}^2$ as the pull-back of $\mathcal{H}^2$ by f. So $\mu(E) = \mathcal{H}^2(f(E))$ for a Borel set $E \subset \mathbb{S}^2$. Using the fact that f is quasisymmetric and that Z is 2-regular, it easy to see that μ is doubling. Moreover, we have $\delta_\mu(x, y) \simeq d(f(x), f(y))$ for $x, y \in \mathbb{S}^2$. This shows that μ is a metric doubling measure, and that $f\colon (\mathbb{S}^2, \delta_\mu) \to (Z, d)$ is bilipschitz.

Conversely, if μ is a metric doubling measure on $\mathbb{S}^2$ with exponent $Q = 2$, then $(\mathbb{S}^2, \delta_\mu)$ is 2-regular. Hence (Z, d) is also 2-regular, because this property is preserved under bilipschitz maps. Since (Z, d) is bilipschitz to $(\mathbb{S}^2, \delta_\mu)$ and the latter space is quasisymmetric to $\mathbb{S}^2$ by the identity map, the spaces (Z, d) and $\mathbb{S}^2$ are quasisymmetric. Linear local contractibility is invariant under quasisymmetries, and since $\mathbb{S}^2$ has this property, so does (Z, d).

(3) A necessary condition for a metric 2-sphere Z to be bilipschitz to $\mathbb{S}^2$ is that Z is 2-regular and linearly locally contractible. By the result in (2) a space satisfying these necessary conditions is bilipschitz to a space $(\mathbb{S}^2, \delta_\mu)$, where μ is a metric doubling measure on $\mathbb{S}^2$ with exponent 2. So the problem of characterizing $\mathbb{S}^2$ up to bilipschitz equivalence is reduced to the question which of the spaces $(\mathbb{S}^2, \delta_\mu)$ are bilipschitz to $\mathbb{S}^2$.

This question is related to the Jacobian problem for quasiconformal mappings on $\mathbb{S}^2$ as follows. If $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a quasiconformal map, we denote by J_f its Jacobian (determinant). The Jacobian problem for quasiconformal maps asks for a characterization of the weights $w\colon \mathbb{S}^2 \to [0, \infty]$ for which there exists a quasiconformal map $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$(1/C)J_f(x) \leq w(x) \leq CJ_f(x) \quad \text{for} \quad \sigma_2\text{-a.e. } x \in \mathbb{S}^2,$$

where C is a constant independent of x. A necessary and sufficient condition for a weight w to be comparable to a Jacobian of a quasiconformal map is that w is a strong A_∞-weight, i.e., the measure μ defined by $d\mu = w\,d\sigma_2$ is a metric doubling measure, and that $(\mathbb{S}^2, \delta_\mu)$ is bilipschitzly equivalent to $\mathbb{S}^2$ (cf. [25]).

From this we see that the Jacobian problem for quasiconformal mappings on $\mathbb{S}^2$ is equivalent with the problem of characterizing $\mathbb{S}^2$ up to bilipschitz equivalence.

(4) The usefulness of Theorem 11.1 depends on whether one can verify its hypotheses in concrete situations. There are some interesting fractal spaces of Hausdorff dimension greater than 2 where this can be done. For example, consider the space $Z \subset \mathbb{R}^3$ obtained as follows. The space Z will be the limit of a sequence of two-dimensional cell complexes Z_n. Each Z_n consists of a union of congruent oriented squares. The orientation of each square is visualized by specifying which of the two directions perpendicular to the square is considered as normal. The sets Z_n are inductively constructed as follows. The cell complex Z_0 is the boundary of the unit cube $I^3 \subset \mathbb{R}^3$, where the 2-cells are the six squares forming the faces of Z_0. We orient the squares of Z_0 by assigning to them the normal pointing outward I^3. Now Z_{n+1} is obtained from Z_n by modifying each of the oriented squares S forming Z_n as follows. Subdivide S into 25 congruent subsquares with the induced orientation. (Actually any fixed number $(2k+1)^2$ with $k \geq 2$ could be taken here. In the case $k = 1$ there are some problems with overlaps in the inductive construction.) On the "central" subsquare S' of S place an appropriately sized cube C in the normal direction so that one of the faces of C agrees with S'. The face squares of C are oriented so that their normals point outward C. The desired modification of S is now obtained by replacing the "central" subsquare S' of S by the oriented faces of C different from S' and keeping all other oriented subsquares. In this way each square of Z_n leads to $24 + 5 = 29$ squares of Z_{n+1}. The limit set Z is equipped with the ambient metric of $\mathbb{R}^3$. It can be shown that Z is homeomorphic to $\mathbb{S}^2$ and Q-regular for some $Q > 2$. Using the symmetry properties of Z and Theorem 11.1, one can show: *Z is quasisymmetric to $\mathbb{S}^2$*. An independent proof of this fact based on the dynamics of rational functions is due to D. Meyer [21].

We hope to explore applications of Theorem 11.1 more systematically in the future.

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