

## QUASITORIC MANIFOLDS OVER A PRODUCT OF SIMPLICES

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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### Abstract

A quasitoric manifold (resp. a small cover) is a  $2n$ -dimensional (resp. an  $n$ -dimensional) smooth closed manifold with an effective locally standard action of  $(S^1)^n$  (resp.  $(\mathbb{Z}_2)^n$ ) whose orbit space is combinatorially an  $n$ -dimensional simple convex polytope  $P$ . In this paper we study them when  $P$  is a product of simplices. A generalized Bott tower over  $\mathbb{F}$ , where  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{R}$ , is a sequence of projective bundles of the Whitney sum of  $\mathbb{F}$ -line bundles starting with a point. Each stage of the tower over  $\mathbb{F}$ , which we call a generalized Bott manifold, provides an example of quasitoric manifolds (when  $\mathbb{F} = \mathbb{C}$ ) and small covers (when  $\mathbb{F} = \mathbb{R}$ ) over a product of simplices. It turns out that every small cover over a product of simplices is equivalent (in the sense of Davis and Januszkiewicz [5]) to a generalized Bott manifold. But this is not the case for quasitoric manifolds and we show that a quasitoric manifold over a product of simplices is equivalent to a generalized Bott manifold if and only if it admits an almost complex structure left invariant under the action. Finally, we show that a quasitoric manifold  $M$  over a product of simplices is homeomorphic to a generalized Bott manifold if  $M$  has the same cohomology ring as a product of complex projective spaces with  $\mathbb{Q}$  coefficients.

### 1. Introduction

Toric varieties in algebraic geometry and Hamiltonian torus actions on symplectic manifolds exhibit fascinating relations between the geometry of algebraic varieties or smooth manifolds and the combinatorics of their orbit spaces. Considering the success of toric theory, it is natural to generalize them to the topological category, and a monumental development in this direction was obtained by the work of Davis and Januszkiewicz in [5]. They defined a topological generalization of toric variety by the name of “toric manifold”, which is a  $2n$ -dimensional closed manifold  $M$  with a locally standard action of  $n$ -torus  $G = (S^1)^n$  whose orbit space is combinatorially an  $n$ -dimensional simple convex polytope  $P$ . In this case  $M$  is said to be a “toric manifold” over  $P$ . They also defined a  $\mathbb{Z}_2$ -analogue of a “toric manifold” called a small cover, which is an  $n$ -dimensional man-

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ifold with an effective action of the  $\mathbb{Z}_2$ -torus of rank  $n$  with an  $n$ -dimensional simple polytope as the orbit space.

Unfortunately the term “toric manifolds” is already well-established among algebraic geometers as “non-singular toric variety”. Moreover there are “toric manifolds” (in the sense of Davis and Januszkiewicz) which are not algebraic varieties, for example  $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ . Because of this reason Buchstaber and Panov introduced the term “quasitoric manifold” as an alias for Davis and Januszkiewicz’s “toric manifold” in [1]. In this paper we adopt Buchstaber and Panov’s “quasitoric manifold” instead of “toric manifold”. We refer the reader to Chapter 5 of [1] for an excellent exposition on quasitoric manifolds including their comparison with (compact non-singular) toric varieties.

This paper is motivated by the work [10] which investigates quasitoric manifold over a cube. A cube is a product of 1-simplices. We take a product of simplices as the simple polytope  $P$  and describe quasitoric manifolds and small covers over  $P$  in terms of matrices with vectors as entries. A typical example of quasitoric manifolds or small covers over a product of simplices appears in a sequence of projective bundles

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where  $B_i$  for  $i = 1, \dots, m$  is the projectivization of the Whitney sum of  $n_i + 1$   $\mathbb{F}$ -line bundles over  $B_{i-1}$  ( $\mathbb{F} = \mathbb{C}$  or  $\mathbb{R}$ ). Grossberg–Karshon [7] considered the sequence above when  $\mathbb{F} = \mathbb{C}$  and  $n_i = 1$  for any  $i$ , and they named it a *Bott tower*. Motivated by this, we call the sequence above a *generalized Bott tower* (over  $\mathbb{F}$ ). The  $j$ -stage  $B_j$  of the tower provides a quasitoric manifold (when  $\mathbb{F} = \mathbb{C}$ ) and a small cover (when  $\mathbb{F} = \mathbb{R}$ ) over  $\prod_{i=1}^j \Delta^{n_i}$  where  $\Delta^{n_i}$  is the  $n_i$ -simplex. We call each  $B_j$  a *generalized Bott manifold* (over  $\mathbb{F}$ ) and especially call it a *Bott manifold* when the tower is a Bott tower. It turns out that any small cover over a product of simplices is equivalent (in particular, homeomorphic) to a generalized Bott manifold (over  $\mathbb{R}$ ) (see Remark 6.5) but this is not the case for quasitoric manifolds. We give a necessary and sufficient condition for a quasitoric manifold over a product of simplices to be equivalent to a generalized Bott manifold (over  $\mathbb{C}$ ) (see Theorem 6.4), where a part of the statement is a particular case of [6, Theorem 6].

This paper is organized as follows. In Section 2 we recall general facts on quasitoric manifolds and small covers over a simple polytope. From Section 3 we restrict our concern to a product of simplices as the simple polytope and treat only quasitoric manifolds because small covers can be treated similarly. In Section 3 we introduce some notation needed for later discussion and associate a matrix with vectors as entries to a quasitoric manifold over a product of simplices. In Section 4 we describe quasitoric manifolds over a product of simplices as the orbit space of a product of odd dimensional spheres by some free torus action. This is done in [7] and [4] when the orbit space is a product of 1-simplices, that is, a cube. The association of the matrix with vectors as entries to a quasitoric manifold over a product of simplices depends on the order of the product of the simplices. We discuss this in Section 5. Generalized Bott

towers are introduced in Section 6 and generalized Bott manifolds are characterized among quasitoric manifolds over a product of simplices (Theorem 6.4). In Section 7 we explicitly describe the cohomology ring of a quasitoric manifold over a product of simplices and prove in Section 8 that such a quasitoric manifold is homeomorphic to a generalized Bott manifold if it has the same cohomology ring as a product of complex projective spaces with  $\mathbb{Q}$  coefficients.

## 2. General facts

An  $n$ -dimensional convex polytope  $P$  is said to be *simple* if precisely  $n$  facets (namely codimension-one faces of  $P$ ) meet at each vertex. Equivalently,  $P$  is simple if the dual of the boundary complex  $\partial P$  of  $P$  is a simplicial complex. It is clear that every simplex is simple and a product of simple convex polytopes is simple. Therefore a product of simplices is simple.

Let  $d = 1$  or  $2$ . We denote by  $S_d$  an order two group  $S^0$  when  $d = 1$  and a circle group  $S^1$  when  $d = 2$ , and by  $G_d$  a group isomorphic to  $(S_d)^n$ . A  $dn$ -dimensional smooth  $G_d$ -manifold  $M_d$  with a projection  $\pi: M_d \rightarrow P$  is called a *small cover* (when  $d = 1$ ) and a *quasitoric manifold* (when  $d = 2$ ) over an  $n$ -dimensional simple convex polytope  $P$  if  $M_d$  is locally isomorphic to a faithful real  $dn$ -dimensional representation of  $G_d$  and each fiber of  $\pi$  is a  $G_d$ -orbit. The orbit space  $M_d/G_d$  can be identified with  $P$ . Two quasitoric manifolds or small covers  $\pi: M_d \rightarrow P$  and  $\pi': M'_d \rightarrow P$  are *equivalent* (in the sense of Davis and Januszkiewicz) if there is a homeomorphism  $f: M_d \rightarrow M'_d$  covering the identity on  $P$  and an automorphism  $\theta: G_d \rightarrow G_d$  such that  $f$  satisfies  $\theta$ -equivariance, i.e.,  $f(gm) = \theta(g)f(m)$  for all  $m \in M_d$  and  $g \in G_d$ . Note that the equivalence is neither weaker nor stronger than  $G_d$ -homeomorphism, because any  $G_d$ -homeomorphism must satisfy  $\theta$ -equivariance with  $\theta = \text{id}$ , but it may not cover the identity on the orbit space.

Let  $\pi: M_d \rightarrow P$  be a small cover or a quasitoric manifold and let  $\mathcal{F}$  be the set of facets of  $P$ . If  $F \in \mathcal{F}$ , then the isotropy subgroup of a point  $x \in \pi^{-1}(\text{int } F)$  is independent of the choice of  $x$ , and is a rank-one subgroup  $G_d(F)$  of  $G_d$ . The group  $\text{Hom}(S_d, G_d)$  of homomorphisms from  $S_d$  to  $G_d$  is isomorphic to  $(R_d)^n$  where  $R_d$  is  $\mathbb{Z}/2$  when  $d = 1$  and  $\mathbb{Z}$  when  $d = 2$ . Each rank-one subgroup of  $G_d$  corresponds uniquely (up to sign) to a primitive vector of  $\text{Hom}(S_d, G_d)$  which generates a rank-one direct summand of  $\text{Hom}(S_d, G_d)$ . Therefore every  $M_d$  defines what is called the *characteristic function* of  $M_d$

$$\lambda: \mathcal{F} \rightarrow \text{Hom}(S_d, G_d)$$

such that the image of  $F \in \mathcal{F}$  is a primitive vector of  $\text{Hom}(S_d, G_d)$  corresponding to the rank-one subgroup  $G_d(F)$ . When  $d = 1$ , such a primitive vector is unique for each  $F$ , but sign ambiguity arises when  $d = 2$ . This sign ambiguity can be resolved if an orientation (see [1]) is assigned to a quasitoric manifold  $M_d$ , in particular if  $M_d$  admits an almost complex structure left invariant under the action (see Lemma 1.5 and 1.10

of [9]). In any case, the characteristic function  $\lambda$  of  $M_d$  must satisfy the following condition, see [5].

CONDITION 2.1. If  $n$  facets  $F_1, \dots, F_n$  of  $P$  intersect at a vertex, then their images  $\lambda(F_1), \dots, \lambda(F_n)$  must form a basis of  $\text{Hom}(S_d, G_d)$ .

Conversely, for a function  $\lambda: \mathcal{F} \rightarrow \text{Hom}(S_d, G_d)$  satisfying Condition 2.1, there exists a unique (up to equivalence) small cover (when  $d = 1$ ) and quasitoric manifold (when  $d = 2$ ) with  $\lambda$  as the characteristic function, see [5] or [2] for details. Therefore in order to classify all small covers or quasitoric manifolds over a simple convex polytope  $P$ , it is necessary and sufficient to understand the functions  $\lambda$  satisfying Condition 2.1.

Let  $F_1, \dots, F_k$  be the all facets of  $P$  and let  $\omega_1, \dots, \omega_k$  be the indeterminates corresponding to the facets. Then it is shown in [5] that the equivariant cohomology ring  $H_{G_d}^*(M_d; R_d)$  is the face ring (or the Stanley–Reisner ring) of  $P$  with  $R_d$  coefficient as graded rings, that is,

$$(2.1) \quad H_{G_d}^*(M_d; R_d) = R_d[\omega_1, \dots, \omega_k]/I,$$

where the degree of  $\omega_i$  is  $d$  for each  $i$  and  $I$  is the homogeneous ideal of the polynomial ring  $R_d[\omega_1, \dots, \omega_k]$  generated by all square-free monomials of the form  $\omega_{i_1} \cdots \omega_{i_s}$  such that the intersection of the corresponding facets  $F_{i_1}, \dots, F_{i_s}$  is empty.

We choose a basis of  $\text{Hom}(S_d, G_d)$  and identify  $\text{Hom}(S_d, G_d)$  with  $(R_d)^n$ . We form a  $k \times n$  matrix whose  $i$ -th row is  $\lambda(F_i) \in (R_d)^n$ , i.e.,

$$(2.2) \quad (\lambda_{ij}) = \begin{pmatrix} \lambda(F_1) \\ \vdots \\ \lambda(F_k) \end{pmatrix}.$$

Let  $\lambda_j = \lambda_{1j}\omega_1 + \cdots + \lambda_{kj}\omega_k$ , and let  $J$  be the ideal of  $R_d[\omega_1, \dots, \omega_k]$  generated by  $\lambda_j$  for  $j = 1, \dots, n$ . Then we have

$$(2.3) \quad H^*(M_d; R_d) = R_d[\omega_1, \dots, \omega_k]/(I + J).$$

REMARK 2.2. In general it would be natural to use a *column* vector to express  $\lambda(F_i)$  (see [1]), but then, as noticed in [10], we need to take a transpose of a matrix at some point to adjust our description to the notation used in [4] and [7]. Therefore we will use a *row* vector to express  $\lambda(F_i)$  in this paper.

As is seen above, most of the arguments for quasitoric manifolds work for small covers with  $S^1$  and  $\mathbb{Z}$  replaced by  $S^0$  and  $\mathbb{Z}/2$  respectively. In fact, the study of small covers is a bit simpler than that of quasitoric manifolds in our case. So we shall treat

only quasitoric manifolds throughout this paper. The main difference between quasitoric manifolds and small covers in our arguments is stated in Remark 6.5, so that the arguments after Section 7 are unnecessary for small covers.

### 3. Vector matrices

From now on, we take

$$P = \prod_{i=1}^m \Delta^{n_i}, \quad \text{with} \quad \sum_{i=1}^m n_i = n,$$

where  $\Delta^{n_i}$  is the  $n_i$ -simplex for  $i = 1, \dots, m$ . Let  $\{v_0^i, \dots, v_{n_i}^i\}$  be the set of vertices of the simplex  $\Delta^{n_i}$ . Then each vertex of  $P$  is the product of vertices of  $\Delta^{n_i}$ 's for  $i = 1, \dots, m$ , hence the set of vertices of  $P$  is

$$\{v_{j_1 \dots j_m} = v_{j_1}^1 \times \dots \times v_{j_m}^m \mid 0 \leq j_i \leq n_i\}.$$

Each facet of  $P$  is the product of a codimension-one face of one of  $\Delta^{n_i}$ 's and the remaining simplices. Therefore the set of facets of  $P$  is

$$\mathcal{F} = \{F_{k_i}^i \mid 0 \leq k_i \leq n_i, i = 1, \dots, m\}$$

where  $F_{k_i}^i = \Delta^{n_1} \times \dots \times \Delta^{n_{i-1}} \times f_{k_i}^i \times \Delta^{n_{i+1}} \times \dots \times \Delta^{n_m}$ , and  $f_{k_i}^i$  is the codimension-one face of the simplex  $\Delta^{n_i}$  which is opposite to the vertex  $v_{k_i}^i$ . Hence there are  $\sum_{i=1}^m (n_i + 1) = n + m$  facets in  $P$ . Since  $P$  is simple, exactly  $n$  facets meet at each vertex. Indeed, at each vertex  $v_{j_1 \dots j_m}$  of  $P$  all  $n$  facets in  $\mathcal{F} - \{F_{j_i}^i \mid i = 1, \dots, m\}$  intersect, in particular, the  $n$  facets in the set

$$\mathcal{F} - \{F_0^i \mid i = 0, \dots, m\} = \{F_1^1, \dots, F_{n_1}^1, \dots, F_1^m, \dots, F_{n_m}^m\}$$

intersect at the vertex  $v_{0 \dots 0}$ .

Let  $\lambda: \mathcal{F} \rightarrow \text{Hom}(S^1, (S^1)^n)$  be the characteristic function of a quasitoric manifold over  $P$ . By Condition 2.1,  $n$  vectors

$$(3.1) \quad \lambda(F_1^1), \dots, \lambda(F_{n_1}^1), \dots, \lambda(F_1^m), \dots, \lambda(F_{n_m}^m)$$

form a basis of  $\text{Hom}(S^1, (S^1)^n)$  and we identify  $\text{Hom}(S^1, (S^1)^n)$  with  $\mathbb{Z}^n$  through this basis. Then the vectors in (3.1) correspond to the standard basis elements

$$\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1)$$

in the given order. For the remaining  $m$  facets  $F_0^i$ , we set

$$\lambda(F_0^i) = \mathbf{a}_i \in \mathbb{Z}^n \quad \text{for} \quad i = 1, \dots, m.$$

In this way, to the characteristic function  $\lambda$  of a quasitoric manifold over  $P$  we have a corresponding  $m \times n$  matrix

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}, \quad \text{where } \mathbf{a}_i \in \mathbb{Z}^n.$$

Each row vector  $\mathbf{a}_i$  can be written as

$$\begin{aligned} \mathbf{a}_i &= (\mathbf{a}_i^1, \dots, \mathbf{a}_i^j, \dots, \mathbf{a}_i^m) \\ &= ([a_{i1}^1, \dots, a_{in_1}^1], \dots, [a_{i1}^j, \dots, a_{in_j}^j], \dots, [a_{i1}^m, \dots, a_{in_m}^m]) \end{aligned}$$

where  $\mathbf{a}_i^j = [a_{i1}^j, \dots, a_{in_j}^j] \in \mathbb{Z}^{n_j}$  for  $j = 1, \dots, m$ . Therefore we may write

$$\begin{aligned} (3.2) \quad A &= \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^1 & \cdots & \mathbf{a}_1^m \\ \vdots & \cdots & \vdots \\ \mathbf{a}_m^1 & \cdots & \mathbf{a}_m^m \end{pmatrix} \\ &= \begin{pmatrix} a_{11}^1 & \cdots & a_{1n_1}^1 & \cdots & a_{11}^m & \cdots & a_{1n_m}^m \\ \vdots & & & & & & \vdots \\ a_{m1}^1 & \cdots & a_{mn_1}^1 & \cdots & a_{m1}^m & \cdots & a_{mn_m}^m \end{pmatrix} \end{aligned}$$

with  $\mathbf{a}_i^j \in \mathbb{Z}^{n_j}$  for all  $i = 1, \dots, m$ . In other words, the  $m \times n$  matrix  $A$  can be viewed as an  $m \times m$  matrix whose entries in the  $j$ -th column are vectors in  $\mathbb{Z}^{n_j}$ . From now on, we shall view the matrix  $A$  this way and call it a *vector matrix*.

Since the characteristic function  $\lambda$  satisfies Condition 2.1, we need to translate this into a condition on the corresponding matrix  $A$ . For this let us fix some more notation. For given  $1 \leq k_j \leq n_j$  with  $j = 1, \dots, m$ , let  $A_{k_1 \cdots k_m}$  be the  $m \times m$  submatrix of  $A$  whose  $j$ -th column is the  $k_j$ -th column of the  $m \times n_j$  matrix

$$\begin{pmatrix} \mathbf{a}_1^j \\ \vdots \\ \mathbf{a}_m^j \end{pmatrix} = \begin{pmatrix} a_{11}^j & \cdots & \overline{a_{1k_j}^j} & \cdots & a_{1n_j}^j \\ \vdots & & \vdots & & \vdots \\ a_{m1}^j & \cdots & \overline{a_{mk_j}^j} & \cdots & a_{mn_j}^j \end{pmatrix}.$$

Thus

$$A_{k_1 \cdots k_m} = \begin{pmatrix} a_{1k_1}^1 & \cdots & a_{1k_m}^m \\ \vdots & & \vdots \\ a_{mk_1}^1 & \cdots & a_{mk_m}^m \end{pmatrix}.$$

EXAMPLE 3.1. Let  $P = \Delta^2 \times \Delta^1$  be a triangular cylinder. Let  $\{v_0^1, v_1^1, v_2^1\}$  be the vertices of  $\Delta^2$  and  $\{v_0^2, v_1^2\}$  the vertices of  $\Delta^1$ . Then

$$\{v_{00}, v_{10}, v_{20}, v_{01}, v_{11}, v_{21}\}$$

is the vertex set of  $P$  where  $v_{ij} = v_i^1 \times v_j^2$ . The set of facets of  $P$  is

$$\{F_0^1, F_1^1, F_2^1, F_0^2, F_1^2\}$$

where  $F_i^1 = f_i^1 \times \Delta^1$  for  $i = 0, 1, 2$  are the side rectangles and  $F_j^2 = \Delta^2 \times f_j^2$  for  $j = 0, 1$  are the top and bottom triangles. The characteristic function  $\lambda: \mathcal{F} \rightarrow \mathbb{Z}^3$  is assigned as follows:

$$\begin{aligned} \lambda(F_0^1) &= \mathbf{a}_1, & \lambda(F_1^1) &= \mathbf{e}_1, & \lambda(F_2^1) &= \mathbf{e}_2, \\ \lambda(F_0^2) &= \mathbf{a}_2, & \lambda(F_1^2) &= \mathbf{e}_3. \end{aligned}$$

The corresponding  $2 \times 3$  matrix  $A$  is

$$\begin{aligned} A &= \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{a}_1^2 \\ \mathbf{a}_2^1 & \mathbf{a}_2^2 \end{pmatrix} \text{ as a } 2 \times 2 \text{ vector matrix} \\ &= \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{11}^2 \\ a_{21}^1 & a_{22}^1 & a_{21}^2 \end{pmatrix}. \end{aligned}$$

Thus the  $2 \times 2$  submatrices  $A_{11}$  and  $A_{21}$  are as follows:

$$A_{11} = \begin{pmatrix} a_{11}^1 & a_{11}^2 \\ a_{21}^1 & a_{21}^2 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} a_{12}^1 & a_{11}^2 \\ a_{22}^1 & a_{21}^2 \end{pmatrix}.$$

Condition 2.1 at a vertex, say  $v_{21}$ , can be translated as follows: since the facets  $F_0^1$ ,  $F_1^1$  and  $F_0^2$  intersect at  $v_{21}$

$$\begin{aligned} \det \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} &= \det \begin{pmatrix} 1 & 0 & 0 \\ a_{11}^1 & a_{12}^1 & a_{11}^2 \\ a_{21}^1 & a_{22}^1 & a_{21}^2 \end{pmatrix} \\ &= \det A_{21} = \pm 1. \end{aligned}$$

Similarly Condition 2.1 at  $v_{01}$  is equivalent to  $a_{21}^2 = \pm 1$ , and that at  $v_{20}$  is equivalent to  $a_{12}^1 = \pm 1$ . These conditions are equivalent to the condition that all principal minors of  $A_{21}$  (including the determinant of  $A_{21}$  itself) are  $\pm 1$ . Similarly Condition 2.1 at other vertices is equivalent to all principal minors of  $A_{11}$  being  $\pm 1$ .

The last statement in Example 3.1 holds in general. A *principal minor* of an  $m \times m$  vector matrix  $A$  of the form (3.2) means a principal minor of an  $m \times m$  matrix  $A_{j_1 \dots j_m}$  for some  $1 \leq j_1 \leq n_1, \dots, 1 \leq j_m \leq n_m$  where the determinant of  $A_{j_1 \dots j_m}$  itself is understood to be a principal minor of  $A_{j_1 \dots j_m}$ .

**Lemma 3.2.** *Let  $P = \prod_{i=1}^m \Delta^{n_i}$ . If an  $m \times m$  vector matrix  $A$  of the form (3.2) is associated with the characteristic function  $\lambda$  of a quasitoric manifold over  $P$ , then Condition 2.1 for  $\lambda$  at all vertices of  $P$  is equivalent to all principal minors of  $A$  being  $\pm 1$ .*

*Proof.* The basic idea of the proof is same as in Example 3.1. Indeed, at a vertex  $v_{j_1 \dots j_m}$  of  $P$  all  $n$  facets in  $\mathcal{F}' = \mathcal{F} - \{F_{j_i}^i \mid i = 1, \dots, m\}$  intersect. Hence Condition 2.1 at  $v_{j_1 \dots j_m}$  is equivalent to the determinant of the  $n \times n$  matrix having  $\lambda(F)$  as its row vectors for all  $F \in \mathcal{F}'$  being  $\pm 1$ . But this determinant is nothing but a principal minor of the  $m \times m$  matrix  $A_{j_1 \dots j_m}$  up to sign. Therefore the lemma follows.  $\square$

**REMARK 3.3.** It follows from the lemma above that each component  $a_{ij}^i$  in the diagonal entry vector  $\mathbf{a}_i^i = (a_{i1}^i, \dots, a_{in_i}^i)$  of the matrix  $A$ , see (3.2), is  $\pm 1$  for  $j = 1, \dots, n_i$ . The characteristic function  $\lambda$  is defined up to sign and if we change the sign of a vector  $\lambda(F_k^j)$  in (3.1) (say  $\lambda(F_k^j) = \mathbf{e}_l$ ), then the column vector corresponding to  $\lambda(F_k^j)$  (the  $l$ -th column) changes the sign; so we can always arrange  $a_{i,j}^i = 1$  for  $i = 1, \dots, m$  and  $j = 1, \dots, n_i$ , i.e.,  $\mathbf{a}_i^i = (1, \dots, 1)$  by an appropriate choice of signs of the vectors in (3.1). In the following we always take  $\mathbf{a}_i^i = (1, \dots, 1)$  for  $i = 1, \dots, m$  for the matrix  $A$  associated with a quasitoric manifold unless otherwise stated.

#### 4. Quotient construction

It is known that any quasitoric manifold over a simple polytope is realized as the orbit space of the moment-angle manifold of the polytope by some free torus action, see [1] and [2]. When the polytope is  $\prod_{i=1}^m \Delta^{n_i}$ , the moment-angle manifold is the product  $\prod_{i=1}^m S^{2n_i+1}$  of odd dimensional spheres. In this section we shall describe the free torus action on it explicitly. We remark that the case where  $n_i = 1$  for all  $i$  (i.e., the polytope is an  $m$ -cube) is treated in [7] and [4].

**Lemma 4.1.** *If  $C = (c_{ij})$  is a unimodular matrix of size  $m$ , then the system of equations*

$$z_1^{c_{i1}} \cdots z_m^{c_{im}} = 1, \quad \text{for } i = 1, \dots, m$$

*has a unique solution  $z_1 = \cdots = z_m = 1$  in  $S^1 \subset \mathbb{C}$ .*



Proof. Write  $z_j = \exp(2\pi\theta_j\sqrt{-1})$  with  $\theta_j \in \mathbb{R}$  for  $j = 1, \dots, m$ . Then the equations in the lemma are equivalent to

$$c_{i1}\theta_1 + \dots + c_{im}\theta_m = k_i \quad \text{for } i = 1, \dots, m$$

for some  $k_i \in \mathbb{Z}$ . Since  $C$  is unimodular and  $k_i$ 's are integers,  $\theta_j$ 's are also integers, which means  $z_j = 1$  for  $j = 1, \dots, m$ .  $\square$

Let  $A$  be an  $m \times m$  vector matrix in (3.2). We construct a quasitoric manifold  $M(A)$  with  $A$  as its corresponding matrix. Consider the subspace  $X = \prod_{i=1}^m S^{2n_i+1}$  of  $\prod_{i=1}^m \mathbb{C}^{n_i+1}$ , which is the moment-angle manifold of  $\prod_{i=1}^m \Delta^{n_i}$ . Let  $K = (S^1)^m$  and define an action of  $K$  on  $X$  by

$$(4.1) \quad \begin{aligned} & (g_1, \dots, g_m) \cdot ((z_0^1, \dots, z_{n_1}^1), \dots, (z_0^m, \dots, z_{n_m}^m)) \\ &= ((g_1 z_0^1, (g_1^{a_{11}} \dots g_m^{a_{m1}}) z_1^1, \dots, (g_1^{a_{1n_1}} \dots g_m^{a_{mn_1}}) z_{n_1}^1), \dots, \\ & \quad (g_m z_0^m, (g_1^{a_{1m}} \dots g_m^{a_{mm}}) z_1^m, \dots, (g_1^{a_{1n_m}} \dots g_m^{a_{mn_m}}) z_{n_m}^m)) \end{aligned}$$

where  $(g_1, \dots, g_m) \in K$  and  $(z_0^i, \dots, z_{n_i}^i) \in S^{2n_i+1} \subset \mathbb{C}^{n_i+1}$  for  $i = 1, \dots, m$ .

**Lemma 4.2.** *The action of  $K$  on  $X$  defined in (4.1) is free if all principal minors of  $A$  are equal to  $\pm 1$ .*

Proof. To prove that the action is free we have to show that the equation

$$(4.2) \quad \begin{aligned} & (g_1, \dots, g_m) \cdot ((z_0^1, \dots, z_{n_1}^1), \dots, (z_0^m, \dots, z_{n_m}^m)) \\ &= ((z_0^1, \dots, z_{n_1}^1), \dots, (z_0^m, \dots, z_{n_m}^m)) \end{aligned}$$

implies  $g_1 = \dots = g_m = 1$ . Since  $(z_0^i, \dots, z_{n_i}^i) \in S^{2n_i+1}$ , at least one component, say  $z_{j_i}^i$ , is nonzero for every  $i = 1, \dots, m$ . If  $z_0^i = 0$  for all  $i = 1, \dots, m$ , then equation (4.2) implies that  $g_1^{a_{1j_i}} \dots g_m^{a_{mj_i}} = 1$  for all  $i = 1, \dots, m$ . Since  $\det A_{j_1 \dots j_m} = \pm 1$  from the hypothesis, Lemma 4.1 implies that  $g_1 = \dots = g_m = 1$ . Now suppose  $z_0^i \neq 0$  for some  $i = 1, \dots, m$ . For simplicity let us assume that there is some  $0 \leq s \leq m$  such that  $z_0^1 = \dots = z_0^s = 0$  and  $z_0^i \neq 0$  for all  $i = s+1, \dots, m$ . Then equation (4.2) implies that  $g_1 = \dots = g_s = 1$  and  $g_{s+1}^{a_{(s+1)j_i}} \dots g_m^{a_{mj_i}} = 1$  for all  $i = s+1, \dots, m$ . Since all principal minors of  $A_{j_1 \dots j_m}$  are  $\pm 1$ , Lemma 4.1 implies that  $g_{s+1} = \dots = g_m = 1$ , which proves the lemma.  $\square$

Since the action  $K$  on  $X$  is free, the orbit space  $X/K$  is a smooth manifold of dimension  $2n$ . Let  $M(A)$  be the orbit space  $X/K$  with the action of  $G = (S^1)^n$  defined by

$$(4.3) \quad \begin{aligned} (t_1, \dots, t_n) \cdot [(z_0^1, \dots, z_{n_1}^1), \dots, (z_0^m, \dots, z_{n_m}^m)] \\ = [(z_0^1, t_1 z_1^1, \dots, t_{n_1} z_{n_1}^1), \dots, (z_0^m, t_{n-n_m+1} z_1^m, \dots, t_n z_{n_m}^m)]. \end{aligned}$$

Then we have the following proposition.

**Proposition 4.3.**  *$M(A)$  is a quasitoric manifold over  $\prod_{i=1}^m \Delta^{n_i}$  with  $A$  as its associated matrix.*

*Proof.* We think of  $q$ -simplex  $\Delta^q$  as

$$\Delta^q = \left\{ (x_0, \dots, x_q) \in \mathbb{R}^{q+1} \mid x_0 \geq 0, \dots, x_q \geq 0, \sum_{i=0}^q x_i = 1 \right\}.$$

Then  $P = \prod_{i=1}^m \Delta^{n_i}$  sits in  $\prod_{i=1}^m \mathbb{R}^{n_i+1}$ . It is easy to see that  $M(A)$  with the action of  $G = (S^1)^n$  is a quasitoric manifold over  $P$  with the projection  $\pi: M(A) \rightarrow P$  defined by

$$\pi([(z_0^1, \dots, z_{n_1}^1), \dots, (z_0^m, \dots, z_{n_m}^m)]) = ( (|z_0^1|, \dots, |z_{n_1}^1|), \dots, (|z_0^m|, \dots, |z_{n_m}^m|) ).$$

The facets  $F_j^i$  of  $P$  are given by  $x_j^i = 0$  for some  $1 \leq i \leq m$  and  $0 \leq j \leq n_i$ , where  $x_j^i$  denotes the  $(j+1)$ -st coordinate of the  $i$ -th factor  $\mathbb{R}^{n_i+1}$ . The isotropy subgroup of a point in  $\pi^{-1}(\text{int } F_j^i)$  is a circle subgroup. One can check that it is the  $(\sum_{k=1}^{i-1} n_k + j)$ -th factor of  $G = (S^1)^n$  when  $j \geq 1$  and the circle subgroup

$$\{ ((g^{a_{i1}^1}, \dots, g^{a_{in_1}^1}), \dots, (g^{a_{i1}^m}, \dots, g^{a_{in_m}^m})) \mid g \in S^1 \}$$

when  $j = 0$ . This shows that if we denote the characteristic function of  $M(A)$  by  $\lambda$ , then

$$\lambda(F_1^1), \dots, \lambda(F_{n_1}^1), \dots, \lambda(F_1^m), \dots, \lambda(F_{n_m}^m)$$

are the standard basis elements of  $\mathbb{Z}^n$  in the given order and

$$\lambda(F_0^i) = ((a_{i1}^1, \dots, a_{in_1}^1), \dots, (a_{i1}^m, \dots, a_{in_m}^m)) \in \mathbb{Z}^n \quad \text{for } i = 1, \dots, m,$$

which is the  $i$ -th row of our matrix  $A$ , proving the lemma.  $\square$

## 5. Conjugation of vector matrices

The correspondence between a quasitoric manifold over  $P = \prod_{i=1}^m \Delta^{n_i}$  and an  $m \times m$  vector matrix  $A$  depends on the order of the simplices  $\Delta^{n_i}$ 's in the product

formula of  $P$ . Namely, if we consider  $P = \prod_{i=1}^m \Delta^{n_{\sigma(i)}}$  for some permutation  $\sigma$  of  $\{1, \dots, m\}$ , then the corresponding  $m \times m$  vector matrix  $A_\sigma$  will be different from  $A$ . In fact it is not difficult to see that if  $E_\sigma$  is the  $m \times m$  permutation matrix of  $\sigma$  obtained from the identity matrix by permuting the  $i$ -th row and column to  $\sigma(i)$ -th row and column respectively for all  $i = 1, \dots, m$ , then  $A_\sigma = E_\sigma A E_\sigma^{-1}$ . One should be cautious that, as an  $m \times m$  vector matrix, the entries in the  $j$ -th column of  $A_\sigma$  are vectors in  $\mathbb{Z}^{n_{\sigma(j)}}$  while the  $j$ -th column of  $A$  are vectors in  $\mathbb{Z}^{n_j}$ .

As an example let us consider  $P$  as in Example 3.1. If we consider  $P = \Delta^1 \times \Delta^2$  instead of  $\Delta^2 \times \Delta^1$  then the corresponding  $2 \times 2$  vector matrix  $A_\sigma$  is given by

$$\begin{aligned} A_\sigma &= \begin{pmatrix} \mathbf{a}_2^2 & \mathbf{a}_2^1 \\ \mathbf{a}_1^2 & \mathbf{a}_1^1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}. \end{aligned}$$

The entries of the first column above are vectors in  $\mathbb{Z}$  and the ones in the second column are in  $\mathbb{Z}^2$ .

We say that two  $m \times m$  vector matrices  $A$  and  $B$  are *conjugate* if there exists an  $m \times m$  permutation matrix  $E_\sigma$  such that  $B = E_\sigma A E_\sigma^{-1}$ . In this case, the quasitoric manifolds  $M(A)$  and  $M(B)$  defined in Proposition 4.3 are equivariantly diffeomorphic.

Let  $A$  be an  $m \times m$  vector matrix of the form (3.2). A *proper principal minor* (resp. *determinant*) of  $A$  means that a proper principal minor (resp. determinant) of  $A_{j_1 \dots j_m}$  for some  $1 \leq j_1 \leq n_1, \dots, 1 \leq j_m \leq n_m$ . The set of proper principal minors or determinants is invariant under the conjugation relation.

**Lemma 5.1.** *Let  $A$  be an  $m \times m$  vector matrix of the form (3.2) such that all the proper principal minors of  $A$  are 1. If all the determinants of  $A$  are 1, then  $A$  is conjugate to a unipotent upper triangular vector matrix of the following form:*

$$(5.1) \quad \begin{pmatrix} \mathbf{1} & \mathbf{b}_1^2 & \mathbf{b}_1^3 & \cdots & \mathbf{b}_1^m \\ \mathbf{0} & \mathbf{1} & \mathbf{b}_2^3 & \cdots & \mathbf{b}_2^m \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{1} & \mathbf{b}_{m-1}^m \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

where  $\mathbf{0} = (0, \dots, 0)$ ,  $\mathbf{1} = (1, \dots, 1)$  of appropriate sizes. If all the determinants of  $A$

are  $\pm 1$  and at least one of them is  $-1$ , then  $A$  is conjugate to a vector matrix of the following form:

$$(5.2) \quad \begin{pmatrix} \mathbf{1} & \mathbf{b}^2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{b}^3 & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{1} & \mathbf{b}^m \\ \mathbf{b}^1 & \cdots & \cdots & \mathbf{0} & \mathbf{1} \end{pmatrix},$$

where  $\mathbf{b}^i$  is non-zero for any  $i$  and  $\prod_{i=1}^m b_i$ , where  $b_i$  is any non-zero component of  $\mathbf{b}^i$ , is  $(-1)^{m2}$ . (Therefore, the non-zero components in  $\mathbf{b}^i$  are all same for each  $i$  and they are  $\pm 1$  or  $\pm 2$ .)

*Proof.* The lemma is proved in [10] when  $A$  is an ordinary  $m \times m$  matrix except the last statement on the components of  $\mathbf{b}^i$ , and the proof for an  $m \times m$  vector matrix is quite similar. So we refer the reader to the cited paper and shall prove only the statement on the components of  $\mathbf{b}^i$ .

Let  $B$  be the vector matrix of the form (5.2). The determinants of  $A$  are  $\pm 1$  and at least one of them is  $-1$  by assumption while any determinant of  $B$  is of the form  $1 + (-1)^{m+1} \prod_{i=1}^m b_i$  where  $b_i$  is a component of  $\mathbf{b}^i$ . Since the set of determinants of  $A$  agrees with that of  $B$  as remarked above, it follows that there is a non-zero  $b_i$  for each  $i$  and  $\prod_{i=1}^m b_i = (-1)^{m2}$  whenever each  $b_i$  is non-zero. This implies the statement on  $b_i$ 's in the lemma.  $\square$

## 6. Generalized Bott towers

A quasitoric manifold over a product of simplices also appears in iterated projective bundles. For a complex vector bundle  $E$ , we denote the total space of its projectivization by  $P(E)$ .

**DEFINITION 6.1.** We call a sequence

$$(6.1) \quad B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where  $B_j = P(\mathbb{C} \oplus \xi_j)$  and  $\xi_j$  is the Whitney sum of complex line bundles over  $B_{j-1}$ , a *generalized Bott tower* and each  $B_j$  for  $j = 1, \dots, m$  a *generalized Bott manifold*.

Each  $B_j$  admits an effective action of  $G_j = (S^1)^{\sum_{i=1}^j \dim \xi_i}$  defined as follows. Assume by induction that  $B_{j-1}$  admits an effective action of  $G_{j-1}$ . Then it lifts to an action on  $\xi_j$  since  $H^1(B_{j-1}) = 0$  although the lifting is not unique, see [8]. On the other hand since  $\xi_j$  is the Whitney sum of complex line bundles, it admits an action of  $(S^1)^{\dim \xi_j}$  by scalar multiplication on fibers. These two actions commute and define

an action of  $G_j$  on  $\xi_j$ , which induces an effective action of  $G_j$  on  $B_j$ . Without much difficulty it can be shown that  $B_j$  with the action of  $G_j$  is a quasitoric manifold over  $\prod_{i=1}^j \Delta^{\dim \xi_i}$ . Furthermore each  $B_j$  is a nonsingular toric variety (i.e., a toric manifold).

**Proposition 6.2.** *Let  $M$  be a quasitoric manifold over  $P = \prod_{i=1}^m \Delta^{n_i}$ , and let  $A$  be an  $m \times m$  vector matrix associated with  $M$ . Then  $M$  is equivalent to a generalized Bott manifold if  $A$  is conjugate to an  $m \times m$  upper triangular vector matrix of the form (5.1).*

REMARK 6.3. We will see later that the “only if” statement in the proposition above also holds, see Lemma 5.1 and Theorem 6.4.

Proof of Proposition 6.2. We may assume that  $M = M(A)$  and  $A$  is of the form (5.1). We recall the quotient construction in Section 3. Let  $X_j = \prod_{i=1}^j S^{2n_i+1}$  for  $j = 1, \dots, m$ , so  $X_m$  agrees with  $X$  in Section 3. The group  $K = (S^1)^m$  is acting on  $X$  as in (4.1) and  $X/K = M(A)$ . We set  $B_j = X_j/K$ , so  $B_m = M(A)$ . In the following we claim that the sequence

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \dots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}$$

induced from the natural projections from  $X_j$  on  $X_{j-1}$  for  $j = m, \dots, 2, 1$  is a generalized Bott tower.

Since  $A$  is of the form (5.1), the last  $(m - j)$  factors of  $K = (S^1)^m$  are acting on  $X_j$  trivially, so the action of  $K$  on  $X_j$  reduces to an action of the product  $K_j$  of the first  $j$  factors of  $K = (S^1)^m$ . This means that  $X_j/K = X_j/K_j$ . Moreover, the last factor of  $K_j$  is acting on the last factor  $S^{2n_j+1}$  of  $X_j$  as scalar multiplication and trivially on the other factors of  $X_j$ . Therefore the map  $\pi_j: B_j = X_j/K_j \rightarrow B_{j-1} = X_{j-1}/K_{j-1}$  is a fibration with  $\mathbb{C}P^{n_j} = S^{2n_j+1}/S^1$  as a fiber and this is actually the projectivization of a complex vector bundle  $\xi_j$  over  $B_{j-1}$ . In fact, the bundle  $\xi_j$  is obtained as follows. Let  $V_j$  be  $\mathbb{C}^{n_j+1}$  with the linear  $K_{j-1}$ -action defined by

$$\begin{aligned} & (g_1, \dots, g_{j-1}) \cdot (z_0^j, \dots, z_{n_j}^j) \\ &= \left( z_0^j, \left( g_1^{b_{11}^j} \dots g_{j-1}^{b_{j-1,1}^j} \right) z_1^j, \dots, \left( g_1^{b_{1n_j}^j} \dots g_{j-1}^{b_{j-1,n_j}^j} \right) z_{n_j}^j \right) \end{aligned}$$

where  $\mathbf{b}_i^j = (b_{i1}^j, \dots, b_{in_j}^j)$  is a vector in (5.1) for  $i = 1, \dots, j - 1$ . Since the action of  $K_{j-1}$  on  $X_{j-1}$  is free, the projection

$$(X_{j-1} \times V_j)/K_{j-1} \rightarrow X_{j-1}/K_{j-1} = B_{j-1}$$

becomes a vector bundle, where the action of  $K_{j-1}$  on  $X_{j-1} \times V_j$  is a diagonal one. This is the desired bundle  $\xi_j$  and since  $V_j$  decomposes into sum of complex one dimensional

$K$ -modules, the bundle  $\xi_j$  decomposes into the Whitney sum of complex line bundles accordingly.  $\square$

One can describe the bundles  $\xi_j$  in the proof of the proposition above more explicitly. For that let us fix some notation. For a vector bundle  $\eta$  and a vector  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$  let  $\eta^{\mathbf{a}}$  denote the bundle  $\eta^{a_1} \oplus \dots \oplus \eta^{a_n}$ . For vector bundles  $\eta_1, \dots, \eta_k$  over a space and vectors  $\mathbf{a}_1 = (a_{11}, \dots, a_{1n}), \dots, \mathbf{a}_k = (a_{k1}, \dots, a_{kn})$  let

$$\begin{aligned} \bigodot_{i=1}^k \eta_i^{\mathbf{a}_i} &= \eta_1^{\mathbf{a}_1} \odot \dots \odot \eta_k^{\mathbf{a}_k} \\ &= (\eta_1^{a_{11}} \otimes \dots \otimes \eta_k^{a_{k1}}) \oplus \dots \oplus (\eta_1^{a_{1n}} \otimes \dots \otimes \eta_k^{a_{kn}}) \end{aligned}$$

where the last expression denotes the Whitney sum of componentwise tensor products.

Let  $\xi_1^2$  denote the canonical line bundle over  $B_1$  and let  $\xi_1^3 = \pi_2^*(\xi_1^2)$  the pull-back bundle of the canonical line bundle over  $B_1$  to  $B_2$  via the projection  $\pi_2: B_2 \rightarrow B_1$ . In general, let  $\xi_{j-1}^j$  be the canonical line bundle over  $B_{j-1}$ , and we inductively define

$$\xi_{j-k}^j = \pi_j^* \circ \dots \circ \pi_{j-k+1}^*(\xi_{j-k}^{j-k+1}) \quad \text{for } k = 2, \dots, j - 1.$$

Then one can see that  $\xi_j = \bigodot_{i=1}^{j-1} (\xi_i^j)^{\mathbf{b}_i^j}$ .

A generalized Bott manifold is not only a quasitoric manifold over a product of simplices but also a complex manifold on which the action preserves the complex structure, in particular, it has an almost complex structure left invariant under the action. The following theorem shows that the converse holds. We remark that the equivalence (1)  $\Leftrightarrow$  (3) is a particular case of [6, Theorem 6].

**Theorem 6.4.** *Let  $M$  be a quasitoric manifold over  $P = \prod_{i=1}^m \Delta^{n_i}$ , and let  $A$  be the  $m \times m$  vector matrix associated with  $M$  which has  $\mathbf{1}$  as the diagonal entries. Then the following are equivalent:*

- (1)  $M$  is equivalent to a generalized Bott manifold.
- (2)  $M$  is equivalent to a quasitoric manifold which admits an invariant almost complex structure under the action.
- (3) All the principal minors of  $A$  are 1.

*Proof.* The implication (1)  $\Rightarrow$  (2) is obvious and the implication (3)  $\Rightarrow$  (1) follows from Proposition 6.2 and Lemma 5.1, so it suffices to prove the implication (2)  $\Rightarrow$  (3).

We may assume that  $M$  itself admits an invariant almost complex structure. As is noted in the paragraph before Condition 2.1 we can define a sign-unambiguous characteristic function  $\lambda$  of  $M$ . Let  $\Lambda$  be the matrix associated with  $\lambda$ . To each cubical face of  $P$ , the submanifold of  $M$  over it inherits an invariant almost complex structure, so it follows from [10, Theorem 3.4] that all principal minors of the restriction of  $-\Lambda$  to each cubical face of  $P$  are equal to 1. Therefore  $A = -\Lambda$  and this proves (3).  $\square$

REMARK 6.5. A difference between quasitoric manifolds and small covers appears here. Namely, not every quasitoric manifold over a product of simplices is equivalent to a generalized Bott manifold as is seen from Theorem 6.4, while it follows from the real version of Proposition 6.2 and the  $\mathbb{Z}/2$  version of the former part of Lemma 5.1 that every small cover over a product of simplices turns out to be equivalent to a generalized Bott manifold (over  $\mathbb{R}$ ).

### 7. Cohomology ring

The connected sum  $\mathbb{C}P^2 \# \mathbb{C}P^2$  is a quasitoric manifold over a square but not homeomorphic to a Bott manifold (or Hirzebruch surface) over a square. In the rest of this paper, we shall give a sufficient condition in terms of cohomology ring for a quasitoric manifold over a product of simplices to be homeomorphic to a generalized Bott manifold (Theorem 8.1). This section is a preliminary section for the purpose.

**Lemma 7.1.** *Let  $M$  be a quasitoric manifold over  $\prod_{i=1}^m \Delta^{n_i}$  and let  $A$  be the vector matrix of the form (3.2) associated with  $M$ . Then*

$$(7.1) \quad H^*(M) = \mathbb{Z}[y_1, \dots, y_m]/L$$

where the ideal  $L$  is generated by the following  $m$  expressions:

$$(7.2) \quad y_k \cdot \prod_{j=1}^{n_k} \left( \sum_{i=1}^m a_{ij}^k y_i \right) \quad \text{for } k = 1, \dots, m.$$

Proof. We will use the result (2.3). In our case, the matrix in (2.2) is of the form

$$(7.3) \quad (\lambda_{ij}) = \begin{pmatrix} A \\ I_n \end{pmatrix}$$

where  $I_n$  is the  $n \times n$  identity matrix. Let

$$\omega_0^1, \dots, \omega_{n_1}^1, \dots, \omega_0^m, \dots, \omega_{n_m}^m$$

be the indeterminates corresponding to the facets

$$F_0^1, \dots, F_{n_1}^1, \dots, F_0^m, \dots, F_{n_m}^m$$

in the given order. Then by (2.3) we have

$$(7.4) \quad H^*(M) \cong \mathbb{Z}[\omega_0^1, \dots, \omega_{n_1}^1, \dots, \omega_0^m, \dots, \omega_{n_m}^m]/(I + J)$$

where  $I$  is the ideal generated by the monomials

$$\omega_0^i \cdots \omega_{n_i}^i \quad \text{for } i = 1, \dots, m$$

because the intersection of facets  $F_0^i, \dots, F_{n_i}^i$  is empty for  $i = 1, \dots, m$ , and  $J$  is the ideal generated by

$$\begin{aligned} \lambda_j &= \lambda_{1j}\omega_0^1 + \dots + \lambda_{mj}\omega_0^m \\ &\quad + \lambda_{(m+1)j}\omega_1^1 + \dots + \lambda_{(m+n_1)j}\omega_{n_1}^1 \\ &\quad + \dots \\ &\quad + \lambda_{(m+\sum_{i=1}^{m-1} n_i+1)j}\omega_1^m + \dots + \lambda_{(m+n)j}\omega_{n_m}^m \end{aligned}$$

for  $j = 1, \dots, m+n$  because the order of the row vectors in (7.3) is

$$\lambda(F_0^1), \dots, \lambda(F_0^m), \lambda(F_1^1), \dots, \lambda(F_{n_1}^1), \dots, \lambda(F_1^m), \dots, \lambda(F_{n_m}^m).$$

If  $j = (\sum_{i=1}^{k-1} n_i) + l$  and  $1 \leq l \leq n_k$ , then

$$\lambda_j = a_{1l}^k\omega_0^1 + a_{2l}^k\omega_0^2 + \dots + a_{ml}^k\omega_0^m + \omega_l^k.$$

Since  $\lambda_j = 0$  in  $H^*(M)$ , we have that

$$(7.5) \quad \omega_l^k = -(a_{1l}^k\omega_0^1 + a_{2l}^k\omega_0^2 + \dots + a_{ml}^k\omega_0^m).$$

Set  $y_k = \omega_0^k$  for  $k = 1, \dots, m$ . Then  $\omega_0^k \cdots \omega_{n_1}^k = 0$  in the cohomology ring implies that

$$y_k \prod_{l=1}^{n_k} (a_{1l}^k y_1 + a_{2l}^k y_2 + \dots + a_{ml}^k y_m) = 0.$$

This proves the relation in the lemma. □

**Lemma 7.2.** *Let  $M$  and  $y_1, \dots, y_m$  be as above. Let  $x = \sum_{j=1}^m b_j y_j$  be an element of  $H^*(M)$  such that  $b_j \neq 0$  for some  $j$ . Then  $x^{n_j} \neq 0$  in  $H^*(M)$ .*

*Proof.* Suppose  $x^{n_j} = 0$  on the contrary. Then  $(\sum_{j=1}^m b_j y_j)^{n_j}$  must be in the ideal  $L$  in (7.2). However,  $y_j^{n_j+1}$  is the least power of  $y_j$  which appears as a term in a polynomial of  $L$  while  $(\sum_{j=1}^m b_j y_j)^{n_j}$  contains a non-zero scalar multiple of  $y_j^{n_j}$  because  $b_j \neq 0$  by assumption. This is a contradiction. □

**Lemma 7.3.** *Let  $M(j)$  be a facial submanifold of  $M$  over  $\prod_{i \neq j}^m \Delta^{n_i}$ . Then  $H^*(M(j))$  is equal to (7.1) with  $y_j = 0$  plugged in.*

*Proof.* Let  $y_1, \dots, y_m$  be the generators of  $H^*(M)$  in Lemma 7.1. We may assume that  $M(j)$  is over  $\prod_{i \neq j}^m \Delta^{n_i} \times \{v\}$  where  $v$  is a vertex of  $\Delta^{n_j}$  and also that  $y_j$  is the



dual of the characteristic submanifold  $M_j$  over  $\prod_{i \neq j}^m \Delta^{n_i} \times \Delta^{n_j-1}(v)$  where  $\Delta^{n_j-1}(v)$  is the facet of  $\Delta^{n_j}$  not containing  $v$ . Since  $M(j)$  and  $M_j$  have no intersection, the restriction of  $y_j$  to  $M(j)$  vanishes.

We know that

$$(7.6) \quad H^*(M) = \mathbb{Z}[y_1, \dots, y_m]/(g_1, \dots, g_m),$$

where  $g_k$  is the polynomial in (7.2). Since  $y_j$  maps to zero in  $H^*(M(j))$  and  $g_j$  contains  $y_j$  as a factor, we have a natural surjective map

$$\mathbb{Z}[y_1, \dots, \widehat{y}_j, \dots, y_m]/(g'_1, \dots, \widehat{g'_j}, \dots, g'_m) \rightarrow H^*(M(j)),$$

where  $g'_k$  denotes  $g_k$  with  $y_j = 0$  plugged in and  $\widehat{\phantom{x}}$  denotes the term there is dropped. The degree of  $g'_k$  for  $k \neq j$  is  $n_k + 1$  and  $g'_k$  contains the term  $y_k^{n_k+1}$ . Therefore, the ranks of the both sides above agree, so that the map is an isomorphism. This proves the lemma.  $\square$

**Lemma 7.4.** *Let  $N$  be the smallest number among  $n_i$ 's. If the vector matrix associated with  $M$  is of the form (5.2) in Lemma 5.1, then there is no non-zero element in  $H^2(M)$  whose  $(N + 1)$ -st power vanishes.*

*Proof.* Let  $y$  be an element of  $H^2(M)$  whose  $(N + 1)$ -st power vanishes. Since  $N$  is smallest among  $n_i$ 's,  $y$  can be expressed as a linear combination of the canonical generators  $y_i$ 's with  $n_i = N$  by Lemma 7.2, say  $y = \sum_{n_i=N} a_i y_i$  with  $a_i \in \mathbb{Z}$ . All relations in  $H^*(M)$  of cohomological degree  $2(N + 1)$  are generated by  $y_i^{k_i+1}(y_i + b_i y_{i-1})^{n_i-k_i}$ 's with  $n_i = N$  over  $\mathbb{Z}$ , where  $y_{i-1}$  with  $i = 1$  is understood to be  $y_m$ ,  $b_i$  is the non-zero component of the vector  $\mathbf{b}_i$  in Lemma 5.1 and  $k_i$  is the number of zero components of  $\mathbf{b}_i$ . Note that  $k_i < N$  when  $n_i = N$  since  $\mathbf{b}_i$  is non-zero. It follows that we obtain a polynomial identity

$$(7.7) \quad \left( \sum_{n_i=N} a_i y_i \right)^{N+1} = \sum_{n_i=N} a_i^{N+1} y_i^{k_i+1} (y_i + b_i y_{i-1})^{N-k_i}.$$

**CASE 1.** The case where  $N = 1$ . In this case  $k_i = 0$  for  $i$  with  $n_i = N = 1$ . Suppose that  $a_i$  is non-zero for some  $i$  with  $n_i = 1$ . Comparing the coefficients of  $y_i^2$  and  $y_i y_{i-1}$  at both sides of the identity (7.7) with an observation that the right-hand side of (7.7) contains a  $y_i y_{i-1}$ -term, we see that  $n_{i-1} = 1$  and  $2a_i a_{i-1} = a_i^2 b_i$ . Since  $a_i$  and  $b_i$  are both non-zero, this shows that  $a_{i-1}$  is also non-zero and  $2a_{i-1} = a_i b_i$ . Since  $n_{i-1} = 1$  and  $a_{i-1}$  is non-zero, the same argument can be applied to  $i - 1$  instead of  $i$ . Repeating this argument, we see that  $n_i = 1$  and  $2a_{i-1} = a_i b_i$  for any  $i$ . It follows that  $\prod_{i=1}^m b_i = 2^m$  which contradicts the fact that  $\prod_{i=1}^m b_i = (-1)^m 2$  in Lemma 5.1.

CASE 2. The case where  $N \geq 2$ . When we expand the right hand side of the identity (7.7), no monomial in more than two variables appears. Since  $N \geq 2$ , this implies that at most two coefficients among  $a_i$ 's are non-zero. Since all  $b_i$ 's are non-zero, it easily follows from (7.7) that the case where only one coefficient among  $a_i$ 's is non-zero does not occur.

Suppose that there are exactly two non-zero coefficients, say  $a_i$  and  $a_j$ . Then only two variables appear at the left hand side. Unless  $m = 2$  and  $n_1 = n_2 = N$ , at least three variables appear at the right hand side of (7.7) which is a contradiction. If  $m = 2$  and  $n_1 = n_2 = N$ , then the identity (7.7) is

$$(a_1y_1 + a_2y_2)^{N+1} = a_1^{N+1}y_1^{k_1+1}(y_1 + b_1y_2)^{N-k_1} + a_2^{N+1}y_2^{k_2+1}(y_2 + b_2y_1)^{N-k_2}.$$

Replacing  $y_2$  by  $-b_2y_1$  above, we obtain an identity

$$|a_1 - a_2b_2|^{N+1} = |a_1|^{N+1}$$

where we used the fact  $b_1b_2 = 2$  in Lemma 5.1. Since  $a_2b_2 \neq 0$ , it follows from the identity above that  $2a_1 = a_2b_2$ . Similarly, replacing  $y_1$  by  $-b_1y_2$  above, we obtain  $2a_2 = a_1b_1$ . These two identities imply that  $b_1b_2 = 4$  which contradicts to  $b_1b_2 = 2$ .

This completes the proof of the lemma. □

### 8. Cohomologically product quasitoric manifolds

We say that a quasitoric manifold  $M$  over  $\prod_{i=1}^m \Delta^{n_i}$  is *cohomologically product* over  $\mathbb{Q}$  if there are elements  $x_1, \dots, x_m$  in  $H^2(M; \mathbb{Q})$  such that

$$(8.1) \quad H^*(M; \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_m]/(x_1^{n_1+1}, \dots, x_m^{n_m+1}).$$

The purpose of this section is to prove the following.

**Theorem 8.1.** *If a quasitoric manifold  $M$  over  $\prod_{i=1}^m \Delta^{n_i}$  is cohomologically product over  $\mathbb{Q}$ , then the vector matrix associated with  $M$  is conjugate to a unipotent upper triangular vector matrix, so that  $M$  is homeomorphic to a generalized Bott manifold.*

REMARK 8.2. We prove in [3] that if a generalized Bott manifold is cohomologically trivial over  $\mathbb{Z}$ , then it is diffeomorphic to a product of complex projective spaces. This together with Theorem 8.1 implies that if a quasitoric manifold over a product of simplices is cohomologically trivial over  $\mathbb{Z}$ , then it is homeomorphic to a product of complex projective spaces.

In the following  $M$  is assumed to be cohomologically product over  $\mathbb{Q}$ . We have another set of generators  $\{y_1, \dots, y_m\}$  in Lemma 7.1. Since both  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_m\}$  are sets of generators of  $H^2(M; \mathbb{Q})$ , one can write

$$(8.2) \quad y_j = \sum_{i=1}^m c_{ji} x_i \quad \text{for } j = 1, \dots, m \quad \text{and } c_{ji} \in \mathbb{Q},$$

where the coefficient matrix  $C = (c_{ji})$  has non-zero determinant.

**Lemma 8.3.** *By an appropriate change of indices in  $x_i$ 's and  $y_j$ 's, we may assume that  $c_{jj} \neq 0$  for any  $j = 1, \dots, m$ .*

*Proof.* We may assume that  $n_1 \geq n_2 \geq \dots \geq n_m$  by an appropriate change of indices. Let  $S = \{N_1, \dots, N_k\}$  be the set of all distinct elements of  $n_1, \dots, n_m$  such that  $N_1 > \dots > N_k$ . We can view  $\{n_1, \dots, n_m\}$  as a function  $\mu: \{1, \dots, m\} \rightarrow \mathbb{N}$  such that  $\mu(j) = n_j$ . Then  $S$  is the image of  $\mu$ . Let  $J_l = \mu^{-1}(N_l)$  for  $l = 1, \dots, k$ . We write

$$(8.3) \quad x_i = \sum_{j=1}^m d_{ij} y_j \quad \text{for } i = 1, \dots, m \quad \text{and } d_{ij} \in \mathbb{Q}.$$

Since  $x_i^{n_i+1} = 0$ ,  $d_{ij} = 0$  if  $n_i < n_j$  by Lemma 7.2. This shows that  $D = (d_{ij})$  is a block upper triangular matrix because we assume  $n_1 \geq n_2 \geq \dots \geq n_m$ . The matrix  $C$  in (8.2) is the inverse of the matrix  $D$ , so  $C$  is also a block upper triangular matrix and of the same type as  $D$ , i.e.,

$$C = \begin{pmatrix} C_{J_1} & & & * \\ & C_{J_2} & & \\ & & \ddots & \\ 0 & & & C_{J_k} \end{pmatrix}$$

where  $C_{J_l}$  ( $l = 1, \dots, k$ ) is a square matrix formed from  $c_{ij}$  with  $i, j \in J_l$ . Since  $\det C \neq 0$ , we have  $\det C_{J_l} \neq 0$  for any  $l$ . By definition of determinant  $\det C_{J_l} = \sum_{\sigma} \text{sgn } \sigma \prod_{j \in J_l} c_{j\sigma(j)}$  where the sum is taken over all permutations  $\sigma$  on  $J_l$ . Therefore there must exist a permutation  $\sigma$  on  $J_l$  such that  $\prod_{j \in J_l} c_{j\sigma(j)} \neq 0$ . This implies the lemma. □

**Lemma 8.4.** *The facial submanifold  $M(j)$  of  $M$  over  $\prod_{i \neq j}^m \Delta^{n_i}$  is also cohomologically product over  $\mathbb{Q}$  for any  $j$ .*

Proof. Since  $H^*(M(j))$  is  $H^*(M)$  with  $y_j = 0$  plugged by Lemma 7.3, it follows from (8.2) that

$$H^*(M(j); \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_m] / \left( x_1^{n_1+1}, \dots, x_m^{n_m+1}, \sum_{i=1}^m c_{ji} x_i \right).$$

Here  $c_{jj} \neq 0$  by Lemma 8.3, so that one can eliminate the variable  $x_j$  using the relation  $\sum_{i=1}^m c_{ji} x_i = 0$ . Therefore a natural map

$$\mathbb{Q}[x_1, \dots, \widehat{x}_j, \dots, x_m] / (x_1^{n_1+1}, \dots, \widehat{x_j^{n_j+1}}, \dots, x_m^{n_m+1}) \rightarrow H^*(M(j); \mathbb{Q})$$

is surjective. Since the dimensions at the both sides above are same, this map is actually an isomorphism, proving the lemma.  $\square$

Now we shall prove Theorem 8.1 by induction on the number  $m$  of factors in  $\prod_{i=1}^m \Delta^{n_i}$ . Suppose that  $M$  is cohomologically product over  $\mathbb{Q}$ . Then any facial submanifold  $M(j)$  is cohomologically product over  $\mathbb{Q}$  by Lemma 8.4. Therefore by induction assumption all the proper principal minors of the vector matrix  $A$  associated with  $M$  are 1. It follows that the vector matrix  $A$  is conjugate to a unipotent upper triangular vector matrix or to a matrix of the form (5.2) in Lemma 5.1. But the latter does not occur because since  $M$  is cohomologically product over  $\mathbb{Q}$ ,  $H^2(M)$  must contain a non-zero element whose  $(N+1)$ -st power vanishes, where  $N$  is the smallest number among  $n_j$ 's, but this fact contradicts Lemma 7.4. This proves Theorem 8.1.

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### References

- [1] V.M. Buchstaber and T.E. Panov: *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture Series **24**, Amer. Math. Soc., Providence, RI, 2002.
- [2] V.M. Buchstaber, T.E. Panov and N. Ray: *Spaces of polytopes and cobordism of quasitoric manifolds*, Mosc. Math. J. **7** (2007), 219–242, arXiv:math/0609346v2.
- [3] S. Choi, M. Masuda and D.Y. Suh: *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. **362** (2010), 1097–1112, arXiv:0807.4334.
- [4] Y. Civan and N. Ray: *Homotopy decompositions and K-theory of Bott towers*, *K-Theory* **34** (2005), 1–33.
- [5] M.W. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, *Duke Math. J.* **62** (1991), 417–451.
- [6] N.È. Dobrinskaya: *Classification problem for quasitoric manifolds over a given simple polytope*, *Funct. Anal. and Appl.* **35** (2001), 83–89.
- [7] M. Grossberg and Y. Karshon: *Bott towers, complete integrability, and the extended character of representations*, *Duke Math. J.* **76** (1994), 23–58.
- [8] A. Hattori and T. Yoshida: *Lifting compact group actions in fiber bundles*, *Japan. J. Math. (N.S.)* **2** (1976), 13–25.
- [9] M. Masuda: *Unitary toric manifolds, multi-fans and equivariant index*, *Tohoku Math. J. (2)* **51** (1999), 237–265.
- [10] M. Masuda and T.E. Panov: *Semifree circle actions, Bott towers, and quasitoric manifolds*, *Sbornik Math.* **199** (2008), 1201–1223, arXiv:math.AT/0607094.

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