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QUATERNION KÄHLERIAN MANIFOLDS AND FIBRED RIEMANNIAN SPACES WITH SASAKIAN 3-STRUCTURE

BY SHIGERU ISHIHARA

In a previous paper [6], we have studied fibred Riemannian spaces with Sasakian 3-structure and showed that there appears a kind of structure in the base space of a fibred Riemannian space with Sasakian 3-structure. In the present paper, we shall show that this kind of structure is what is called a quaternion Kählerian structure (See [1], [2], [3], [5], [7] and [9]).

In § 1, we recall definitions and some properties of a fibred Riemannian space with Sasakian 3-structure for later use. In § 2, we show that the base space of a fibred Riemannian space with Sasakian 3-structure admits a quaternion Kählerian structure defined in [5]. The last section is devoted to state some properties of a quaternion Kählerian manifold. Quaternion Kählerian manifolds will be studied a little bit in detail in [5].

Manifolds, mappings and geometric objects we consider are assumed to be differentiable and of class C^∞. The indices h, i, j, k run over the range $\{1, 2, \cdots, n\}$, the indices a, b, c, d, e over the range $\{1, 2, \cdots, n-3\}$ and the indices $\alpha, \beta, \gamma, \delta, \varepsilon$ over the range $\{1, 2, 3\}$. The summation convention will be used with respect to these three systems of indices.

§ 1. Fibred Riemannian spaces with Sasakian 3-structure.

In a Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension n with metric tensor $\tilde{g}$, let there be given a Killing vector ξ of unit length satisfying the condition

$$(1.1) \qquad \tilde{\nabla}_j \tilde{\nabla}_i \xi^h = \xi_i \delta_j^h - \xi^h \tilde{g}_{ji},$$

ξ^h being components of ξ and $\tilde{g}_{ji}$ components of $\tilde{g}$, where $\xi_i = \xi^h \tilde{g}_{hi}$ and $\tilde{\nabla}_j$ denote the Riemannian connection of $(\tilde{M}, \tilde{g})$. Then ξ is called a *Sasakian structure* or a *normal contact metric structure* in $(\tilde{M}, \tilde{g})$ (See [4] and [8]).

We now assume that $(\tilde{M}, \tilde{g})$ admits three Sasakian structures ξ, η and ζ which are mutually orthogonal and satisfy the conditions

$$[\eta, \zeta] = 2\xi, \qquad [\zeta, \xi] = 2\eta, \qquad [\xi, \eta] = 2\zeta.$$

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Then the set $\{\xi, \eta, \zeta\}$ is called a *Sasakian* 3-*structure* or a *normal contact metric* 3-*structure* in $(\tilde{M}, \tilde{g})$. In such a case, $\tilde{M}$ is necessarily of dimension $n=4m+3$ $(m\geqq 0)$. Moreover, the distribution D spanned by ξ, η and ζ is integrable and every integral manifold of D is totally geodesic and of constant curvature 1 (See [6]).

Next, we assume that in $(\tilde{M}, \tilde{g})$ with Sasakian 3-ststructure $\{\xi, \eta, \zeta\}$ the distribution D is regular. Then, denoting by M the set of all maximal integral submanifolds of D and by $\pi: \tilde{M}\to M$ the natural projection, we see that M becomes a differentiable manifold of dimension $4m(=n-3)$, if M is naturally topologized. That is to say, M is the quotient space $\tilde{M}/D$ and $\pi: \tilde{M}\to M$ is differentiable and of rank $4m$ everywhere. In such a case, $(\tilde{M}, \tilde{g})$ is called a *fibred Riemannian space with Sasakian* 3-*structure* $\{\xi, \eta, \zeta\}$ and each of maximal integral manifold of D is called a *fibre*. Then each fibre is coonected. In the sequel, let $(\tilde{M}, \tilde{g})$ be a fibred Riemannian space with Sasakian 3-structure $\{\xi, \eta, \zeta\}$ and assume that $\dim M\geqq 7$ (i.e., $m\geqq 1$).

We take coordinate neighborhoods $\{\tilde{U}; x^h\}$ of $\tilde{M}$ such that $\pi(\tilde{U})=U$ are coordinate neighborhoods of M with local coordinates (v^a). Then the projection $\pi: \tilde{M}\to M$ may be expressed, with respect to $\{\tilde{U}; x^h\}$ and $\{U; v^a\}$, by certain equations of the form

$$v^a=v^a(x^1, \cdots, x^n), \tag{1.2}$$

$v^a(x^1, \cdots, x^n)$ denoting coordinates in U of the projection $P=\pi(\sigma)$ of a point σ with coordinates x^h in $\tilde{U}$, where $v^a(x^1, \cdots, x^n)$ are differentiable functions of variables x^h with Jacobian $(\partial v^a/\partial x^h)$ of the maximum rank $4m(=n-3)$. We take a fibre F such that $F\cap\tilde{U}\neq\phi$. Then we may assume that $F\cap\tilde{U}$ is connected. We can introduce local coordinates (u^α) in $F\cap\tilde{U}$ in such a way that (v^a, u^α) is a system of local coordinates in $\tilde{U}$, (v^a) being coordinates of $\pi(F)$ in U. Differentiating (1.2) with respect to x^i, we put $E_i{}^a=\partial_i v^a$, where $\partial_i=\partial/\partial x^i$. We denote by E^a local covector fields with components $E_i{}^a$ in $\tilde{U}$. On the other hand, $C_\alpha=\partial/\partial u^\alpha$ form a natural frame tangent to each fibre F in $F\cap\tilde{U}$. Denoting by $C^h{}_\alpha$ components of C_α in $\tilde{U}$, we put $C_i{}^\alpha=\tilde{g}_{ih}\tilde{g}^{\alpha\beta}C^h{}_\beta$, where $\tilde{g}_{ji}$ are components of $\tilde{g}$ in $\tilde{U}$, $\tilde{g}_{\gamma\beta}=\tilde{g}_{ji}C^j{}_\gamma C^i{}_\beta$ and $(\tilde{g}^{\gamma\beta})=(\tilde{g}_{\gamma\beta})^{-1}$. We now denote by C^α local covector fields with components $C_i{}^\alpha$ in $\tilde{U}$. We next define $E^h{}_a$ by $(E^h{}_a, C^h{}_\alpha)=(E_i{}^a, C_i{}^\alpha)^{-1}$ and denote by E_a local vector fields with components $E^h{}_a$ in $\tilde{U}$. Then $\{E_b, C_\beta\}$ is a local frame in $\tilde{U}$ and $\{E^a, C^\alpha\}$ the coframe dual to $\{E_b, C_\beta\}$ in $\tilde{U}$. we now obtain

$$\begin{aligned} &\mathcal{L}_{C_\beta}E^a=0, \qquad \mathcal{L}_{C_\beta}E_b=-P_{b\beta}{}^\alpha C_\alpha, \\ &\mathcal{L}_{C_\beta}C_\alpha=0, \qquad \mathcal{L}_{C_\beta}C^\alpha=P_{b\beta}{}^\alpha E^b, \end{aligned} \tag{1.3}$$

$\mathcal{L}_{\tilde{X}}$ denoting the Lie derivation with respect to a vector field $\tilde{X}$ in $\tilde{M}$, where $P_{b\beta}{}^\alpha$ are local functions given in $\tilde{U}$ by

$$P_{b\beta}{}^\alpha=(\partial_b a_\beta)a^\alpha+(\partial_b b_\beta)b^\alpha+(\partial_b c_\beta)c^\alpha, \tag{1.4}$$

∂_b being defined by $\partial_b=E^i{}_b\partial_i$ in $\tilde{M}$, and $\xi=a^\alpha C_\alpha$, $\eta=b^\alpha C_\alpha$, $\zeta=c^\alpha C_\alpha$, $a_\beta=\tilde{g}_{\beta\alpha}a^\alpha$, $b_\beta=\tilde{g}_{\beta\alpha}b^\alpha$, $c_\beta=\tilde{g}_{\beta\alpha}c^\alpha$ in $\tilde{U}$ (See [6]).

A tensor field, say $\tilde{T}$ of type (1, 2), in $\tilde{M}$ is represented in $\tilde{U}$ as

$$\tilde{T}=T_{cb}{}^{a}E^{c}\otimes E^{b}\otimes E_{a}+T_{cb}{}^{\alpha}E^{c}\otimes E^{b}\otimes C_{\alpha}+\cdots$$

$$+T_{\gamma\beta}{}^{a}C^{\gamma}\otimes C^{\beta}\otimes E_{a}+T_{\gamma\beta}{}^{\alpha}C^{\gamma}\otimes C^{\beta}\otimes C_{\alpha},$$

where $T_{cb}{}^{a}$, $T_{cb}{}^{\alpha}$, $\cdots$, $T_{\gamma\beta}{}^{a}$ and $T_{\gamma\beta}{}^{\alpha}$ are local functions in $\tilde{U}$. In the right hand side, the first term $T_{cb}{}^{a}E^{c}\otimes E^{b}\otimes E_{a}$ determines a global tensor field in $\tilde{M}$, which is called the *horizontal part* of $\tilde{T}$ and denoted by $\tilde{T}^{H}$. When $\tilde{T}=\tilde{T}^{H}$, $\tilde{T}$ is said to be *horizontal*. For a function $\tilde{f}$ in $\tilde{M}$, its *horizontal part* $\tilde{f}^{H}$ is defined by $\tilde{f}^{H}=\tilde{f}$.

A tensor field $\tilde{T}$ in $\tilde{M}$ is said to be *projectable* if it satisfies $(\mathcal{L}_{\tilde{X}}\tilde{T}^{H})^{H}=0$, for any vertical vector field $\tilde{X}$, i.e., for any vector field $\tilde{X}$ tangent to the fibre at each point. Then a tensor field $\tilde{T}$ in $\tilde{M}$ is projectable if $\mathcal{L}_{\xi}\tilde{T}=0$, $\mathcal{L}_{\eta}\tilde{T}=0$ and $\mathcal{L}_{\zeta}\tilde{T}=0$. A function $\tilde{f}$ in $\tilde{M}$ is projectable if $\mathcal{L}_{\tilde{X}}\tilde{f}=0$ for any vertical vector field $\tilde{X}$. Thus a function $\tilde{f}$ in $\tilde{M}$ is projectable if and only if it is constant along each fibre. A tensor field, say $\tilde{T}$ of type (1, 2), in $\tilde{M}$ is projectable if and only if the local functions $T_{cb}{}^{a}$ are all contant along $F\cap\tilde{U}, F$ being an arbitrary fibre, where $\tilde{T}^{H}=T_{cb}{}^{a}E^{c}\otimes E^{b}\otimes E_{a}$ in $\tilde{U}$. When $\tilde{f}$ is a projectable function in $\tilde{M}$, there is in M a function f such that $\tilde{f}=f\circ\pi$. The function f is called the *proiection* of $\tilde{f}$ and denoted by $f=p\tilde{f}$. In the sequel, we identify any projectable function $\tilde{f}$, local or global, in $\tilde{M}$ with its projection $p\tilde{f}$. When a tensor field, say $\tilde{T}$ of type (1, 2), in $\tilde{M}$ is projectable, there is in M a tensor field T of the same type as that of $\tilde{T}$ with components $T_{cb}{}^{a}$, which are identified with their projection, where $\tilde{T}^{H}=T_{cb}{}^{a}E^{c}\otimes E^{b}\otimes E_{a}$ in $\tilde{U}$. We call the tensor field T the *projection* of $\tilde{T}$ and denoted it by $T=p\tilde{T}$ (See [6]). Given in $\tilde{M}$ a projectable function $\tilde{f}$, local or global, the local functions $\partial_{b}\tilde{f}=E^{i}{}_{b}\partial_{i}\tilde{f}$ in $\tilde{U}$ is projectable and its projection is $\partial_{b}f=\partial f/\partial v^{b}$ in U, where $f=p\tilde{f}$. In the sequel, we put $\partial_{b}=E^{i}{}_{b}(\partial/\partial x^{i})$ in $\tilde{U}$ and $\partial_{b}=\partial/\partial v^{b}$ in U.

Since ξ,η and ζ are Killing vectors in $(\tilde{M},\tilde{g})$, we have $\mathcal{L}_{\xi}\tilde{g}=0$, $\mathcal{L}_{\eta}\tilde{g}=0$ and $\mathcal{L}_{\zeta}\tilde{g}=0$. Thus $\tilde{g}$ is projectable. We denote by g the projection $p\tilde{g}$ of $\tilde{g}$. Thus we obtain a Riemannian manifold (M,g), which is called the *vase space*. If we put $\tilde{g}^{H}=g_{cb}E^{c}\otimes E^{b}$, then g_{cb} are projectable functions in $\tilde{U}$. Thus g has components g_{cb} in $\{U;v^{a}\}$.

Let $\tilde{T}$ be a projectable tensor field in $\tilde{M}$. Then $\tilde{\nabla}\tilde{T}$ is projectable, $\tilde{\nabla}$ being the the Riemannian connection of $(\tilde{M},\tilde{g})$, and its projection is given by

$$p(\tilde{\nabla}\tilde{T})=\nabla T, \tag{1.5}$$

where $T=p\tilde{T}$ and ∇ denotes the Riemannian connection of the base space (M,g) (See [6]).

We now denote by α,β and γ the 1-forms associated with ξ,η and ζ respectively, for example $\alpha(\tilde{X})=\tilde{g}(\tilde{X},\xi)$ for any vector field $\tilde{X}$ in $\tilde{M}$. If we put

$$\phi=\tilde{\nabla}\xi,\qquad \psi=\tilde{\nabla}\eta,\qquad \theta=\tilde{\nabla}\zeta,$$

$$\Phi=\tilde{\nabla}\alpha,\qquad \Psi=\tilde{\nabla}\beta,\qquad \Theta=\tilde{\nabla}\gamma,$$

then Φ,Ψ and Θ are skew-symmetric tensor fields, i.e., 2-forms in $\tilde{M}$. Moreover, we have

$$\phi\xi=0, \qquad \psi\eta=0, \qquad \theta\zeta=0,$$

$$\theta\eta=-\psi\zeta=\xi, \qquad \phi\zeta=-\theta\xi=\eta, \qquad \psi\xi=-\phi\eta=\zeta,$$

from which,

$$\phi=\phi^H+\gamma\otimes\eta-\beta\otimes\zeta,\ \psi=\psi^H+\alpha\otimes\zeta-\gamma\otimes\xi,\ \theta=\theta^H+\beta\otimes\xi-\alpha\otimes\eta. \tag{1.6}$$

We also have

$$\begin{gathered}(\phi^H)^2=-I^H, \qquad (\psi^H)^2=-I^H, \qquad (\theta^H)^2=-I^H,\\ \theta^H\psi^H=-\psi^H\theta^H=\phi^H,\ \phi^H\theta^H=-\theta^H\phi^H=\psi^H,\ \psi^H\phi^H=-\phi^H\psi^H=\theta^H,\end{gathered} \tag{1.7}$$

where I is the identity tensor field of type (1, 1) in M (See [6]). We have obtained in [6]

$$\begin{gathered}(\mathcal{L}_\xi\varphi^H)^H=0, \qquad (\mathcal{L}_\eta\varphi^H)^H=-2\theta^H, \qquad (\mathcal{L}_\zeta\varphi^H)^H=2\psi^H,\\ (\mathcal{L}_\xi\psi^H)^H=2\theta^H, \qquad (\mathcal{L}_\eta\psi^H)^H=0, \qquad (\mathcal{L}_\zeta\psi^H)^H=-2\phi^H,\\ (\mathcal{L}_\xi\theta^H)^H=-2\psi^H, \qquad (\mathcal{L}_\eta\theta^H)^H=2\phi^H, \qquad (\mathcal{L}_\zeta\theta^H)^H=0.\end{gathered} \tag{1.8}$$

If we put in $\tilde{U}$

$$\phi^H=\phi_b{}^aE^b\otimes E_a,\ \psi^H=\psi_b{}^aE^b\otimes E_a,\ \theta^H=\theta_b{}^aE^b\otimes E_a, \tag{1.9}$$

where $\phi_b{}^a, \psi_b{}^a$ and $\theta_b{}^a$ are local functions in $\tilde{U}$, then we have

$$\Phi^H=\phi_{ba}E^b\otimes E^a,\ \Psi^H=\psi_{ba}E^b\otimes E^a,\ \Theta^H=\theta_{ba}E^b\otimes E^a, \tag{1.10}$$

where $\phi_{ba}=-\phi_{ab}=\phi_b{}^cg_{ca}$, $\psi_{ba}=-\psi_{ab}=\psi_b{}^cg_{ca}$, $\theta_{ba}=-\theta_{ab}=\theta_b{}^cg_{ca}$.

We have already proved in [6] the formulas

$$\begin{gathered}\tilde{\nabla}_jE^h{}_b=\left\{{a \atop c\ b}\right\}E_j{}^cE^h{}_a+h_{cb}{}^\alpha E_j{}^cC^h{}_\alpha-h_b{}^a{}_\beta C_j{}^\beta E^h{}_a,\\ \tilde{\nabla}_jC^h{}_\beta=-h_c{}^a{}_\beta E_j{}^cE^h{}_a+P_{c\beta}{}^\alpha E_j{}^cC^h{}_\alpha+\left\{{\alpha \atop \gamma\ \beta}\right\}C_j{}^\gamma C^h{}_\alpha\end{gathered} \tag{1.11}$$

and

$$\begin{gathered}\tilde{\nabla}_jE_i{}^a=-\left\{{a \atop c\ b}\right\}E_j{}^cE_i{}^b+h_b{}^a{}_\beta(E_j{}^bC_i{}^\beta+C_j{}^\beta E_i{}^b),\\ \tilde{\nabla}_jC_i{}^\alpha=-h_{cb}{}^\alpha E_j{}^cE_i{}^b-P_{c\beta}{}^\alpha E_j{}^cC_i{}^\beta-\left\{{\alpha \atop \gamma\ \beta}\right\}C_j{}^\gamma C_i{}^\beta,\end{gathered} \tag{1.12}$$

where we have put in $\tilde{U}$

$$\begin{Bmatrix} a \\ c\ b \end{Bmatrix} = \frac{1}{2} g^{ae}(\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{cb}),$$

$$\begin{Bmatrix} \alpha \\ \gamma\ \beta \end{Bmatrix} = \frac{1}{2} g^{\alpha\varepsilon}(\partial_\gamma g_{\beta\varepsilon} + \partial_\beta g_{\gamma\varepsilon} - \partial_\varepsilon g_{\gamma\beta}),$$

∂_β being defined by $\partial_\beta = C^i{}_\beta(\partial/\partial x^i) = \partial/\partial u^\beta$ in $\tilde{U}$, and

$$h_{cb}{}^\alpha = -(a^\alpha \phi_{cb} + b^\alpha \psi_{cb} + c^\alpha \theta_{cb}). \tag{1.13}$$

On the other hand, we have from (1.1)

$$\tilde{V}\phi = \alpha \otimes I - \tilde{g} \otimes \xi. \tag{1.14}$$

If we substitute (1.6) into (1.14) and use (1.9), (1.11) and (1.12), then we find

$$\begin{aligned} &\partial_c \phi_b{}^a + \begin{Bmatrix} a \\ c\ e \end{Bmatrix} \phi_b{}^e - \begin{Bmatrix} e \\ c\ b \end{Bmatrix} \phi_e{}^a = 0, \\ &\partial_\gamma \phi_b{}^a + h_b{}^e{}_\gamma \phi_e{}^a - h_e{}^a{}_\gamma \phi_b{}^e = 0. \end{aligned} \tag{1.15}$$

Similarly, we obtain

$$\begin{aligned} &\partial_c \psi_b{}^a + \begin{Bmatrix} a \\ c\ e \end{Bmatrix} \psi_b{}^e - \begin{Bmatrix} e \\ c\ b \end{Bmatrix} \psi_e{}^a = 0,\ \partial_c \theta_b{}^a + \begin{Bmatrix} a \\ c\ e \end{Bmatrix} \theta_b{}^e - \begin{Bmatrix} e \\ c\ b \end{Bmatrix} \theta_e{}^a = 0, \\ &\partial_\gamma \psi_b{}^a + h_b{}^e{}_\gamma \psi_e{}^a - h_e{}^a{}_\gamma \psi_b{}^e = 0,\ \partial_\gamma \theta_b{}^a + h_b{}^e{}_\gamma \theta_e{}^a - h_e{}^a{}_\gamma \theta_b{}^e = 0. \end{aligned} \tag{1.16}$$

If we now take account of (1.6), we find

$$\begin{aligned} &\phi = \phi_b{}^a E^b \otimes E_a + \phi_\beta{}^\alpha C^\beta \otimes C_\alpha,\ \psi = \psi_b{}^a E^b \otimes E_a + \psi_\beta{}^\alpha C^\beta \otimes C_\alpha, \\ &\theta = \theta_b{}^a E^b \otimes E_a + \theta_\beta{}^\alpha C^\beta \otimes C_\alpha, \end{aligned} \tag{1.17}$$

where we have put $\phi_\beta{}^\alpha = c_\beta b^\alpha - b_\beta c^\alpha$, $\psi_\beta{}^\alpha = a_\beta c^\alpha - c_\beta a^\alpha$, $\theta_\beta{}^\alpha = b_\beta a^\alpha - a_\beta b^\alpha$.

§ 2. A structure induced in the base space.

Consider a point P of the base space M and a point σ of $\tilde{M}$ such that $\pi(\sigma) = \mathrm{P}$. We denote by ϕ_σ, ψ_σ and θ_σ respectively the values of ϕ, ψ and θ at σ. Then we can define tensors $\bar{F}_\sigma, \bar{G}_\sigma$ and $\bar{H}_\sigma$ of type (1, 1) at $\mathrm{P} \in M$ respectively by

$$\bar{F}_\sigma A = d\pi(\phi_\sigma A^L),\ \bar{G}_\sigma A = d\pi(\psi_\sigma A^L),\ \bar{H}_\sigma A = d\pi(\theta_\sigma A^L) \tag{2.1}$$

for any vector A tagent to M at P, $d\pi$ being the differential of $\pi: \tilde{M} \to M$, where A^L denotes the horizontal lift of A at σ, i.e., the unique horizontal vector tagent to $\tilde{M}$ at σ such that $d\pi(A^L) = A$. We now denote by V_{P} the linear closure of the set

$$(\bigcup_{\sigma\in\pi^{-1}(P)}\bar{F}_\sigma)\cup(\bigcup_{\sigma\in\pi^{-1}(P)}\bar{G}_\sigma)\cup(\bigcup_{\sigma\in\pi^{-1}(P)}\bar{H}_\sigma)$$

of tensors of type (1, 1) at $P\in M$ and put $V=\cup_{P\in M}V_P$, which is a linear subbundle of the tensor bundle of type (1, 1) over M. Any element L of V, if $L\in V_P$, satisfies $g_P(LA, B)+g_P(LB, A)=0$ for any vectors A and B tangent to M at P, where g_P is the value of g at P, because Φ, Ψ and Θ appearing in § 1 are skew-symmetric.

Take a coordinate neighborhood $\{U, v^a\}$ of M and consider a local cross-section τ of $\tilde{M}$ over U, that is, a mapping τ: $U\to\tilde{M}$ such that $\pi\circ\tau$ is the identity mapping of U. If we put

$$F_P=\bar{F}_{\tau(P)},\qquad G_P=\bar{G}_{\tau(P)},\qquad H_P=\bar{H}_{\tau(P)},\qquad P\in U, \tag{2.2}$$

then the correspondences $P\to F_P$, $P\to G_P$ and $P\to H_P$ $(P\in U)$ define respectively local tensor fields F, G and H of type (1, 1) U. If we take account of (1, 7), we obtain

$$\begin{gathered} F^2=-I,\qquad G^2=-I,\qquad H^2=-I,\\ HG=-GH=F,\qquad FH=-HF=G,\qquad GF=-FG=H \end{gathered} \tag{2.3}$$

in U, where I denotes the identity tensor field of type (1, 1) in M. Since Φ, Ψ and Θ appearing in § 1 are skew-symmetric, F, G and H are almost Hermitian structures in U with respect to g. Summing up, we see that there is a triple $\{F, G, H\}$ of local almost Hermitian structures in (U, g) which satisfies (2. 3) if there is given a local cross-section τ of $\tilde{M}$ over U. Moreover, if we take account of (2. 4), which will be given later, we see that $\{F, G, H\}$ is in U a local base of the bundle V.

We take another local cross-section τ' of $\tilde{M}$ in U'. Then we can construct a triple $\{F', G', H'\}$ of local almost Hermitian structures in (U', g) in the same way as above, i.e., $F'_P=\bar{F}_{\tau'(P)}, G'_P=\bar{G}_{\tau'(P)}, H'_P=\bar{H}_{\tau'(P)}, P\in U$. Thus, if $U\cap U'\neq\phi$, taking account of (1. 8), we find in $U\cap U'$

$$\begin{aligned} F'&=s_{11}F+s_{12}G+s_{13}H,\\ G'&=s_{21}F+s_{22}G+s_{23}H,\\ H'&=s_{31}F+s_{32}G+s_{33}H \end{aligned} \tag{2.4}$$

with functions $s_{\gamma\beta}$ in $U\cap U'$, where the matrix $S'_{U,U'}=(s_{\gamma\beta})$ at each point of $U\cap U'$ belongs to the proper orthogonal groups $SO(3)$ of dimension 3, because both of $\{F, G, H\}$ and $\{F', G', H'\}$ satisfy (2. 3).

Using a local cross-section τ: $U\to\tilde{M}$, we construct in $\{U, v^a\}$ a local base $\{F, G, H\}$ of V in the same as above. If we assume that $\tau(U)\subset\tilde{U}$ and that x^h are local coordinates in $\tilde{U}$, then we may assume that the local cross-section τ is expressed as $x^h=\tau^h(v^a)$ with differentiable functions $\tau^h(v^a)$, where $(\tau^h(v^a))$ denote coordinates of the point τ(P) and (v^a) those of $P\in U$. Thus we have

$$(\partial_b\tau^h)E_h{}^a=\delta_b^a \tag{2.5}$$

along $\tau(U)$, because $\pi\circ\tau$ is the identity mapping of U.

Next, taking account of (1.17) and (2.1), we have from (2.2)

$$F_b{}^a(v)=\phi_b{}^a(\tau^h(v)), \tag{2.6}$$

where $F_b{}^a(v)$ denote components of F defined by (2.2) at a point $\mathrm{P}\in U$ having coordinates (v^a). Differentiating (2.6) with respect to v^c and using (2.5), we find

$$\begin{aligned}\partial_c F_b{}^a&=(\partial_h\phi_b{}^a)(\partial_c\tau^h)=((\partial_e\phi_b{}^a)E_h{}^e+(\partial_\gamma\phi_b{}^a)C_h{}^\gamma)(\partial_c\tau^h)\\&=\partial_c\phi_b{}^a+(\partial_c\tau^h)C_h{}^\gamma\partial_\gamma\phi_b{}^a,\end{aligned}$$

from which, using (1.13), (1.15) and (2.6),

$$\begin{aligned}\nabla_c F_b{}^c&=\partial_c F_b{}^a+\left\{{a \atop c\ e}\right\}F_b{}^e-\left\{{e \atop c\ b}\right\}F_e{}^a\\&=r_c G_b{}^a-q_c H_b{}^a,\end{aligned}$$

where we have put $q_c=-b_\gamma C_h{}^\gamma\partial_c\tau^h$ and $r_c=-c_\gamma C_h{}^\gamma\partial_c\tau^h$. Thus we find

$$\nabla_X F=r(X)G-q(X)H$$

for any vector field X in U, where q and r are certain local 1-forms defind in U. Similarly, using (1.15) and (1.16), we obtain in U

$$\begin{aligned}\nabla_X F&=\qquad\qquad r(X)G-q(X)H,\\ \nabla_X G&=-r(X)F\qquad\qquad+p(X)H,\\ \nabla_X H&=q(X)F\quad-p(X)H\end{aligned}\tag{2.7}$$

for any vector field X in M, where p, q and r are local 1-forms defined in U.

§ 3. Quaternion Kählerian maniforlds.

We are now going to define a structure which we call a quaternion Kählerian structure. Let (M, g) be a Riemannian manifold. Assume that there is over M a vector bundle V consisting of tensors of type (1, 1) such that any element L of V, if L belongs to the fibre V_{P} of V at $\mathrm{P}\in M$, satisfies $g_{\mathrm{P}}(LA, B)+g_{\mathrm{P}}(LB, A)=0$ for any vectors A and B tangent to M at P, where g_{P} denotes the value of g at P. Moreover, we suppose that the bundle V satisfies the following condition:

(a) In any coordinate neighborhood U and M, there is a local base $\{F, G, H\}$ of V such that F, G and H satisfy the condition (2.3).

Such a local base $\{F, G, H\}$ of V is called *canonical local base* of V in U. Then the set $\{g, V\}$ is called an *almost quaternion metric structure*. In such a case, M is necessarily of dimension $n=4m (m\geqq 1)$ (See [5]).

In a Riemannian manifold (M, g) with almost quaternion metric structure $\{g, V\}$, we take intersecting coordinate neighborhoods U and U'. Let $\{F, G, H\}$ and

$\{F', G', H'\}$ be canonical local bases of V in U and in U', respectively. Then $\{F, G, H\}$ and $\{F', G', H'\}$ satisfy in $U\cap U'$ the condition (2.4) with $S_{U,U'}=(s_{\gamma\beta})\in SO(3)$, because F', G' and H' are linear combinations of F, G and H. And both of $\{F, G, H\}$ and $\{F', G', H'\}$ satisfy (2.3). Thus, taking account of the arguments developed in § 2, we have

PROPOSITION 1. *The base space (M, g) of a fibred Riemannian space with Sasakian 3-structure admits an almost quaternion metric structure $\{g, V\}$.*

When the Riemannian connection ∇ of a Riemannian manifold (M, g) with almost quaternion metric structure $\{g, V\}$ satisfies (2.7) for any local base $\{F, G, H\}$ of V and for any vector field X in M, $\{g, V\}$ is called a *quaternion Kählerian structure* and a set (M, g, V) of such a manifold M and such an almost quaternion metric structure $\{g, V\}$ a *quaternion Kählerian manifold* (See [5]). Thus we have

PROPOSITION 2. *The base space (M, g) of a fibred Riemannian space with Sasakian 3-structure admits a quaternion Kählerian structure $\{g, V\}$, that is, (M, g, V) is a quaternion Kählerian manifold.*

We now give a tipical example of quaternion Kählerian manifolds. Let S^{4m+3} be a unit sphere of curvature 1 and of dimension $4m+3 (m\geqq 1)$ and $\pi: S^{4m+3}\to HP(m)$ the natural projection of S^{4m+3} onto a quaternion projective space $HP(m)$. As is well known, S^{4m+3} admito a Sasakian 3-structure $\{\xi, \eta, \zeta\}$ and any fibre $\pi^{-1}(\mathrm{P})$, $\mathrm{P}\in HP(m)$, is a maximal integral manifold of the distribution D spanned by ξ, η and ζ. Thus, $HP(m)$ is the base space of a fibred Riemannian space with Sasakian 3-structure. Therefore $HP(m)$ admits the induced quaternion Kählerian structure $\{g, V\}$. We have already seen in [5] that the curvature tensor K of $HP(m)$ has local components of the form

$$
\begin{aligned}
K_{dcb}{}^{a} = {} & \delta_d^a g_{cb} - \delta_c^a g_{db} + F_d{}^a F_{cb} - F_c{}^a F_{db} - 2F_{dc}F_b{}^a \\
& + G_d{}^a G_{cb} - G_c{}^a G_{db} - 2G_{dc}G_b{}^a + H_d{}^a H_{cb} - H_c{}^a H_{db} - 2H_{dc}H_b{}^a,
\end{aligned} \tag{3.1}
$$

$g_{cb}, F_b{}^a, G_b{}^a$ and $H_b{}^a$ being respectively components of g, F, G and H, where $F_{cb} = F_c{}^e g_{eb}$, $G_{cb}=G_c{}^e g_{eb}$, and $H_{cb}=H_c{}^e g_{eb}$. The F, G and H are locally defined, but the right-hand side of (3.1) is globally defined (See [5]). The linear holonomy group of $HP(m)$ coincides with $Sp(m)\cdot Sp(1)$ itself (See [1], [2], [3] and [5]).

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TOKYO INSTITUTE OF TECHNOLOGY.