Calibrations in hyperkähler geometry

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Abstract

We describe a family of calibrations arising naturally on a hyperkähler manifold M. These calibrations calibrate the holomorphic Lagrangian, holomorphic isotropic and holomorphic coisotropic subvarieties. When M is an HKT (hyperkähler with torsion) manifold with holonomy $SL(n, \mathbb{H})$, we construct another family of calibrations Φ_i, which calibrates holomorphic Lagrangian and holomorphic coisotropic subvarieties. The calibrations Φ_i are (generally speaking) not parallel with respect to any torsion-free connection on M.

Contents

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1 Introduction

The theory of calibrations was developed by R. Harvey and B. Lawson in [HL], and proved to be very useful in describing the geometric structures associated with special holonomies. Since then calibrations have become a central notion in many geometric developments in string physics and M-theory. Up to dimension 8, the calibrations are thoroughly studied and pretty much understood ([DHM]), but in the higher dimensions, the classification problem seems to be immense. Even in more special situations, such as in hyperkähler geometry, the problem of classification of natural[1] calibrations is unsolved.

On a Kähler manifold, the normalized power of the Kähler form $\frac{\omega^p}{p!}$ is a calibration. A subvariety is complex analytic if and only if it is calibrated. This is actually very easy to see, because a subspace $V \subset TM$ is a face of $\frac{\omega^p}{p!}$ if and only if V is complex linear (this follows from the so-called "Wirtinger inequalities", see e.g. [HL]).

In this paper we study a family of calibrations which appear naturally in quaternionic geometry, and describe the corresponding calibrated subvarieties. These calibrations are in many ways analogous to the powers of the Kähler form. We define several new calibrations, for hyperkähler, hypercomplex and HKT-geometry. From the calibration-theoretic point of view, the last of these is most interesting, because it is (generally speaking) not preserved by *any* torsionless connection on M. Some of these forms were considered previously in [V6, AV2, V7].

In hyperkähler geometry, the role of a Kähler form is played by a 4-form $\Theta := \omega_I^2 + \omega_J^2 + \omega_K^2$. In Section 5.2 we show that the normalized powers Θ^p are calibrations. It is easy to see that $V \subset TM$ is a face of Θ if and only if V is a quaternionic subspace (Theorem 5.3).

[1]For hyperkähler geometry, "natural" would mean "$Sp(n)$-invariant".

The corresponding calibrated subvarieties are those which are complex analytic with respect to I, J and K. Such subvarieties are called **trianalytic**. In [V1, V2], the theory of trianalytic subvarieties was developed to some extent. It was shown that the trianalytic subvarieties admit a canonical desingularizaton, which is hyperkähler. Also it was shown that any complex analytic subvariety of (M, I) is trianalytic, if the complex structure I is generic in its twistor family.

Any homogeneous polynomial $P(x, y, z)$ of degree p gives a closed $2p$-form $P(\omega_I, \omega_J, \omega_K)$ on M, and (when the holonomy of M is maximal) all parallel differential forms on M are obtained this way. When $P(x, y, z) = \frac{x^p}{p!}$, it is a Kähler calibration, when $P(x, y, z) = c_p(x^2 + y^2 + z^2)^p$, where $c_p = \sum_{k=0}^{p} \frac{(p!)^2}{(k!)^2}(2k)!4^{p-k}$, it is the trianalytic calibration defined above(Theorem 5.3). It would be interesting to classify all calibrations obtained this way.

The calibrations Ψ_k and Φ_{n+k} we study in this paper are also polynomials on $\omega_I, \omega_J, \omega_K$. These calibrations are called **holomorphic Lagrangian**, **holomorphic isotropic** and **holomorphic coisotropic calibrations**. The form Ψ_k is obtained as a (k, k)-component of $\operatorname{Re}(\omega_I - \sqrt{-1}\,\omega_K)^k$, normalized in appropriate way, where $\omega_I - \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, J), and the (k, k)-part is taken with respect to the complex structure I. In [V6, AV2] it was proven that this form is closed and weakly positive.

We show in Section 5.4 that a subvariety $Z \subset M$ is calibrated by Ψ_k if and only if Z is holomorphic Lagrangian in (M, I) (for $k = \frac{1}{2}\dim_{\mathbb{C}} M$) and isotropic (for $k < \frac{1}{2}\dim_{\mathbb{C}} M$) (Proposition 5.5, Proposition 5.8). Note that holomorphic Lagrangian calibrations have been found previously in [BrH] in dimension eight.

In [F] a different holomorphic Lagrangian calibration in any dimension was constructed as part of an investigation relating the faces of some calibrations to intersecting supersymmetric branes in M-theory. In String Theory the holomorphic Lagrangian submanifolds were related to 3-dimensional topological field theory with target hyperkähler manifold [KRS]. In Section 5.6 we provide some examples of holomorphic Lagrangian subvarieties of hypercomplex manifolds which are not hyperkähler.

The proof of this result relies on a particular partial order defined on the set of precalibrations. We say that $\eta \preceq \eta_1$ if all faces of η are also faces of η_1. For instance, the calibrations $c_p\Theta^p$, $c_p = \sum_{k=0}^{p} \frac{(p!)^2}{(k!)^2}(2k)!4^{p-k}$, and $\frac{\omega_I^k}{k!}$ defined above can be compared:

$$c_p\Theta^p \preceq \frac{\omega_I^{2p}}{(2p)!}$$

because the faces of $c_p\Theta^p$ are quaternionic subspaces in TM, and the faces of $\frac{\omega_I^{2p}}{(2p)!}$ are complex subspaces (Theorem 5.3).

Let ρ be a precalibration on a complex manifold (Definition 2.2), and $\rho^{p,p}$ be its (p,p)-part. We show that a plane $V \subset TM$ is a face of $\rho^{p,p}$ if and only for $\zeta(V)$ is a face of ρ for all $\zeta \in U(1)$, for the standard $U(1)$-action on TM (Theorem 5.2).

Applying this result to the special Lagrangian calibration on (M,J) defined in [HL] (see also [McL]), we obtain the form Ψ_n, $n = \dim_{\mathbb{H}} M$, which calibrates complex analytic Lagrangian subvarieties on (M,I) (these subvarieties are known to be special Lagrangian on (M,J); see e.g. [Hit]). This argument is not hard to generalize to arbitrary dimension.

In most cases listed in [HL] and elsewhere, a calibration form is parallel with respect to the Levi-Civita connection. An interesting side effect of our construction of holomorphic Lagrangian calibrations is an appearance of a family of calibrations which are not parallel, under any torsionless connection (Claim 6.6). These calibrations are associated with the so-called HKT structures in hypercomplex geometry. In physics the HKT manifolds appear as target manifolds with $N = (4,0)$ supersymmetric σ-models with Wess-Zumino term [HP].

We construct calibrations on a special class of hypercomplex manifolds with holonomy of its Obata connection in $SL(n,\mathbb{H})$, the commutator subgroup of $GL(n,\mathbb{H})$. Such manifolds are called **$SL(n,\mathbb{H})$-manifolds**. For more examples and an introduction to $SL(n,\mathbb{H})$-geometry, see Section 3. For any $SL(n,\mathbb{H})$-manifold M, and an induced complex structure I, there is a holomorphic volume form $\Phi \in \Lambda^{2n,0}(M,J)$, which is parallel with respect to the Obata connection ([V5], [BDV]). The space V of parallel holomorphic volume forms is 1-dimensional. A choice of an auxiliary induced complex structure such that $I \circ J = -J \circ I$ endows V with a real structure and a positive direction (Subsection 4.2). We choose Φ to be real and positive. Denote by $\Pi_I^{n,n}$ the projection to (n,n)-component with respect to the complex structure I, such that $I \circ J = -J \circ I$.

In Section 6 we show that $\mathrm{Re}(\Pi_I^{n,n}\Phi)$ is a calibration for any quaternionic Hermitian metric g for which $|\Phi| = 2^n$ (Theorem 6.1). This calibration calibrates complex subvarieties of $Z \subset (M,I)$ which are Lagrangian with respect to the $(2,0)$-form $\Omega = \omega_J + \sqrt{-1}\,\omega_K$, defined as in (2.2).

This calibration is defined for any quaternionic Hermitian metric, subject to the condition $|\Phi| = 1$ (and there are always many). When (M, I, J, K, Φ, g) is an HKT manifold with $\operatorname{Hol}(M) \subset SL(n, \mathbb{H})$, more calibrations can be defined.

We choose Φ to be positive, real $(2n, 0)$-form on (M, J), and let $\Phi_n := \operatorname{Re}\Pi_I^{n,n}(\Phi)$. In [V7] it was shown that the form $\Phi_{n+k} := \frac{1}{2^k k!}\Phi_n \wedge \omega_I^k$ is always closed and positive (Proposition 4.7). In Theorem 6.2, we prove that this form is a calibration, for a metric $g' := g \cdot \left|\frac{\Phi_{n+k}}{2^n}\right|^{(2n+2k)^{-1}}$, conformally equivalent to g. When g is also balanced, $|\Phi| = const$, the conformal weight $\left|\frac{\Phi_{n+k}}{2^n}\right|^{(2n+2k)^{-1}}$ is constant (Theorem 6.1), and g' is also HKT, but otherwise g' is not an HKT metric. In either case, the calibration Φ_{n+k} is (generally speaking) not parallel with respect to any connection on M (Claim 6.6).

We show that Φ_{n+k} calibrates complex subvarieties of (M, I) which are coisotropic with respect to the (2,0)-form $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ (Theorem 6.4). The situation with *isotropic* subvarieties is completely different. Using the examples from Section 5.6, we notice in Remark 6.5 that complex isotropic submanifolds in this case do not have to be calibrated by any form, since they could be homologous to zero.

2 Preliminaries

2.1 Calibrations in Riemannian geometry

We provide here the basic definitions of the theory of calibrations which we use in the paper. The standard reference for this material is [HL] and the reader may also consult [J2] for recent progress and developments related to manifolds with restricted holonomy.

Definition 2.1: Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $\operatorname{Vol}(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let **comass** $\mathsf{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all p-tuples $(v_1, ..., v_p)$ of vectors in V and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \mathsf{comass}(\eta)$.

Definition 2.2: A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

Definition 2.3: A **calibration** is a precalibration which is closed.

Definition 2.4: Let η be a k-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a k-dimensional subvariety (we usually assume that the Hausdorff dimension of the set of singular points of Z is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over Z). We say that Z is **calibrated by** η if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration η.

Remark 2.5: Clearly, for any precalibration η,

$$\mathrm{Vol}(Z) \geqslant \int_Z \eta, \tag{2.1}$$

where $\mathrm{Vol}(Z)$ denotes the Riemannian volume of a compact Z, and the equality happens iff Z is calibrated by η. If, in addition, η is closed, $\int_Z \eta$ is a cohomological invariant, and the inequality (2.1) implies that Z minimizes the Riemannian volume in its homology class.

2.2 Hyperkähler manifolds and calibrations

The following definitions are standard.

Definition 2.6: A manifold M is called **hypercomplex** if M is equipped with a triple of complex structures I, J, K, satisfying the quaternionic relations $I \circ J = -J \circ I = K$. If, in addition, M is equipped with a Riemannian metric g which is Kähler with respect to I, J, K, (M, I, J, K, g) is called **hyperkähler**. This is equivalent to $\nabla I = \nabla J = \nabla K = 0$, where ∇ is the Levi-Civita connection of g; see [Bes].

Remark 2.7: It has been known since 1955 that any hypercomplex manifold admits a torsion-free connection preserving I, J and K, which is necessarily unique. This connection is called **the Obata connection**, after M. Obata, who discovered it in [Ob]. Any almost complex structure which is preserved by a torsion-free connection is necessarily integrable (this is an easy consequence of Newlander-Nirenberg theorem). Therefore, for any $a, b, c \in \mathbb{R}$, with $a^2 + b^2 + c^2 = 1$, the almost complex structure $aI + bJ + cK$ is in fact integrable. We denote by (M, L) the manifold M considered as a complex manifold with the complex structure induced by $L = aI + bJ + cK$.

Definition 2.8: Such complex structures are called **induced by quaternions**, and the corresponding family, parametrized by S^2 – **the twistor family**, or **the hypercomplex family**. This family is holomorphic, and its total space (fibered over $\mathbb{C}P^1$) is called **the twistor space of** M. It is a complex analytic space, non-Kähler even in simplest cases (for M a torus or a K3 surface).

Hyperkähler geometry has a long history and is already well established. For more details and background definitions, please see [Bes, J2]. In algebraic geometry, the word *hyperkähler* is essentialy synonymous with "holomorphic symplectic". The reason is that any hyperkähler manifold is equipped with a complex-valued form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$.[1] This form has Hodge type (2,0) on (M, I) and is closed, hence holomorphically symplectic.

The converse follows from the Yau's proof of Calabi's conjecture: a holomorphically symplectic, Kähler manifold admits a unique hyperkähler metric in a given Kähler class ([Bes]). For survey of recent advances in hyperkähler geometry see [H1, H2].

Some of the main objects of this paper are holomorphic Lagrangian, isotropic and coisotropic subvarieties of (M, I), where (M, I, J, K, g) is hyperkähler.

Definition 2.9: A complex analytic subvariety Z of a holomorphically symplectic manifold (M, Ω) is called **holomorphic Lagrangian** if $\Omega\big|_Z = 0$, and $\dim_{\mathbb{C}} Z = \frac{1}{2}\dim_{\mathbb{C}} M$, and **isotropic** if $\Omega\big|_Z = 0$, and $\dim_{\mathbb{C}} Z < \frac{1}{2}\dim_{\mathbb{C}} M$. It is called **coisotropic** if Ω has rank $\frac{1}{2}\dim_{\mathbb{C}} M - \operatorname{codim}_{\mathbb{C}} Z$ on TZ in all smooth points of Z, which is the minimal possible rank for a $2n-p$-dimensional subspace in a $2n$-dimensional symplectic space.

2.3 Calibrations in HKT-geometry

Let (M, I, J, K) be a hypercomplex manifold. Then the tangent bundle TM is equipped with a natural quaternionic action. In particular, the group $SU(2)$ of unitary quaternions acts on TM, in a canonical way. A Riemannian metric on M is called **quaternionic Hermitian** if it is $SU(2)$-invariant. A hyperkähler metric is obviously quaternionic Hermitian, but the converse is manifestly false, as we shall explain presently.

[1] We always write $\omega_I, \omega_J, \omega_K$ for the corresponding Kähler forms.

With every quaternionic Hermitian metric g we associate 2-forms $\omega_I := g(I\cdot,\cdot), \omega_J := g(J\cdot,\cdot)$ and $\omega_K := g(K\cdot,\cdot)$ which are clearly antisymmetric, because g is $SU(2)$-invariant. It is easy to check that

$$\Omega := \omega_J + \sqrt{-1}\,\omega_K \tag{2.2}$$

is a (2,0)-form on (M, I). This form is closed if and only if (M, I, J, K, g) is hyperkähler ([Bes]).

For a weaker form of this condition, consider the $(1,0)$-part of the de Rham differential,

$$\partial : \ \Lambda^{p,q}(M, I) \longrightarrow \Lambda^{p+1,q}(M).$$

A quaternionic Hermitian hypercomplex manifold is called **HKT** (short for "hyperkähler with torsion") if $\partial\Omega = 0$.

The theory of HKT-manifolds is a rapidly developing subfield of quaternionic geometry. Originally this notion appeared in physics ([HP]), but mathematicians found it very useful. For an early survey of HKT-geometry, please see [GP].

Another ingredient of an HKT calibration theory is the notion of Obata connection (Remark 2.7). Since this connection preserves the quaternionic structure, its holonomy $\operatorname{Hol}(M)$ lies in $GL(n, \mathbb{H})$. The holonomy of the Obata connection is one of the most important invariants of a hypercomplex manifold. Many properties of M can be related directly to its holonomy group. In particular, the group $\operatorname{Hol}(M)$ is compact if and only if (M, I, J, K) admits a hyperkähler metric.

There seems to be no holonomy characterization of HKT structures. In fact the holonomy of Obata connection is rarely known explicitly, except on hyperkähler manifolds, where it is equal to the Levi-Civita connection. However the knowledge of holonomy is still quite useful for the study of HKT geometry. For many examples of compact hypercomplex manifolds, the group $\operatorname{Hol}(M) \subset GL(n, \mathbb{H})$ is strictly smaller than $GL(n, \mathbb{H})$. Only recently it was found that the group $SU(3)$ with the left-invariant hypercomplex structure has $GL(n, \mathbb{H})$ as its holonomy group ([Sol]).

An important subgroup inside $GL(n, \mathbb{H})$ is its commutator $SL(n, \mathbb{H})$. This group can be defined as a group of quaternionic matrices $A \subset \operatorname{End}(\mathbb{H}^n)$ preserving a non-zero complex-valued form $\Phi \in \Lambda^{2n,0}_{\mathbb{C}}(\mathbb{H}^n_I)$, where $\mathbb{H}^n_I$ is $\mathbb{H}^n$ considered as a $2n$-dimensional complex space, with the complex structure I induced by quaternions. The coefficient $\lambda := \frac{A(\Phi)}{\Phi}$ is called **the Moore determinant** of the matrix A ([A], [AV1]); it is always a positive real number, with λ^4 equal to the determinant of A, considered as an element of $GL(4n, \mathbb{R})$. The group $SL(n, \mathbb{H})$ is a group of quaternionic matrices with Moore determinant 1.

3 $SL(n,\mathbb{H})$-manifolds

3.1 An introduction to $SL(n,\mathbb{H})$-geometry

As Obata has shown ([Ob]), a hypercomplex manifold (M,I,J,K) admits a necessarily unique torsion-free connection, preserving I,J,K. The converse is also true: if a manifold M equipped with an action of $\mathbb{H}$ admits a torsion-free connection preserving the quaternionic action, it is hypercomplex. This implies that a hypercomplex structure on a manifold can be defined as a torsion-free connection with holonomy in $GL(n,\mathbb{H})$. This connection is called **the Obata connection** on a hypercomplex manifold.

Connections with restricted holonomy are one of the central notions in Riemannian geometry, due to Berger's classification of irreducible holonomy of Riemannian manifolds. However, a similar classification exists for general torsion-free connections ([MS]). In the Merkulov-Schwachhöfer list, only three subroups of $GL(n,\mathbb{H})$ occur. In addition to the compact group $Sp(n)$ (which defines hyperkähler geometry), also $GL(n,\mathbb{H})$ and its commutator $SL(n,\mathbb{H})$ appear, corresponding to hypercomplex manifolds and hypercomplex manifolds with trivial determinant bundle, respectively. Both of these geometries are interesting, rich in structure and examples, and deserve detailed study.

It is easy to see that (M,I) has holomorphically trivial canonical bundle, for any $SL(n,\mathbb{H})$-manifold (M,I,J,K) ([V5]). For a hypercomplex manifold with trivial canonical bundle admitting an HKT metric, a version of Hodge theory was constructed ([V3]). Using this result, it was shown that a compact hypercomplex manifold with trivial canonical bundle has holonomy in $SL(n,\mathbb{H})$, if it admits an HKT-structure ([V5]).

In [BDV], it was shown that holonomy of all hypercomplex nilmanifolds lies in $SL(n,\mathbb{H})$. Many working examples of hypercomplex manifolds are in fact nilmanifolds, and by this result they all belong to the class of $SL(n,\mathbb{H})$-manifolds.

The $SL(n,\mathbb{H})$-manifolds were studied in [AV2] and [V6], because on such manifolds the quaternionic Dolbeault complex is identified with a part of de Rham complex (Proposition 4.7). Under this identification, $\mathbb{H}$-positive forms become positive in the usual sense, and ∂, ∂_J-closed or exact forms become $\partial,\overline{\partial}$-closed or exact (see Section 3.1). This linear-algebraic identification is especially useful in the study of the quaternionic Monge-Ampère equation ([AV2]).

3.2 Balanced HKT-manifolds

The following lemma is contained in [BDV] (Theorem 3.2; see also [V7], Lemma 4.3). Recall that the map $\eta \longrightarrow J(\overline{\eta})$ defines a real structure on $\Lambda^{2p,0}(M,I)$. A $(p,0)$-form η is called **$\mathbb{H}$-real** if $J(\eta) = \overline{\eta}$.

Lemma 3.1: Let (M,I,J,K) be a hypercomplex manifold, and η a top degree $(2n,0)$-form, which is $\mathbb{H}$-real and holomorphic. Then η is Obata-parallel.

■

Definition 3.2: Let (M,I,g) be a complex Hermitian manifold, $\dim_{\mathbb{C}} M = n$, and $\omega \in \Lambda^{1,1}(M)$ its Hermitian form. One says that M is **balanced** if $d(\omega^{n-1}) = 0$.

Remark 3.3: It is easy to see that $d(\omega^m) = 0$ for $1 \leqslant m \leqslant n-2$ implies that ω is Kähler; the balancedness makes sense as the only non-trivial condition of form $d(\omega^m) = 0$ which is not equivalent to the Kähler property.

Theorem 3.4: Let (M,I,J,K,Ω) be an HKT-manifold as in Section 2.3, $\dim_{\mathbb{H}} M = n$. If $\overline{\partial}$ is the standard Dolbeault operator on (M,I), then the following conditions are equivalent.

(i) $\overline{\partial}(\Omega^n) = 0$

(ii) $\nabla(\Omega^n) = 0$, where ∇ is the Obata connection

(iii) The manifold (M,I) with the induced quaternionic Hermitian metric is balanced as a Hermitian manifold:

$$d(\omega_I^{2n-1}) = 0.$$

Proof: [V7], Theorem 4.8. ■

Remark 3.5: A balanced HKT-manifold has holonomy in $SL(n,\mathbb{H})$. This statement follows immediately from the implication (iii) ⇒ (ii) of Theorem 3.4. However the balanced HKT condition is a little stronger. It is shown in [IP] that an HKT manifold has (restricted) holonomy of the Obata connection in $SL(n,\mathbb{H})$ if and only if it is (locally) conformally balanced.

Remark 3.6: The condition $\nabla(\Omega^n) = 0$ is independent from the choice of a basis I, J, K, $IJ = -JI = K$ of $\mathbb{H}$. Indeed, suppose that $g \in SU(n)$, and $(I_1, J_1, K_1) = (g(I), g(J), g(K))$ is a new basis in $\mathbb{H}$. The corresponding HKT-form $\Omega_1 = \omega_{J_1} + \sqrt{-1}\,\omega_{K_1}$ can be expressed as $\Omega_1 = g(\Omega)$, hence

$$\nabla(\Omega_1^n) = \nabla(g(\Omega_1^n)) = g(\nabla(\Omega^n)) = 0.$$

Therefore, Theorem 3.4 leads to the following corollary.

Corollary 3.7: Let (M, I, J, K, Ω) be an HKT-manifold, such that the corresponding complex Hermitian manifold (M, I) is balanced. Then (M, I_1) is balanced for any complex structrure I_1 induced by the quaternions. Moreover, (M, I, J, K, Ω) is an $SL(\mathbb{H}, n)$-manifold. ■

4 Differential forms on hypercomplex manifolds

In this section, we give an introduction to the linear algebraic structures on the de Rham algebra of a hypercomplex manifold. We follow [V6] and [V7].

4.1 The quaternionic Dolbeault complex

It is well-known that any irreducible representation of $SU(2)$ over $\mathbb{C}$ can be obtained as a symmetric power $S^i(V_1)$, where V_1 is a fundamental 2-dimensional representation. We say that a representation W **has weight** i if it is isomorphic to $S^i(V_1)$. A representation is said to be **pure of weight** i if all its irreducible components have weight i.

Remark 4.1: The Clebsch-Gordan formula (see [Hu]) claims that the weight is *multiplicative*, in the following sense: if $i \leqslant j$, then

$$V_i \otimes V_j = \bigoplus_{k=0}^{i} V_{i+j-2k},$$

where $V_i = S^i(V_1)$ denotes the irreducible representation of weight i.

Let M be a hypercomplex manifold, $\dim_{\mathbb{H}} M = n$. There is a natural multiplicative action of $SU(2) \subset \mathbb{H}^*$ on $\Lambda^*(M)$, associated with the hypercomplex structure.

Let $V^i \subset \Lambda^i(M)$ be a maximal $SU(2)$-invariant subspace of weight $< i$. The space V^i is well defined, because it is a sum of all irreducible representations $W \subset \Lambda^i(M)$ of weight $< i$. Since the weight is multiplicative (Remark 4.1), $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$.

It is easy to see that the de Rham differential d increases the weight by 1 at most. Therefore, $dV^i \subset V^{i+1}$, and $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

Definition 4.2: Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$. It is called **the quaternionic Dolbeault algebra of** M, or **the quaternionic Dolbeault complex** (qD-algebra or qD-complex for short).

Remark 4.3: The complex $(\Lambda^*_+(M), d_+)$ was constructed earlier by Capria and Salamon ([CS]) in a different (and more general) situation, and much studied since then.

The Hodge bigrading is compatible with the weight decomposition of $\Lambda^*(M)$, and gives a Hodge decomposition of $\Lambda^*_+(M)$ ([V3]):

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M).$$

The spaces $\Lambda^{p,q}_{+,I}(M)$ are the weight spaces for a particular choice of a Cartan subalgebra in $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of the weight spaces within an irreducible representation. This gives the following result.

Proposition 4.4: Let (M, I, J, K) be a hypercomplex manifold and

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$$

the Hodge decomposition of qD-complex defined above. Then there is a natural isomorphism

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}(M, I). \tag{4.1}$$

Proof: See [V3]. ■

This isomorphism is compatible with a natural algebraic structure on

$$\bigoplus_{p+q=i} \Lambda^{p+q,0}(M, I),$$

and with the Dolbeault differentials, in the following way.

Let (M, I, J, K) be a hypercomplex manifold. We extend

$$J: \ \Lambda^1(M) \longrightarrow \Lambda^1(M)$$

to $\Lambda^*(M)$ by multiplicativity. Recall that

$$J(\Lambda^{p,q}(M, I)) = \Lambda^{q,p}(M, I),$$

because I and J anticommute on $\Lambda^1(M)$. Denote by

$$\partial_J: \ \Lambda^{p,q}(M, I) \longrightarrow \Lambda^{p+1,q}(M, I)$$

the operator $J \circ \overline{\partial} \circ J$, where $\overline{\partial}: \ \Lambda^{p,q}(M, I) \longrightarrow \Lambda^{p,q+1}(M, I)$ is the standard Dolbeault operator on (M, I), that is, the $(0,1)$-part of the de Rham differential. Since $\overline{\partial}^2 = 0$, we have $\partial_J^2 = 0$. In [V3] it was shown that ∂ and ∂_J anticommute:

$$\{\partial_J, \partial\} = 0. \tag{4.2}$$

Consider the quaternionic Dolbeault complex $(\Lambda^*_+(M), d_+)$ constructed in Definition 4.2. Using the Hodge bigrading, we can decompose this complex, obtaining a bicomplex

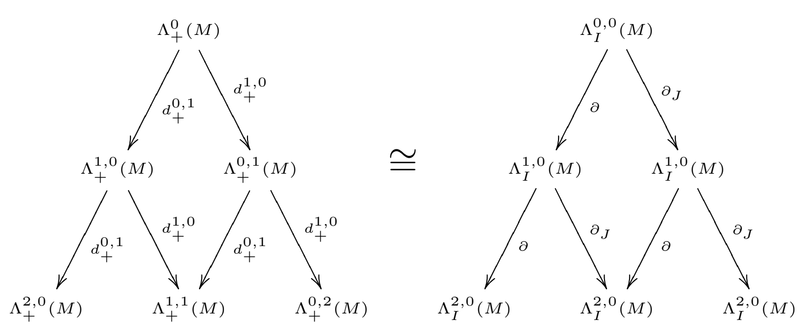
$$\Lambda^{*,*}_{+,I}(M) \xrightarrow{d^{1,0}_{+,I}, d^{0,1}_{+,I}} \Lambda^{*,*}_{+,I}(M)$$

where $d^{1,0}_{+,I}$, $d^{0,1}_{+,I}$ are the Hodge components of the quaternionic Dolbeault differential d_+, taken with respect to I.

Theorem 4.5: Under the multiplicative isomorphism

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}(M, I)$$

constructed in Proposition 4.4, $d^{1,0}_+$ corresponds to ∂ and $d^{0,1}_+$ to ∂_J:



(4.3)

Moreover, under this isomorphism, the form $\omega_I \in \Lambda^{1,1}_{+,I}(M)$ corresponds to $\Omega \in \Lambda^{2,0}_I(M)$.

Proof: See [V3] or [V4]. ■

4.2 Positive $(2,0)$-forms on hypercomplex manifolds

The notion of positive $(2p,0)$-forms on hypercomplex manifolds (sometimes called q-positive, or $\mathbb{H}$-positive) was developed in [AV1] (see also [AV2] and [V6]).

Let $\eta \in \Lambda^{p,q}_I(M)$ be a differential form. Since I and J anticommute, $J(\eta)$ lies in $\Lambda^{q,p}_I(M)$. Clearly, $J^2\Big|_{\Lambda^{p,q}_I(M)} = (-1)^{p+q}$. For $p+q$ even, $J\Big|_{\Lambda^{p,q}_I(M)}$ is an anticomplex involution, that is, a real structure on $\Lambda^{p,q}_I(M)$. A form $\eta \in \Lambda^{2p,0}_I(M)$ is called **real** if $J(\overline{\eta}) = \eta$.

For a real $(2,0)$-form η,

$$\eta\left(x, J(\overline{x})\right)) = \overline{\eta}\left(J(x), J^2(\overline{x})\right) = \overline{\eta}\left(\overline{x}, J(x)\right),$$

for any $x \in T^{1,0}_I(M)$. From the definition of a real form, we obtain that the scalar $\eta\left(x, J(\overline{x})\right)$ is always real.

Definition 4.6: A real $(2,0)$-form η on a hypercomplex manifold is called **positive** if $\eta\left(x, J(\overline{x})\right) \geqslant 0$ for any $x \in T^{1,0}_I(M)$, and **strictly positive** if this inequality is strict, for all $x \neq 0$.

An HKT-form $\Omega \in \Lambda^{2,0}_I(M)$ of any HKT-structure is strictly positive. Moreover, HKT-structures on a hypercomplex manifold are in one-to-one correspondence with ∂-closed, strictly positive $(2,0)$-forms.

The analogy between Kähler forms and HKT-forms can be pushed further; it turns out that any HKT-form $\Omega \in \Lambda^{2,0}_I(M)$ has a local potential $\varphi \in C^\infty(M)$, in such a way that $\partial\partial_J\varphi = \Omega$ ([AV1]). Here $\partial\partial_J$ is a composition of ∂ and ∂_J defined on quaternionic Dolbeault complex as above (these operators anticommute).

4.3 The map $\mathcal{V}_{p,q}:\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{n+p,n+q}_I(M)$ on $SL(n,\mathbb{H})$-manifolds

Let (M,I,J,K) be an $SL(n,\mathbb{H})$-manifold, $\dim_{\mathbb{R}} M = 4n$, and

$$\mathcal{R}_{p,q}:\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{p,q}_{I,+}(M)$$

the isomorphism induced by $\mathfrak{su}(2)$-action as in Theorem 4.5. Consider the projection

$$\Lambda^{p,q}_I(M) \longrightarrow \Lambda^{p,q}_{I,+}(M), \tag{4.4}$$

and let $R:\ \Lambda^{p,q}_I(M) \longrightarrow \Lambda^{p+q,0}_I(M)$ denote the composition of (4.4) and $\mathcal{R}^{-1}_{p,q}$.

Let Φ_I be a nowhere degenerate holomorphic section of $\Lambda^{2n,0}_I(M)$. Assume that Φ_I is real, that is, $J(\Phi_I) = \overline{\Phi}_I$, and positive. Existence of such a form is equivalent to $\operatorname{Hol}(M) \subset SL(n, \mathbb{H})$ (Lemma 3.1). It is often convenient to define $SL(n, \mathbb{H})$-structure by fixing the quaternionic action and the holomorphic form Φ_I.

Define the map

$$\mathcal{V}_{p,q}:\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{n+p,n+q}_I(M)$$

by the relation

$$\mathcal{V}_{p,q}(\eta)\wedge\alpha = \eta\wedge R(\alpha)\wedge\overline{\Phi}_I, \tag{4.5}$$

for any test form $\alpha \in \Lambda^{n-p,n-q}_I(M)$.

The map $\mathcal{V}_{p,p}$ is especially remarkable, because it maps closed, positive $(2p,0)$-forms to closed, positive $(n+p,n+p)$-forms, as the following proposition implies.

Proposition 4.7: Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, and

$$\mathcal{V}_{p,q}:\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{n+p,n+q}_I(M)$$

the map defined above. Then

(i) $\mathcal{V}_{p,q}(\eta) = \mathcal{R}_{p,q}(\eta)\wedge\mathcal{V}_{0,0}(1)$.

(ii) The map $\mathcal{V}_{p,q}$ is injective, for all p, q.

(iii) $(\sqrt{-1}\,)^{(n-p)^2}\mathcal{V}_{p,p}(\eta)$ is real if and only $\eta \in \Lambda^{2p,0}_I(M)$ is real, and positive if and only if η is positive.

(iv) $\mathcal{V}_{p,q}(\partial\eta) = \partial\mathcal{V}_{p-1,q}(\eta)$, and $\mathcal{V}_{p,q}(\partial_J\eta) = \overline{\partial}\mathcal{V}_{p,q-1}(\eta)$.

(v) $\mathcal{V}_{0,0}(1) = \lambda\mathcal{R}_{n,n}(\Phi_I)$, where λ is a positive rational number, depending only on the dimension n.

Proof: See [V6], Proposition 4.2, or [AV2], Theorem 3.6. ■

Remark 4.8: For the purposes of the present paper, we are interested in Proposition 4.7 for the case $\eta = \Omega^k$, where Ω is an HKT-form. In this case, $\mathcal{R}_{p,p}(\Omega^k)$ is a projection of ω_I^k to the component of maximal weight (see Proposition 4.9 below). Now, $\mathcal{V}_{p,q}(\Omega^k) = \mathcal{R}_{p,q}(\Omega^k) \wedge \mathcal{V}_{0,0}(1)$, as follows from Proposition 4.7 (i). However, $\mathcal{V}_{0,0}(1)$ has weight $2n$, by Proposition 4.7 (v), and ω_I^k has weight $\leqslant 2k$, hence their product is of weight $\geqslant 2n - 2k$. Since this product is $(2n-2k)$-form, it is pure of weight $(2n-2k)$, and components of ω_I^k of weight $< 2k$ do not contribute to the product $\omega_I^k \wedge \mathcal{V}_{0,0}(1)$. We obtain that the closed, positive form $\mathcal{V}_{k,k}(\Omega^k)$ is proportional to $\omega_I^k \wedge \mathcal{V}_{0,0}(1)$, with positive coefficient.

4.4 Algebra generated by ω_I, ω_J, ω_K

Let (M, I, J, K, g) be a quaternionic Hermitian manifold. Consider the algebra $A^* = \oplus A^{2i}$ generated by ω_I, ω_J, and ω_K. In [V1], this algebra was computed explicitly. It was shown that, up to the middle degree, A^* is a symmetric algebra with generators ω_I, ω_J, ω_K. The algebra A^* has Hodge bigrading $A^k = \bigoplus_{p+q=k} A^{p,q}$. From the Clebsch-Gordan formula, we obtain that $A^{2i}_+ := \Lambda^{2i}_+(M) \cap A^{2i}$, for $i \leqslant n$, is an orthogonal complement to $Q(A^{2i-4})$, where $Q(\eta) = \eta \wedge (\omega_I^2 + \omega_J^2 + \omega_K^2)$. Moreover, A^{2i}_+ is irreducible as a representation of $SU(2)$. Therefore, the space $A^{p,p}_+ = \ker Q^*\big|_{A^{p,p}}$ is 1-dimensional. This argument also implies that the form $\mathcal{V}_{0,0}(1)$ is proportional to $\Phi_J|_I^{n,n}$, where Φ_J is a holomorphic volume form on (M, J), obtained as a top power of the appropriate holomorphic symplectic form, and $\Phi_J|_I^{n,n}$ its (n,n)-part, taken with respect to I.

Proposition 4.9: Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, equipped with an HKT-structure Ω. Assume that $\Omega^n = \Phi_I$. Let

$$\Pi_+ : \ \Lambda_I^{n+k,n+k}(M) \longrightarrow \Lambda_{I,+}^{n+k,n+k}(M)$$

be the projection to the component of maximal weight with respect to the $SU(2)$-action. Then $\Xi_k := \Pi_+(\omega_I^{n+k,n+k})$ is a closed, weakly positive $(n+k, n+k)$-form, which is proportional to $\omega_I^k \wedge \Phi_J|_I^{n,n}$ and to $\omega_I^k \wedge \mathcal{V}_{0,0}(1)$.

Proof: The form $\omega_I^k \wedge \Phi_J|_I^{n,n}$ is proportional to $\omega_I^k \wedge \mathcal{V}_{0,0}(1)$ as indicated above. Consider the algebra $A^* = \oplus A^{2i}$ generated by ω_I, ω_J, and ω_K. The map $R^{p,q}$ is induced by the $SU(2)$-action, hence it maps $A^{*,*}$ to itself. Since

$\mathcal{V}_{p,q}(\eta) = \mathcal{R}_{p,q}(\eta) \wedge \mathcal{V}_{0,0}(1)$, and $\mathcal{V}_{0,0}(1)$ is proportional to $\mathcal{R}_{n,n}(\Phi_I) \in A^*$, we obtain

$$\mathcal{V}_{p,q}(A^{p+q,0}) \subset A^{n+p,n+q}.$$

Since $\mathcal{V}_{p,p}(\Omega^p) \subset A_+^{n+p,n+p}$, the 1-dimensional space $A_+^{n+p,n+p}$ is generated by $\mathcal{V}_{p,p}(\Omega^p)$. This form is closed and positive by Proposition 4.7. Therefore, the projection of ω_I^{n+p} to $A_+^{n+p,n+p}$ is closed and positive (see Remark 4.8). ■

5 Calibrations on hyperkähler manifolds

5.1 Hodge decomposition and $U(1)$-action

Let I be a complex structure on a vector space V and $\rho: U(1) \longrightarrow \operatorname{End}(V)$ a real $U(1)$-representation given by $\rho(t)(X) = (\cos t + \sin tI)X$. This is extended by multiplicativity to a representation in the tensor powers of V with $\rho(t)(\alpha)(X) = \alpha(\rho(t)X)$ for a 1-form α. In the usual fashion, we define the weight decomposition associated with this $U(1)$-action: the tensor z has weight p if $\rho(t)z = (\cos pt)z + \sqrt{-1}\,(\sin pt)z$. We need also the definition of average over $U(1)$ of Y:

$$\mathsf{Av}_\rho Y = \frac{1}{2\pi}\int_0^{2\pi} \rho(t)Y\,dt.$$

Note that $\rho(t)Y = Y$ for every t implies $IY = Y$ for any tensor Y and that $I\,\mathsf{Av}_\rho Y = \mathsf{Av}_\rho Y$.

Lemma 5.1: Let ρ be a U(1)-action on W, and $W = \bigoplus W^i$ the corresponding weight decomposition. Then the projection to W^0 along the sum of other W^i, $i \neq 0$, coincides with taking the average over $U(1)$.

Proof: For each $\eta \in W^i$, $i \neq 0$, one has $\int_{U(1)} \rho(t)\eta dt = 0$, because $\int_0^{2\pi} \cos(t)dt = 0$. ■

Theorem 5.2: Let η be a $2p$-form on a complex vector space W, with $\mathsf{comass}(\eta) \leqslant 1$, and $\eta^{p,p} = \mathsf{Av}_\rho\,\eta$ be the (p,p)-part of η. Then $\mathsf{comass}(\eta^{p,p}) \leqslant 1$. Moreover, a $2p$-dimensional plane V is a face of $\eta^{p,p}$ if and only if $\rho(t)(V)$ is a face of η for all $t \in \mathbb{R}$.

Proof: For any decomposable $2p$-vector ξ, its image $\rho(t)(\xi)$ is again decomposable for any t and $|\rho(t)(\xi)| = |\xi|$. Then

$$\eta^{(p,p)}(\xi) = (\mathsf{Av}_\rho(\eta))(\xi) = \frac{1}{2\pi}\int_0^{2\pi} \eta(\rho(t)(\xi)) \leqslant 1$$

since $\eta(\rho(t)\xi) \leqslant 1$ for every t. The equality holds iff $\eta(\rho(t)\xi) = 1$ for every t. ■

5.2 An $SU(2)$-invariant calibration

The most obvious example of a calibration on a hyperkähler manifold is provided by the following theorem (see [Ber] for similar statement about a quaternionic Wirtinger's inequality).

Theorem 5.3: Let (M, I, J, K, g) be a hyperkähler manifold, $\omega_I, \omega_J, \omega_K$ the corresponding symplectic forms, and $\Theta_p := \frac{(\omega_I^2+\omega_J^2+\omega_K^2)^p}{c_p}$ the standard $SU(2)$-invariant $4p$-form normalized by $c_p = \sum_{k=1}^p \frac{(p!)^2}{(k!)^2}(2k)!4^{p-k}$. Then Θ_p is a calibration, and its faces are p-dimensional quaternionic subspaces of TM. Moreover, the form $\Xi_p := \frac{(\omega_J^2+\omega_K^2)^p}{(p!)^2 4^p}$ is also a calibration, with the same faces.

Proof: Consider the form $\tilde{\Xi}_p := \frac{\omega_J^{2p}}{(2p)!}$. By Lemma 5.1, $(\tilde{\Xi}_p)_I^{2p,2p} = \Xi_p$, where $(\cdot)_I^{2p,2p}$ is an operation of taking $(2p, 2p)$-part under the complex structure I. Indeed, $\omega_J^{2p} = \frac{(\Omega+\overline{\Omega})^{2p}}{4^p}$, where Ω is the standard $(2, 0)$ form on (M, I). Then the $(2p, 2p)$-part of ω_J^{2p} is equal to

$$\frac{(2p)!}{(p!)^2}\frac{\Omega^p \wedge \overline{\Omega}^p}{4^p} = \frac{(2p)!(\omega_J^2 + \omega_K^2)^p}{(p!)^2 4^p}.$$

By Theorem 5.2, a subspace $V \subset TM$ is a face of Ξ_p if and only if $\rho_I(t)(V)$ is a face of $\tilde{\Xi}_p$ for all t, with $\rho_I(t)$ the $U(1)$-action associated with I. The form $\tilde{\Xi}_p$ is a standard Kähler calibration associated with J; it follows from [HL] that $V \subset TM$ is a face of $\tilde{\Xi}_p$ if and only if it is J-linear, that is, $\mathbb{C}$-linear with respect to the action of $\mathbb{C}$ induced by J. Since $\rho(t)(V)$ is J-linear for all t, it remains J-linear if we act on V by a group G generated by ρ_I and ρ_J, with ρ_J a $U(1)$-action associated with J. Clearly, $G \cong SU(2)$ is the group of unitary quaternions acting on $\Lambda^* M$. Therefore, V is a face of Ξ_p if and only if V is $g(J)$-linear, for all $g \in SU(2)$. This is equivalent to V being a quaternionic subspace. Taking the average of Ξ_p with respect to $SU(2)$ will not change its faces, because they are already $SU(2)$-invariant. Therefore, $\mathsf{Av}_{SU(2)}(\Xi_p)$ is a calibration with its faces quaternionic subspaces. Moreover it is $Sp(n)Sp(1)$-invariant $4p$-form, so it is proportional to $(\omega_I^2 + \omega_J^2 + \omega_K^2)^p$. Then, using Lemma 5.12 below, we obtain that that $\mathsf{Av}_{SU(2)}(\Xi_p) = \Theta_p$ by evaluating both forms on a fixed quaternionic subspace. ■

Remark 5.4: Subvarieties calibrated by Θ_p are called **trianalytic subvarieties**. They were studied, at some length, in [V1] and [V2].

5.3 A holomorphic Lagrangian calibration

Proposition 5.5: Let (V^{4p}, I, J, K, g) be a quaternionic Hermitian vector space with fundamental forms $\omega_I, \omega_J, \omega_K$, and $\Psi \in \Lambda^{2p}(V)$ a $2p$-form which is the real part of $\frac{1}{p!}(\omega_I - \sqrt{-1}\,\omega_K)^p$ (it is a $(2p, 0)$-form with respect to J). Denote by $\Psi_I^{p,p}$ the (p,p)-part of Ψ with respect to I. Then $\Psi_I^{p,p}$ has comass 1. Moreover, a $2p$-dimensional subspace $W \subset V$ is calibrated by $\Psi_I^{p,p}$ if and only if W is complex I-linear and calibrated by Ψ.

Proof: The real part of $\frac{1}{p!}(\omega_I - \sqrt{-1}\,\omega_K)^p$ calibrates special Lagrangian subspaces taken with respect to the symplectic form ω_J (see [HL]). Therefore, any face of $\frac{1}{p!}(\omega_I - \sqrt{-1}\,\omega_K)^p$ is ω_J-Lagrangian. By Theorem 5.2, a $2p$-dimensional plane W is a face of $\Psi_I^{p,p}$ if and only if $\rho(t)(W)$ is a face of Ψ for all $t \in \mathbb{R}$. It follows by taking $t = 0$ that W is ω_J-Lagrangian and by taking $t = \pi/2$ that $I(W)$ is ω_J-Lagrangian too. But $I(W)$ is ω_J-Lagrangian iff W is ω_K-Lagrangian. By [Hit] (see also Remark 5.6 below) W determines an I-complex subspace. ■

Remark 5.6: Let V be a quaternionic Hermitian space, $\dim_{\mathbb{H}} V = p$, and $\xi \in \Lambda^{2p} V$ a decomposable $2p$-vector which is associated with a $2p$-dimensional subspace $W \subset V$. Clearly, W is Lagrangian with respect to ω_J if and only if $L_{\omega_J}\xi = 0$ and $\Lambda_{\omega_J}\xi = 0$, where L_{ω_J}, Λ_{ω_J} are the corresponding Hodge operators, $L_{\omega_J}(\eta) := \eta \wedge \omega_J$, and $\Lambda_{\omega_J} = *L_{\omega_J}*$ its Hermitian adjoint. If W is Lagrangian with respect to J and K, one has

$$[L_{\omega_J}, \Lambda_{\omega_K}]\xi = 0. \tag{5.1}$$

However, the commutator $[L_{\omega_J}, \Lambda_{\omega_K}]$ acts on forms of type (p, q) with respect to I as a multiplication by $(p - q)\sqrt{-1}$ (see [V0]). Then (5.1) implies that ξ is of type (p, p) with respect to I.

Claim 5.7: Let V be an n-dimensional quaternionic Hermitian space, and $\mathcal{V}^{0,0} : \mathbb{R} \longrightarrow \Lambda_I^{n,n}(V)$ be a map defined in Subsection 4.3 (in Subsection 4.3 it was defined for $SL(n, \mathbb{H})$-manifolds, but the definition can be repeated for quaternionic spaces word by word). Then $\mathcal{V}^{0,0}(1) = \Psi_I^{n,n}$, where $\Psi_I^{n,n}$ is a form defined as in Proposition 5.5.

Proof: From Proposition 4.7 (v), we know that $\mathcal{V}^{0,0}(1)$ and $\Psi_I^{n,n}$ are proportional and we only have to calculate the coefficient of proportionality. For this we use $\mathcal{V}^{0,0}(1) \wedge \alpha = R(\alpha) \wedge \overline{\Phi}_I$ for a particular choice of α as

$$\alpha = \xi_1 \wedge ... \wedge \xi_n \wedge \overline{\xi_{n+1}} \wedge ... \overline{\xi_{2n}},$$

where ξ_i are orthogonal and of unit norm. Then

$$R(\alpha) = \xi_1 \wedge ... \wedge J\overline{\xi_{n+1}} \wedge ... J\overline{\xi_{2n}}.$$

From here if $\mathcal{V}^{0,0}(1) = \lambda \Psi_I^{n,n}$, then $\lambda = 1$. ■

Comparing Proposition 4.7 and Claim 5.7, we find that the form $\Psi_I^{n,n}$ is positive.

5.4 Isotropic and coisotropic calibrations

A similar argument can be applied to other powers of Ω_J.

Proposition 5.8: Consider an n-dimensional quaternionic Hermitian space V, and let $\Omega_J := \omega_I - \sqrt{-1}\,\omega_K$ be the usual $(2,0)$-form on the complex space (V, J). When $p \leqslant n$ denote by $\Psi_p := \frac{1}{p!}\operatorname{Re}(\Omega_J^p)$, and let $\Psi_I^{p,p}$ be its (p,p)-part taken with respect to I. Then $\Psi_I^{p,p}$ has comass 1, and its faces are complex isotropic subspaces of (V, I)

Proof: Let $W \subset V$ be a real $2p$-dimensional subspace, and W_1 be the smallest complex subspace of (V, J) containing W. Adding more vectors if necessary, we can always assume that $\dim_{\mathbb{C}} W_1 = 2p$. Denote by ξ the decomposable $4p$-vector associated with W_1, and $I(\xi)$ its image under the action of a quaternion I. Then $\frac{1}{p!}\Omega_J^p$ is a $(2p,0)$-form on W_1, proportional to the unit holomorphic volume form $\operatorname{Vol}^{2p,0}(W_1)$ with a coefficient κ which satisfies

$$|\kappa| = \frac{(\xi, I(\xi))}{|\xi|^2}$$

where $(,)$ is the induced scalar product. By Cauchy-Schwarz inequality $|\xi| \leqslant 1$, where the equality holds iff $I\xi = \xi$ or, equivalently, W_1 is quaternionic. Since $\operatorname{Vol}^{2p,0}(W_1)$ has comass 1,

$$\mathsf{comass}\left(\frac{1}{p!}\Omega_J^p\right) \leqslant 1$$

with equality if and only if W_1 is quaternionic. In the latter case, W is a face of $\frac{1}{p!}\Omega_J^p$ if and only if W is complex Lagrangian in W_1, as follows from Proposition 5.5. ■

We provide also an expression of $\Psi^{p,p}$ as a polynomial of ω_I, ω_J and ω_K for even p.

Proposition 5.9: Let $\Psi^{p,p}$ be the (p,p) part with respect to I of $Re(\omega_I - \sqrt{-1}\,\omega_K)^p$. Then

$$\Psi^{p,p} = \sum_{k=0}^{q} \frac{(-1)^k}{4^k}\binom{p}{2k}\binom{2k}{k}\omega_I^{p-2k}\wedge(\omega_K^2+\omega_J^2)^k$$

where $q = \llcorner\frac{p}{2}\lrcorner$ is the greatest integer not exceeding $\frac{p}{2}$.

Proof: First we notice that

$$Re(\omega_I - \sqrt{-1}\,\omega_K)^p = \sum_{k=0}^{\llcorner\frac{p}{2}\lrcorner}(-1)^k\binom{p}{2k}\omega_I^{p-2k}\wedge\omega_K^{2k}.$$

Since ω_I^{p-2k} is of type $(p-2k, p-2k)$ with respect to I we need to determine the type of ω_K^{2k}. To do this we use the fact that $\omega_K = \frac{1}{2}\Omega + \frac{1}{2}\overline{\Omega}$ is the decomposition of ω_K in $(2,0)+(0,2)$ parts with respect to I where $\Omega = \omega_K + \sqrt{-1}\,\omega_J$. Then

$$\omega_K^{2k} = \frac{1}{4^k}\sum_{s=0}^{p-2k}\binom{2k}{s}\Omega^s\wedge\overline{\Omega}^{2k-s}$$

and each term in the sum has degree $(2s, 4k-s)$ with respect to I. So the only term which will contribute to $\Psi^{p,p}$ above will be when $s=k$. Obviously the term is $\frac{1}{4^k}\binom{2k}{k}\Omega^k\wedge\overline{\Omega}^k$. Then the proposition follows from the fact that $\Omega\wedge\overline{\Omega} = \omega_K^2+\omega_J^2$. ■

Notice that one can take the imaginary part of Ω_J^p instead of the real part. The resulting calibrated subspaces are again complex isotropic. To identify the complex coisotropic subspaces, however, one has to be more careful.

Proposition 5.10: Consider an n-dimensional quaternionic Hermitian space V, and let $\Omega_J := \omega_I - \sqrt{-1}\omega_K$ be the usual $(2,0)$-form on the complex space

(V,J). Let $\Phi_p + \sqrt{-1}\Phi'_p := \frac{1}{2^p p! n!}(\Omega_J)^n \wedge \omega_I^p$, and $\Phi_I^{p,p}$ (resp. $\Phi_I'^{p,p}$) be the corresponding $(n+p, n+p)$-parts taken with respect to I. Then $\Phi_I^{p,p}$ (resp. $\Phi_I'^{p,p}$) have comass 1 and their faces are complex coisotropic subspaces of (V,I)

Proof: First we notice that if a form α is calibration, then its Hodge dual $*\alpha$ is again calibration and its faces are orthogonal complements to the faces of α. Then the form $*\Psi^{p,p}$ is a calibration with faces I-complex Ω_J-coisotropic subspaces. The same is true also if we consider the imaginary part of Ω_J^p instead of Ψ^p. Then it remains to check that the complex form in the proposition is Hodge dual to Ω_J^p up to a real constant. To this end we first notice that $*\Omega_J^{n-p} = c_1\overline{\Omega_J^n} \wedge \Omega_J^p$ for a real positive constant c_1. Then $\Phi^{p,p} + \sqrt{-1}\Phi'^{p,p}$ and $\overline{\Omega_J^n} \wedge \Omega_J^p$ are both highest vectors in an irreducible representation A^{2n+2p} of $SU(2)$ (see Subsection 4.4), hence they are proportional up to a complex constant. More explicitly we have:

$$(\omega_I - \sqrt{-1}\,\omega_K)^n \wedge (\omega_I - \sqrt{-1}\,\omega_K)^p = \Omega_J^n \wedge (2\omega_I - \Omega_J)^p$$

$$= (\Omega_J)^n \wedge \sum_{s=0}^{p} \binom{p}{s} (-\Omega_J)^s \wedge 2^{p-s}\omega_I^{p-s}$$

Since $\Omega_J^{n+s} = 0$ for $s > 0$ all terms in the sum above vanish except the first one. Then

$$(\omega_I - \sqrt{-1}\,\omega_K)^n \wedge (\omega_I + \sqrt{-1}\,\omega_K)^p = (\omega_I - \sqrt{-1}\,\omega_K)^n \wedge 2^p\omega_I^p$$

From here and Lemma 5.12 *ii*) the proposition follows. ■

To calculate the comass of the forms above we need the following well-known preliminary Lemma:

Lemma 5.11: If (V^{2n}, I, g) is an Hermitian vector space and ω is the fundamental 2-form, then for any subset $X_1, ..., X_{2k}$ of a given unitary basis $(e_1, Ie_1, ..., e_n, Ie_n)$ we have:

i) $\omega^k(X_1, ..., X_{2k}) = \pm k!$ if $span\{X_1, ..., X_{2k}\}$ is complex and

ii) $\omega^k(X_1, ..., X_{2k}) = 0$ otherwise.

The proof of *i*) is standard, while *ii*) follows from the definition of wedge product and the fact that $\omega(X_i, X_j) \neq 0$ only if $IX_i = \pm X_j$.

Lemma 5.12: Let (V^{4n}, I, J, K, g) be a real vector space with anti-commuting complex structures I, J, K compatible with the positive scalar product

g. Denote by $\omega_I, \omega_J, \omega_k$ the fundamental 2-forms corresponding to I, J and K respectively and $\Omega_I = \omega_J + \sqrt{-1}\omega_K$ be the standard I-complex symplectic 2-form. Consider the form $\Psi_I^n = Re(\omega_I + \sqrt{-1}\omega_J)^n|_I^{(n,n)}$, where $|_I^{(n,n)}$ denotes the (n,n) component with respect to I. Then:

i) $\Omega_I^n \wedge \overline{\Omega_I}^n = 4^n (n!)^2 \operatorname{Vol}$ for the volume form Vol on V.

ii) $(\omega_I^2 + \omega_J^2 + \omega_K^2)^n = c_n \operatorname{Vol}$ where $c_n = \sum_{k=0}^n \frac{(n!)^2}{(k!)^2}(2k)!4^{n-k}$

iii) $\omega_I^k \wedge \Psi^n = 2^k k! n! \operatorname{Vol}_{E_{n+k}}$, where E_{n+k} is an $(n+k)$-dimensional I-complex and ω_J-coisotropic subspace.

Proof: Fix a quaternionic-Hermitian co-basis

$$(e^1, Ie^1, Je^1, Ke^1, e^2, Ie^2, ..., Ke^n)$$

of V^* so that $\operatorname{Vol} = e^1 \wedge ... \wedge Ke^n$ and let $e_1, Ie_1, ..., Ke_n$ be the dual basis of V. From the fact that $\Omega_I = \sum_i dz^i \wedge dw^i$ for coordinates $dz_i = e^i + \sqrt{-1}Ie^i$ and $dw_i = Je^i + \sqrt{-1}Ke^i$, follows that $\Omega_I^n = n! dz^1 \wedge dw^1 ... dz^n \wedge dw^n$. Then to obtain $i)$ we notice that $dz_i \wedge d\overline{z_i} = -2\sqrt{-1}e^i \wedge Ie^i$ and $dw_i \wedge d\overline{w_i} = -2\sqrt{-1}Je^i \wedge Ke^i$.

To prove $ii)$ we write

$$(\omega_I^2 + \omega_J^2 + \omega_K^2)^n = (\omega_I^2 + \Omega_I \wedge \overline{\Omega_I})^n = \sum_{k=0}^n \binom{n}{k} \omega_I^{2k} \wedge \Omega_I^{n-k} \wedge \overline{\Omega_I}^{n-k}.$$

Then we consider the term $\omega_I^{2k} \wedge \Omega_I^{n-k} \wedge \overline{\Omega_I}^{n-k}$. Let $s_i = e^i \wedge Ie^i + Je^i \wedge Ke^i$ and $t_j = dz^j \wedge dw^j$, so $\omega_I = \sum s_i$ and $\Omega_I = \sum t_j$. Then $s_i^3 = s_i t_i = t_i^2 = 0$ s_i, t_j commute and $s_i^2 = 2 \operatorname{Vol}_i, t_i \overline{t_i} = 4 \operatorname{Vol}_i$, where $\operatorname{Vol}_i = e^i \wedge Ie^i \wedge Je^i \wedge Ke^i$. Fix $n-k$ indexes $(i_{k+1}, i_{k+2}, ..., i_n)$. Then notice that in the product $\omega_I^{2k} \wedge t_{i_{k+1}} t_{i_{k+2}} ... t_{i_n} \wedge \overline{\Omega_I}^{n-k}$ the only non-vanishing terms are of the form

$$s_{i_1}^2 s_{i_2}^2 ... s_{i_k}^2 t_{i_{k+1}} t_{i_{k+2}} ... t_{i_n} \overline{t_{i_{k+1}} t_{i_{k+2}} ... t_{i_n}}$$

for the complementary indexes $(i_1, ..., i_k)$, such that $(i_1, ..., i_n)$ is a permutation of $(1, 2..., n)$. Every such product is equal to $2^k 4^{n-k} \operatorname{Vol}$. Then we may select $i_1 = 1, .., i_k = k, i_{k+1} = k+1, ..., i_n = n$ and count the number of terms corresponding to it; clearly, this number does not depend on the choice of the permutation. The number is the product of the coefficients in front of $s_1^2 ... s_k^2$ $t_{k+1} ... t_n$ and $\overline{t_{k+1}} ... \overline{t_n}$ in the expansions of $(s_1 + ... s_k)^{2k}$ $(t_{k+1} + ... + t_n)^{n-k}$

and $(\overline{t_{k+1}}+...+\overline{t_n})^{n-k}$ respectively, which is $\frac{(2k)!}{2^k}((n-k)!)^2$. Since there are $\frac{n!}{k!(n-k)!}$ different choices for $n-k$ indexes, we obtain

$$(\omega_I^2+\omega_J^2+\omega_K^2)^n=\sum_{k=0}^{n}\frac{(n!)^2}{(k!)^2}(2k)!4^{n-k}\,\mathrm{Vol}$$

and *ii*) follows.

To prove *iii*) we notice that $Sp(n)$ acts transitively on complex coisotropic subspaces of fixed dimension. Then we choose the coisotropic subspace L spanned by $e_1, Ie_1, ..., e_n, Ie_n, Je_1, Ke_1, ..., Je_k, Ke_k$. Let $\Omega_K=\omega_I+\sqrt{-1}\omega_J$, $\alpha\in L$ a subspace spanned by $2n$ vectors and β be a subspace generated by $2k$ vectors among $e_1, Ie_1, ..., e_n, Ie_n, Je_1, Ke_1, ..., Je_k, Ke_k$. Since $\Psi^n = Re(\Omega_K)|_I^{n,n}$, then $\Psi^n|_\alpha = 0$ if α contains a quaternionic subspace or is not I-invariant. Similarly, $\omega_I^k\Big|_\beta = 0$ if β is not I-invariant as follows from Lemma Lemma 5.11.

From the calculations in [HL] p. 88, we have

$$\Psi^n(e_1, Ie_1, ..., e_n, Ie_n)=n!Re(dz_1\wedge ....\wedge dw_n)(e_1, Ie_1, ...e_n, Ie_n)=n!,$$

and from Lemma 5.11 above, $\omega_I^K(Je_1, Ke_1, ..., Je_k, Ke_k)=k!$. Then in the expression for $\Psi^n\wedge\omega_I^k(e_1, Ie_1, ...Je_k, Ke_k)$ the only non-vanishing summands are $\omega_I^K(Je_1, Ke_1, ..., Je_k, Ke_k)$ and the terms where one or more pairs e_i, Ie_i are interchanged with Je_i, Ke_i. If we have exactly s pairs interchanged, then there will be $\binom{l}{s}$ terms each with value $n!k!$. So

$$\Psi^n\wedge\omega_I^k(e_1, Ie_1, ...Je_k, Ke_k)=n!k!\left(1+k+\binom{k}{2}+...+\binom{k}{k}\right)=2^n n!k!,$$

which proves the Lemma. Note that for $k=n$ the result fits with the case *i*) and the calculations in Proposition 5.10. ■

5.5 Holomorphic Lagrangian calibrations of degree two

The calibration 4-forms with constant coefficients in $\mathbb{R}^8$ were studied systematically in [DHM]. Also various 4-forms which are calibrations in $\mathbb{H}^n$ or any hyperkähler manifold are considered in [BrH]. We want to relate our results to these works.

If $p = 2$, from Proposition 5.9 we obtain

$$\Psi_I^{2,2} = \frac{1}{2}\operatorname{Re}\left(\omega_K + \sqrt{-1}\omega_I\right)^2\Big|_I^{2,2}$$
$$= \left(-\frac{1}{2}\omega_I^2 + \frac{1}{2}\omega_K^2\right)\Big|_I^{2,2} = -\frac{1}{2}\omega_I^2 + \frac{1}{4}(\omega_J^2 + \omega_K^2).$$

In [BrH] R.Bryant and R. Harvey considered the forms $\Psi_{\lambda,\mu,\nu} = \frac{\lambda}{2}\omega_I^2 + \frac{\mu}{2}\omega_J^2 + \frac{\nu}{2}\omega_K^2$ and showed that they are calibrations iff $-1 \leqslant \nu, \lambda, \mu \leqslant 1$ and $-1 \leqslant \nu + \lambda + \mu \leqslant 1$. We show here that the "generic" form of this type calibrates either quaternionic or complex isotropic subspaces.

Proposition 5.13: For the forms $\Psi_{\lambda,\mu,\nu}$ the following is valid:

i) If $\lambda, \mu, \nu \geqslant 0$ and $\lambda + \mu + \nu = 1$ with at least two of λ, μ, ν non-zero, the form $\Psi_{\lambda,\mu,\nu}$ has comass 1 and the faces are the quaternionic ones.

ii) If $\mu, \nu \leqslant 0$ and $\mu + \nu \geqslant -1$ with at least two of the inequalities being strict, then $\Psi_{1,\mu,\nu}$ has comass 1 and the faces are the I-complex Ω_I-isotropic subspaces of $\mathbb{H}^n = \mathbb{C}^{2n}$.

Proof: First we note that a convex hull of calibrations is a calibration. In case i), for any unit 4-vector ψ,

$$\Psi_{\lambda,\mu,\nu}(\psi) = \frac{\lambda}{2}\omega_I^2(\psi) + \frac{\mu}{2}\omega_J^2(\psi) + \frac{\nu}{2}\omega_K^2(\psi) \leqslant (\lambda + \mu + \nu)|\psi| = |\psi|,$$

and the equality is achieved only when ψ spans a subspace which is invariant with respect to at least two of I, J and K, hence quaternionic.

For ii) we note that

$$\frac{1}{2}\omega_I^2 + \frac{\mu}{2}\omega_J^2 + \frac{\nu}{2}\omega_K^2 = \frac{1 + \mu + \nu}{2}\omega_I^2 - \frac{\mu}{2}(\omega_I^2 - \omega_J^2) - \frac{\nu}{2}(\omega_I^2 - \omega_K^2)$$

Then according to [BrH], Theorem 2.38, $\frac{1}{2}(\omega_I^2 - \omega_J^2)$ and $\frac{1}{2}(\omega_I^2 - \omega_K^2)$ are calibrations with comass 1 and faces which are ω_K or ω_J isotropic and contained in 2-dimensional quaternionic subspaces. So as in i) if ψ is a unit 4-vector, then $\Psi_{1,\mu,\nu}(\psi) \leqslant |\psi|$ with equality if and only if ψ is a face for all terms with nonvanishing coefficients on the right-hand-side above. If the span of ψ satisfies at least two of the following:

i) ψ is I-complex

ii) ψ is ω_J isotropic and

iii) ψ is ω_K isotropic

then ψ satisfies also the third one and the Proposition follows.

■

In [BrH, Theorem 6.4], Proposition 5.13 is implicit. We note also that in String Theory, the holomorphic Lagrangian submanifolds in 8-dimensional manifolds were related to the notion of intersecting branes [F].

5.6 Examples

Examples of complex Lagrangian submanifolds in hyper-Kähler manifolds are given by many authors. In [Vo], C. Voisin has proven a result about the stability of such submanifolds under small deformation of the complex structure of the ambient space; she gave also several classes of examples. N. Hitchin noticed the fact that such subspaces are coming in complete families ([Hit]). In [M], D. Matsushita has shown that the families of holomorphic Lagrangian fibrations on a hyperkaehler manifold always deform with a deformation of a manifold, if the cohomology class of a fiber remains of Hodge type (n, n). Existence of such families is postulated by a conjecture called "SYZ conjecture", or, sometimes, the "Huybrechts-Sawon conjecture". It is also known as a hyperkähler version of *abundance conjecture,* related to the minimal model program. For a survey of related questions, please see [Saw]. Recently in String Theory the holomorphic Lagrangian submanifolds were related to 3-dimensional topological field theory with target hyperkähler manifold [KRS].

In this section we provide examples of complex Lagrangian submanifolds of hypercomplex manifolds with holonomy $SL(n, \mathbb{H})$.

The known examples of manifolds with holonomy $SL(n, \mathbb{H})$ are either nilmanifolds ([BDV]) or obtained via the twist construction of A. Swann [S], which is based on previous examples by D. Joyce. The later construction provides also simply-connected examples. We describe briefly a simplified version of it.

Let (X, I, J, K, g) be a compact hyper-Kähler manifold. By definition, an anti-self-dual 2-form on it is a form which is of type (1,1) with respect to I and J and hence with respect to all complex structures of the hypercomplex family. Let $\alpha_1, ..., \alpha_{4k}$ be anti-self-dual closed 2-forms representing integral cohomology classes on X (instatons). Consider the principal T^{4k}-bundle $\pi : M \to X$ with characteristic classes determined by $\alpha_1, ..., \alpha_{4k}$. It admits an instanton connection A given by $4k$ 1-forms θ_i s.t. $d\theta_i = \pi^*(\alpha_i)$. Then M carries a hypercomplex structure determined in the following way: on the horizontal spaces of A we have the pull-backs of I, J, K and on the vertical spaces we fix a linear hypercomplex structure of the $4k$-torus. The structures $\mathcal{I}, \mathcal{J}, \mathcal{K}$ on M are extended to act on the cotangent bundle T^*M using the following relations:

$$\mathcal{I}(\theta_{4i+1}) = \theta_{4i+2}, \mathcal{I}(\theta_{4i+3}) = \theta_{4i+4}, \quad \mathcal{J}(\theta_{4i+1}) = \theta_{4i+3}, \mathcal{J}(\theta_{4i+2}) = -\theta_{4i+4},$$
$$\mathcal{I}(\pi^*\alpha) = \pi^*(I\alpha), \mathcal{J}(\pi^*\alpha) = \pi^*(J\alpha)$$

for any 1-form α on X and $i = 0, 1, ...k-1$. Similarly one can define a hyperhermitian (or quaternion-Hermitian) metric on M from g and a fixed hyper-Kähler metric on T^{4k} using the splitting of TM in horizontal and vertical subspaces. As A. Swann [S] has shown the structure is HKT and has a holonomy $SL(n, \mathbb{H})$.

Suppose now that Y is a complex Lagrangian subspace in X with respect to I. Consider the T^{2k}-bundle over X determined by $\alpha_{4i+1}, \alpha_{4i+3}$. Suppose that N is its restriction to Y i.e N is a principal T^{2k}-subbundle of M over Y determined by $\alpha_{4i+1}|_Y, \alpha_{4i+3}|_Y$. Then N is naturally embedded in M and by the definiton above N is $\mathcal{J}$-invariant and Lagrangian with respect to the fundamental 2-form of $\mathcal{I}$. Notice that in general the complex Lagrangian subspace could be Kähler or non-Kähler depending on whether $\alpha_1|Y$ and $\alpha_3|Y$ define zero classes or not.

As a particular case assume X to be a $K3$ surface with large enough Picard group such that there are 4 independent anti-self-dual integral classes defining a principal T^4-bundle M over $X = K3$ with finite fundamental group. After passing to a finite cover we may assume that M is simply-connected. Now if vol denotes the volume form on X, then we can choose representatives $\alpha_1, ..., \alpha_4$ in the characteristic classes of M such that $\alpha_i^2 = -F\,\mathrm{Vol}$ where F is a function and $F > 0$ almost everywhere. We want to see what is the structure of an arbitrary complex Lagrangian subspace N of M. Since N is 4-dimensional and $\mathcal{J}$-complex, we claim that its intersection with a generic fiber of $\pi : M \longrightarrow X$ is at least complex 1-dimensional. Indeed, otherwise N would be a multisection of M and will intersect a generic fiber transversally. However then $\int_N \pi^*(\alpha_1^2) < 0$ since $\alpha_1^2 = -vol$ on one hand, and $\int_N \pi^*(\alpha_1^2) = 0$ since $\pi^*(\alpha_1) = d\theta_1$ for some connection form θ_1 on the other. The contradiction proves the claim and we have:

Proposition 5.14: If M is a principal instanton T^4-bundle over a $K3$ surface then any complex Lagrangian subspace is fibered by complex Lagrangian curves of the fibers of M over a Lagrangian curve of the base $K3$. ∎

Remark 5.15: Notice that any complex curve is *a priori* Lagrangian in a K3 surface.

In general one can use a similar construction to obtain complex isotropic and coisotropic subspaces of the instanton bundle M.

6 Calibrations on $SL(n, \mathbb{H})$-manifolds

Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, that is, a hypercomplex manifold with Φ_I a holomorphic volume form on (M, I) preserved by the Obata connection. Clearly, $\overline{\Phi}_I$ is proportional to $J(\Phi_I)$. After a rescaling to $e^{\sqrt{-1}t}\Phi_I$ if necessary, we can assume that Φ_I is $\mathbb{H}$-real, i.e. $J(\Phi_I) = \overline{\Phi}_I$, and $\mathbb{H}$-positive (Subsection 4.2). A number of interesting calibrations can be constructed in this situation.

Theorem 6.1: Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, and $(\Phi_I)^{n,n}_J$ the (n, n)-part of Φ_I taken with respect to J. Pick a quaternionic Hermitian metric on M. Using a conformal change, we may assume that $|\Phi_I|_g = 2^n$. Then $Re((\Phi_I)^{n,n}_J)$ is a calibration, and it calibrates complex subvarieties of (M, J) which are Lagrangian with respect to the $(2, 0)$-form $\omega_K + \sqrt{-1}\,\omega_I$.

Proof: It follows from the assumptions of Theorem 6.1 that

$$\Phi_I = \lambda\frac{(\omega_J + \sqrt{-1}\omega_K)^n}{n!}.$$

Since both forms are real and $\mathbb{H}$-positive, λ is real and positive. It is easy to check that in local quaternionic Hermitian frame $(dz_1, dw_1, ..., dz_n, dw_n)$ the norm is calculated as

$$\left|\frac{(\omega_J + \sqrt{-1}\omega_K)^n}{n!}\right|^2 = |dz_1|^2|dw_1|^2...|dz_n|^2|dw_n|^2 = 4^n.$$

Then $\left|\frac{(\omega_J+\sqrt{-1}\omega_K)^n}{n!}\right| = 2^n = |\Phi_I|$ and $\lambda = 1$. Now the proof follows from the fact that $Re(\Phi_I)$ and $Re(\Phi_I)^{n,n}_J$ are both closed,[1] and Proposition 5.5. ■

Theorem 6.2: Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, and $(\Phi_I)^{n,n}_J$ the (n, n)-part of Φ_I taken with respect to J. Assume that (M, I, J, K) is equipped with an HKT metric g which is balanced and $|\Phi_I| = 2^n$. Then

[1] The form $(\Phi_I)^{n,n}_J$ is parallel with respect to the Obata connection, which is torsion-free.

$V_{n+i,n+i} := \frac{1}{2^i i!} Re((\Phi_I)_J^{n,n} \wedge \omega_J^i)$ is a calibration, which calibrates complex subvarieties of (M, J) which are coisotropic with respect to the $(2,0)$-form $\omega_K + \sqrt{-1}\,\omega_I$.

Proof: As in the previous proof, $\Phi_I = \frac{(\omega_J+\sqrt{-1}\omega_K)^n}{n!}$, so the form $V_{n+i,n+i}$ is a pre-calibration by Proposition 5.10. It is closed, as follows from Proposition 4.9. ■

Remark 6.3: Notice that the form $V_{n+i,n+i}$ is not parallel with respect to any torsion-free connection on M (Claim 6.6), unless M is hyperkähler.

Existence of a balanced HKT metric is a hard problem, which is equivalent to a quaternionic version of a Calabi-Yau theorem ([V7]). However, even if g is not balanced, an analogue of the calibration $V_{n+i,n+i}$ is possible to construct.

Theorem 6.4: Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, and $(\Phi_I)_J^{n,n}$ the (n,n)-part of Φ_I taken with respect to J, and g an HKT metric. Then there exists a function $c_i(m)$ on M, such that $V_{n+i,n+i} := (\Phi_I)_J^{n,n} \wedge \omega_J^i$ is a calibration with respect to the conformal metric $\widetilde{g} = c_i g$, calibrating complex subvarieties of (M, J) which are coisotropic with respect to the $(2,0)$-form $\widetilde{\omega}_K + \sqrt{-1}\,\widetilde{\omega}_I$.

Proof: Since Φ_I is $\mathbb{H}$-positive and Obata parallel, the form $(\Phi_I)_J^{n,n}$ is closed. Then Proposition 4.9 implies that $V_{n+i,n+i}$ is also closed. If we denote by $\widetilde{\omega_J}$ and $\widetilde{\Omega_I}$ the corresponding forms after the conformal change $\widetilde{g} = c_i(m)g$, then we can find the function $c_i(m)$ such that

$$V_{n+i,n+i} = \frac{1}{2^i n! i!} (\widetilde{\Omega}_I^n)_J^{n,n} \wedge \widetilde{\omega}_J^i.$$

Theorem 6.4 then follows from Proposition 5.10. ■

Remark 6.5: Similarly to the hyperkähler case, it is a natural question to ask whether the complex isotropic submanifolds are also calibrated in $SL(n, \mathbb{H})$-manifolds with an HKT structure. However we can see in the examples from Section 4.6 that this is not the case. Consider again a toric bundle M over $K3$-surface which has 4-dimensional fiber and is simply-connected. Such fiber contains a 2-torus which will be a complex isotropic curve with respect to some of the structures. By a spectral sequence argument as in Lemma 4.7 of [S], one can see that all second cohomology classes of M are pull-backs from classes on the base $K3$-surface. Then such a torus is homologous to zero, since the integral of any closed 2-form on it vanishes. Therefore, it can not be calibrated by any form.

Claim 6.6: Let M be an $SL(n, \mathbb{H})$-manifold, Ω an HKT-form, and $V_{n+i,n+i}$ the corresponding calibration, constructed above. Assume that Ω is not hyperkähler. Then, the form $V_{n+i,n+i}$ is not preserved by any torsion-free connection, for any $0 < i < n$.

Proof: It is easy to check that the stabilizer $St_{GL(4n,\mathbb{R})}(V_{n+i,n+i})$ is equal to the group $Sp(n)$ of quaternionic Hermitian matrices. Therefore, any connection preserving $V_{n+i,n+i}$ would also preserve an $Sp(n)$-structure. However, a torsion-free connection preserving $Sp(n)$-structure is hyperkähler. ■

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