Quaternions and special relativity

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We reformulate Special Relativity by a quaternionic algebra on reals. Using *real linear quaternions*, we show that previous difficulties, concerning the appropriate transformations on the 3+1 space–time, may be overcome. This implies that a complexified quaternionic version of Special Relativity is a choice and not a necessity. © *1996 American Institute of Physics.* [S0022-2488(96)01106-1]

I. INTRODUCTION

''The most remarkable formula in mathematics is:

$$e^{i\theta}=\cos\,\theta+i\,\sin\,\theta. \tag{1}$$

This is our jewel. We may relate the geometry to the algebra by representing complex numbers in a plane

$$x+iy=re^{i\theta}.$$

This is the unification of algebra and geometry.''—Feynman.[1]

We know that a rotation of α-angle around the z axis, can be represented by $e^{i\alpha}$, in fact,

$$e^{i\alpha}(x+iy)=re^{i(\theta+\alpha)}.$$

In 1843, Hamilton in the attempt to generalize the complex field in order to describe the rotation in the three-dimensional space, discovered quaternions. Quaternions, as used in this paper, will always mean ''real quaternions''

$$q=a+ib+jc+kd,\quad a,b,c,d\in\mathscr{R}.$$

Today a rotation about an axis passing trough the origin and parallel to a given unitary vector $\mathbf{u}\equiv(u_x,u_y,u_z)$ by an angle α can be obtained taking the transformation

$$e^{(iu_x+ju_y+ku_z)\alpha/2}(ix+jy+kz)e^{-(iu_x+ju_y+ku_z)\alpha/2}. \tag{2}$$

Therefore, if we wish to represent rotations in the three-dimensional space and complete ''*the unification of algebra and geometry*,'' we need quaternions.

The quaternionic algebra has been expounded in a series of papers[2] and books[3] with particular reference to quantum mechanics; the reader may refer to these for further details. For convenience we repeat and develop the relevant points in the following section, where the terminology is also defined.

Nothing that U$(1,q)$ is algebraically isomorphic to SU$(2,c)$, the imaginary units i,j,k can be realized by means of the 2×2 Pauli matrices through

$$(i,j,k)\leftrightarrow(i\sigma_3,-i\sigma_2,-i\sigma_1)$$

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(this particular representation of the imaginary units i,j,k has been introduced in Ref. 4). So a quaternion q can be represented by a 2×2 complex matrix

$$q\leftrightarrow\mathcal{Q}=\begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}, \tag{3}$$

where

$$z_1=a+ib, \quad z_2=c-id\in\mathcal{C}(1,i),$$

$$z_1^*=a-ib, \quad z_2^*=c+id.$$

It follows that a quaternion with unitary norm is identified by a unitary 2×2 matrix with unit determinant. This gives the correspondence between unitary quaternions $\mathrm{U}(1,q)$ and $\mathrm{Su}(2,c)$ [in a recent paper[5] the representation theory of the group $\mathrm{U}(1,q)$ has been discussed in detail]. Let us consider the transformation law of a spinor (two-dimensional representations of the rotation group)

$$\psi'=\mathcal{U}\psi, \tag{4}$$

where

$$\psi=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \mathcal{U}\in\mathrm{SU}(2,c).$$

We can immediately verify that

$$\tilde{\psi}=\begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}$$

transforms as follows,

$$\tilde{\psi}'=\mathcal{U}^*\tilde{\psi}, \tag{5}$$

so

$$\begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}'=\mathcal{U}\begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}$$

represents again the transformation law of a spinor.

Thanks to the identification (3) we can write the previous transformations by real quaternions as follows

$$q'=\mathcal{U}q,$$

with $q=z_1+jz_2$ and $\mathcal{U}$ quaternion with unitary norm $[N(\mathcal{U})=\mathcal{U}^+\mathcal{U}=1]$. Note that we do not need right operators to indicate the transformation law of a spinor.

Now we can obtain the transformation law of a three-dimensional vector $\mathbf{r}\equiv(x,y,z)$ by product of spinors; in fact, if we consider the purely imaginary quaternion

$$\omega=qiq^+=ix+jy+kz, \quad (i,j,k)^+\equiv-(i,j,k),$$

or the corresponding traceless 2×2 complex matrix

$$\Omega = \psi i \psi^+ = \begin{pmatrix} ix & -y-iz \\ y-iz & -ix \end{pmatrix},$$

a rotation in the three-dimensional space can be written as follows:[6]

$$\omega' = \mathscr{U}\omega\mathscr{U}^+ \quad \text{(quaternions)},$$

$$\Omega' = \mathscr{U}\Omega\mathscr{U}^+ \quad (2\times 2 \ \ \text{complex matrices}).$$

For infinitesimal transformations, $\mathscr{U}=1+\mathbf{Q}\cdot\boldsymbol{\theta}$, we find

$$\mathbf{Q}\cdot\mathbf{r}' = \mathbf{Q}\cdot\mathbf{r} + \mathbf{Q}\cdot\boldsymbol{\theta}\,\mathbf{Q}\cdot\mathbf{r} - \mathbf{Q}\cdot\mathbf{r}\mathbf{Q}\cdot\boldsymbol{\theta},$$

where

$$\mathbf{Q}\equiv(i,j,k), \quad \boldsymbol{\theta}\equiv(\alpha,\beta,\gamma).$$

If we rewrite the above mentioned transformation in the following form,

$$\mathbf{Q}\cdot\mathbf{r}' = [1+\boldsymbol{\theta}\cdot(\mathbf{Q}-1|\mathbf{Q})]\mathbf{Q}\cdot\mathbf{r}, \tag{6}$$

barred operators $\mathscr{O}|q$ act on quaternionic objects Φ as in $(\mathscr{O}|q)\Phi=\mathscr{O}\Phi q$.
We identify

$$\frac{i-1|i}{2}, \quad \frac{j-1|j}{2}, \quad \frac{k-1|k}{2},$$

as the generators for rotations in the three-dimensional space. The factor $\frac{1}{2}$ guarantees that our generators satisfy the usual algebra:

$$[A_m\,,\,A_n] = \epsilon_{mnp}A_p\,, \quad m,n,p=1,2,3.$$

Up until now, we have considered only particular operations on quaternions. A quaternion q can also be multiplied by unitary quaternions $\mathscr{V}$ from the right. A possible transformation which preserves the norm is given by

$$q' = \mathscr{U}q\mathscr{V}, \quad (\mathscr{U}^+\mathscr{U}=\mathscr{V}^+\mathscr{V}=1). \tag{7}$$

Since left and right multiplications commute, the group is locally isomorphic to SU(2)$\times$SU(2), and so to O(4), the four-dimensional Euclidean rotation group.

As far as here we can recognize only particular real linear quaternions, namely,

$$1, \quad i, \quad j, \quad k, \quad 1|i, \quad 1|j, \quad 1|k.$$

Real linear and complex linear quaternion operators were first systematically discussed in the paper by Horwitz and Biedenharn.[7]

We have to hope of describing the Lorentz group if we use only previous objects. Analyzing the most general transformation on quaternions (see Sec. IV), we introduce new real linear quaternions which allow us to overcome the above difficulty and so obtain a quaternionic version of the Lorentz group, without the use of complexified quaternions. This result appears, to the best of our knowledge, for the first time in print.

First we briefly recall the standard way to rewrite special relativity by a quaternionic algebra on complex (see Sec. III).

In Sec. V, we present a quaternionic version of the special group SL$(2,c)$, which is as well-known collected to the Lorentz group. Our conclusions are drawn in the final section.

II. QUATERNIONIC ALGEBRAS

A quaternionic algebra over a field $\mathscr{F}$ is a set

$$\mathscr{H}=\{\alpha+i\beta+j\gamma+k\delta|\alpha,\beta,\gamma,\delta\in\mathscr{F}\},$$

with multiplication operations defined by following rules for imaginary units i,j,k:

$$i^2=j^2=k^2=-1,\quad jk=-kj=i,\quad ki=-ik=j,\quad ij=-ji=k.$$

In our paper we will work with quaternionic algebras defined on reals and complex, so in this section we give a panoramic review of such algebras.

We start with a quaternionic algebra on reals

$$\mathscr{H}_R=\{\alpha+i\beta+j\gamma+k\delta|\alpha,\beta,\gamma,\delta\in\mathscr{R}\}.$$

We introduce the quaternion conjugation denoted by $^+$ and defined by

$$q^+=\alpha-i\beta-j\gamma-k\delta.$$

The previous definition implies

$$(\psi\varphi)^+=\varphi^+\psi^+,$$

for ψ, φ quaternionic functions. A conjugation operation which does not reverse the order of ψ, φ factors is given, for example, by

$$\tilde{q}=\alpha-i\beta+j\gamma-k\delta.$$

An important difference between quaternions and complexified quaternions, as remarked by Adler in his recent book[8] (pag. 8), is based on the concept of *division algebra*, which is a finite-dimensional algebra for which $a\neq0$, $b\neq0$ implies $ab\neq0$, in others words, which has no nonzero divisors of zero. A classical theorem[9] states that the only division algebras over the reals are algebras of dimension 1, 2, 4, and 8; the only associative algebras over the reals are $\mathscr{R}$, $\mathscr{C}$, and $\mathscr{H}_{\mathscr{R}}$;[10] the nonassociative division algebras include the octonions $\mathscr{O}$ (but there are others as well; see Ref. 11).

A simple example of a *nondivision* algebra is provided by the algebra of complexified quaternions

$$\mathscr{H}_{\mathscr{C}}=\{\alpha+i\beta+j\gamma+k\delta|\alpha,\beta,\gamma,\delta\in\mathscr{C}(1,\mathscr{I})\},$$

$$[\mathscr{I},\ i]=[\mathscr{I},\ j]=[\mathscr{I},\ k]=0.$$

In fact, since

$$(1+i\mathscr{I})(1-i\mathscr{I})=0,$$

there are nonzero divisors of zero.

For complexified quaternions we have different opportunities to define conjugation operations; we shall use the following terminology:

(1) The *complex* conjugate of $q_{\mathscr{C}}$ is

$$q_{\mathscr{C}}^{*}=\alpha^{*}+i\beta^{*}+j\gamma^{*}+k\delta^{*}.$$

Under this operation

$$(\mathscr{T},i,j,k)\rightarrow(-\mathscr{T},i,j,k)$$

and

$$(q_{\mathscr{C}}p_{\mathscr{C}})^{*}=q_{\mathscr{C}}^{*}p_{\mathscr{C}}^{*}.$$

(2) The *quaternion* conjugate of $q_{\mathscr{C}}$ is

$$q_{\mathscr{C}}^{\star}=\alpha-i\beta-j\gamma-k\delta.$$

Here

$$(\mathscr{T},i,j,k)\rightarrow(\mathscr{T},-i,-j,-k)$$

and

$$(q_{\mathscr{C}}p_{\mathscr{C}})^{\star}=p_{\mathscr{C}}^{\star}q_{\mathscr{C}}^{\star}.$$

(3) In the absence of standard terminology, we call that formed by combining these operations the *full* conjugate:

$$q_{\mathscr{C}}^{+}=\alpha^{*}-i\beta^{*}-j\gamma^{*}-k\delta^{*}.$$

Under this operation

$$(\mathscr{T},i,j,k)\rightarrow-(\mathscr{T},i,j,k)$$

and

$$(q_{\mathscr{C}}p_{\mathscr{C}})^{+}=p_{\mathscr{C}}^{+}q_{\mathscr{C}}^{+}.$$

Note that for real quaternions we have

$$q^{\star}\equiv q^{+}.$$

III. COMPLEXIFIED QUATERNIONS AND SPECIAL RELATIVITY

We begin this section by recalling a sentence of Anderson and Joshi[12] about the quaternionic reformulation of special relativity:

''*There has been a long tradition of using quaternions for Special Relativity... The use of quaternions in special relativity, however, is not entirely straightforward. Since the field of quaternions is a four-dimensional Euclidean space, complex components for the quaternions are required for the 3+1 space–time of special relativity.*''

In the following section, we will demonstrate that a reformulation of special relativity by a quaternionic algebra on reals is possible.

In the present section, we use complexified quaternions to reformulate special relativity (for further details the reader may consult the papers of Edmonds,[13] Gough,[14] Abonyi,[15] Gürsey,[16] and the book of Synge[17]).

A space–time point can be represented by complexified quaternions as follows:

$$\mathscr{X}=\mathscr{T}ct+ix+jy+kz. \tag{8}$$

The Lorentz invariant in this formalism is given by

$$\mathscr{X}^{*}\mathscr{X}=(ct)^2-x^2-y^2-z^2. \tag{9}$$

If we consider the standard Lorentz transformation (boost $ct-x$)

$$ct'=\gamma(ct-\beta x),\quad x'=\gamma(x-\beta ct),\quad y'=y,\quad z'=z$$

and note that the first two equations may be rewritten as

$$ct'=ct\ \cosh\ \theta-x\ \sinh\ \theta,$$

$$x'=x\ \cosh\ \theta-ct\ \sinh\ \theta,$$

where $\cosh\ \theta=\gamma$ and $\sinh\ \theta=\beta\gamma$.

We can represent an infinitesimal transformation by

$$\mathscr{X}'=\mathscr{T}(ct-x\theta)+i(x-ct\theta)+jy+kz=\mathscr{X}+\mathscr{T}\,\frac{i+1|i}{2}\ \theta\mathscr{X}.$$

We thus recognize, in the previous transformation, the generator

$$\mathscr{T}\,\frac{i+1|i}{2}.$$

It is now very simple to complete the translation. The set of generators of the Lorentz group is provided with

$$\text{boost}\ \ (ct,x)\quad \mathscr{T}\,\frac{i+1|i}{2},$$

$$\text{boost}\ \ (ct,y)\quad \mathscr{T}\,\frac{j+1|j}{2},$$

$$\text{boost}\ \ (ct,z)\quad \mathscr{T}\,\frac{k+1|k}{2},$$

$$\text{rotation around}\ x\quad \frac{i-1|i}{2},$$

$$\text{rotation around}\ y\quad \frac{j-1|j}{2},$$

$$\text{rotation around}\ z\quad \frac{k-1|k}{2}.$$

Therefore a general finite Lorentz transformation is given by

$$e^{\mathscr{T}(i\alpha_b+j\beta_b+k\gamma_b)+i\alpha_r+j\beta_r+k\gamma_r}(\mathscr{T}ct+ix+jy+kz)e^{\mathscr{T}(i\alpha_b+j\beta_b+k\gamma_b)-i\alpha_r-j\beta_r-k\gamma_r}.$$

The previous results can be elegantly summarized by the relation

$$\mathscr{X}'=\Lambda\mathscr{X}\Lambda^{+},\quad \Lambda^{\star}\Lambda=1, \tag{10}$$

where Λ is obviously a complexified quaternion. In this or a similar way many authors have reformulated special relativity with complex quaternions.

We remark that the complex component for the quaternions represent a choice and not a necessity.

IV. A NEW POSSIBILITY

We think that quaternions are the natural candidates to describe special relativity. It is simple to understand why: quaternions are characterized by four real numbers (whereas complexified quaternions by eight), thus we can collect these four real quantities with a point (ct,x,y,z) in the space–time. In quaternionic notation we have

$$\mathscr{X}=ct+ix+jy+kz. \tag{11}$$

In the first section we have introduced particular *real linear quaternions*, namely,

$$1, \quad \mathbf{Q}, \quad 1|\mathbf{Q},$$

where

$$\mathbf{Q}\equiv(i,j,k).$$

In order to write the most general real linear quaternions we must consider the following quantities:

$$\mathbf{Q}|i, \quad \mathbf{Q}|j, \quad \mathbf{Q}|k.$$

In fact, the most general transformation on quaternions is represented by

$$q+p|i+r|j+s|k, \tag{12}$$

with

$$q,p,r,s\in\mathscr{H}_{\mathscr{R}}.$$

New objects like

$$k|j, \quad j|k, \quad i|k, \quad k|i, \quad j|i, \quad i|j$$

will be essential to reformulate special relativity with real quaternions. They represent the wedges which permit us to overcome the difficulties which in the past did not allow a (real) quaternionic version of special relativity.

Returning to Lorentz transformations, let us start with the following infinitesimal transformation (boost $ct-x$):

$$\mathscr{X}'=ct-x\theta+i(x-ct\theta)+jy+kz=\mathscr{X}+\frac{k|j-j|k}{2}\,\theta\mathscr{X}.$$

We can immediately note that the generator which substitutes

$$\mathscr{T}\,\frac{i+1|i}{2}$$

is

$$\frac{k|j-j|k}{2}.$$

So we have the possibility of listing the generators of the Lorentz group without the need to work with complexified quaternions:

$$\text{boost} \ \ (ct,x) \quad \frac{k|j-j|k}{2},$$

$$\text{boost} \ \ (ct,y) \quad \frac{i|k-k|i}{2},$$

$$\text{boost} \ \ (ct,z) \quad \frac{j|i-i|j}{2},$$

$$\text{rotation around } x \quad \frac{i-1|i}{2},$$

$$\text{rotation around } y \quad \frac{j-1|j}{2},$$

$$\text{rotation around } z \quad \frac{k-1|k}{2}.$$

In Appendix A we explicitly prove that the action of previous generators leaves

$$\text{Re}\ \mathscr{X}^2=(ct)^2-x^2-y^2-z^2 \tag{13}$$

invariant.

In Appendix B we will give an alternate but equivalent presentation of special relativity by a quaternionic algebra on reals. There we introduce a real linear quaternion g which substitutes the metric tensor $g^{\mu\nu}$.

V. A QUATERNIONIC VERSION OF THE COMPLEX GROUP SL(2)

In analogy to the connection between the rotation group O(3) to the special unitary group SU(2), there is a natural correspondence[18] between the Lorentz group O(3,1) and the special linear group SL(2). In fact, SL(2) is the universal covering group of O(3,1) in the same way that SU(2) is of O(3).

The aim of this Section is to give, by extending the consideration with which we collect the special unitary group SU(2) with unitary real quaternions (as shown in Sec. I), a quaternionic version of the special linear group SL(2). Once more the aim will be achieved with help of real linear quaternions.

A Lorentz spinor is a complex object which transforms under Lorentz transformations as

$$\psi'=\mathscr{A}\psi,$$

where $\mathscr{A}$ is a SL(2) matrix. When we restrict ourselves to the three-dimensional space and to rotations, this definition gives the usual Pauli spinors

$$\psi'=\mathscr{U}\psi,$$

where $\mathscr{U}$ is a SU(2) matrix.

Now we shall derive the generators of rotations and Lorentz boosts in the spinor representation by using real linear quaternions.

The action of generators of the special group SL(2),

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

on the spinor

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

can be represented by the action of real linear quaternions

$$i, \quad j, \quad k, \quad i|i, \quad j|i, \quad k|i$$

on the quaternion

$$q = \xi + j\eta.$$

In Sec. I we have obtained a three-dimensional vector (x,y,z) by product of Pauli spinors $q_{\mathscr{P}}$:

$$q_{\mathscr{P}}\, i q_{\mathscr{P}}^{+} = ix + jy + kz \quad (q'_{\mathscr{P}} = \mathscr{U} q_{\mathscr{P}}, \quad \mathscr{U}^{+}\mathscr{U} = 1).$$

Consequently, we have written its transformation law as follows:

$$(q_{\mathscr{P}}\, i q_{\mathscr{P}}^{+})' = \mathscr{U} q_{\mathscr{P}} i q_{\mathscr{P}}^{+} \mathscr{U}^{+}.$$

Now we start with a Lorentz spinor $q_{\mathscr{L}}$

$$q'_{\mathscr{L}} = \mathscr{A} q_{\mathscr{L}},$$

and construct a four-vector (ct,x,y,z) by-product of such spinors:

$$q_{\mathscr{L}}(1+i)q_{\mathscr{L}}^{+} = ct + ix + jy + kz.$$

The transformation law is then given by

$$(q_{\mathscr{L}}(1+i)q_{\mathscr{L}}^{+})' = (\mathscr{A} q_{\mathscr{L}})(1+i)(\mathscr{A} q_{\mathscr{L}})^{+}.$$

If we consider infinitesimal transformations

$$\mathscr{A} = 1 + \frac{\mathbf{Q}}{2} \cdot (\boldsymbol{\theta} + \boldsymbol{\zeta}|i),$$

with $\boldsymbol{\theta} \equiv (\alpha,\beta,\gamma)$ and $\boldsymbol{\zeta} \equiv (\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$,
we have

$$\mathscr{T}' = \mathscr{T} + \frac{\alpha}{2}[i, \mathscr{T}] + \frac{\beta}{2}[j, \mathscr{T}] + \frac{\gamma}{2}[k, \mathscr{T}] + \frac{\tilde{\alpha}}{2}\{i, \widetilde{\mathscr{T}}\} + \frac{\tilde{\beta}}{2}\{j, \widetilde{\mathscr{T}}\} + \frac{\tilde{\gamma}}{2}\{k, \overline{\mathscr{T}}\},$$

where

$$\mathscr{T}=q_{\mathscr{L}}(1+i)q_{\mathscr{L}}^{+}$$

and

$$\tilde{\mathscr{T}}=q_{\mathscr{L}}i(1+i)q_{\mathscr{L}}^{+}=\mathscr{T}-2q_{\mathscr{L}}q_{\mathscr{L}}^{+}.$$

In order to simplify next considerations we pose

$$\mathscr{T}=ix+jy+kz+ct=\mathscr{T}_i+\mathscr{T}_j+\mathscr{T}_k+\mathscr{T}_1,$$

$$\tilde{\mathscr{T}}=ix+jy+kz-ct=\mathscr{T}_i+\mathscr{T}_j+\mathscr{T}_k-\mathscr{T}_1,$$

so the standard Lorentz transformations are given by

$$\mathscr{T}_1\to\mathscr{T}_1+\tilde{\alpha}i\mathscr{T}_i+\tilde{\beta}j\mathscr{T}_j+\tilde{\gamma}k\mathscr{T}_k, \quad \mathscr{T}_i\to\mathscr{T}_i-\tilde{\alpha}i\mathscr{T}_1+\beta j\mathscr{T}_k-\gamma k\mathscr{T}_j,$$

$$\mathscr{T}_j\to\mathscr{T}_j-\tilde{\beta}j\mathscr{T}_1-\alpha i\mathscr{T}_k+\gamma k\mathscr{T}_i, \quad \mathscr{T}_k\to\mathscr{T}_k-\tilde{\gamma}k\mathscr{T}_1+\alpha i\mathscr{T}_j-\beta j\mathscr{T}_i.$$

In this way we obtain a quaternionic version of the special group SL(2) and demonstrate (in contrast with the opinion of Penrose)[6] that, if real linear quaternions appear, a ''trick'' similar to that one of rotations works to relate the full four-vector (ct,x,y,z) with real quaternions.

VI. CONCLUSIONS

The study of special relativity with a quaternionic algebra on reals has yielded a result of interest. While we cannot demonstrate in this paper that one number system (quaternions) is preferable to another (complexified quaternions), we have pointed out the advantages of using real linear quaternions which naturally appear when we work with a noncommutative number system, such as the quaternionic field. As seen in this paper these objects are very useful if we wish to rewrite special relativity by a quaternionic algebra on reals. The complexified quaternionic reformulation of special relativity is thus a choice and not a necessity. This affirmation is in contrast with the standard folklore (see, for example, Ref. 12).

Our principal aim in this work is to underline the potentialities of real linear quaternions. We wish to remember that many difficulties have been overcome thanks to these objects (which in our colorful language we have named generalized objects).[4]

To remark on their potentialities let us list the situations which have requested their use.

(i) The need of such objects naturally appears, for example, in the construction of quaternion group theory and tensor product group representations.[5] Also starting with only standard quaternions i,j,k in order to represent the generators of the group U(1,q), we find generalized quaternions when we analyze quaternionic tensor products.

$$\text{Spin } \frac{1}{2} \text{ generators:} \quad \frac{i}{2}, \quad \frac{j}{2}, \quad \frac{k}{2}.$$

$$\text{Spin } 1\oplus 0 \text{ generators:} \begin{pmatrix} \frac{i+1|i}{2} & 0 \\ 0 & \frac{i-1|i}{2} \end{pmatrix}, \quad \begin{pmatrix} j & 1|i \\ 1|i & j \end{pmatrix}, \quad \begin{pmatrix} k & -1 \\ 1 & k \end{pmatrix}.$$

(ii) If we desire to extend the isomorphism of SU(2,c) with U(1,q) to the group U(2,c), we must introduce the additional real linear quaternion ''$1|i$.'' In this way there exists at least one version of quaternionic quantum mechanics in which a ''partial'' set of translations may be defined;[4] in fact, thanks to real linear operators, a translation between $2n\times 2n$ complex and $n\times n$ quaternionic matrices is possible.

(iii) In the work of Ref. 19 a quaternion version of the Dirac equation was derived in the form

$$\gamma_\mu \partial^\mu \psi i = m\psi,$$

where the γ_μ are two-by-two quaternionic matrices satisfying the Dirac condition

$$\{\gamma_\mu , \gamma_\nu\} = 2g_{\mu\nu}.$$

In Rotelli's formalism the momentum operator must be defined as

$$p^\mu = \partial^\mu | i,$$

which is also a generalized object.

(iv) In this paper, contrary to the common opinion, we have given a real quaternionic formulation of special relativity. In order to obtain that, we have introduced the following real linear quaternions:

$$\mathbf{Q}|i, \quad \mathbf{Q}|j, \quad \mathbf{Q}|k, \quad \mathbf{Q}\equiv(i,j,k).$$

A quaternionic version of the special group SL(2) has also been given.

We finally note that the process of generalization can be extended also to complexified quaternions. In a recent paper[20] we gave an elegant one-component formulation of the Dirac equation and, thanks to our generalization, we overcame previous difficulties concerning the doubling of solutions[12–14] in the complexified quaternionic Dirac equation.

In seeking a better understanding of the success of mathematical abstraction in physics and in particular of the wide applicability of quaternionic numbers in theories of physical phenomena, we found that generalized quaternions should *not* be undervalued. We think that there are good reasons to hope that these generalized structures provide new possibilities concerning physical applications of quaternions.

''The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities...''—Dirac.[21]

APPENDIX A: QUATERNIONIC LORENTZ INVARIANT

In this Appendix we prove that the Lorentz invariant[8] is

$$\mathrm{Re}\ \mathscr{X}'^2 = \mathrm{Re}\ \mathscr{X}^2, \tag{A1}$$

where

$$\mathscr{X} = ct + ix + jy + kz.$$

Under an infinitesimal transformation, we have

$$\mathscr{X}' = \left(1 + \theta\, \frac{k|j - j|k}{2} + \alpha\, \frac{i - 1|i}{2} + \cdots \right) \mathscr{X},$$

so, neglecting second-order terms,

$$\mathscr{X}'^2 = \mathscr{X}^2 + \frac{\theta}{2}\{\mathscr{X}, k\mathscr{X}j - j\mathscr{X}k\} + \frac{\alpha}{2}\{\mathscr{X}, i\mathscr{X} - \mathscr{X}i\} + \cdots .$$

Equation (14) is then satisfied since

$$\{\mathscr{X}, i\mathscr{X} - \mathscr{X}i\} = (i - 1|i)\mathscr{X}^2,$$

$$\{\mathscr{X}, k\mathscr{X}j - j\mathscr{X}k\} = (1|j - j)\mathscr{X}k\mathscr{X} + (k - 1|k)\mathscr{X}j\mathscr{X}$$

are purely imaginary quaternions.

Obviously we can derive the generators of the Lorentz group by starting from the infinitesimal transformation

$$\mathscr{X}' = \mathscr{X} + \mathscr{A}\mathscr{X}$$

and imposing that they satisfy the relation

$$\mathrm{Re}\{\mathscr{X}, \mathscr{A}\mathscr{X}\} = 0$$

$$(\mathrm{Re}\ \mathscr{X}'^2 = \mathrm{Re}\ \mathscr{X}^2 \Rightarrow \mathrm{Re}\{\mathscr{X}, \mathscr{A}\mathscr{X}\} = 0). \tag{A2}$$

With straightforward mathematical calculus we can find the generators requested. In order to simplify the following considerations let us pose

$$\mathscr{X} = a + ib + jc + kd, \quad \mathscr{A} = q_0 + q_1|i + q_2|j + q_3|k$$

where $q_m = \alpha_m + i\beta_m + j\gamma_m + k\delta_m$ ($m = 0,1,2,3$) are real quaternions.

The only quantities which we must calculate are

$$\mathrm{Re}\{\mathscr{X}, \mathscr{X}\}, \quad \mathrm{Re}\{\mathscr{X}, i\mathscr{X}i\}, \quad \mathrm{Re}\{\mathscr{X}, i\mathscr{X}\}, \quad \mathrm{Re}\{\mathscr{X}, k\mathscr{X}j\};$$

in fact, the other quantities can be obtained from previous ones, by simple manipulations:

$$\mathrm{Re}\{\mathscr{X}, \mathscr{X}\} = 2(+a^2 - b^2 - c^2 - d^2), \quad \mathrm{Re}\{\mathscr{X}, i\mathscr{X}i\} = 2(-a^2 + b^2 - c^2 - d^2),$$

$$\mathrm{Re}\{\mathscr{X}, j\mathscr{X}j\} = 2(-a^2 - b^2 + c^2 - d^2), \quad \mathrm{Re}\{\mathscr{X}, k\mathscr{X}k\} = 2(-a^2 - b^2 - c^2 + d^2),$$

$$\mathrm{Re}\{\mathscr{X}, i\mathscr{X}\} = \mathrm{Re}\{\mathscr{X}, \mathscr{X}i\} = -4ab, \quad \mathrm{Re}\{\mathscr{X}, k\mathscr{X}j\} = \mathrm{Re}\{\mathscr{X}, j\mathscr{X}k\} = 4cd,$$

$$\mathrm{Re}\{\mathscr{X}, j\mathscr{X}\} = \mathrm{Re}\{\mathscr{X}, \mathscr{X}j\} = -4ac, \quad \mathrm{Re}\{\mathscr{X}, j\mathscr{X}i\} = \mathrm{Re}\{\mathscr{X}, i\mathscr{X}j\} = 4bc,$$

$$\mathrm{Re}\{\mathscr{X}, k\mathscr{X}\} = \mathrm{Re}\{\mathscr{X}, \mathscr{X}k\} = -4ad, \quad \mathrm{Re}\{\mathscr{X}, i\mathscr{X}k\} = \mathrm{Re}\{\mathscr{X}, k\mathscr{X}i\} = 4bd.$$

The previous relations imply the following conditions on the real parameters of the generator $\mathscr{A}$:

$$\alpha_0 = 0, \quad \beta_1 = 0,$$

$$\gamma_2 = 0, \quad \delta_3 = 0,$$

$$\beta_0 = -\alpha_1 = \alpha, \quad \gamma_0 = -\alpha_2 = \beta,$$

$$\delta_0 = -\alpha_3 = \gamma, \quad \delta_2 = -\gamma_3 = \theta,$$

$$\gamma_1 = -\beta_2 = \varphi, \quad \beta_3 = -\delta_1 = \eta.$$

We can immediately recognize the Lorentz generators given in Sec. IV.

APPENDIX B: QUATERNIONIC METRIC TENSOR

We introduce the usual four-vector x^{μ} by the following quaternion,

$$\mathscr{X}=x^0+ix^1+jx^2+kx^3,$$

and define a scalar product of two vectors $\mathscr{X}$, $\mathscr{Y}$ by

$$(\mathscr{X},g\mathscr{Y})_{\mathscr{R}}=\mathrm{Re}(\mathscr{X}^{+}g\mathscr{Y})=x^{\mu}g_{\mu\nu}y^{\nu}, \tag{B1}$$

where g is the generalized quaternion

$$-\tfrac{1}{2}(1+i|i+j|j+k|k).$$

We can define a real norm (or metric)

$$(\mathscr{X},g\mathscr{X})_{\mathscr{R}}=\mathrm{Re}(\mathscr{X}^{+}g\mathscr{X})=x^{\mu}g_{\mu\nu}x^{\nu}.$$

The vectors which transform under a Lorentz transformation $\mathscr{L}$ will be denoted by

$$\mathscr{X}'=\mathscr{L}\mathscr{X},$$

with $\mathscr{L}$ real linear operators [see Eq. (12)]. From the postulated invariance of the norm we can deduce the generators of Lorentz group.

If we consider infinitesimal transformations

$$\mathscr{L}=1+\mathscr{A},$$

we have

$$\mathrm{Re}(\mathscr{X}'^{+}g\mathscr{X}')=\mathrm{Re}\big(\mathscr{X}^{+}g\mathscr{X}+\mathscr{X}^{+}(\mathscr{A}^{+}g+g\mathscr{A})\mathscr{X}\big)=\mathrm{Re}(\mathscr{X}^{+}g\mathscr{X}),$$

and therefore

$$\mathscr{A}^{+}g+g\mathscr{A}=0. \tag{B2}$$

Using real scalar products, given an operator

$$\mathscr{A}=q+p|i+r|j+s|k, \quad q,p,r,s\in\mathscr{H}_{\mathscr{R}},$$

we can write its Hermitian conjugate as follows:

$$\mathscr{A}^{+}=q^{+}-p^{+}|i-r^{+}|j-s^{+}|k.$$

Then Eq. (17) can be rewritten as

$$g\mathscr{A}+h.c.=0.$$

If we pose

$$g\mathscr{A}=B=\tilde{q}+\tilde{p}|i+\tilde{r}\,|j+\tilde{s}\,|k,$$

we obtain the following conditions on the operator B:

$$\mathrm{Re}\ \tilde{q}=\mathrm{Vec}\ \tilde{p}=\mathrm{Vec}\ \tilde{r}=\mathrm{Vec}\ \tilde{s}=0.$$

Noting that $\mathscr{A}=gB$ we can quickly write the generators of Lorentz group. We give explicitly an example

$$\mathscr{A}_1=g(1|i)=-\tfrac{1}{2}(-i+1|i+j|k-k|j),$$

$$\mathscr{A}_2=gi=-\tfrac{1}{2}(i-1|i+j|k-k|j),$$

$$\mathscr{A}=\mathscr{A}_1-\mathscr{A}_2=\frac{i-1|i}{2}, \quad \tilde{\mathscr{A}}=\mathscr{A}_1+\mathscr{A}_2=\frac{k|j-j|k}{2}.$$

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