QUENCHED INVARIANCE PRINCIPLE FOR RANDOM WALKS WITH TIME-DEPENDENT ERGODIC DEGENERATE WEIGHTS

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ABSTRACT. We study a continuous-time random walk, X, on $\mathbb{Z}^d$ in an environment of dynamic random conductances taking values in $(0, \infty)$. We assume that the law of the conductances is ergodic with respect to space-time shifts. We prove a quenched invariance principle for the Markov process X under some moment conditions on the environment. The key result on the sublinearity of the corrector is obtained by Moser's iteration scheme.

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1. INTRODUCTION

Random walks in random environment is a topic of major interest in probability theory. A specific model for such a random walks that has been intensively studied during the last decade is the Random Conductance Model (RCM). The question

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whether a quenched invariance principle or quenched functional central limit theorem (QFCLT) holds is of particular interest. In the case of an environment generated by static i.i.d. random variables this question has been object of very active research (see [2, 12] and references therein). Recently, in [5] a QFCLT has been proven for random walks under general ergodic conductances satisfying a certain moment condition.

Quenched invariance principles have also been shown for various models for random walks evolving in dynamic random environments (see [1, 8, 14, 17, 24, 33, 32]). Here analytic, probabilistic and ergodic techniques were invoked, but assumptions on the ellipticity and the mixing behaviour of the environment remained a pivotal requirement. For instance, the QFCLT for the time-dynamic RCM in [1] required strict ellipticity, i.e. the conductances are almost surely uniformly bounded and bounded away from zero, as well as polynomial mixing, i.e. the polynomial decay of the correlations of the conductances in space and time. In this paper we significantly relax these assumptions and show a QFLCT for the dynamic RCM with degenerate space-time ergodic conductances that only need to satisfy a moment condition. In contrast to the earlier results mentioned above the environment is *not* assumed to be strictly elliptic or mixing or Markovian in time and we also do not require any regularity with respect to the time parameter.

1.1. **The setting.** Consider the d-dimensional Euclidean lattice, $(\mathbb{Z}^d, E_d)$, for $d \geq 2$, those edge set, E_d, is given by the set of all non-oriented nearest neighbor bonds, that is $E_d = \{\{x,y\} : x, y \in \mathbb{Z}^d,\ |x-y| = 1\}$. For any $A \subset \mathbb{Z}^d$ we denote by $|A|$ the cardinality of the set A.

The graph $(\mathbb{Z}^d, E_d)$ is endowed with time-dependent positive weights, that is, we consider a family $\omega = \{\omega_t(e) : e \in E_d,\, t \in \mathbb{R}\} \in \Omega := (0,\infty)^{\mathbb{R}\times E_d}$. We refer to $\omega_t(e)$ as the *conductance* on an edge e at time t. To simplify notation, for $x, y \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ we set $\omega_t(x,y) = \omega_t(y,x) = \omega_t(\{x,y\})$ if $\{x,y\} \in E_d$ and $\omega_t(x,y) = 0$ otherwise. A *space-time shift* by $(s,z) \in \mathbb{R} \times \mathbb{Z}^d$ is a map $\tau_{s,z} \colon \Omega \to \Omega$ defined by

$$\big(\tau_{s,z}\, \omega\big)_t(\{x,y\}) \;:=\; \omega_{t+s}(\{x+z, y+z\}), \qquad \forall\, t \in \mathbb{R},\ \{x,y\} \in E_d. \tag{1.1}$$

The set $\{\tau_{t,x} : x \in \mathbb{Z}^d, t \in \mathbb{R}\}$ together with the operation $\tau_{t,x} \circ \tau_{s,y} := \tau_{t+s,x+y}$ defines the *group of space-time shifts*.

Finally, let Ω be equipped with a σ-algebra, $\mathcal{F}$, and a probability measure, $\mathbb{P}$, so that $(\Omega, \mathcal{F}, \mathbb{P})$ becomes a probability space. We also write $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$.

Assumption 1.1. *Assume that $\mathbb{P}$ satisfies the following conditions:*

(i) $\mathbb{E}\big[\omega_t(e)\big] < \infty$ *and* $\mathbb{E}\big[\omega_t(e)^{-1}\big] < \infty$ *for all* $e \in E_d$ *and* $t \in \mathbb{R}$.

(ii) $\mathbb{P}$ *is ergodic and stationary with respect to space-time shifts, that is* $\mathbb{P} \circ \tau_{t,x}^{-1} = \mathbb{P}$ *for all* $x \in \mathbb{Z}^d$, $t \in \mathbb{R}$, *and* $\mathbb{P}[A] \in \{0,1\}$ *for any* $A \in \mathcal{F}$ *such that* $\mathbb{P}[A \triangle \tau_{t,x}(A)] = 0$ *for all* $x \in \mathbb{Z}^d$, $t \in \mathbb{R}$.

Remark 1.2. (i) Note that Assumption 1.1(i) implies that $\mathbb{P}\big[0 < \omega_t(e) < \infty\big] = 1$ for all $e \in E_d$ and almost all $t \in \mathbb{R}$.

(ii) The static model where the conductances are constant in time and ergodic with respect to space shifts is included as a special case.

Remark 1.3. Let $T_t : L^2(\Omega, \mathbb{P}) \to L^2(\Omega, \mathbb{P})$ be the map defined by $T_t f := f \circ \tau_{t,0}$, for $f \in L^2(\Omega, \mathbb{P})$. Then Assumption 1.1 (ii) implies that $\{T_t : t \in \mathbb{R}\}$ is a strongly continuous contraction group (SCCS) on $L^2(\Omega, \mathbb{P})$, cf. [23, Section 7.1].

We denote by $D(\mathbb{R}, \mathbb{Z}^d)$ the space of $\mathbb{Z}^d$-valued càdlàg functions on $\mathbb{R}$. We will study the dynamic nearest-neighbour *random conductance model*. For a given $\omega \in \Omega$ and for $s \in \mathbb{R}$ and $x \in \mathbb{Z}^d$, let $\mathrm{P}^\omega_{s,x}$ be the probability measure on $D(\mathbb{R}, \mathbb{Z}^d)$, under which the coordinate process $\{X_t : t \in \mathbb{R}\}$ is the continuous-time Markov chain on $\mathbb{Z}^d$ starting in x at time $t = s$ with time-dependent generator acting on bounded functions $f \colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\mathcal{L}^\omega_t f(x) \;=\; \sum_{y \sim x} \omega_t(x,y)\big(f(y) \,-\, f(x)\big). \tag{1.2}$$

That is, X is the time-inhomogeneous random walk, whose time-dependent jump rates are given by the conductances. Note that the counting measure, independent of t, is an invariant measure for X. Further, the total jump rate out of any site x is not normalised, in particular the sojourn time at site x depends on x. Therefore, the random walk X is sometimes called the *variable speed random walk (VSRW)*.

1.2. **Main Results.** We are interested in the $\mathbb{P}$-almost sure or quenched long time behaviour of this process. Our main objective is to establish a quenched functional central limit theorem for the process X in the sense of the following definition.

Definition 1.4. Set $X^{(n)}_t := \frac{1}{n} X_{n^2 t}$, $t \geq 0$. We say that the *Quenched Functional CLT* (QFCLT) or *quenched invariance principle* holds for X if for $\mathbb{P}$-a.e. ω under $\mathrm{P}^\omega_{0,0}$, $X^{(n)}$ converges in law to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $\Sigma^2 = \Sigma \cdot \Sigma^T$. That is, for every $T > 0$ and every bounded continuous function F on the Skorohod space $D([0,T], \mathbb{R}^d)$, setting $\psi_n = \mathrm{E}^\omega_0[F(X^{(n)})]$ and $\psi_\infty = \mathrm{E}^{\mathrm{BM}}_0[F(\Sigma \cdot W)]$ with $(W, \mathrm{P}^{\mathrm{BM}}_0)$ being a Brownian motion started at 0, we have that $\psi_n \to \psi_\infty$ $\mathbb{P}$-a.s.

As our main result we establish a QFCLT for X under some additional moment conditions on the conductances. In order to formulate this moment condition we first define measures μ^ω_t and ν^ω_t on $\mathbb{Z}^d$ by

$$\mu^\omega_t(x) \;:=\; \sum_{x \sim y} \omega_t(x,y) \qquad \text{and} \qquad \nu^\omega_t(x) \;:=\; \sum_{x \sim y} \frac{1}{\omega_t(x,y)}.$$

In addition, for arbitrary numbers $p, p' \geq 1$ and any non-empty compact interval $I \subset \mathbb{R}$ and any finite $B \subset \mathbb{Z}^d$ let us introduce a space-time averaged $L^{p,p'}$-norm on functions $u\colon \mathbb{R}\times\mathbb{Z}^d \to \mathbb{R}$ by

$$\|u\|_{p,p',I\times B} := \left(\frac{1}{|I|}\int_I \left(\frac{1}{|B|}\sum_{x\in B}|u(t,x)|^p\right)^{p'/p} \mathrm{d}t\right)^{1/p'}.$$

Note that by Jensen's inequality $\|u\|_{p,p',I\times B} \leq \|u\|_{q,q',I\times B}$ if $q \geq p$ and $q' \geq q$. Further, we denote by $B(x,r)$ the closed ball with center x and radius r with respect to the natural graph distance d, that is $B(x,r) := \{y \in \mathbb{Z}^d : d(x,y) \leq \lfloor r \rfloor\}$, and we write $B(r) := B(0,r)$.

Assumption 1.5. *There exist $p,p',q,q' \in (1,\infty]$ satisfying*

$$\frac{1}{p}\cdot\frac{p'}{p'-1}\cdot\frac{q'+1}{q'} + \frac{1}{q} < \frac{2}{d} \tag{1.3}$$

such that

$$\limsup_{n\to\infty}\|\mu^\omega\|_{p,p',Q(n)} < \infty, \qquad \limsup_{n\to\infty}\|\nu^\omega\|_{q,q',Q(n)} < \infty, \tag{1.4}$$

where $Q(n) := [0,n^2]\times B(n)$.

Remark 1.6. In the special case $p' = p$ and $q' = q$ Assumption 1.5 directly translates into a moment condition. More precisely, if there exist $p,q \in (1,\infty]$ satisfying

$$\frac{1}{p-1} + \frac{1}{(p-1)q} + \frac{1}{q} < \frac{2}{d}$$

such that

$$\mathbb{E}\big[\omega_t(e)^p\big] < \infty \quad \text{and} \quad \mathbb{E}\big[\omega_t(e)^{-q}\big] < \infty$$

for any $e \in E_d$ and $t \in \mathbb{R}$, then Assumption 1.5 holds by the ergodic theorem.

Theorem 1.7. *Suppose that $d \geq 2$ and Assumptions 1.1 and 1.5 hold. Then, the QFCLT holds for X with a deterministic non-degenerate covariance matrix Σ^2.*

For the static RCM a QFCLT is proven in [5] for stationary ergodic conductances $\{\omega(e), e \in E_d\}$ satisfying $\mathbb{E}[\omega(e)^p] < \infty$ and $\mathbb{E}[\omega(e)^{-q}] < \infty$ for $p,q > 1$ such that $1/p + 1/q < 2/d$. Since in the static case we can choose $p' = q' = \infty$, the moment condition for the static model can be recovered in (1.3).

In the setting of general ergodic environments it is natural to expect that some moment conditions are needed in view of the results in [9], where Barlow, Burdzy and Timar give an example for a static RCM on $\mathbb{Z}^2$ for which the QFCLT fails but a weak moment condition is fulfilled.

One motivation to study the dynamic RCM is to consider random walks in an environment generated by some interacting particle systems like zero-range or exclusion processes (cf. [16, 30]). Recently, some on-diagonal upper bounds for the transition kernel of a degenerate time-dependent conductances model are obtained in [30], where the conductances are uniformly bounded from above but they are allowed to be zero at a a given time satisfying a lower moment condition. In [22] it is shown that for uniformly elliptic dynamic RCM in discrete time – in contrast to the time-static case – two-sided Gaussian heat kernel estimates are not stable under perturbations. In a time dynamic balanced environment a QFCLT under moment conditions has been recently shown in [16].

An annealed FCLT has been obtained for strictly elliptic conductances in [1], for non-elliptic conductances generated by an exclusion process in [6] and for a similar one-dimensional model allowing some local drift in [7] and recently for environments generated by random walks in [21]. In [11, 29] random walks on the backbone of an oriented percolation cluster are considered, which are interpreted as the ancestral lines in a population model.

Finally, let us remark that there is a link between the time dynamic RCM and Ginzburg-Landau interface models as such random walks appear in the so-called Helffer-Sjöstrand representation of the space-time covariance in these models (cf. [15, 1]). However, in this context the annealed FCLT is relevant.

1.3. **The method.** We follow the most common approach to prove a QFLCT for the RCM and introduce the so-called harmonic coordinates, that is we construct a *corrector* $\chi\colon \Omega \times \mathbb{R} \times \mathbb{Z}^d \to \mathbb{R}^d$ such that

$$y(\omega, t, x) \;=\; x - \chi(\omega, t, x)$$

is a space-time harmonic function. In other words,

$$\partial_t y(\omega, t, x) + \mathcal{L}_t^\omega y(\omega, t, x) \;=\; 0. \tag{1.5}$$

This can be rephrased by saying that χ is a solution of the time-inhomogeneous Poisson equation

$$\partial_t u + \mathcal{L}_t^\omega u \;=\; \nabla^* V_t^\omega, \tag{1.6}$$

where $V_t^\omega\colon E_d \to \mathbb{R}^d$ is the local drift at time t given by $V_t^\omega(x,y) := \omega_t(x,y)\,(y-x)$ and ∇^* denotes the divergence operator associated with the discrete gradient. Recall that one property of the *static* RCM – being one its main differences to other models for random walks in random media – is the reversibility of the random walk w.r.t. its speed measure. In our setting, the generator $(\partial_t + \mathcal{L}_t^\omega)$ of the space-time process (t, X_t) is asymmetric and the construction of the corrector as carried out for instance in [2, 12] fails, since it is based on a simple projection argument using the symmetry of the generator and an integration by parts. In [1] it was possible to construct the corrector by techniques close to the original method by Kipnis and Varadhan, since in the case of strictly elliptic conductances the asymmetric part can be controlled and a sector condition holds. In our degenerate situation, following the approach in [19], we first solve a regularised corrector equation by an application of the Lax-Milgram lemma and then we obtain the harmonic coordinates by taking limits in a suitable distribution space. The resulting corrector function consists of two parts, one part χ_0 being time-homogeneous and invariant w.r.t. space shift in the sense that for every fixed t it satisfies $\mathbb{P}$-a.s. the *cocycle property*

$$\chi_0(\omega, t, x+y) - \chi_0(\omega, t, x) \;=\; \chi_0(\tau_{0,x}\omega, t, y), \qquad x, y \in \mathbb{Z}^d,$$

and a second part which is only depending on the time variable and which therefore does not appear in the corrector for the time-static model.

Given the harmonic coordinates as a solution of (1.5) the process

$$M_t \;=\; X_t - \chi(\omega, t, X_t)$$

is a martingale under $\mathrm{P}^\omega_{0,0}$ for $\mathbb{P}$-a.e. ω, and a QFCLT for the martingale part M can be easily shown by standard arguments. We thus get a QFCLT for X once we verify that $\mathbb{P}$-almost surely the corrector is sublinear:

$$\lim_{n\to\infty} \max_{(t,x)\in Q(n)} \frac{|\chi(\omega,t,x)|}{n} \;=\; 0. \tag{1.7}$$

This control on the corrector implies that for any $T > 0$ and $\mathbb{P}$-a.e ω,

$$\sup_{0\le t\le T} \frac{1}{n}\Big|\chi\big(\omega, n^2 t, n\, X^{(n)}_t\big)\Big| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } \mathrm{P}^\omega_{0,0}\text{-probability}$$

(see Proposition 4.5 below). Combined with the QFCLT for the martingale part this gives Theorem 1.7.

The main challenge in the proof of the QFCLT is to prove (1.7). In a first step we show that the rescaled corrector converges in the space-time averaged $\|\cdot\|_{1,1,Q(n)}$-norm to zero (see Proposition 3.3 below). This is based on some input from ergodic theory, see Section 3 and Appendix B and C for more details . In a second step we establish a maximal inequality for the corrector as a solution of (1.6) using Moser iteration, that is we show that the maximum of the rescaled corrector in (1.7) can be controlled by its $\|\cdot\|_{1,1,Q(n)}$-norm (see Proposition 3.2 below). In the case of static conductances Moser iteration has already been implemented in order to show the QFCLT in [5], but also to obtain a local limit theorem and elliptic and parabolic Harnack inequalities in [4] as well as upper Gaussian estimates on the heat kernel in [3]. In the present time-inhomogeneous setting involving a time-dependent operator $\mathcal{L}^\omega_t$ a space-time version of the Sobolev inequality in [5] is needed and the actual iteration procedure has to be carried out in both the space and the time parameter of the space-time averaged norm (cf. [26]).

The paper is organised as follows: In Section 2 we construct the corrector and show some of its properties. Then, in Section 3 we prove the sublinearity of the corrector (1.7) and complete the proof of the QFCLT in Section 4. The maximal inequality for the time-inhomogeneous Poisson equation in (1.6) is proven in a more general context in Section 5. Finally, the Appendix contains a collection of some elementary estimates and a version of Maker's theorem as well as certain ergodic theorems needed in the proofs.

Throughout the paper, we write c to denote a positive constant which may change on each appearance. Constants denoted by C_i will be the same through each argument. Further, we write B_r, $r \geq 0$, for closed balls in $\mathbb{R}^d$ w.r.t. the Euclidean distance with center at the origin and radius r, while $B(r)$ defined above denotes the ball in $\mathbb{Z}^d$ w.r.t. the graph distance. The canonical basis vectors in $\mathbb{R}^d$ will be denoted by $e_1, \dots, e_d$.

2. Harmonic embedding and the corrector

Throughout this section we suppose that Assumption 1.1 holds.

2.1. **Setup and Preliminaries.** We consider the Hilbert space $L^2(\Omega \times \mathbb{Z}^d, m)$ with respect to the measure m given by

$$m(\mathrm{d}\omega, \mathrm{d}z) \;:=\; \frac{1}{2}\sum_{x\in\mathbb{Z}^d} \omega_0(0,x)\,\mathrm{d}\mathbb{P}(\omega)\otimes\delta_x(z),$$

which comes naturally with the norm

$$\|\phi\|_m^2 \;:=\; \frac{1}{2}\sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[\omega_0(0,x)\,|\phi(\cdot,x)|^2\Big].$$

Definition 2.1. (i) A measurable function $\Psi\colon \Omega\times\mathbb{Z}^d\to\mathbb{R}^d$, also called random field, satisfies the *cocycle property* (in space), if for $\mathbb{P}$-a.e. ω,

$$\Psi(\tau_{0,x}\omega, y-x) \;=\; \Psi(\omega,y) \;-\; \Psi(\omega,x), \qquad \forall\, x,y\in\mathbb{Z}^d.$$

(ii) A measurable function $\Psi : \Omega\times\mathbb{R}\times\mathbb{Z}^d \to \mathbb{R}^d$ is space-time homogeneous, if there exists a measurable function $\psi\colon\Omega\to\mathbb{R}^d$ such that

$$\Psi(\omega,t,x) \;=\; \psi(\tau_{t,x}\omega), \qquad (t,x)\in\mathbb{R}\times\mathbb{Z}^d.$$

Remark 2.2. If Ψ satisfies the cocycle property, then $\Psi(\omega,0) = 0$ and for any $x_0,x_1,\dots,x_n\in\mathbb{Z}^d$,

$$\sum_{i=1}^n \Psi(\tau_{0,x_{i-1}}\omega, x_i - x_{i-1}) \;=\; \Psi(\omega,x_n) \;-\; \Psi(\omega,x_0).$$

For functions $\phi\colon\Omega\to\mathbb{R}^d$ we define the horizontal gradient $\mathrm{D}\phi\colon\Omega\times\mathbb{Z}^d\to\mathbb{R}$ as $\mathrm{D}\phi(\omega,x) := \phi(\tau_{0,x}\omega) - \phi(\omega)$. We will often also write $\mathrm{D}_x\phi(\omega)$ for $\mathrm{D}\phi(\omega,x)$. Further, we set

$$L^2_{\mathrm{pot}} \;:=\; \overline{\{\mathrm{D}\phi \,:\, \phi : \Omega\to\mathbb{R} \text{ bounded}\}}^{\|\cdot\|_{L^2(m)}}$$

to be the closure of the set of gradients in $L^2(m)$ and let L^2_{sol} be its orthogonal, i.e.

$$L^2(m) \;=\; L^2_{\mathrm{pot}} \,\oplus\, L^2_{\mathrm{sol}}.$$

Note that for every $\phi : \Omega\to\mathbb{R}$, $\mathrm{D}\phi$ satisfies the cocycle property. Then, by approximation, for every $\Psi\in L^2_{\mathrm{pot}}$ and $x_0,x_1,\dots,x_n\in\mathbb{Z}^d$,

$$\sum_{i=1}^n \Psi(\tau_{0,x_{i-1}}\omega, x_i - x_{i-1}) \;=\; 0, \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

whenever $n>1$, $x_0 = x_n$ and $\omega_0(x_i,x_{i-1})>0$, $\mathbb{P}$-a.e. for all $i=1,\dots,n$.

The discrete gradient of functions on $\mathbb{Z}^d$ will be denoted by ∇, that is $\nabla_y g(x) = g(x+y) - g(x)$ for any $x,y\in\mathbb{Z}^d$. Note that for a space-time homogeneous function Ψ it holds that

$$\nabla_y\Psi(\omega,t,x) \;=\; \mathrm{D}_y\Psi(\omega,t,x), \qquad y\in\mathbb{Z}^d. \tag{2.2}$$

By Remark 1.3 the group $\{T_t\}_{t\in\mathbb{R}}$ is a SCCG on $L^2(\Omega,\mathbb{P})$, therefore it has an infinitesimal generator D_0, whose domain $\mathcal{D}(\mathrm{D}_0)$ is dense in $L^2(\Omega,\mathbb{P})$,

$$\mathrm{D}_0\phi \;:=\; \lim_{h\to 0}\frac{T_h\phi-\phi}{h},$$

whenever the limit exists in $L^2(\Omega,\mathbb{P})$.

Lemma 2.3. (i) *The operator D_0 is antisymmetric in $L^2(\mathbb{P})$, that is*

$$\langle\phi,\mathrm{D}_0\psi\rangle_{\mathbb{P}} \;=\; -\langle\psi,\mathrm{D}_0\phi\rangle_{\mathbb{P}}, \qquad \forall\,\phi,\psi\in\mathcal{D}(\mathrm{D}_0). \tag{2.3}$$

In particular $\langle\phi,D_0\phi\rangle_{\mathbb{P}}=0$ and $\langle 1,D_0\phi\rangle_{\mathbb{P}}=0$.

(ii) *For every $x\in\mathbb{Z}^d$ the operators D_x and D_0 commute, that is*

$$\mathrm{D}_0\mathrm{D}_x\phi \;=\; \mathrm{D}_x\mathrm{D}_0\phi, \qquad \forall\,\phi\in\mathcal{D}(\mathrm{D}_0). \tag{2.4}$$

(iii) *For every $x\in\mathbb{Z}^d$ the adjoint of the operator D_x is given by D_{-x},*

$$\langle\phi,\mathrm{D}_x\psi\rangle_{\mathbb{P}} \;=\; \langle\mathrm{D}_{-x}\phi,\psi\rangle_{\mathbb{P}}, \qquad \forall\,\phi,\psi\in L^2(\mathbb{P}). \tag{2.5}$$

(iv) *For every $\xi\in L^2(\mathbb{P})$ the function $t\mapsto\xi(\tau_{t,0}\omega)$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R})$ $\mathbb{P}$-almost surely.*

(v) *For any $\zeta\in C^1(\mathbb{R})$ with compact support, $\phi\in\mathcal{D}(\mathrm{D}_0)$ and $\psi\in L^2(\mathbb{P})$,*

$$\int_{\mathbb{R}}\zeta(t)\,\langle\mathrm{D}_0\phi\circ\tau_{-t,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}t \;=\; \int_{\mathbb{R}}\zeta'(t)\,\langle\phi\circ\tau_{-t,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}t. \tag{2.6}$$

(vi) *For any $\phi\in\mathcal{D}(\mathrm{D}_0)$, the function $t\mapsto\phi(\tau_{t,0}\omega)$ is weakly differentiable $\mathbb{P}$-almost surely. In particular*

$$\mathrm{D}_0\phi(\tau_{t,0}\omega) \;=\; \phi'(\tau_{\cdot,0}\omega)(t) \tag{2.7}$$

for almost all t, $\mathbb{P}$-almost surely.

(vii) *For every $\xi\in L^2(\mathbb{P})$ and every $\phi\in L^2_{\mathrm{pot}}$,*

$$\langle\phi,\mathrm{D}\xi\rangle_m \;=\; -2\langle\phi,\xi\rangle_m. \tag{2.8}$$

Proof. (i) By the shift-invariance of $\mathbb{P}$ we have

$$\langle\phi,\mathrm{D}_0\psi\rangle_{\mathbb{P}} \;=\; \lim_{t\to0}t^{-1}\langle\phi,T_t\psi-\psi\rangle_{\mathbb{P}} \;=\; -\lim_{t\to0}t^{-1}\langle\psi,T_{-t}\phi-\phi\rangle_{\mathbb{P}} \;=\; -\langle\psi,\mathrm{D}_0\phi\rangle_{\mathbb{P}}.$$

The second statement is trivial.

(ii) This follows directly from the linearity of D_0 as

$$\mathrm{D}_0\mathrm{D}_x\phi(\omega) \;=\; \mathrm{D}_0\big(\phi(\tau_{0,x})-\phi(\omega)\big) \;=\; \mathrm{D}_0\phi(\tau_{0,x}\omega)-\mathrm{D}_0\phi(\omega) \;=\; \mathrm{D}_x\mathrm{D}_0\phi(\omega).$$

(iii) Again by the shift invariance of $\mathbb{P}$ we have

$$\langle\phi,\mathrm{D}_x\psi\rangle_{\mathbb{P}} \;=\; \langle\phi,\psi\circ\tau_{0,x}\rangle_{\mathbb{P}} - \langle\phi,\psi\rangle_{\mathbb{P}} \;=\; \langle\phi\circ\tau_{0,-x},\psi\rangle_{\mathbb{P}} - \langle\phi,\psi\rangle_{\mathbb{P}} \;=\; \langle\mathrm{D}_{-x}\phi,\psi\rangle_{\mathbb{P}}.$$

(iv) For any compact $I\subset\mathbb{R}$ and $\xi\in L^2(\mathbb{P})$

$$\mathbb{E}\Big[\int_I(\xi\circ\tau_{t,0})^2\,\mathrm{d}t\Big] \;=\; \int_I\mathbb{E}\big[(\xi\circ\tau_{t,0})^2\big]\,\mathrm{d}t \;=\; |I|\;\mathbb{E}\big[\xi^2\big] \;<\; \infty.$$

Thus, for $\mathbb{P}$-a.e. ω it holds that

$$\int_I \xi(\tau_{t,0}\,\omega)^2 \,\mathrm{d}t \;<\; \infty.$$

(v) A simple change of variables gives

$$\begin{aligned}
&\int_{\mathbb{R}} \zeta(t)\,\langle \mathrm{D}_0\phi\circ\tau_{-t,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}t \\
&\quad= \lim_{h\to 0} h^{-1}\Big(\int_{\mathbb{R}} \zeta(t)\,\langle \phi\circ\tau_{-t+h,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}t \;-\; \int_{\mathbb{R}} \zeta(t)\,\langle \phi\circ\tau_{-t,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}t\Big) \\
&\quad= \lim_{h\to 0} h^{-1}\Big(\int_{\mathbb{R}} \zeta(s+h)\,\langle \phi\circ\tau_{-s,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}s \;-\; \int_{\mathbb{R}} \zeta(s)\,\langle \phi\circ\tau_{-s,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}s\Big) \\
&\quad= \int_{\mathbb{R}} \zeta'(s)\,\langle \phi\circ\tau_{-s,0},\psi\rangle_{\mathbb{P}}\,\mathrm{d}s.
\end{aligned}$$

(vi) It follows by (iv) that $t\mapsto\phi(\tau_{t,0}\omega)$ and $t\mapsto \mathrm{D}_0\phi(\tau_{t,0}\omega)$ belong to $L^2_{\mathrm{loc}}(\mathbb{R})$ $\mathbb{P}$-almost surely. By definition of weak differentiability, it suffices to show that for $\mathbb{P}$-a.a ω

$$\int_{\mathbb{R}} \zeta(t)\,\mathrm{D}_0\phi\circ\tau_{t,0}\,\mathrm{d}t \;=\; -\int_{\mathbb{R}} \zeta'(t)\phi\circ\tau_{t,0}\mathrm{d}t, \tag{2.9}$$

for all $\zeta\in C_0^\infty(\mathbb{R})$. From (v), Fubini theorem and the fact that (v) holds for all $\psi\in L^2(\mathbb{P})$, (2.9) follows for any fixed ζ $\mathbb{P}$-a.s. The null-set where (2.9) does not hold may depend on ζ. We can remove this ambiguity using that $C_0^\infty(\mathbb{R})$ is separable.
(vii) By the shift invariance of $\mathbb{P}$ we have

$$\begin{aligned}
\langle\phi,\mathrm{D}\xi\rangle_m &= \frac12\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\omega_0(0,x)\phi(\omega,x)\left(\xi(\tau_{0,x}\omega)-\xi(\omega)\right)\right] \\
&= \frac12\sum_{x\in\mathbb{Z}^d}\left(\mathbb{E}\left[\omega_0(-x,0)\phi(\tau_{0,-x}\omega,x)\xi(\omega)\right]-\mathbb{E}\left[\omega_0(0,x)\phi(\omega,x)\xi(\omega)\right]\right) \\
&= \frac12\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\omega_0(0,x)\big(\phi(\tau_{0,x}\omega,-x)-\phi(\omega,x)\big)\,\xi(\omega)\right] = -2\langle\phi,\xi\rangle_m,
\end{aligned}$$

where we used in the last step that $\phi(\tau_{0,x}\omega,-x)=-\phi(\omega,x)$ by the property (2.1). □

2.2. **Construction of the corrector.** In this subsection we construct the corrector. We introduce the position field $\Pi:\Omega\times\mathbb{Z}^d\to\mathbb{R}^d$ with $\Pi(\omega,x)=x$. We write Π^k for the k-th coordinate of Π. Obviously, Π satisfies the cocycle property as $\Pi(\omega,y-x)=\Pi(\omega,y)-\Pi(\omega,x)$. Moreover, for every k,

$$\|\Pi^k\|_m^2 \;=\; \mathbb{E}\Big[\textstyle\sum_{x\in\mathbb{Z}^d}\omega_0(0,x)|x^k|^2\Big] \;\le\; \mathbb{E}[\mu_0^\omega(0)] \;<\; \infty.$$

Next we state the main result of this subsection.

Theorem 2.4. *Suppose that Assumption 1.5 holds. There exists $\Phi\colon \Omega\times\mathbb{R}\times\mathbb{Z}^d\to\mathbb{R}^d$ called the* harmonic coordinates *such that for $\mathbb{P}$-a.e. ω, a.e. t and every $x\in\mathbb{Z}^d$ the following hold.*

(i) The harmonic coordinates satisfy the equation

$$\partial_t\Phi(\omega,t,x) \;+\; \mathcal{L}_t^\omega\Phi(\omega,t,x) \;=\; 0. \tag{2.10}$$

(ii) The harmonic coordinates can be written as

$$\Phi(\omega,t,x) \;=\; \Phi_0(\omega,t,x) \;-\; \int_0^t \mathcal{L}_s^\omega\Phi_0(\omega,s,0)\,\mathrm{d}s,$$

where for every t the mapping $(\omega,x)\mapsto\Phi_0(\omega,t,x)-x$ is a cocycle in L^2_{pot} and for every $x\in\mathbb{Z}^d$ the mapping $(\omega,t)\mapsto\Phi_0(\omega,t,x)$ is time-homogeneous. In particular, for every $y\in\mathbb{Z}^d$,

$$\nabla_y\Phi(\omega,t,x) \;=\; \nabla_y\Phi_0(\omega,t,x). \tag{2.11}$$

(iii) The harmonic coordinates have the asymptotics

$$\lim_{n\to\infty}\max_{(t,x)\in Q(n)}\frac{|\Phi(\omega,t,x)-x|}{n} \;=\; 0.$$

Before we prove Theorem 2.4 we define the corrector and collect some of its properties.

Definition 2.5. The corrector $\chi\colon\Omega\times\mathbb{R}\times\mathbb{Z}^d\to\mathbb{R}^d$ is defined as

$$\chi(\omega,t,x) \;:=\; \Pi(\omega,x) \;-\; \Phi(\omega,t,x).$$

Corollary 2.6. *For $\mathbb{P}$-a.e. ω, a.e. $t\in\mathbb{R}$ and every $x\in\mathbb{Z}^d$ the following hold.*

(i) The corrector satisfies the equation

$$\partial_t\chi(\omega,t,x) \;+\; \mathcal{L}_t^\omega\chi(\omega,t,x) \;=\; \nabla^* V_t^\omega, \tag{2.12}$$

where $V_t^\omega(x,y)=\omega_t(x,y)(y-x)$, $\{x,y\}\in E_d$, denotes the local drift and ∇^* the divergence operator of the discrete gradient.

(ii) The corrector can be written as

$$\chi(\omega,t,x) \;=\; \chi_0(\omega,t,x) \;+\; \int_0^t \mathcal{L}_s^\omega\Phi_0(\omega,s,0)\,\mathrm{d}s, \tag{2.13}$$

where $\chi_0(\omega,t,x):=x-\Phi_0(\omega,t,x)$.

(iii) For every t the mapping $(\omega,x)\mapsto\chi_0(\omega,t,x)$ is a cocycle in L^2_{pot} and for every $x\in\mathbb{Z}^d$ the mapping $(\omega,t)\mapsto\chi_0(\omega,t,x)$ is time-homogeneous, that is

$$\chi_0(\omega,t,x) \;=\; \chi_0(\tau_{t,0}\omega,0,x). \tag{2.14}$$

In particular,

$$\chi(\omega,t,x) \;=\; \chi_0(\omega,t,x) \;+\; \chi(\omega,t,0). \tag{2.15}$$

Proof. These are immediate consequences from Theorem 2.4. Note that (2.15) follows from (ii) since $\chi_0(\omega,t,0)=0$ by the cocycle property. □

The rest of this section is devoted to the construction of the harmonic coordinates and the proof of Theorem 2.4 (i) and (ii). Statement (iii) is equivalent to the sublinearity of the corrector and will be proven in Section 3 below.

Let $\mathcal{H}^1 := \{\phi \in \mathcal{D}(\mathrm{D}_0) \,:\, \mathrm{D}\phi \in L^2_{\mathrm{pot}}\}$ equipped with the norm given by

$$\|\phi\|^2_{\mathcal{H}^1} \;:=\; \|\phi\|^2_{\mathbb{P}} \,+\, \|\mathrm{D}_0\phi\|^2_{\mathbb{P}} \,+\, \|\mathrm{D}\phi\|^2_m,$$

and a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}^1}$ defined by polarisation. It is easy to see that $\mathcal{H}^1$ is a Hilbert space.

We want to solve the following equation

$$Q^\beta(\phi, \xi) \;=\; B^k(\xi), \qquad \forall\, \xi \in \mathcal{H}^1,\; k = 1, \dots, d, \tag{2.16}$$

where

$$Q^\beta(\phi, \xi) \;:=\; -\langle \mathrm{D}_0\phi, \xi\rangle_{\mathbb{P}} \,+\, \langle \mathrm{D}\phi, \mathrm{D}\xi\rangle_m \,+\, \beta\, \langle \mathrm{D}_0\phi, \mathrm{D}_0\xi\rangle_{\mathbb{P}} \,+\, \beta\, \langle \phi, \xi\rangle_{\mathbb{P}}$$

and

$$B^k(\xi) \;:=\; \langle \Pi^k, \mathrm{D}\xi\rangle_m.$$

Lemma 2.7. *For all $\beta > 0$, $Q^\beta : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{R}$ is a coercive bounded bilinear form, and for all $k = 1, \dots, d$, B^k is a bounded and linear operator on $\mathcal{H}^1$.*

Proof. The statement is true basically by definition and Lemma 2.3 (i). Indeed, we have

$$Q^\beta(\phi, \phi) \;\geq\; (1 \wedge \beta)\, \|\phi\|^2_{\mathcal{H}^1}$$

and by the Cauchy-Schwarz inequality

$$\begin{aligned} \left|Q^\beta(\phi, \xi)\right| \;&\leq\; (1 \vee \beta)\Big(2\|\mathrm{D}_0\phi\|^2_{\mathbb{P}} + \|\mathrm{D}\phi\|^2_m + \|\phi\|^2_{\mathbb{P}}\Big)^{1/2}\Big(2\|\xi\|^2_{\mathbb{P}} + \|\mathrm{D}\xi\|^2_m + \|\mathrm{D}_0\xi\|^2_{\mathbb{P}}\Big)^{1/2} \\ &\leq\; 2(1 \vee \beta)\|\phi\|_{\mathcal{H}^1}\|\xi\|_{\mathcal{H}^1}. \end{aligned}$$

Similarly, since $\mathbb{E}[\omega_0(0, e)] < \infty$ it follows that B^k is bounded for all k. □

By an application of Lax-Milgram Lemma it follows that for every $\beta > 0$ there exists $\phi^{\beta,k} \in \mathcal{H}^1$ such that $Q^\beta(\phi^{\beta,k}, \xi) = B^k(\xi)$ holds for all $\xi \in \mathcal{H}$. In particular, the equation is satisfied for $\xi = \phi^{\beta,k}$. We use this information to obtain a first energy bound.

Lemma 2.8. *For all $\beta > 0$ and $k = 1, \dots, d$,*

$$\|\mathrm{D}\phi^{\beta,k}\|^2_m \,+\, \beta\, \|\mathrm{D}_0\phi^{\beta,k}\|^2_{\mathbb{P}} \,+\, \beta\, \|\phi^{\beta,k}\|^2_{\mathbb{P}} \;\leq\; \frac{1}{2}\, \mathbb{E}[\mu_0(0)]. \tag{2.17}$$

Moreover, for $\beta \in (0, 1]$ and all $k = 1, \dots, d$,

$$\left|\langle \mathrm{D}_0\phi^{\beta,k}, \xi\rangle_{\mathbb{P}}\right| \;\leq\; \sqrt{2}\, \mathbb{E}[\mu_0(0)]^{1/2}\, \|\xi\|_{\mathcal{H}^1}. \tag{2.18}$$

Proof. Since $\langle \phi^{\beta,k}, \mathrm{D}_0\phi^{\beta,k}\rangle_{\mathbb{P}} = 0$ we obtain from $Q^\beta(\phi^{\beta,k}, \phi^{\beta,k}) = B^k(\phi^{\beta,k})$ that

$$\|\mathrm{D}\phi^{\beta,k}\|_m^2 + \beta\,\|\mathrm{D}_0\phi^{\beta,k}\|_{\mathbb{P}}^2 + \beta\,\|\phi^{\beta,k}\|_{\mathbb{P}}^2 = \langle \Pi^k, \mathrm{D}\phi^{\beta,k}\rangle_m \leq \frac{1}{\sqrt{2}}\,\mathbb{E}[\mu_0(0)]^{1/2}\,\|\mathrm{D}\phi^{\beta,k}\|_m, \tag{2.19}$$

where we used the Cauchy-Schwarz inequality in the last step. By dropping the positive terms with β in front, we obtain

$$\|\mathrm{D}\phi^{\beta,k}\|_m^2 \leq \frac{1}{2}\,\mathbb{E}\left[\mu_0(0)\right]. \tag{2.20}$$

By combining this with (2.19) we immediately get (2.17).

In order to prove (2.18) we use (2.16), the triangle inequality and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&\left|\langle \mathrm{D}_0\phi^{\beta,k}, \xi\rangle_{\mathbb{P}}\right| \\
&\quad\leq \left|\langle \mathrm{D}\phi^{\beta,k}, \mathrm{D}\xi\rangle_m\right| + \beta\left|\langle \mathrm{D}_0\phi^{\beta,k}, \mathrm{D}_0\xi\rangle_{\mathbb{P}}\right| + \beta\left|\langle \phi^{\beta,k}, \xi\rangle_{\mathbb{P}}\right| + |B^k(\xi)| \\
&\quad\leq \sqrt{2}\left(\|\mathrm{D}\phi^{\beta,k}\|_m^2 + \beta^2\,\|\mathrm{D}_0\phi^{\beta,k}\|_{\mathbb{P}}^2 + \beta^2\,\|\phi^{\beta,k}\|_{\mathbb{P}}^2 + \frac{1}{2}\,\mathbb{E}[\mu_0(0)]\right)^{1/2} \|\xi\|_{\mathcal{H}^1}.
\end{aligned}$$

Finally, since $\beta^2 \leq \beta$ for $\beta \in (0,1]$ the bound (2.17) gives (2.18). □

As a consequence of Lemma 2.8 we have that $\mathrm{D}\phi^{\beta,k}$ are uniformly bounded in $L^2(\Omega \times \mathbb{Z}^d, m)$. Therefore there exist $\Psi^k \in L^2(\Omega \times \mathbb{Z}^d, m)$ such that

$$\mathrm{D}\phi^{\beta,k} \rightharpoonup \Psi^k$$

weakly in $L^2(\Omega \times \mathbb{Z}^d, m)$ along some subsequence $\beta \downarrow 0$. In fact $\Psi^k \in L^2_{\mathrm{pot}}$, since for all $\xi \in L^2_{\mathrm{sol}}$ we have $\langle \Psi^k, \xi\rangle_m = \lim_{\beta\to 0}\langle \mathrm{D}\phi^{\beta,k}, \xi\rangle_m$ and $\langle \mathrm{D}\phi^{\beta,k}, \xi\rangle_m = 0$ for all $\beta > 0$.

As a further consequence of Lemma 2.8 we observe that the linear functional $F^{\beta,k}\colon \mathcal{H}^1 \to \mathbb{R}$ defined by

$$F^{\beta,k}(\xi) := -\langle \mathrm{D}_0\phi^{\beta,k}, \xi\rangle_{\mathbb{P}}$$

are uniformly bounded in $\mathcal{H}^{-1}$, the dual of $\mathcal{H}^1$. It follows that there exist $F^k \in \mathcal{H}^{-1}$ such that

$$F^{\beta,k} \rightharpoonup F^k$$

weakly in $\mathcal{H}^{-1}$ along a subsequence. Recalling that weak convergence in $\mathcal{H}^{-1}$ implies that $F^{\beta,k}(\xi) \to F^k(\xi)$ for all $\xi \in \mathcal{H}^1$, we can take the limit in (2.16) as $\beta \to 0$ along some subsequence and get

$$F^k(\xi) + \langle \Psi^k, \mathrm{D}\xi\rangle_m = B^k(\xi), \qquad \forall\, \xi \in \mathcal{H}^1. \tag{2.21}$$

The first term on the left of (2.21) is implicit. We want to identify it at least for a class of functions $\xi \in \mathcal{H}^1$. This is the content of the next lemma.

Lemma 2.9. *Consider the class*

$$\mathcal{H}_b^1 \ := \ \{\xi \in L^\infty(\Omega,\mathbb{P}) \cap \mathcal{D}(\mathrm{D}_0) \ : \ \mathrm{D}_0\xi \in L^\infty(\Omega,\mathbb{P})\}$$

For any $\xi \in \mathcal{H}_b^1$ and $x \in \mathbb{Z}^d$ we have

$$F^k(\mathrm{D}_x\xi) \ = \ \langle \Psi^k(\cdot,-x), \mathrm{D}_0\xi\rangle_{\mathbb{P}}. \tag{2.22}$$

Proof. By (2.4) and (2.5) we get for $\xi \in \mathcal{H}_b^1$,

$$\begin{aligned} -\langle \mathrm{D}_0\phi^{\beta,k}, \mathrm{D}_x\xi\rangle_{\mathbb{P}} \ &= \ \langle \phi^{\beta,k}, \mathrm{D}_x\mathrm{D}_0\xi\rangle_{\mathbb{P}} \ = \ \langle \mathrm{D}_{-x}\phi^{\beta,k}, \mathrm{D}_0\xi\rangle_{\mathbb{P}} \\ &= \ \Big\langle \omega_0(-x,0)\mathrm{D}_{-x}\phi^{\beta,k}, \omega_0(-x,0)^{-1}\mathrm{D}_0\xi\Big\rangle_{\mathbb{P}} \ = \ \langle \mathrm{D}\phi^{\beta,k}, \Xi^x\rangle_m, \end{aligned} \tag{2.23}$$

where $\Xi^x \colon \Omega \times \mathbb{Z}^d \to \mathbb{R}$ is defined by

$$\Xi^x(\omega,y) \ := \ 2\delta_{-x}(y)\,\omega_0(-x,0)^{-1}\,\mathrm{D}_0\xi(\omega).$$

Observe that, since $\xi \in \mathcal{H}_b^1$, $\mathrm{D}_x\xi \in \mathcal{H}^1$ and Ξ^x belongs to $L^2(\Omega\times\mathbb{Z}^d, m)$, since by Assumption 1.1(i) $\mathbb{E}[\omega_0(0,x)^{-1}] < \infty$. Using the weak convergence along a subsequence as $\beta \downarrow 0$ in (2.23) we finally get

$$\begin{aligned} F^k(\mathrm{D}_x\xi) \ &= \ \langle \Psi^k, \Xi^x\rangle_m \\ &= \ \big\langle \omega_0(-x,0)\Psi^k(\cdot,-x), \omega_0(-x,0)^{-1}\mathrm{D}_0\xi\big\rangle_{\mathbb{P}} \ = \ \langle \Psi^k(\cdot,-x), \mathrm{D}_0\xi\rangle_{\mathbb{P}}, \end{aligned}$$

which is the claim. □

Proof of Theorem 2.4 (i) and (ii). By means of (2.22) for $\xi \in \mathcal{H}_b^1$ we have for any $x \in \mathbb{Z}^d$,

$$\langle \Psi^k(\cdot,x), \mathrm{D}_0\xi\rangle_{\mathbb{P}} \ + \ \langle \Psi^k, \mathrm{D}\mathrm{D}_{-x}\xi\rangle_m \ = \ B^k(\mathrm{D}_{-x}\xi),$$

which can be rewritten as

$$\langle \Psi^k(\cdot,x), \mathrm{D}_0\xi\rangle_{\mathbb{P}} \ + \ \langle \Psi^k - \Pi^k, \mathrm{D}\mathrm{D}_{-x}\xi\rangle_m \ = \ 0. \tag{2.24}$$

Now we define

$$\Phi_0^k(\omega,t,x) \ := \ x^k \ - \ \sum_{i=1}^n \Psi^k(\tau_{t,x_{i-1}}\omega, x_i - x_{i-1}),$$

where $x_0,\dots,x_n \in \mathbb{Z}^d$ are such that $x_0 = 0$, $x_n = x$ and $|x_i - x_{i-1}| = 1$ for all $i = 1,\dots,n$. We observe that by (2.1) the definition does not depend on the particular path we choose. Moreover, note that $\Phi_0^k(\omega,t,0) = 0$ and $\nabla_y\Phi_0^k(\omega,t,x) = y^k - \Psi^k(\tau_{t,x}\omega,y)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{Z}^d$ with $|y| = 1$ by (2.1). Further, $\Phi_0^k(\omega,t,x)$ satisfies the cocycle property in space. Using $\mathbb{E}[y^k\mathrm{D}_0\xi] = 0$ we rewrite (2.24) as

$$\langle \nabla_y\Phi_0^k(\cdot,0,0), \mathrm{D}_0\xi\rangle_{\mathbb{P}} \ + \ \langle \nabla\Phi_0^k(\cdot,0,0), \mathrm{D}\mathrm{D}_{-y}\xi\rangle_m \ = \ 0. \tag{2.25}$$

For any $z \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ we have that $\xi' = \xi \circ \tau_{-t,-z} \in \mathcal{H}_b^1$ for all $\xi \in \mathcal{H}_b^1$. We replace ξ by ξ' in (2.25), integrate with respect to t against a function $\zeta \in C^1(\mathbb{R})$ with compact support and use (2.8) to obtain

$$\int_{\mathbb{R}} \zeta(t)\Big(-\langle \nabla_y \Phi_0^k(\cdot,0,0), \mathrm{D}_0(\xi \circ \tau_{-t,-z})\rangle_{\mathbb{P}} + 2\langle \mathrm{D}_{-y}(\xi \circ \tau_{-t,-z}), \nabla \Phi_0^k(\cdot,0,0)\rangle_m\Big)\,\mathrm{d}t = 0.$$

Further, by (2.5), (2.6), Fubini and the shift invariance of $\mathbb{P}$,

$$\mathbb{E}\bigg[\xi \int_{\mathbb{R}} \Big(-\zeta'(t)\, \nabla_y \Phi_0^k(\cdot,t,z) \,+\, \mathrm{D}_y\big(\mathcal{L}_0^\omega \Phi_0^k(\cdot,0,0) \circ \tau_{t,z}\big)\,\zeta(t)\Big)\,\mathrm{d}t\bigg] \;=\; 0.$$

We apply (2.2) and since $\mathcal{H}_b^1$ is dense in $L^p(\Omega,\mathbb{P})$ for all $p \geq 1$ this implies

$$\nabla_y\bigg(\int_{\mathbb{R}} -\zeta'(t)\, \Phi_0^k(\omega,t,z) \,+\, \mathcal{L}_t^\omega \Phi_0^k(\omega,t,z)\,\zeta(t)\,\mathrm{d}t\bigg) \;=\; 0$$

for all $y,z \in \mathbb{Z}^d$ and all $\zeta \in C_0^\infty(\mathbb{R})$, $\mathbb{P}$-a.s. In particular, the term in brackets is constant in z and since $\Phi_0(\omega,t,0) = 0$ we get that

$$\int_{\mathbb{R}} -\zeta'(t)\, \Phi_0^k(\omega,t,z) \,+\, \mathcal{L}_t^\omega \Phi_0^k(\omega,t,z)\,\zeta(t)\,\mathrm{d}t \;=\; \int_{\mathbb{R}} \mathcal{L}_t^\omega \Phi_0^k(\omega,t,0)\,\zeta(t)\,\mathrm{d}t. \tag{2.26}$$

From this equation it follows in particular that $t \mapsto \Phi_0^k(\omega,t,z)$ is weakly differentiable in time, hence by Sobolev's embedding it is also absolutely continuous in time for all $x \in \mathbb{Z}^d$, $\mathbb{P}$-a.s. and differentiable for almost all $t \in \mathbb{R}$. In particular $\Phi_0^k(\omega,t,z) - \Phi_0^k(\omega,0,z) = \int_0^t \partial_t \Phi_0^k(\omega,s,z)\,\mathrm{d}s$ and for almost all $t \in \mathbb{R}$, all $z \in \mathbb{Z}^d$

$$\partial_t \Phi_0^k(\omega,t,z) \,+\, \mathcal{L}_t^\omega \Phi_0^k(\omega,t,z) \;=\; \mathcal{L}_t^\omega \Phi_0^k(\omega,t,0).$$

We define

$$\Phi^k(\omega,t,z) \;:=\; \Phi_0^k(\omega,t,z) \,-\, \int_0^t \mathcal{L}_s^\omega \Phi_0^k(\omega,s,0)\,\mathrm{d}s.$$

Using (2.26) it is easy to see that Φ^k solves (2.10). We postpone the proof of (iii) to Proposition 3.1 below. □

3. Sublinearity of the corrector

The key ingredient in the proof of Theorem 1.7 is the sublinearity of the corrector as stated in the following proposition, which we prove as the main result in this section.

Proposition 3.1. *Let $d \geq 2$ and suppose that Assumptions 1.1 and 1.5 hold. Then,*

$$\lim_{n\to\infty} \max_{(t,x)\in Q(n)} \frac{|\chi(\omega,t,x)|}{n} \;=\; 0, \qquad \mathbb{P}\text{- a.s.} \tag{3.1}$$

The proof is based on both ergodic theory and purely analytic tools. First we state the maximum inequality, which we establish in a more general context in Section 5 below, to bound from above the maximum of the rescaled corrector in $Q(n)$ in terms of its $\|\cdot\|_{1,1,Q(n)}$-norm.

Proposition 3.2. *Let $p, p', q, q' \in [1, \infty)$ be as in Assumption 1.5. Then, for every $\alpha > 0$, there exist $\gamma > 0$ and $\kappa > 0$ and $c \equiv c(p, q, q', d) < \infty$ such that*

$$\max_{(t,x)\in Q(n)} \frac{\big|\chi^j(\omega,t,x)\big|}{n} \le c \Big(\|1 \vee \mu^\omega\|_{p,p',Q(2n)} \, \|1 \vee \nu^\omega\|_{q,q',Q(2n)} \Big)^{\kappa} \, \big\|\tfrac{1}{n}\chi^j(\omega,\cdot)\big\|_{\alpha,\alpha,Q(2n)}^{\gamma}$$

for $j = 1, \dots, d$.

Proof. It is obvious that $\mathbb{Z}^d$ satisfies the properties of the general graphs considered in Section 5. Then the assertion for $n^{-1}\chi^j$ follows directly from Theorem 5.5 with $\sigma = 1$, $\sigma' = 1/2$ and n replaced by $2n$. Note that, in view of (2.12) the function V_t^ω appearing in Theorem 5.5 is given $V_t^\omega(x,y) = \frac{1}{n}\,\omega_t(x,y)\,(y^j - x^j)$. □

Proposition 3.1 is now immediate from Proposition 3.2 with the choice $\alpha = 1$, Assumption 1.5 and the following proposition.

Proposition 3.3. *Suppose Assumption 1.1 holds. Then for $\mathbb{P}$-a.e. ω,*

$$\lim_{n\to\infty} \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x\in B(n)} \big|\chi(\omega,t,x)\big| \, \mathrm{d}t \;=\; 0. \tag{3.2}$$

Proof. Recall the decomposition of the corrector in (2.15). Hence,

$$\begin{aligned} &\frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x\in B(n)} \big|\chi(\omega,t,x)\big| \, \mathrm{d}t \\ &\qquad\le\; \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x\in B(n)} \big|\chi_0(\omega,t,x)\big| \, \mathrm{d}t \;+\; \frac{1}{n^3} \int_0^{n^2} \big|\chi(\omega,t,0)\big| \, \mathrm{d}t. \end{aligned}$$

In view of Lemma 3.5 the first term on the right hand side converges to zero for $\mathbb{P}$-a.e. ω. In order to deal with the second term we define for $f \in C^1(\mathbb{R})$ with $\operatorname{supp} f \subset [-1/2, 1/2]$,

$$F_n \colon \mathbb{Z}^d \to \mathbb{R}, \qquad x \;\longmapsto\; \prod_{i=1}^{d} f(x^i/n), \qquad n \in \mathbb{N} \tag{3.3}$$

and set $c_n := \big(n^{-d} \sum_{y\in B(n)} F_n(y)\big)^{-1}$. In particular, $\operatorname{supp} F_n \subset B(n)$ and $c_n \in [0,1]$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} &\frac{1}{n^3} \int_0^{n^2} \big|\chi(\omega,t,0)\big| \, \mathrm{d}t \\ &\quad\le\; \frac{c_n}{n^{d+3}} \int_0^{n^2} \Big| \sum_{y\in B(n)} F_n(y)\, \chi(\omega,t,y) \Big| \, \mathrm{d}t \;+\; \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{y\in B(n)} \big|\chi_0(\omega,t,y)\big| \, \mathrm{d}t, \end{aligned}$$

where we used again (2.15) and $F_n \le 1$ in the second step. Again by Lemma 3.5, the second term converges to zero for $\mathbb{P}$-a.e. ω. Notice that by (2.12),

$$\chi(\omega,t,y) \;=\; \chi(\omega,0,y) \;+\; \int_0^t \partial_t \chi(\omega,s,y) \, \mathrm{d}s \;=\; \chi_0(\omega,0,y) \;+\; \int_0^t \mathcal{L}_s^\omega \Phi(\omega,s,y) \, \mathrm{d}s,$$

so that

$$\frac{c_n}{n^{d+3}} \int_0^{n^2} \bigg| \sum_{y \in B(n)} F_n(y) \chi(\omega, t, y) \bigg| \,\mathrm{d}t$$

$$\leq \frac{c_n}{n^{d+1}} \bigg| \sum_{y \in B(n)} F_n(y) \chi_0(\omega, 0, y) \bigg| + \frac{1}{n^{d+3}} \int_0^{n^2} \!\! \int_0^t \bigg| \sum_{y \in B(n)} F_n(y) \, \mathcal{L}_s^\omega \Phi(\omega, s, x) \bigg| \,\mathrm{d}s \,\mathrm{d}t.$$

The first term on the right hand side converges to zero for $\mathbb{P}$-a.e. ω by Lemma 3.6. Thus, it remains to show that for $\mathbb{P}$-a.e. ω,

$$\lim_{n \to \infty} \frac{1}{n^{d+3}} \int_0^{n^2} \!\! \int_0^t \bigg| \sum_{y \in B(n)} F_n(y) \, \mathcal{L}_s^\omega \Phi(\omega, s, x) \bigg| \,\mathrm{d}s \,\mathrm{d}t \;=\; 0. \tag{3.4}$$

But a summation by parts together with Fubini's theorem and (2.11) yields

$$\frac{1}{n^{d+3}} \int_0^{n^2} \!\! \int_0^t \bigg| \sum_{y \in B(n)} F_n(y) \, \mathcal{L}_s^\omega \Phi(\omega, s, x) \bigg| \,\mathrm{d}s \,\mathrm{d}t$$

$$= \frac{1}{n^{d+3}} \int_0^{n^2} \!\! \int_0^{n^2} \mathbb{1}_{\{s<t\}} \bigg| \sum_{y \in B(n)} \sum_{|e|=1} \omega_s(y, y+e) \, \nabla_e F_n(y) \, \nabla_e \Phi(\omega, s, y) \bigg| \,\mathrm{d}s \,\mathrm{d}t$$

$$= \frac{1}{n^{d+2}} \int_0^{n^2} \bigg| \sum_{y \in B(n)} \sum_{|e|=1} \omega_s(y, y+e) \, g_e(y/n) \, \nabla_e \Phi_0(\omega, s, y) \bigg| \, \frac{n^2 - s}{n^2} \,\mathrm{d}s,$$

where for $i = 1, \ldots, d$ the function g_{e_i} is defined by

$$g_{e_i} \colon \mathbb{R}^d \longrightarrow \mathbb{R}, \qquad u \;\longmapsto\; f'(u_i) \prod_{j \neq i} f(u_j)$$

and $g_{-e_i} := -g_{e_i}$ with f defined as in (3.3). In particular, note that $\int_{B_1} g_e(u) \,\mathrm{d}u = 0$ for every $|e| = 1$. Obviously, $(n^2 - s)/n^2 \leq 1$ for $s \in [0, n^2]$. Further, by the cocycle property and the time homogeneity of Φ_0, it holds that $\nabla_e \Phi_0(\omega, s, y) = \Phi_0(\tau_{s,y}\omega, 0, e)$. Therefore,

$$\frac{1}{n^{d+3}} \int_0^{n^2} \!\! \int_0^t \bigg| \sum_{y \in B(n)} F_n(y) \, \mathcal{L}_s^\omega \Phi(\omega, s, x) \bigg| \,\mathrm{d}s \,\mathrm{d}t$$

$$\leq \frac{1}{n^{d+2}} \sum_{|e|=1} \int_0^{n^2} \bigg| \sum_{y \in B(n)} (\tau_{s,y}\omega)_0(0, e) \, \Phi_0(\tau_{s,y}\,\omega, 0, e) \, g_e(y/n) \bigg| \,\mathrm{d}s.$$

Since $\Phi_0 \in L^2(m)$ and $\mathbb{E}\big[\omega_0(0, e)\big] < \infty$, an application of the Cauchy-Schwarz inequality implies that $\omega_0(0, e)\Phi_0(\omega, 0, e) \in L^1(\mathbb{P})$ for every e. Thus, (3.4) follows from Lemma 3.4. □

The rest of this section is devoted to the proofs of Lemmas 3.4, 3.5 and 3.6 that have been used in the proof of Proposition 3.3 above. We start with Lemma 3.4 which is a consequence from Maker's theorem.

Lemma 3.4. *Suppose that Assumption 1.1 holds. Let $\phi \in L^1(\Omega, \mathbb{P})$ and $g \in C(\mathbb{R}^d)$ be such that $\int_{B_1} g(u)\,\mathrm{d}u = 0$. Then,*

$$\lim_{n\to\infty} \frac{1}{n^{d+2}} \int_0^{n^2} \Big| \sum_{x\in B(n)} g(x/n)\, \phi\circ\tau_{t,x} \Big| \,\mathrm{d}t \;=\; 0, \qquad \mathbb{P}\text{-a.s.}$$

Proof. Let $\varepsilon > 0$ be given. First notice that by the ergodic theorem we have $\mathbb{P}$-a.s. for every $m > 0$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n^{d+2}} \int_0^{n^2} \Big| \sum_{x\in B(n)} g(x/n)\, \phi\circ\tau_{t,x}\; \mathbb{1}_{\{|\phi\circ\tau_{t,x}|>m\}} \Big| \,\mathrm{d}t \\ &\quad\le\; \limsup_{n\to\infty} \frac{1}{n^{d+2}} \int_0^{n^2} \sum_{x\in B(n)} \big|g(x/n)\big|\, \big|\phi\circ\tau_{t,x}\big|\; \mathbb{1}_{\{|\phi\circ\tau_{t,x}|>m\}} \,\mathrm{d}t \\ &\quad\le\; \|g\|_\infty\; \mathbb{E}\big[|\phi|\; \mathbb{1}_{\{|\phi|>m\}}\big]. \end{aligned}$$

Since $\phi \in L^1(\mathbb{P})$ there exists m such that $\mathbb{E}\left[|\phi| \mathbb{1}_{\{|\phi|>m\}}\right] \le \varepsilon/\|g\|_\infty$. Thus,

$$\limsup_{n\to\infty} \frac{1}{n^{d+2}} \int_0^{n^2} \Big| \sum_{x\in B(n)} g(x/n)\, \phi\circ\tau_{t,x}\; \mathbb{1}_{\{|\phi\circ\tau_{t,x}|>m\}} \Big| \,\mathrm{d}t \;\le\; \varepsilon, \qquad \mathbb{P}\text{-a.s.} \tag{3.5}$$

On the other hand, by the ergodic theorem (see C.1 in the Appendix), $\mathbb{P}$-a.s.

$$\begin{aligned} \phi_n \;&:=\; \frac{1}{n^d} \Big| \sum_{x\in B(n)} g(x/n)\, \phi\circ\tau_{0,x}\; \mathbb{1}_{\{|\phi\circ\tau_{0,x}|\le m\}} \Big| \\ &\longrightarrow\; \Big| \Big(\int_{B_1} g(u)\,\mathrm{d}u \Big)\; \mathbb{E}\big[\phi\; \mathbb{1}_{\{|\phi|\le m\}} \,\big|\, \mathcal{I}\big] \Big| \;=\; 0 \end{aligned}$$

as $n \to \infty$ with $\mathcal{I}$ being the σ-algebra of invariant sets w.r.t. the space shifts $\{\tau_{0,x} : x \in \mathbb{Z}^d\}$. Further, note that $\sup_n \phi_n \in L^1(\mathbb{P})$ since we have the trivial $\mathbb{P}$-a.s. bound $\sup_n \phi_n \le c\, m\, \|g\|_\infty$. Thus, we may apply the version of Maker's theorem in Proposition B.2 to obtain that $\mathbb{P}$-a.s.

$$\frac{1}{n^{d+2}} \int_0^{n^2} \Big| \sum_{x\in B(n)} g(x/n)\, \phi\circ\tau_{t,x}\; \mathbb{1}_{\{|\phi\circ\tau_{t,x}|\le m\}} \Big| \,\mathrm{d}t \;=\; \frac{1}{n^2} \int_0^{n^2} \phi_n\circ\tau_{t,0}\,\mathrm{d}t \;\longrightarrow\; 0$$

as $n \to \infty$. Combining this and (3.5) gives the claim. □

Lemma 3.5. *Suppose that Assumption 1.1 holds. Then, for $\mathbb{P}$-a.e. ω,*

$$\lim_{n\to\infty} \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x\in B(n)} \big|\chi_0(\omega, t, x)\big| \,\mathrm{d}t \;=\; 0, \tag{3.6}$$

and the convergence also takes place in $L^1(\mathbb{P})$.

Proof. Recall that the mapping $(\omega, x) \mapsto \chi_0(\omega, 0, x)$ is a cocycle in L^2_{pot}. Thus, there exists a sequence of bounded functions $\varphi_k : \Omega \to \mathbb{R}^d$ such that $\mathrm{D}\varphi_k \to \chi_0(\cdot, 0, \cdot)$ in $L^2(m)$ as $k \to \infty$. For abbreviation we set

$$\lambda_k(\omega, t, x) \;:=\; \chi_0(\omega, t, x) - \mathrm{D}\varphi_k(\omega, t, x).$$

Thus,

$$\frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x \in B(n)} \big|\chi_0(\omega, t, x)\big| \,\mathrm{d}t \;\le\; \frac{2\|\varphi_k\|_{L^\infty(\Omega)}}{n} + \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x \in B(n)} \big|\lambda_k(\omega, t, x)\big| \,\mathrm{d}t. \tag{3.7}$$

For any $x \in B(n)$, we denote by γ_{0x} the lattice approximation of the line segment $[0, x]$, that is $\gamma_{0,x}$ is a path from the origin to x those distance to the line segment is at most $1/\sqrt{d}$. Then,

$$\begin{aligned} &\frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x \in B(n)} \big|\lambda_k(\omega, t, x)\big| \,\mathrm{d}t \\ &\quad\le \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x \in B(n)} \sum_{(y,y') \in \gamma_{0,x}} \big|\nabla_{y'-y} \lambda_k(\omega, t, y)\big| \frac{n^d}{(1 + |y|)^{d-1}} \mathrm{d}t \\ &\quad\le \frac{1}{n^{d+2}} \int_0^{n^2} \sum_{y \in B(n)} \sum_{|e|=1} \big|\lambda_k(\tau_{t,y}\omega, 0, e)\big| \frac{n^{d-1}}{(1 + |y|)^{d-1}} \mathrm{d}t, \end{aligned}$$

where we used in the first step that for any given edge $(y, y') \in E_d$ with $y \in B(n)$ we have $|\{x \in B(n) : (y, y') \in \gamma_{0x}\}| = cn^d/(1 + |y|^{d-1})$ (see [34, page 373] for a proof in $d = 2$, which can be generalised to any dimension). In the second step we used both the cocyle property and the fact that the time-homogeneity of χ_0 in (2.14) implies that $\lambda_k(\omega, t, x) = \lambda_k(\tau_{t,0}\omega, 0, x)$. Thus, in view of Theorem C.2, we obtain that

$$\limsup_{n \to \infty} \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x \in B(n)} \big|\chi_0(\omega, t, x)\big| \,\mathrm{d}t \;\le\; c \sum_{|e|=1} \mathbb{E}\big[|\lambda_k(\omega, 0, e)|\big] \int_{B_1} \frac{1}{|y|^{d-1}} \,\mathrm{d}y.$$

Since,

$$\sum_{|e|=1} \mathbb{E}\big[|\lambda_k(\omega, 0, e)|\big] \;\le\; \mathbb{E}\big[\nu_0^\omega(0)\big]^{1/2} \, \|\lambda_k(\cdot, 0, \cdot)\|_m$$

and, by construction, $\lambda_k(\cdot, 0, \cdot) \to 0$ in $L^2(m)$ as $k \to \infty$, the almost sure convergence of (3.6) follows. By taking the expectation of both sides of (3.7), the convergence in $L^1(\mathbb{P})$ from the stationarity of $\mathbb{P}$ with respect to space-time shifts. □

Lemma 3.6. *Suppose that Assumption 1.1 holds and let F_n be defined as in 3.3. Then, for $\mathbb{P}$-a.e. ω,*

$$\lim_{n \to \infty} \frac{1}{n^{d+1}} \sum_{x \in B(n)} F_n(x) \, \chi_0(\omega, 0, x) \;=\; 0.$$

Proof. We divide the proof into two steps. First, we show that the limit exists $\mathbb{P}$-a.e. ω. Then, in the second step, we prove that it is actually zero. Since for any $n \in \mathbb{N}$ we have $[-n,n]^d \cap \mathbb{Z}^d \subset B(2n) \subset [-2n,2n]^d \cap \mathbb{Z}^d$, it suffices to prove the lemma for boxes in place of balls.

STEP 1: We show the existence of the limit similarly to [5, Lemma 2.8] based on arguments in [36]. By symmetry it suffices to show the existence of

$$\lim_{n\to\infty} \frac{1}{n^{d+1}} \sum_{x \in C(n)} F_n(x)\, \chi(\omega, 0, x)\,,$$

where $C(n) := [0,n]^d \cap \mathbb{Z}^d$. Let us denote by $C^j(n) := [0,n]^j \times \{0\}^{d-j}$, $j = 1, \dots, d$. When $x = (x^1, \dots, x^d) \in \mathbb{Z}^d$, we write $x = (y, x^d)$ with $y = (x^1, \dots, x^{d-1}) \in \mathbb{Z}^{d-1}$, and we identify $\mathbb{Z}^{d-1}$ with $\mathbb{Z}^{d-1} \times \{0\} \subseteq \mathbb{Z}^d$. We also denote by $F_n^j(x) = \prod_{i=1}^j f(x^i/n)$ and $F_n^0 \equiv 1$. Then, using the fact that $\chi_0(\cdot, 0, \cdot)$ is a cocycle we obtain

$$\begin{aligned}
&\frac{1}{n^{d+1}} \sum_{x \in C^d(n)} F_n(x)\, \chi_0(\omega, 0, x) \\
&\quad = \frac{1}{n^{d+1}} \sum_{\substack{y \in C^{d-1}(n) \\ 0 \le x^d \le n}} F_n(x) \left(\chi_0(\omega, 0, y) \,+\, \sum_{k=0}^{x^d-1} \chi_0(\tau_{0,y+ke_d}\omega, 0, e_d) \right) \\
&\quad = \left(\frac{1}{n} \sum_{x^d=0}^{n} f(x^d/n) \right) \left(\frac{1}{n^d} \sum_{y \in C^{d-1}(n)} F_n^{d-1}(y)\, \chi_0(\omega, 0, y) \right) \\
&\qquad + \frac{1}{n^d} \sum_{x \in C^d(n)} \chi_0(\tau_{0,x}\omega, 0, e_d)\, G_n^d(x),
\end{aligned} \tag{3.8}$$

where

$$G_n^j(x) \;:=\; \mathbb{1}_{\{0 \le x^j < n\}}\, F_n^{j-1}(x^1, \dots, x^{j-1}) \left(\frac{1}{n} \sum_{k=x^j+1}^{n} f(k/n) \right), \qquad x \in C^j(n).$$

Since f is bounded and continuous, we observe that

$$\lim_{n\to\infty} \sup_{j=0,\dots,n} \left| \int_{j/n}^1 f(s)\, \mathrm{d}s \,-\, \frac{1}{n} \sum_{k=j}^{n} f(k/n) \right| \;=\; 0.$$

Define $\tilde{G}_n^j(x) := F_n^{j-1}(x^1, \dots, x^{j-1}) \int_{x^j/n}^1 f(s)\, \mathrm{d}s$ and notice that $\tilde{G}_n^j$ is continuous for all $j = 1, \dots, d$. It is easy to see that we can replace $G_n^d(x)$ by $\tilde{G}_n^d(x)$ in (3.8) and obtain the same limit.

Since $\mathbb{E}[|\chi_0(\omega, 0, x)|] < \infty$ for all $x \in \mathbb{Z}^d$, an application of the spatial ergodic theorem (see Theorem C.1 in the Appendix) gives that

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x \in C^d(n)} \chi_0(\tau_{0,x}\omega, 0, e_d)\, \tilde{G}_n^d(x)$$

exists $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$ and is finite. Further, since $f \in L^1([0,1])$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x^j=0}^{n} f(x^j/n) \;=\; \int_{[0,1]} f(s)\,\mathrm{d}s.$$

The claim follows now by induction. Indeed, in each step we use the spatial ergodic theorem with respect to the subgroup of space shifts to obtain that the limit

$$\lim_{n\to\infty} \frac{1}{n^{d-k}} \sum_{x\in C^{d-k}(n)} \chi_0(\tau_{0,x}\omega, 0, e_{d-k})\, \tilde{G}_n^{d-k}(x)$$

exists $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$ and is finite for every $k = 0, \dots, d-1$.

STEP 2: Using the shift invariance of $\mathbb{P}$, the time homogeneity of χ_0 in (2.14) and afterwards Lemma 3.5 we get

$$\frac{1}{n^{d+1}} \sum_{x\in B(n)} \mathbb{E}\big[|\chi_0(\omega,0,x)|\big] \;=\; \frac{1}{n^{d+3}} \int_0^{n^2} \sum_{x\in B(n)} \mathbb{E}\big[|\chi_0(\omega,t,x)|\big]\,\mathrm{d}t \;\underset{n\to\infty}{\longrightarrow}\; 0.$$

Recall that F_n is bounded, so this implies that $n^{-(d+1)} \sum_{x\in C(n)} F_n(x)\, \chi_0(\omega,0,x)$ converges to zero in $L^1(\mathbb{P})$ and therefore also $\mathbb{P}$-a.s. along some subsequence. Since the limit exists as shown in STEP 1, the proof is complete. □

4. QUENCHED INVARIANCE PRINCIPLE

Throughout this section, which is devoted to the proof of our main result in Theorem 1.7, we suppose that Assumption 1.1 holds. We start with some comments on the construction of the VSRW X and its stochastic completeness as they are not totally obvious in the present time-dependent degenerate situation.

We follow the construction of time-inhomogeneous Markov processes in [37]. Let $\{E_k : k \geq 1\}$ be a sequence of independent $\mathrm{Exp}(1)$-distributed random variables. In order to construct the random walk X under the law $\mathrm{P}^\omega_{s,x}$ we specify its jump times $s < J_1 < J_2 < \dots$ inductively. Set $J_0 = s$ and $X_s = x$ and suppose that for any $k \geq 1$ the process X is constructed on $[s, J_{k-1}]$. Then, J_k is given by

$$J_k \;=\; \inf\Big\{t \geq 0 \,:\, \int_{J_{k-1}}^{J_{k-1}+t} \mu^\omega_s(X_{J_{k-1}})\,\mathrm{d}s \geq E_k\Big\},$$

and at the jump time $t = J_k$ the random walk X jumps according to the transition probabilities $\{\omega_t(X_{J_{k-1}}, y)/\mu^\omega_t(X_{J_{k-1}}),\, y \sim X_{J_{k-1}}\}$.

Lemma 4.1. *For $\mathbb{P}$-a.e. ω, $\mathrm{P}^\omega_{0,0}$-a.s. the process $\{X_t : t \geq 0\}$ does not explode, that is there are only finitely many jumps in finite time.*

Proof. We will follow the approach in [16, Section 5] and consider first a slowed-down process. Let $\{(T_t, Y_t) : t \geq 0\}$ be the Markov process on $\mathbb{R}\times\mathbb{Z}^d$ with generator $\mathcal{L}^\omega_Y$ acting on functions $u : \mathbb{R}\times\mathbb{Z}^d \to \mathbb{R}$ defined by

$$\mathcal{L}^\omega_Y u(t,x) \;=\; \frac{1}{1 \vee \mu^\omega_t(x)} \big(\partial_t u(t,x) + (\mathcal{L}^\omega_t u(t,\cdot))(x)\big)$$

with $\mu_t^\omega(x) = \sum_{y\sim x} \omega_t(x,y)$. At point (t,x) the slowed-down process $\{Y_t : t \geq 0\}$ will jump to $y \sim x$ with rate $\omega_{T_t}(x,y)/(1 \vee \mu_{T_t}^\omega(x))$ and at time t the time process $\{T_t : t \geq 0\}$ will increase at rate $(1 \vee \mu_t^\omega(x))^{-1}$, more precisely

$$T_t \;=\; \int_0^t \frac{1}{1 \vee \mu_{T_s}^\omega(Y_s)}\, \mathrm{d}s.$$

Further, notice that the process Y can be obtained from X by a time change, namely

$$X_{T_t} \;=\; Y_t, \tag{4.1}$$

which will allow us to infer non-explosion of the process X from that of Y. Clearly, the process $\{(T_t, Y_t) : t \geq 0\}$ is non-explosive since $T_t \leq t$ and the jump-rates of Y bounded from above by one.

On the other hand, under Assumption 1.1 using the irreducibility of the process Y it can be easily seen that the measure

$$\frac{1 \vee \mu_0^\omega(0)}{\mathbb{E}[1 \vee \mu_0^\omega(0)]}\, d\,\mathbb{P}$$

is stationary and ergodic for the environment process $\{\tau_{T_t, Y_t}\omega : t \geq 0\}$ (cf. e.g. [1, Proposition 2.1]). Thus, we may apply the ergodic theorem to obtain that

$$\lim_{t\to\infty} \frac{T_t}{t} \;=\; \frac{1}{\mathbb{E}[1 \vee \mu_0^\omega(0)]}, \qquad (\mathbb{P} \otimes P_{0,0}^\omega)\text{-a.s.}$$

In particular, by (4.1) the process $(X_t)_{t\geq 0}$ is non-explosive for $\mathbb{P}$-almost all ω, $P_{0,0}^\omega$-almost surely. □

For our purposes the main reason to construct the harmonic coordinates in Section 2 is that they allow to decompose the random walk X into a martingale part and a corrector part. We now state this decomposition as a Corollary.

Corollary 4.2. *For $\mathbb{P}$-a.e. ω, the process*

$$M_t \;:=\; \Phi(\omega, t, X_t), \qquad t \geq 0,$$

is a $\mathrm{P}_{0,0}^\omega$-martingale and

$$X_t \;=\; M_t + \chi(\omega, t, X_t), \qquad t \geq 0. \tag{4.2}$$

Moreover, for every $v \in \mathbb{R}^d$, $v \cdot M$ is a $\mathrm{P}_{0,0}^\omega$-martingale and its quadratic variation process is given by

$$\langle v \cdot M \rangle_t \;=\; \int_0^t \sum_{y\in\mathbb{Z}^d} \omega_s(X_s, y)\, \big(v \cdot \big(\Phi_0(\omega, s, y) - \Phi_0(\omega, s, X_s)\big)\big)^2\, \mathrm{d}s. \tag{4.3}$$

Proof. From (2.10) it is immediate that M and hence also $v \cdot M$ are $\mathrm{P}_{0,0}^\omega$-martingales. The decomposition in (4.2) follows directly from the definition of χ. It remains to show (4.3). First note that the opérateur carré du champ associated with $\partial_t + \mathcal{L}_t^\omega$ is given by

$$\begin{aligned}\big(\partial_t + \mathcal{L}_t^\omega\big) f^2 \;-\; 2f\big(\partial_t + \mathcal{L}_t^\omega\big) f \;&=\; \big(\partial_t(f^2) - 2f\partial_t f\big) \;+\; \big(\mathcal{L}_t^\omega(f^2) - 2f\mathcal{L}_t^\omega f\big)\\ &=\; \mathcal{L}_t^\omega(f^2) \;-\; 2f\mathcal{L}_t^\omega f\end{aligned}$$

and

$$\big(\mathcal{L}_t^\omega f^2 - 2f\mathcal{L}_t^\omega f\big)(t,x) \;=\; \sum_{y\in\mathbb{Z}^d} \omega_t(x,y)\big(f(t,y) - f(t,x)\big)^2.$$

Hence,

$$\langle v\cdot M\rangle_t \;=\; \int_0^t \sum_{y\in\mathbb{Z}^d} \omega_s(X_s,y)\;\big(v\cdot\big(\Phi(\omega,s,y) - \Phi(\omega,s,X_s)\big)\big)^2\,\mathrm{d}s$$

and (4.3) follows by (2.11). □

Lemma 4.3. *The measure $\mathbb{P}$ is stationary, reversible and ergodic for the environment process $\{\tau_{t,X_t}\omega : t\geq 0\}$.*

Proof. This follows from the ergodicity of the environment and the irreducibility of the process. See [5, Lemma 2.4] and [1, Proposition 2.1] for detailed proofs. □

Proposition 4.4. *Let $M_t^{(n)} := \frac{1}{n}M_{n^2t}$, $t\geq 0$. Then, for $\mathbb{P}$-a.e. ω, the sequence of processes $\{M^{(n)}\}$ converges in law in the Skorohod topology to a Brownian motion with a non-degenerate covariance matrix Σ^2 given by*

$$\Sigma_{ij}^2 \;=\; \mathbb{E}\Big[\textstyle\sum_{x\in\mathbb{Z}^d}\omega_0(0,x)\,\Phi_0^i(\omega,0,x)\,\Phi_0^j(\omega,0,x)\Big].$$

Proof. The proof is based on the martingale convergence theorem by Helland (see Theorem 5.1a) in [20]); the proofs in [2] or [27] can be easily transferred into the time dynamic setting. The argument is based on the fact that the quadratic variation of $M^{(n)}$ converges by an application of the ergodic theorem, since it can be written in terms of the environment process (cf. (4.3)), which is ergodic by Lemma 4.3. Finally, we refer to Proposition 4.1 in [12] for a proof that Σ^2 is nondegenerate. □

In order to conclude the proof of the invariance principle, an almost sure uniform control of the corrector is required, which is a direct consequence from the sublinearity of corrector established in Proposition 3.1.

Proposition 4.5. *Suppose that Assumption 1.5 holds and let $T>0$ be arbitrary. Then, for $\mathbb{P}$-a.e. ω,*

$$\sup_{0\leq t\leq T}\frac{1}{n}\Big|\chi\big(\omega,n^2t,n\,X_t^{(n)}\big)\Big| \;\underset{n\to\infty}{\longrightarrow}\; 0 \quad \text{in } \mathrm{P}_{0,0}^\omega\text{-probability}. \tag{4.4}$$

Proof. Given Proposition 3.1 this follows by similar arguments as in [5, 18, 19]. □

Theorem 1.7 now follows from Proposition 4.4 and Proposition 4.5.

5. Mean value inequality for time-inhomogeneous Poisson equation

5.1. **Setup and preliminaries.** Let $G = (V, E)$ be an infinite, connected, locally finite graph with vertex set V and (non-oriented) edge set E. We will write $x \sim y$ if $\{x, y\} \in E$. The graph G is endowed with the counting measure that assigns to any $A \subset V$ simply the number $|A|$ of elements in A. Further, we denote by $B(x,r)$ the closed ball with center x and radius r with respect to the natural graph distance d, that is $B(x,r) := \{y \in V \mid d(x,y) \leq \lfloor r \rfloor\}$.

Throughout this section we will make the following assumption on G.

Assumption 5.1. *The graph G satisfies the following conditions:*

(i) *volume regularity of order d for large balls, that is there exists $d \geq 2$ and $C_{\mathrm{reg}} \in (0,\infty)$ such that for all $x \in V$ there exists $N_1(x) < \infty$ with*

$$C_{\mathrm{reg}}^{-1}\, n^d \;\leq\; |B(x,n)| \;\leq\; C_{\mathrm{reg}}\, n^d, \qquad \forall\, n \geq N_1(x). \tag{5.1}$$

(ii) *local Sobolev inequality $(S^1_{d'})$ for large balls, that is there exists $d' \geq d$ and $C_{\mathrm{S}_1} \in (0,\infty)$ such that for all $x \in V$ the following holds. There exists $N_2(x) < \infty$ such that for all $n \geq N_2(x)$,*

$$\Bigg(\sum_{y \in B(x,n)} |u(y)|^{\frac{d'}{d'-1}}\Bigg)^{\frac{d'-1}{d'}} \;\leq\; C_{\mathrm{S}_1}\, n^{1-\frac{d}{d'}} \sum_{\substack{z \vee z' \in B(x,n)\\ \{z,z'\} \in E}} \big|u(z) - u(z')\big| \tag{5.2}$$

for all $u\colon V \to \mathbb{R}$ with $\operatorname{supp} u \subset B(x,n)$.

Remark 5.2. The Euclidean lattice, $(\mathbb{Z}^d, E_d)$, satisfies Assumption 5.1 with $d' = d$ and $N_1(x) = N_2(x) = 1$, where (ii) follows from the isoperimetric inequality. For random graphs, e.g. supercritical Bernoulli percolation clusters, such an inequality is only satisfied for large sets: There exists $\theta \in (0,1)$ and $N(x) < \infty$, $\mathbb{P}$-a.s., such that for all $n \geq N(x)$,

$$|\partial A| \geq C_{\mathrm{iso}} |A|^{(d-1)/d}$$

for all connected $A \subset B(x,n)$ with $|A| \geq n^{\theta}$, see [10, 28]. As it was pointed out by M. Barlow, in such a case Assumption 5.1 (ii) holds with $d' = d/(1-\theta)$, see [31].

For functions $f\colon A \to \mathbb{R}$, where either $A \subseteq V$ or $A \subseteq E$, the ℓ^p-norm $\|f\|_{\ell^p(A)}$ will be taken with respect to the counting measure. The corresponding scalar products in $\ell^2(V)$ and $\ell^2(E)$ are denoted by $\langle \cdot,\cdot\rangle_{\ell^2(V)}$ and $\langle \cdot,\cdot\rangle_{\ell^2(E)}$, respectively. For any non-empty, finite $B \subset V$ and $p \in [1,\infty)$, we introduce space-averaged norms on functions $f\colon B \to \mathbb{R}$ by

$$\|f\|_{p,B} \;:=\; \Bigg(\frac{1}{|B|} \sum_{x \in B} |f(x)|^p\Bigg)^{1/p}.$$

Moreover, for any non-empty compact interval $I \subset \mathbb{R}$ and any finite $B \subset \mathbb{Z}^d$ and $p, p' \geq 1$, we define space-time-averaged norms on functions $u \colon I \times B \to \mathbb{R}$ by

$$\|u\|_{p,p',I\times B} \;:=\; \left(\frac{1}{|I|}\int_I \|u_t\|_{p,B}^{p'}\,\mathrm{d}t\right)^{1/p'} \qquad \text{and} \qquad \|u\|_{p,\infty,I\times B} \;:=\; \max_{t\in I}\|u_t\|_{p,B}\,,$$

where $u_t(\cdot) := u(t,\cdot)$ for any $t \in I$.

Lemma 5.3. *Suppose that $\rho > 1$ and $q' \in [1,\infty]$ are given and $Q \subset \mathbb{R}\times V$. Then, for every $1 < \gamma_1 \leq \rho$ and $q'/(q'+1) \leq \gamma_2 < \infty$ such that*

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\left(1 - \frac{1}{\rho}\right)\frac{q'}{q'+1} \;=\; 1 \tag{5.3}$$

the following estimate holds

$$\|u\|_{\gamma_1,\gamma_2,Q} \;\leq\; \|u\|_{1,\infty,Q} + \|u\|_{\rho,q'/(q'+1),Q}\,. \tag{5.4}$$

Proof. This follows by an application of Hölder's and Young's inequality, as in [26, Lemma 1.1] □

Let us endow the graph G with positive, time-dependent weights, that is we consider a family $\omega = \{\omega_t(e) : t \in \mathbb{R}, e \in E\} \subset (0,\infty)^{\mathbb{R}\times E}$. Further, we define for any $t \in \mathbb{R}$ measures μ_t^ω and ν_t^ω on V by

$$\mu_t^\omega(x) \;:=\; 1 \vee \sum_{x\sim y}\omega_t(x,y) \qquad \text{and} \qquad \nu_t^\omega(x) \;:=\; 1 \vee \sum_{x\sim y}\frac{1}{\omega_t(x,y)}. \tag{5.5}$$

It is convenient to introduce a potential theoretic setup. First, for $f : V \to \mathbb{R}$ and $F \colon E \to \mathbb{R}$ we define the operators $\nabla f \colon E \to \mathbb{R}$ and $\nabla^* F \colon V \to \mathbb{R}$ by

$$\nabla f(e) \;:=\; f(e^+) - f(e^-), \qquad \text{and} \qquad \nabla^* F(x) \;:=\; \sum_{e:e^+=x} F(e) - \sum_{e:e^-=x} F(e),$$

where for each non-oriented edge $e \in E$ we specify one of its two endpoints as its initial vertex e^+ and the other one as its terminal vertex e^-. Nothing of what will follow depends on the particular choice. Since $\langle \nabla f, F\rangle_{\ell^2(E)} = \langle f, \nabla^* F\rangle_{\ell^2(V)}$ for all $f \in \ell^2(V)$ and $F \in \ell^2(E)$, ∇^* can be seen as the adjoint of ∇. Notice that in the discrete setting the product rule reads

$$\nabla(fg) \;=\; \mathrm{av}(f)\nabla g + \mathrm{av}(g)\nabla f, \tag{5.6}$$

where $\mathrm{av}(f)(e) := \frac{1}{2}(f(e^+) + f(e^-))$. Moreover, we denote by $\mathcal{L}_t^\omega$ the following time-dependent operator acting on bounded functions $f \colon V \to \mathbb{R}$ as

$$\big(\mathcal{L}_t^\omega f\big)(x) \;:=\; \sum_{x\sim y}\omega_t(x,y)\,\big(f(y) - f(x)\big) \;=\; -\nabla^*(\omega_t \nabla f)(x).$$

For any $t \in \mathbb{R}$, the *time-dependent Dirichlet form* associated to $\mathcal{L}_t^\omega$ is given by

$$\mathcal{E}_t^\omega(f,g) \;:=\; \langle f, -\mathcal{L}_t^\omega g\rangle_{\ell^2(V)} \;=\; \langle \nabla f, \omega_t \nabla g\rangle_{\ell^2(E)}, \tag{5.7}$$

and we set $\mathcal{E}_t^\omega(f) := \mathcal{E}_t^\omega(f,f)$.

Note that (5.2) is a Sobolev inequality on an unweighted graph, while for our purposes we need a version involving the time-dependent weights.

Proposition 5.4 (local space-time Sobolev inequality)**.** *Consider a graph (V, E) that satisfies the Assumption 1.1 with $d' \geq d \geq 2$ and set*

$$\rho \;\equiv\; \rho(d', q) \;=\; \frac{d'}{d' - 2 + d'/q}. \tag{5.8}$$

Further, let $Q = I \times B$, where $I \subset \mathbb{R}$ is a compact interval and $B \subset V$ is finite and connected. Then, for any $q, q' \in [1, \infty]$, there exists $C_{\mathrm{S}} \equiv C_{\mathrm{S}}(d, q) < \infty$ such that for every $u : \mathbb{R} \times V \to \mathbb{R}$ with $\operatorname{supp} u \subset I \times B$,

$$\big\| u^2 \big\|_{\rho, q'/(q'+1), Q} \;\leq\; C_{\mathrm{S}}\, |B|^{2/d}\, \|\nu^{\omega}\|_{q,q',Q} \left(\frac{1}{|I|} \int_I \frac{\mathcal{E}_t^{\omega}(u_t)}{|B|}\, \mathrm{d}t \right). \tag{5.9}$$

Proof. Proceeding as in the proof of [5, Proposition 3.5], we deduce from (5.2) that

$$\big\| u_t^2 \big\|_{\rho, B} \;\leq\; C_{\mathrm{S}}\, |B|^{2/d}\, \|\nu_t^{\omega}\|_{q,B}\, \frac{\mathcal{E}_t^{\omega}(u_t)}{|B|}.$$

Thus, for any $q' \geq 1$ the assertion follows by Hölder's inequality,. □

5.2. **Maximal inequality via Moser iteration.** In this section, our main objective is to establish a maximum inequality for the solution of a particular Poisson equation having a right-hand side in divergence form. More precisely, we denote by $u : \mathbb{R} \times V \to \mathbb{R}$ the solution of

$$\partial_t u + \mathcal{L}_t^{\omega} u \;=\; \nabla^* V_t^{\omega}, \qquad \text{on} \quad Q = I \times B, \tag{5.10}$$

where $I = [s_1, s_2] \subset \mathbb{R}$ is an interval, $B \subset V$ is a finite, connected subset of V and $V_t^{\omega} : \mathbb{R} \times E \to \mathbb{R}$ is given by

$$V_t^{\omega}(e) \;:=\; \omega_t(e)\, \nabla f(e) \tag{5.11}$$

for some function $f : V \to \mathbb{R}$.

For any $x_0 \in V$, $t_0 \geq 0$ and $n \geq 1$, we denote by $Q(n) \equiv [t_0, t_0 + n^2] \times B(x_0, n)$ the corresponding time-space cylinder, and we set

$$Q(\sigma n) \;=\; [t_0, t_0 + \sigma n^2] \times B(x_0, \sigma n).$$

Now we are ready to state the main result of this section.

Theorem 5.5. *Suppose that $\partial_t u + \mathcal{L}_t^{\omega} u = \nabla^* V_t^{\omega}$ on $Q(n)$ and assume that the function f in (5.11) satisfies $|\nabla f(e)| \leq 1/n$ for all $e \in E$. Then, for any $p, p', q, q' \in [1, \infty]$ with*

$$\frac{1}{p} \cdot \frac{p'}{p' - 1} \cdot \frac{q' + 1}{q'} + \frac{1}{q} \;<\; \frac{2}{d'} \tag{5.12}$$

there exist $\gamma \in (0, 1]$, $\kappa \equiv \kappa(d, p, q) \in (1, \infty)$ and $C_1 \equiv C_1(d) < \infty$ such that for all $\alpha \in (0, \infty)$ and for all $1/2 \leq \sigma' < \sigma \leq 1$,

$$\max_{(t,x) \in Q(\sigma' n)} |u(t, x)| \;\leq\; C_1 \left(\frac{\|\mu^{\omega}\|_{p,p',Q(n)}\, \|\nu^{\omega}\|_{q,q',Q(n)}}{(\sigma - \sigma')^2} \right)^{\kappa} \|u\|_{\alpha,\alpha,Q(\sigma n)}^{\gamma}. \tag{5.13}$$

As a first step we prove the following energy estimate for solutions of (5.10).

Lemma 5.6. *Suppose that $Q = I \times B$, where $I = [s_1, s_2]$ is an interval and B is a finite, connected subset of V. Consider a smooth function $\zeta \colon \mathbb{R} \to [0,1]$ with $\zeta = 0$ on $[s_2, \infty)$ and a function $\eta \colon V \to [0,1]$ such that*

$$\operatorname{supp} \eta \subset B \qquad \text{and} \qquad \eta \equiv 0 \quad \text{on} \quad \partial B.$$

Further, let u be a solution of (5.10) on Q. Then, there exists $C_2 < \infty$ such that for all $\alpha \geq 1$ and $p, p', p_*, p'_* \in (1, \infty)$ with $1/p + 1/p_* = 1$ and $1/p' + 1/p'_* = 1$,

$$\begin{aligned} &\frac{1}{|I|}\, \big\| \zeta (\eta\, \tilde{u}^{\alpha})^2 \big\|_{1,\infty,Q} \,+\, \frac{1}{|I|} \int_I \zeta(t)\, \frac{\mathcal{E}^{\omega}_{t,\eta^2}(\tilde{u}^{\alpha}_t)}{|B|}\, \mathrm{d}t \\ &\qquad \leq\; C_2\, \alpha^2\, \|\mu^{\omega}\|_{p,p',Q}\, \Big(\|\nabla \eta\|^2_{\ell^{\infty}(E)} + \|\zeta'\|_{L^{\infty}(I)} \Big)\, \big\| |u|^{2\alpha} \big\|_{p_*,p'_*,Q} \\ &\qquad\qquad +\, C_2\, \alpha^2\, \|\mu^{\omega}\|_{p,p',Q}\, \|(\nabla\eta)(\nabla f)\|_{\ell^{\infty}(E)}\, \big\| |u|^{2\alpha-1} \big\|_{p_*,p'_*,Q} \\ &\qquad\qquad +\, C_2\, \alpha^2\, \|\mu^{\omega}\|_{p,p',Q}\, \|\nabla f\|^2_{\ell^{\infty}(E)}\, \big\| |u|^{2\alpha-2} \big\|_{p_*,p'_*,Q} \end{aligned} \tag{5.14}$$

where $\tilde{u}^{\alpha} := |u|^{\alpha} \cdot \operatorname{sign} u$ and $\mathcal{E}^{\omega}_{t,\eta^2}(f) := \big\langle \nabla f, \operatorname{av}(\eta^2)\, \omega_t \nabla f \big\rangle_{\ell^2(E)}$.

Proof. Let us consider a function u such that $\partial_t u + \mathcal{L}^{\omega}_t\, u = \nabla^* V^{\omega}_t$ on $Q = I \times B$. Then, for any $t \in I$, a summation by parts yields

$$\frac{1}{2\alpha}\, \partial_t\, \|\eta\, \tilde{u}^{\alpha}_t\|^2_{\ell^2(V)} \;=\; \big\langle \nabla(\eta^2 \tilde{u}^{2\alpha-1}_t), \omega_t \nabla u_t \big\rangle_{\ell^2(E)} \,+\, \big\langle \nabla(\eta^2 \tilde{u}^{2\alpha-1}_t), V^{\omega}_t \big\rangle_{\ell^2(E)}. \tag{5.15}$$

Proceeding as in the proof of [3, Lemma 3.8], we will estimate the terms on the right-hand side of (5.15) separately. Let us point out that the constants $c \in (0, \infty)$ appearing in the computations below, is independent of α, but may change from line to line. In view of (A.2), we have that

$$\big\langle \operatorname{av}(\eta^2) \nabla \tilde{u}^{2\alpha-1}_t, \omega_t \nabla u_t \big\rangle_{\ell^2(E)} \;\geq\; \frac{2\alpha - 1}{\alpha^2}\, \mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}^{\alpha}_t\big) \;\geq\; \frac{1}{\alpha}\, \mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}^{\alpha}_t\big).$$

On the other hand, by (A.3) and Young's inequality, that reads $|ab| \leq \frac{1}{2}(\varepsilon a^2 + b^2/\varepsilon)$ for $\varepsilon \in (0, \infty)$, we obtain that

$$\begin{aligned} \big\langle \operatorname{av}(\tilde{u}^{2\alpha-1}_t) \nabla \eta^2, \omega_t \nabla u_t \big\rangle_{\ell^2(E)} \;&\geq\; -c\, \big\| \omega_t (\nabla \tilde{u}^{\alpha}_t)(\nabla \eta^2) \operatorname{av}(|u_t|^{\alpha}) \big\|_{\ell^1(E)} \\ &\geq\; -c\, \varepsilon\, \mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}^{\alpha}_t\big) \,-\, \frac{c}{\varepsilon}\, \|\nabla\eta\|^2_{\ell^{\infty}(E)}\, \big\| |u_t|^{2\alpha} \mu^{\omega}_t \big\|_{\ell^1(B)}, \end{aligned}$$

where we used that $\nabla \eta^2 = 2 \operatorname{av}(\eta)(\nabla\eta)$ and $\operatorname{av}(\eta)^2 \leq 2 \operatorname{av}(\eta^2)$. Hence, by the above estimates the first term on the right-hand side of (5.15) is bounded from below by

$$\begin{aligned} &\big\langle \nabla(\eta^2 \tilde{u}^{2\alpha-1}_t), \omega_t \nabla u_t \big\rangle_{\ell^2(E)} \\ &\qquad \geq\; \left(\frac{1}{\alpha} - c\, \varepsilon \right) \mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}^{\alpha}_t\big) \,-\, \frac{c}{\varepsilon}\, \|\nabla\eta\|^2_{\ell^{\infty}(E)}\, \big\| |u_t|^{2\alpha} \mu^{\omega}_t \big\|_{\ell^1(B)}. \end{aligned} \tag{5.16}$$

Next, we consider the second term on the right-hand side of (5.15). Since $\eta \in [0,1]$,

$$\big\langle \operatorname{av}(\tilde{u}^{2\alpha-1}_t) \nabla \eta^2, V^{\omega}_t \big\rangle_{\ell^2(E)} \;\geq\; -c\, \|(\nabla\eta)(\nabla f)\|_{\ell^{\infty}(E)}\, \big\| |u|^{2\alpha-1} \mu^{\omega}_t \big\|_{\ell^1(B)}.$$

By applying (A.1) and Young's inequality, we find for any $\alpha \geq 1$,

$$\big\langle \operatorname{av}(\eta^2) \nabla \tilde{u}_t^{2\alpha-1}, \omega_t \nabla f \big\rangle_{\ell^2(E)} \;\geq\; -c\, \big\| \omega_t \operatorname{av}(\eta^2) \operatorname{av}(|u|^{\alpha-1}) (\nabla \tilde{u}_t^{\alpha}) (\nabla f) \big\|_{\ell^1(E)}$$
$$\geq\; -c\,\varepsilon\, \mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}_t^{\alpha}\big) \;-\; \frac{c}{\varepsilon}\, \|\nabla f\|^2_{\ell^\infty(E)}\, \big\| |u_t|^{2\alpha-2} \mu^{\omega}_t \big\|_{\ell^1(B)}.$$

Hence, by combining these estimates, we obtain that the second term on the right-hand side of (5.15) is bounded from below by

$$\big\langle \nabla(\eta^2 \tilde{u}_t^{2\alpha-1}), V^{\omega}_t \big\rangle_{\ell^2(E)} \;\geq\; -\,c\,\varepsilon\, \mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}_t^{\alpha}\big) \;-\; \frac{c}{\varepsilon}\, \|\nabla f\|^2_{\ell^\infty(E)}\, \big\| |u_t|^{2\alpha-2} \mu^{\omega}_t \big\|_{\ell^1(B)}$$
$$-\; c\, \|(\nabla \eta)(\nabla f)\|_{\ell^\infty(E)}\, \big\| |u_t|^{2\alpha-1} \mu^{\omega}_t \big\|_{\ell^1(E)}. \tag{5.17}$$

Thus, in view of (5.16) and (5.17) and by choosing $\varepsilon = 1/(c\,\alpha)$, we deduce from (5.15) that there exists $C_2 < \infty$ such that

$$-\partial_t \, \|(\eta\, \tilde{u}_t^{\alpha})\|^2_{2,B} + \frac{\mathcal{E}^{\omega}_{t,\eta^2}\big(\tilde{u}^{\alpha}\big)}{|B|} \;\leq\; C_2\, \alpha^2\, \|\nabla \eta\|^2_{\ell^\infty(E)}\, \big\| |u|^{2\alpha} \mu^{\omega}_t \big\|_{1,B}$$
$$+\; C_2\, \alpha^2\, \|\nabla f\|^2_{\ell^\infty(E)}\, \big\| |u|^{2\alpha-2} \mu^{\omega}_t \big\|_{1,B}$$
$$+\; C_2\, \alpha^2\, \|(\nabla \eta)(\nabla f)\|_{\ell^\infty(E)}\, \big\| |u|^{2\alpha-1} \mu^{\omega}_t \big\|_{1,B}. \tag{5.18}$$

Moreover, since $\zeta(s_2) = 0$,

$$\int_s^{s_2} -\zeta(t)\, \partial_t \, \|(\eta\, \tilde{u}_t^{\alpha})\|^2_{2,B} \;\mathrm{d}t \;=\; \int_s^{s_2} \Big(-\,\partial_t\big(\zeta(t)\, \|(\eta\, \tilde{u}_t^{\alpha})\|^2_{2,B}\,\big) + \zeta'(t)\, \|(\eta\, \tilde{u}_t^{\alpha})\|^2_{2,B} \Big)\, \mathrm{d}t$$
$$\geq\; \zeta(s)\, \|(\eta\, \tilde{u}_s^{\alpha})\|^2_{2,B} \;-\; \|\zeta'\|_{L^\infty(I)} \int_{s_1}^{s_2} \big\| |u_t|^{2\alpha} \big\|_{1,B} \;\mathrm{d}t$$

for any $s \in [s_1, s_2)$. Thus, by multiplying both sides of (5.18) with ζ and integrating the resulting inequality over $[s, s_2]$ for any $s \in I$, the assertion (5.14) follows by an application of the Hölder and Jensen inequality. $\square$

Proposition 5.7. *Suppose that the assumptions of Theorem 5.5 hold. Then, there exist $\gamma \in (0, 1]$, $\kappa \equiv \kappa(d, p, q) \in (1, \infty)$ and $C_2 \equiv C_2(d) < \infty$ such that for all $1/2 \leq \sigma' < \sigma \leq 1$,*

$$\max_{(t,x) \in Q(\sigma' n)} |u(t,x)| \;\leq\; C_2 \left(\frac{\|\mu^{\omega}\|_{p,p',Q(n)}\; \|\nu^{\omega}\|_{q,q',Q(n)}}{(\sigma - \sigma')^2} \right)^{\kappa} \|u\|^{\gamma}_{2\rho, 2\rho p'_*/p_*, Q(\sigma n)}. \tag{5.19}$$

Proof. For fixed $1/2 \leq \sigma' \leq \sigma \leq 1$, consider a sequence $\{Q(\sigma_k n) : k \in \mathbb{N}_0\}$ of space-time cylinders, where

$$\sigma_k \;=\; \sigma' + 2^{-k}(\sigma - \sigma') \qquad \text{and} \qquad \tau_k \;=\; 2^{-k-1}(\sigma - \sigma'), \quad k \in \mathbb{N}_0.$$

In particular, we have that $\sigma_k = \sigma_{k+1} + \tau_k$ and $\sigma_0 = \sigma$. To lighten notation we write $I_k := [t_0, t_0 + \sigma_k n]$, $B_k := B(x_0, \sigma_k n)$ and $Q_k := I_k \times B_k$. Note that $|I_k|/|I_{k+1}| \leq 2$

and $|B_k|/|B_{k+1}| \leq C_{\text{reg}}^2 2^d$. Further, we set

$$\alpha \;:=\; \frac{1}{p_*} + \frac{1}{p'_*}\left(1 - \frac{1}{\rho}\right)\frac{q'}{q'+1} \qquad \text{and} \qquad \alpha_k \;:=\; \alpha^k,$$

where ρ is defined in (5.8), and for any $p, p' \in (1, \infty)$, let $p_* := p/(p-1)$ and $p'_* := p'/(p'-1)$ be the Hölder conjugate of p and p', respectively. Notice that for any $p, p', q, q' \in (1, \infty]$ for which (5.12) is satisfied, $\alpha > 1$ and therefore $\alpha_k \geq 1$ for every $k \in \mathbb{N}_0$. In particular, $\alpha > 1$ implies that $\alpha p'_* > q'/(q'+1)$ and $\alpha p_* \leq \rho$.

Consider a sequence $\{\eta_k : k \in \mathbb{N}_0\}$ of cut-off functions in space having the properties that $\operatorname{supp} \eta_k \subset B_k$, $\eta_k \equiv 1$ on B_{k+1}, $\eta_k \equiv 0$ on ∂B_k and $\|\nabla \eta_k\|_{\ell^\infty(E)} \leq 1/\tau_k n$. Moreover, let $\{\zeta_k \in C^\infty(\mathbb{R}) : k \in \mathbb{N}_0\}$ be a sequence of cut-off functions in time such that $\zeta_k \equiv 1$ on I_{k+1}, $\zeta_k \equiv 0$ on $[t_0 + \sigma_k n, \infty)$ and $\|\zeta'_k\|_{L^\infty(\mathbb{R})} \leq 1/\tau_k n^2$. First, in view of (5.4) we have that

$$\big\|\tilde{u}^{2\alpha_k}\big\|_{\alpha p_*, \alpha p'_*, Q_{k+1}} \;\leq\; \big\|\tilde{u}^{2\alpha_k}\big\|_{1,\infty,Q_{k+1}} + \big\|\tilde{u}^{2\alpha_k}\big\|_{\rho, q'/(q'+1), Q_{k+1}}. \tag{5.20}$$

By applying the space-time Sobolev inequality (5.9) to $\eta_k \tilde{u}_t^{\alpha_k}$ and using that

$$\mathcal{E}_t^\omega(\eta_k \tilde{u}_t^{\alpha_k}) \;\leq\; 2\,\mathcal{E}_{t,\eta_k^2}(\tilde{u}_t^{\alpha_k}) + 2\,\|\nabla \eta_k\|_{\ell^\infty(E)}^2 \,\big\||u_t|^{2\alpha_k} \mu_t^\omega\big\|_{\ell^1(B_k)}$$

we obtain

$$\begin{aligned}&\big\|\tilde{u}^{2\alpha_k}\big\|_{\rho, q'/(q'+1), Q_{k+1}} \\ &\quad\leq\; c\,n^2\,\|\nu^\omega\|_{q,q',Q_k}\left(\frac{1}{|I_k|}\int_{I_k}\zeta_k(t)\,\frac{\mathcal{E}^\omega_{t,\eta_k^2}(\tilde{u}_t^{\alpha_k})}{|B_k|}\,\mathrm{d}t + \frac{1}{(\tau_k n)^2}\,\big\||u|^{2\alpha_k}\mu^\omega\big\|_{1,1,Q_k}\right).\end{aligned}$$

Moreover, by means of Jensen's inequality, the energy estimate (5.14) implies that

$$\begin{aligned}&\frac{1}{|I_k|}\,\big\|\tilde{u}^{2\alpha_k}\big\|_{1,\infty,Q_{k+1}} + \frac{1}{|I_k|}\int_{I_k}\zeta_k(t)\,\frac{\mathcal{E}^\omega_{t,\eta_k^2}(\tilde{u}_t^{\alpha_k})}{|B_k|}\,\mathrm{d}t \\ &\qquad\qquad\leq\; c\,\|\mu^\omega\|_{p,p',Q_k}\left(\frac{\alpha_k}{\tau_k n}\right)^2 \|u\|_{2\alpha_k p_*, 2\alpha_k p'_*, Q_k}^{2\alpha_k\gamma_k},\end{aligned}$$

where $\gamma_k = 1$ if $\|u\|_{2\alpha_k p_*, 2\alpha_k p'_*, Q_k} \geq 1$ and $\gamma_k = 1 - 1/\alpha_k$ if $\|u\|_{2\alpha_k p_*, 2\alpha_k p'_*, Q_k} < 1$. Thus, by combining these two estimates with (5.20), we find that

$$\begin{aligned}&\|u\|_{2\alpha_{k+1}p_*, 2\alpha_{k+1}p'_*, Q_{k+1}} \\ &\quad\leq\; \left(c\,\frac{2^{2k}\alpha_k^2}{(\sigma-\sigma')^2}\,\|\mu^\omega\|_{p,p',Q(n)}\,\|\nu^\omega\|_{q,q',Q(n)}\right)^{1/(2\alpha_k)} \|u\|_{2\alpha_k p_*, 2\alpha_k p'_*, Q_k}^{\gamma_k}.\end{aligned} \tag{5.21}$$

Observe that $|B_{k+1}|^{1/2\alpha_K} \leq c$ uniformly in n for any K such that $\alpha_K \geq \ln n$. Hence, a further application of (5.20) yields

$$\begin{aligned}\max_{(t,x)\in Q(\sigma' n)}|u(t,x)| \;&\leq\; \max_{(t,x)\in Q_K}|u(t,x)| \;\leq\; |B_K|^{1/(2\alpha_K)}\,\big\|\tilde{u}^{2\alpha_K}\big\|_{1,\infty,Q_K}^{1/(2\alpha_K)} \\ &\leq\; c\left(\frac{2^{2K}\alpha_K^2}{(\sigma-\sigma')^2}\,\|\mu^\omega\|_{p,p',Q(n)}\right)^{1/(2\alpha_K)} \|u\|_{2\alpha_K p_*, 2\alpha_K p'_*, Q_K}^{\gamma_K}.\end{aligned}$$

By iterating the inequality (5.21), we get

$$\max_{(t,x)\in Q(\sigma' n)} |u(t,x)| \;\le\; C_2 \prod_{k=1}^{K} \left(\frac{\|\mu^\omega\|_{p,p',Q(n)}\; \|\nu^\omega\|_{q,q',Q(n)}}{(\sigma-\sigma')^2} \right)^{1/(2\alpha_k)} \|u\|_{2\alpha p_*, 2\alpha p_*', Q(\sigma n)}^{\gamma},$$

where $0 < \gamma = \prod_{k=1}^K \gamma_k \le 1$ and $C_2 < \infty$ is a constant independent of k, since $\sum_{k=0}^\infty k/\alpha_k < \infty$. Finally, by choosing $\kappa = \frac{1}{2}\sum_{k=0}^\infty 1/\alpha_k < \infty$ and using the fact that $\alpha p_* \le \rho$, the claim follows by means of Jensen's inequality. □

Proof of Theorem 5.5. In view of (5.19) for any $\alpha > 2\rho \max\{1, p_*'/p_*\} =: \beta$ the statement (5.13) is an immediate consequence of Jensen's inequality. Thus, it remains to consider the case $\alpha \in (0, \beta)$. But from (5.19) we have for any $1/2 \le \sigma' < \sigma \le 1$,

$$\|u\|_{\infty,\infty,Q(\sigma' n)} \;\le\; C_2 \left(\frac{\|\mu^\omega\|_{p,p',Q(n)}\; \|\nu^\omega\|_{q,q',Q(n)}}{(\sigma-\sigma')^2} \right)^{\kappa} \|u\|_{\beta,\beta,Q(\sigma n)}^{\gamma}. \tag{5.22}$$

The remaining part of the proof follows the arguments in [35, Theorem 2.2.3]. In the sequel, let $1/2 \le \sigma' < \sigma \le 1$ be arbitrary but fixed and set $\sigma_k = \sigma - 2^{-k}(\sigma - \sigma')$ for any $k \in \mathbb{N}_0$. Then, by Hölder's inequality we have for any $\alpha \in (0, \beta)$,

$$\|u\|_{\beta,\beta,Q(\sigma_k n)} \;\le\; \|u\|_{\alpha,\alpha,Q(\sigma_k n)}^{\theta}\; \|u\|_{\infty,\infty,Q(\sigma_k n)}^{1-\theta},$$

where $\theta = \alpha/\beta$. Recall that $|Q(\sigma n)|/|Q(\sigma' n)| \le 2C_{\mathrm{reg}}^2 2^d$ by the volume regularity. In view of (5.22), we get

$$\|u\|_{\infty,\infty,Q(\sigma_{k-1} n)} \;\le\; 2^{2\kappa k}\, J\; \|u\|_{\alpha,\alpha,Q(\sigma n)}^{\gamma\theta}\; \|u\|_{\infty,\infty,Q(\sigma_k n)}^{\gamma-\gamma\theta},$$

where we introduced $J = c\,\big(\|\mu^\omega\|_{p,p',Q(n)}\; \|\nu^\omega\|_{q,q',Q(n)} / (\sigma - \sigma')^2\big)^{\kappa}$ to simplify notation. By iterating the inequality above, we get

$$\|u\|_{\infty,\infty,Q(\sigma' n)} \;\le\; 2^{2\kappa \sum_{k=0}^{i-1}(k+1)(\gamma-\gamma\theta)^k} \Big(J\; \|u\|_{\alpha,\alpha,Q(\sigma n)}^{\gamma\theta} \Big)^{\sum_{k=0}^{i-1}(\gamma-\gamma\theta)^k} \|u\|_{\infty,\infty,Q(\sigma_i n)}^{(\gamma-\gamma\theta)^i}.$$

Note that $\gamma(1-\theta) \in (0,1)$. Hence, in the limit when i tends to infinity, we obtain

$$\max_{(t,x)\in Q(\sigma' n)} |u(t,x)| \;\le\; 2^{2\kappa/(1-\gamma+\gamma\theta)^2}\, J^{1/(1-\gamma+\gamma\theta)}\; \|u\|_{\alpha,\alpha,Q(\sigma n)}^{\gamma\theta/(1-\gamma+\gamma\theta)},$$

which gives (5.13). □

Appendix A. Technical estimates

In this section, we provide proofs of some technical estimates needed in the proof of the Moser iteration proven in [5, Appendix A]. In a sense, some of them may be seen as a replacement for a discrete chain rule. Some extra care is required since the solution of the Poisson equation may be negative.

Lemma A.1. *For $a \in \mathbb{R}$, we write $\tilde{a}^\alpha := |a|^\alpha \cdot \operatorname{sign} a$ for any $\alpha \in \mathbb{R} \setminus \{0\}$.*

(i) For all $a, b \in \mathbb{R}$ and any $\alpha, \beta \neq 0$,

$$\big|\tilde{a}^\alpha - \tilde{b}^\alpha\big| \;\le\; \Big(1 \vee \Big|\frac{\alpha}{\beta}\Big|\Big)\, \big|\tilde{a}^\beta - \tilde{b}^\beta\big| \,\big(|a|^{\alpha-\beta} + |b|^{\alpha-\beta}\big). \tag{A.1}$$

(ii) *For all $a, b \in \mathbb{R}$ and any $\alpha > 1/2$,*

$$\big(\tilde{a}^{\alpha} - \tilde{b}^{\alpha}\big)^2 \;\leq\; \left|\frac{\alpha^2}{2\alpha - 1}\right| \big(a - b\big)\, \big(\tilde{a}^{2\alpha-1} - \tilde{b}^{2\alpha-1}\big). \tag{A.2}$$

In particular, if $a, b \in \mathbb{R}_+$ then (A.2) holds for all $\alpha \notin \{0, 1/2\}$.

(iii) *For all $a, b \in \mathbb{R}$ and any $\alpha \geq 1/2$,*

$$\big(|a|^{2\alpha-1} + |b|^{2\alpha-1}\big)\, \big|a - b\big| \;\leq\; 4\, \big|\tilde{a}^{\alpha} - \tilde{b}^{\alpha}\big|\, \big(|a|^{\alpha} + |b|^{\alpha}\big). \tag{A.3}$$

Appendix B. Maker's Theorem

In this section we briefly recall Maker's theorem and transfer it into the continuous time setting as needed in the present paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta : \Omega \to \Omega$ be a measure-preserving transformation.

Theorem B.1. [25, Theorem 1.7.5] *Let ϕ in $L^1(\mathbb{P})$ and $\{\phi_n : n \in \mathbb{N}\}$ be sequence of functions in $L^1(\mathbb{P})$ such that $\sup_{n\in\mathbb{N}} |\phi_n| \in L^1(\mathbb{P})$ and $\phi_n \to \phi$ $\mathbb{P}$-a.s. as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi_n \circ \theta^k \;=\; \mathbb{E}[\phi \,|\, \mathcal{I}] \qquad \mathbb{P}\text{-a.s.}, \tag{B.1}$$

where $\mathcal{I}$ is the σ-algebra of θ-invariant sets.

Now let $\{\theta_t : t \geq 0\}$ be a semigroup of measure-preserving transformations, that is $\theta_t : \Omega \to \Omega$ is measure-preserving for each $t \geq 0$, θ_0 is the identity and $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \geq 0$. Moreover for each $A \in \mathcal{F}$ the mapping $(\omega, t) \to \mathbb{1}_A(\theta_t \omega)$ is jointly measurable with respect to the σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

Proposition B.2. *Let ϕ in $L^1(\mathbb{P})$ and $\{\phi_n : n \in \mathbb{N}\}$ be a sequence of functions in $L^1(\mathbb{P})$ such that $\sup_{n\in\mathbb{N}} |\phi_n| \in L^1(\mathbb{P})$ and $\phi_n \to \phi$ $\mathbb{P}$-a.s. as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \phi_n \circ \theta_s \,\mathrm{d}s \;=\; \int_0^1 \mathbb{E}[\phi \circ \theta_s \,|\, \mathcal{J}]\, \mathrm{d}s \qquad \mathbb{P}\text{-a.s.}, \tag{B.2}$$

where $\mathcal{J}$ is the σ-algebra of θ_1-invariant sets.

Proof. We will deduce the result from its time-discrete analogue in Theorem B.1. For this purpose, set $\Phi_n(\omega) := \int_0^1 \phi_n(\theta_s \omega)\, \mathrm{d}s$. Then, for every $n \in \mathbb{N}$, we have that

$$\int_0^n \phi_n \circ \theta_s \, \mathrm{d}s \;=\; \sum_{k=0}^{n-1} \Phi_n \circ \theta_1^k.$$

Notice that $\sup_{n\in\mathbb{N}} |\Phi_n| \in L^1(\mathbb{P})$. Moreover, by setting $\Phi := \int_0^1 \phi \circ \theta_s \, \mathrm{d}s$, an application of Fubini's theorem and the shift-invariance of $\mathbb{P}$ with respect to θ_s yields

$$\mathbb{E}\Big[\sup_{m\geq n} \big|\Phi_m - \Phi\big|\Big] \;\leq\; \int_0^1 \mathbb{E}\Big[\sup_{m\geq n} \big|\phi_m \circ \theta_s - \phi \circ \theta_s\big|\Big]\, \mathrm{d}s \;=\; \mathbb{E}\Big[\sup_{m\geq n} \big|\phi_m - \phi\big|\Big],$$

which converges to zero as $n \to \infty$ by the dominated convergence theorem. Thus, by the Markov inequality, $\sup_{m \geq n} |\Phi_m - \Phi| \to 0$ in $\mathbb{P}$-probability as $n \to \infty$ which implies that $\Phi_n \to \Phi$ $\mathbb{P}$-a.s. as $n \to \infty$. Hence, by Theorem B.1, we have that $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_n \circ \theta_1^k \;=\; \mathbb{E}[\Phi \,|\, \mathcal{J}] \;=\; \int_0^1 \mathbb{E}[\phi \circ \theta_s \,|\, \mathcal{J}]\, \mathrm{d}s,$$

which completes the proof. □

Appendix C. Some pointwise ergodic theorems

In this Appendix we provide an extension to [13, Theorem 3] to address non-ergodic situations. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a group of measure preserving transformations $\{\tau_x : x \in \mathbb{Z}^d\}$ on Ω such that $\tau_{x+y} = \tau_x \circ \tau_y$ and $\tau_0 = \mathrm{id}_\Omega$.

Theorem C.1. *For all $\varphi \in L^1(\mathbb{P})$ and $f \in C(\overline{B_1})$, $\mathbb{P}$-a.s.*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x \in B(n)} f(x/n)\, \varphi(\tau_x\omega) \;=\; \left(\int_{B_1} f(x)\, \mathrm{d}x \right) \mathbb{E}\big[\varphi \,|\, \mathcal{I}\big], \tag{C.1}$$

where $\mathcal{I}$ denotes the σ-algebra of invariant sets with respect to $\{\tau_x : x \in \mathbb{Z}^d\}$. In particular, the null set does not depend on f.

Proof. The proof follows literally by the arguments given in [13, Theorem 3]. □

Theorem C.2. *Let $\varphi \in L^1(\mathbb{P})$ and $\varepsilon \in (0, d)$. Then, for $\mathbb{P}$-a.e. ω,*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x \in B(n)}{}' \frac{\varphi(\tau_x\omega)}{|x/n|^{d-\varepsilon}} \;=\; \left(\int_{B_1} |x|^{-(d-\varepsilon)}\, \mathrm{d}x \right) \mathbb{E}\big[\varphi \,|\, \mathcal{I}\big], \tag{C.2}$$

where the summation is taken over all $x \in B(n) \setminus \{0\}$ and $\mathcal{I}$ denotes the σ-algebra of invariant sets with respect to $\{\tau_x : x \in \mathbb{Z}^d\}$.

Proof. To start with, notice that the ergodic theorem, see Theorem C.1, implies that for $\mathbb{P}$-a.e. ω,

$$\begin{aligned} &\lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{n^d} \sum_{x \in B(n)}{}' \left(k \wedge |x/n|^{-(d-\varepsilon)} \right) \varphi(\tau_x\omega) \\ &\quad= \lim_{k\to\infty} \left(\int_{B_1} k \wedge |x|^{-(d-\varepsilon)}\, \mathrm{d}x \right) \mathbb{E}\big[\varphi \,|\, \mathcal{I}\big] \;=\; \left(\int_{B_1} |x|^{-(d-\varepsilon)}\, \mathrm{d}x \right) \mathbb{E}\big[\varphi \,|\, \mathcal{I}\big]. \end{aligned} \tag{C.3}$$

On the other hand, by means of Abel's summation formula, we have that

$$\begin{aligned} \left| \frac{1}{n^\varepsilon} \sum_{x \in B(n)}{}' \frac{\varphi(\tau_x\omega)}{|x|^{d-\varepsilon}} \right| \;&=\; \left| \frac{1}{n^\varepsilon} \sum_{k=1}^{n} \frac{1}{k^{d-\varepsilon}} \sum_{|x|=k} \varphi(\tau_x\omega) \right| \\ &\leq \left| \frac{1}{n^d} \sum_{x \in B(n)} \varphi(\tau_x\omega) \right| + \frac{d-\varepsilon}{n^\varepsilon} \sum_{k=1}^{n-1} \frac{1}{k^{1-\varepsilon}} \left| \frac{1}{k^d} \sum_{x \in B(k)} \varphi(\tau_x\omega) \right|, \end{aligned}$$

where we used that $k^{-(d-\varepsilon)} - (k+1)^{-(d-\varepsilon)} \leq (d-\varepsilon) k^{-(d+1-\varepsilon)}$. Using this estimate and the maximal inequality we get that for any $\varphi \in L^1(\mathbb{P})$ and $\mathbb{P}$-a.e. ω,

$$\sup_{n \geq 1} \left| \frac{1}{n^d} \sum_{x \in B(n)}' \frac{\varphi(\tau_x \omega)}{|x/n|^{d-\varepsilon}} \right| \leq \frac{Cd}{\varepsilon} \sup_{n \geq 0} \frac{1}{|B(n)|} \sum_{x \in B(n)} \bigl|\varphi(\tau_x \omega)\bigr| < \infty \tag{C.4}$$

with $C := \sup_{n \geq 1} |B(n)|/n^d < \infty$. In particular, since

$$\frac{1}{n^d} \left| \sum_{x \in B(n)}' \left(\frac{1}{|x/n|^{d-\varepsilon}} - k \wedge \frac{1}{|x/n|^{d-\varepsilon}} \right) \varphi(\tau_x \omega) \right| \leq \frac{C}{k} \sup_{n \geq 1} \left| \frac{1}{n^d} \sum_{x \in B(n)}' \frac{\varphi(\tau_x \omega)}{|x/n|^{d-\varepsilon}} \right|,$$

we conclude in view of (C.4) that for $\mathbb{P}$-a.e. ω,

$$\lim_{k \to \infty} \frac{1}{n^d} \left| \sum_{x \in B(n)}' |x/n|^{-(d-\varepsilon)} \varphi(\tau_x \omega) - \sum_{x \in B(n)}' \Bigl(k \wedge |x/n|^{-(d-\varepsilon)} \Bigr) \varphi(\tau_x \omega) \right| = 0 \tag{C.5}$$

uniformly in n. The assertion follows by combining (C.3) and (C.5). □

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