Quenched invariance principles for random walks with random conductances.

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Abstract

We prove an almost sure invariance principle for a random walker among i.i.d. conductances in $\mathbb{Z}^d$, $d \geq 2$. We assume conductances are bounded from above but we do not require that they are bounded from below.

1 Introduction

We consider continuous-time, nearest-neighbor random walks among random (i.i.d.) conductances in $\mathbb{Z}^d$, $d \geq 2$ and prove that they satisfy an almost sure invariance principle.

1.1 Random walks and environments

For $x, y \in \mathbb{Z}^d$, we write: $x \sim y$ if x and y are neighbors in the grid $\mathbb{Z}^d$ and let $\mathbb{E}_d$ be the set of non-oriented nearest-neighbor pairs (x, y).
An *environment* is a function $\omega : \mathbb{E}_d \to [0, +\infty[$. Since edges in $\mathbb{E}_d$ are not oriented, i.e. we identified the edge (x, y) with the reversed edge (y, x), it is implicit in the definition that environments are symmetric i.e. $\omega(x, y) = \omega(y, x)$ for any pair of neighbors x and y.
We let $(\tau_z\,,\ z \in \mathbb{Z}^d)$ be the group of transformations of environments defined by $\tau_z\omega(x, y) = \omega(z + x, z + y)$.

We shall always assume that our environments are uniformly bounded from above. Without loss of generality, we may assume that $\omega(x, y) \leq 1$ for any edge. Thus, for the rest of this paper, an environment will rather be a function $\omega : \mathbb{E}_d \to [0, 1]$. We use the notation $\Omega = [0, 1]^{E_d}$ for the set of environments (endowed with the product topology and the corresponding Borel structure). The value of an environment ω at a given edge is called the *conductance*.

Let $\omega \in \Omega$. We are interested in the behavior of the random walk in the environment ω. We denote with $D(\mathbb{R}_+, \mathbb{Z}^d)$ the space of càd-làg $\mathbb{Z}^d$-valued functions on $\mathbb{R}_+$ and let $X(t)$, $t \in \mathbb{R}_+$, be the coordinate maps from $D(\mathbb{R}_+, \mathbb{Z}^d)$ to $\mathbb{Z}^d$. The space $D(\mathbb{R}_+, \mathbb{Z}^d)$ is endowed with the Skorokhod topology, see [6] or [13]. For a given $\omega \in [0, 1]^{\mathbb{E}_d}$ and for $x \in \mathbb{Z}^d$, let P^ω_x be

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the probability measure on $D(\mathbb{R}_+, \mathbb{Z}^d)$ under which the coordinate process is the Markov chain starting at $X(0) = x$ and with generator

$$\mathcal{L}^\omega f(x) = \frac{1}{n^\omega(x)} \sum_{y \sim x} \omega(x, y)(f(y) - f(x)), \tag{1.1}$$

where $n^\omega(x) = \sum_{y \sim x} \omega(x, y)$. If $n^\omega(x) = 0$, let $\mathcal{L}^\omega f(x) = 0$ for any function f.

The behavior of $X(t)$ under P_x^ω can be described as follows: starting from point x, the random walker waits for an exponential time of parameter 1 and then chooses at random one of its neighbors to jump to according to the probability law $\omega(x, .)/n^\omega(x)$. This procedure is then iterated with independent hopping times.

We have allowed environments to take the value 0 and it is clear from the definition of the random walk that X will only travel along edges with positive conductances. This remark motivates the following definitions: call a *cluster* of the environment ω a connected component of the graph $(\mathbb{Z}^d, \{e \in E_d\,;\, \omega(e) > 0\})$. By construction, our random walker never leaves the cluster of ω it started from. Since edges are not oriented, the measures with weights $n^\omega(x)$ on the possibly different clusters of ω are reversible.

1.2 Random environments

Let Q be a product probability measure on Ω. In other words, we will now pick environments at random, in such a way that the conductances of the different edges form a family of independent identically distributed random variables. Q is of course invariant under the action of τ_z for any $z \in \mathbb{Z}^d$.

The random variables $(\mathbf{1}_{\omega(e)>0}\,;\, e \in E_d)$ are independent Bernoulli variables with common parameter $q = Q(\omega(e) > 0)$. Depending on the value of q, a typical environment chosen w.r.t. Q may or may not have infinite clusters. More precisely, it is known from percolation theory that there is a critical value p_c, that depends on the dimension d, such that for $q < p_c$, Q.a.s. all clusters of ω are finite and for $q > p_c$, Q.a.s. there is a unique infinite cluster. In the first case the random walk is almost surely confined to a finite set and therefore does not satisfy the invariance principle (or satisfies a degenerate version of it with vanishing asymptotic variance). We shall therefore assume that the law Q is *super-critical* i.e. that

$$q = Q(\omega(e) > 0) > p_c\,.$$

Then the event 'the origin belongs to the infinite cluster' has a non vanishing Q probability and we may define the conditional law:

$$Q_0(.) = Q(.\,|\, 0 \text{ belongs to the infinite cluster})\,.$$

1.3 Annealed results

Part of the analysis of the behavior of random walks in random environments can be done using the *point of view of the particle*: we consider the random walk X started at the origin and look at the random process describing the environment shifted by the position of the random walker i.e. we let $\omega(t) = \tau_{X(t)}\omega$. Thus $(\omega(t)\,,\, t \in \mathbb{R}_+)$ is a random process taking its values in Ω.

Let us also introduce the measure

$$\tilde{Q}_0(A) = \frac{\int_A n^\omega(0) dQ_0(\omega)}{\int n^\omega(0) dQ_0(\omega)} .$$

Observe that $\tilde{Q}_0$ is obviously absolutely continuous with respect to Q_0.

We list some of the properties of the process $\omega(.)$ as proved in [8]:

Proposition 1.1 *(Lemmata 4.3 and 4.9 in [8])*
The random process $\omega(t)$ is Markovian under P_0^ω. The measure $\tilde{Q}_0$ is reversible, invariant and ergodic with respect to $\omega(t)$.

Based on this proposition, the authors of [8] could deduce that the random walk $X(t)$ satisfies the invariance principle *in the mean.* Let us define the so-called *annealed* semi-direct product measure

$$Q_0.P_x^\omega[\, F(\omega, X(.))\,] = \int P_x^\omega[\, F(\omega, X(.))\,]\, dQ_0(\omega) .$$

Theorem 1.2 *(Annealed invariance principle, [8])*
Consider a random walk with i.i.d. super-critical conductances. Under $Q_0.P_0^\omega$, the process $(X^\varepsilon(t) = \varepsilon X(\frac{t}{\varepsilon^2}), t \in \mathbb{R}_+)$ converges in law to a non-degenerate Brownian motion with covariance matrix $\sigma^2 Id$ where σ^2 is positive.

It should be pointed out that the result of [8] is in fact much more general. On one hand, [8] deals with random walks with unbounded jumps, under a mild second moment condition. Besides, a similar annealed invariance principle is in fact proved for any stationary law Q rather than just product measures.

The positivity of σ^2 is not ensured by the general results of [8]) but it can be proved using comparison with the Bernoulli case, see Remark 2.3.

1.4 The almost sure invariance principle

The annealed invariance principle is not enough to give a completely satisfactory description of the long time behavior of the random walk. It is for instance clear that the annealed measure $Q_0.P_0^\omega$ retains all the symmetries of the grid. In particular it is invariant under reflections through hyperplanes passing through the origin. This is not true anymore for the law of the random walk in a given environment. Still, one would expect symmetries to be restored in the large scale, for a given realization of ω.

Our main result is the following almost sure version of Theorem 1.2:

Theorem 1.3 *(Quenched invariance principle)*
Consider a random walk with i.i.d. super-critical conductances. Q_0 almost surely, under P_0^ω, the process $(X^\varepsilon(t) = \varepsilon X(\frac{t}{\varepsilon^2}), t \in \mathbb{R}_+)$ converges in law as ε tends to 0 to a non-degenerate Brownian motion with covariance matrix $\sigma^2 Id$ where σ^2 is positive and does not depend on ω.

1.5 The Bernoulli case and other cases

The main difficulty in proving Theorem 1.3 is the lack of assumption on a lower bound for the values of the conductances. Indeed, if one assumes that almost any environment is bounded from below by a fixed constant i.e. there exists a $\delta > 0$ such that $Q(\omega(e) < \delta) = 0$ then the conclusion of Theorem 1.3 was already proved in [18] using the classical 'corrector approach' adapted from [14].

Another special case recently solved is the Bernoulli case: let us assume that only the values 0 and 1 are allowed for the conductances i.e. Q is a product of Bernoulli measures of parameter q. Remember that we assume that we are in the supercritical regime $q > p_c$. An environment can then be also thought of as a (unweighted) random sub-graph of the grid and our random walk is the simple symmetric random walk on the clusters of the environment, i.e. jumps are performed according to the uniform law on the neighbors of the current position in the graph ω.

In the Bernoulli case, quenched invariance principles have been obtained by various authors in [4], [15] and [18]. These three works develop different approaches to handle the lack of a positive lower bound for the conductances. They have in common the use of quantitative bounds on the transition probabilities of the random walk. It is indeed known from [2] that the kernel of the simple random walk on an infinite percolation cluster satisfies Gaussian bounds. A careful analysis of the proofs shows that a necessary condition to obtain the invariance principle using any of the three approaches in [4], [15] or [18] is a Poincaré inequality of the correct scaling (and in fact [15] shows that the Poincaré inequality is 'almost' sufficient.) To be more precise, let A_n be the Poincaré constant on a box of size n centered at the origin. In other words, A_n is the inverse spectral gap of the operator $\mathcal{L}^\omega$ restricted to the connected component at the origin of the graph $\omega \cap [-n, n]^d$ and with reflection boundary conditions. Then one needs know that Q_0 almost surely,

$$\limsup n^{-2} A_n < \infty \,. \tag{1.2}$$

Such a statement was originally proved in [16] for the Bernoulli case.

It turns out that (1.2) is false in the general case of i.i.d. conductances, even if one assumes that conductances are always positive. We can choose for instance a product law with a polynomial tail at the origin i.e. we assume that there exists a positive parameter γ such that $Q(\omega(e) \leq a) \sim a^\gamma$ as a tends to 0. Then it is not difficult to prove that, for small values of γ,

$$\liminf \frac{\log A_n}{\log n} > 2 \,.$$

In [11], we considered a slightly different model of symmetric random walks with random conductances with a polynomial tail but non i.i.d. (although with finite range dependency only) and we proved that

$$\frac{\log A_n}{\log n} \to 2 \vee \frac{d}{\gamma} \,,$$

showing that, at least in the case $\gamma < d/2$, the Poincaré constant is too big to be directly used to prove the diffusive behavior of the random walk and one needs some new ingredient to prove Theorem 1.3.

Remark 1.4 *In [11], we derived annealed estimates on the decay of the return probability of the random walk. More interestingly, in the very recent work [5], the authors could also obtain quenched bounds on the decay of the return probability for quite general random walks with random conductances. Their results in particular show that anomalous decays do occur in high dimension. In such situations, although the almost sure invariance principle holds, see Theorem 1.3, the local CLT fails.*

Our proof of Theorem 1.3 uses a time change argument that we describe in the next part of the paper.

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Note: after this paper was posted on the Arxiv, M. Biskup and T. Prescott wrote a preprint with a different proof of Theorem 1.3, see [7]. Their approach is based on ideas from [4] when we prefer to invoke [15]. They also need a time change argument, as here, and percolation results like Lemma 5.3.

2 A time changed process

In this section, we introduce a time changed process, X^ξ, and state an invariance principle for it: Theorem 2.2.

Choose a threshold parameter $\xi > 0$ such that $Q(\omega(e) \geq \xi) > p_c$. For Q almost any environment ω, the percolation graph $(\mathbb{Z}^d, \{e \in E_d\,;\, \omega(e) \geq \xi\})$ has a unique infinite cluster that we denote with $\mathcal{C}^\xi(\omega)$.

By construction $\mathcal{C}^\xi(\omega)$ is a subset of $\mathcal{C}(\omega)$. We will refer to the connected components of the complement of $\mathcal{C}^\xi(\omega)$ in $\mathcal{C}(\omega)$ as *holes*. By definition, holes are connected sub-graphs of the grid. Let $\mathcal{H}^\xi(\omega)$ be the collection of all holes. Note that holes may contain edges such that $\omega(e) \geq \xi$.

We also define the conditioned measure

$$Q_0^\xi(.) = Q(.|0 \in \mathcal{C}^\xi(\omega)) .$$

Consider the following additive functional of the random walk:

$$A^\xi(t) = \int_0^t \mathbf{1}_{X(s) \in \mathcal{C}^\xi(\omega)}\, ds \,,$$

its inverse $(A^\xi)^{-1}(t) = \inf\{s\,;\, A^\xi(s) > t\}$ and define the corresponding time changed process

$$X^\xi(t) = X((A^\xi)^{-1}(t)) .$$

Thus the process X^ξ is obtained by suppressing in the trajectory of X all the visits to the holes. Note that, unlike X, the process X^ξ may perform long jumps when straddling holes.

As X performs the random walk in the environment ω, the behavior of the random process X^ξ is described in the next

Proposition 2.1 *Assume that the origin belongs to $\mathcal{C}^\xi(\omega)$. Then, under P_0^ω, the random process X^ξ is a symmetric Markov process on $\mathcal{C}^\xi(\omega)$.*

The Markov property, which is not difficult to prove, follows from a very general argument about time changed Markov processes. The reversibility of X^ξ is a consequence of the reversibility of X itself as will be discussed after equation (2.2).

The generator of the process X^ξ has the form

$$\mathcal{L}^{\xi,\omega} f(x) = \frac{1}{n^\omega(x)} \sum_y \omega^\xi(x,y)(f(y)-f(x))\,, \tag{2.1}$$

where

$$\begin{aligned} \frac{\omega^\xi(x,y)}{n^\omega(x)} &= \lim_{t\to 0} \frac{1}{t} P_x^\omega(X^\xi(t)=y) \\ &= P_x^\omega(\ y \text{ is the next point in } \mathcal{C}^\xi(\omega) \text{ visited by the random walk } X)\,, \end{aligned} \tag{2.2}$$

if both x and y belong to $\mathcal{C}^\xi(\omega)$ and $\omega^\xi(x,y)=0$ otherwise.

The function ω^ξ is symmetric: $\omega^\xi(x,y)=\omega^\xi(y,x)$ as follows from the reversibility of X and formula (2.2), but it is no longer of nearest-neighbor type i.e. it might happen that $\omega^\xi(x,y)\neq 0$ although x and y are not neighbors. More precisely, one has the following picture: $\omega^\xi(x,y)=0$ unless either x and y are neighbors and $\omega(x,y)\geq\xi$, or there exists a hole, h, such that both x and y have neighbors in h. (Both conditions may be fulfilled by the same pair (x,y).)

Consider a pair of neighboring points x and y, both of them belonging to the infinite cluster $\mathcal{C}^\xi(\omega)$ and such that $\omega(x,y)\geq\xi$, then

$$\omega^\xi(x,y)\geq\xi\,. \tag{2.3}$$

This simple remark will play an important role. It implies, in a sense to be made precise later, that the parts of the trajectory of X^ξ that consist in nearest-neighbors jumps are similar to what the simple symmetric random walk on $\mathcal{C}^\xi(\omega)$ does.

Finally observe that the environment ω^ξ is stationary i.e. the law of ω^ξ under Q is invariant with respect to τ_z for all $z\in\mathbb{Z}^d$ as can be immediately seen from formula 2.2.

Theorem 2.2 *(Quenched invariance principle for X^ξ)*
There exists a value $\xi_0>0$ such that for any $0<\xi\leq\xi_0$ the following holds. For Q_0 almost any environment, under P_0^ω, the process $(X^{\xi,\varepsilon}(t)=\varepsilon X^\xi(\frac{t}{\varepsilon^2}), t\in\mathbb{R}_+)$ converges in law as ε tends to 0 to a non-degenerate Brownian motion with covariance matrix $\sigma^2(\xi)Id$ where $\sigma^2(\xi)$ is positive and does not depend on ω.

The proof of Theorem 2.2 will be given in part 4. It very closely mimics the arguments of [15]. Indeed, one uses the lower bound (2.3) to bound the Dirichlet form of the process X^ξ in terms of the Dirichlet form of the simple symmetric random walk on $\mathcal{C}^\xi(\omega)$ and thus get the Poincaré inequality of the correct order. It is then not difficult to adapt the approach of [16] and [2] to derive the tightness of the family $X^{\xi,\varepsilon}$ and the invariance principle follows as in [15].

Remark 2.3 *The positivity of σ^2 in Theorem 1.3 and the positivity of $\sigma^2(\xi)$ in Theorem 2.2 can be checked using comparison arguments from [8]. Indeed it follows from the expression of the effective diffusivity, see Theorem 4.5 part (iii) of [8], and from the discussion on monotonicity in part 3 of [8] that σ^2 is an increasing function of the probability law Q (up to some multiplicative factor). Therefore, if Q stochastically dominates Q' and the effective diffusivity under Q' is positive, then the effective diffusivity under Q is also positive. Here Q stochastically dominates the law of the environment with conductances $\omega'(e) = \xi \mathbf{1}_{\omega(e)\geq\xi}$. The random walk in the environment ω' is the simple random walk on a percolation cluster which is known to have a positive asymptotic diffusivity, see [2] or the references in [15]. The same argument shows that $\sigma^2(\xi) > 0$ for any ξ such that $Q(\omega(e) \geq \xi) > p_c$.*

To derive Theorem 1.3 from Theorem 2.2, we will compare the processes X and X^ξ, for small values of ξ. The large time asymptotic of the time change A^ξ is easily deduced from the ergodic theorem, as shown in Lemma 2.4 below and it implies that the asymptotic variance $\sigma^2(\xi)$ is continuous at $\xi = 0$, see Lemma 2.5.

Let

$$c(\xi) = \tilde{Q}_0(0 \in \mathcal{C}^\xi(\omega)) .$$

Lemma 2.4

$$\frac{A^\xi(t)}{t} \to c(\xi) \;\; Q_0 \;\; a.s.$$

as t tends to ∞ and

$$c(\xi) \to 1 , \tag{2.4}$$

as ξ tends to 0.

Proof: remember the notation $\omega(t) = \tau_{X(t-)}\omega$. The additive functional $A^\xi(t)$ can also be written in the form $A^\xi(t) = \int_0^t \mathbf{1}_{0\in\mathcal{C}^\xi(\omega(s))}\, ds$.

From Proposition 1.1, we know that $\tilde{Q}_0$ is an invariant and ergodic measure for the process $\omega(t) = \tau_{X(t-)}\omega$ and that it is absolutely continuous with respect to Q_0.

Thus the existence of the limit $\lim_{t\to+\infty} \frac{A^\xi(t)}{t}$ follows from the ergodic theorem and the limit is $c(\xi) = \tilde{Q}_0(0 \in \mathcal{C}^\xi(\omega))$. To check (2.4), note that $\mathbf{1}_{0\in\mathcal{C}^\xi(\omega)}$ almost surely converges to $\mathbf{1}_{0\in\mathcal{C}(\omega)}$ as ξ tends to 0. Since $\tilde{Q}_0(0 \in \mathcal{C}(\omega)) = 1$, we get that $c(\xi)$ converges to 1.

■

Lemma 2.5 *The asymptotic variances σ^2 in Theorem 1.2 and $\sigma^2(\xi)$ from Theorem 2.2, and the constant $c(\xi)$ from Lemma 2.4 satisfy the equality*

$$c(\xi)\sigma^2(\xi) = \sigma^2 . \tag{2.5}$$

As a consequence, $\sigma^2(\xi)$ converges to σ^2 as ξ tends to 0.

Proof: formula (2.5) is deduced from Lemma 2.4. One can, for instance, compute the law of the exit times from a large slab for both processes X and X^ξ. Let $\tau(r)$ (resp. $\tau^\xi(r)$) be the exit time of X (resp. X^ξ) from the set $[-r,r]\times\mathbb{R}^{d-1}$. Under the annealed measure, the Laplace transform of $\tau(r)/r^2$ converges to $E(\exp(-\lambda T/\sigma^2))$ where T is the exit time of $[-1,1]$ by a Brownian motion. This is a consequence of the invariance principle of Theorem 1.2. Theorem 2.2 implies that the Laplace transform of $\tau^\xi(r)/r^2$ converges to $E(\exp(-\lambda T/\sigma^2(\xi)))$. (The convergence holds for Q_0 almost any environment and, by dominated convergence, under the annealed measure.)
On the other hand, we have $\tau^\xi(r) = A^\xi(\tau(r))$ and therefore Lemma 2.4 implies that the Laplace transform of $\tau^\xi(r)/r^2$ has the same limit as the Laplace transform of $c(\xi)\tau^\xi(r)/r^2$ and therefore converges to $E(\exp(-\lambda c(\xi)T/\sigma^2))$. We deduce from these computations that

$$E(\exp(-\lambda c(\xi)T/\sigma^2)) = E(\exp(-\lambda T/\sigma^2(\xi)))\,,$$

and, since this is true for any $\lambda \geq 0$, we must have $c(\xi)\sigma^2(\xi) = \sigma^2$.
The continuity of $\sigma^2(\xi)$ for $\xi = 0$ is ensured by the continuity of $c(\xi)$. ∎

3 How to deduce Theorem 1.3 from Theorem 2.2

We start stating a percolation lemma that will be useful to control the contribution of holes to the behavior of the random walk.

Lemma 3.1 *There exists a value $\xi_0 > 0$ such that for any $0 < \xi \leq \xi_0$ the following holds. There exists a constant a such that, Q almost surely, for large enough n, the volume of any hole $h \in \mathcal{H}^\xi(\omega)$ intersecting the box $[-n,n]^d$ is bounded from above by $(\log n)^a$. ($a = 7$ would do.)*

The proof of Lemma 3.1 is postponed to part 5.

3.1 Tightness

In this section, we derive the tightness of the sequence of processes X^ε from Theorem 2.2.

Lemma 3.2 *Under the assumptions of Theorem 1.3, Q_0 almost surely, under P_0^ω, the family of processes $(X^\varepsilon(t) = \varepsilon X(\frac{t}{\varepsilon^2}), t \in \mathbb{R}_+)$ is tight in the Skorokhod topology.*

Proof: we read from [13], paragraph 3.26, page 315 that a sequence of processes x^ε is tight if and only if the following two estimates hold:
(i) for any T, any $\delta > 0$, there exist ε_0 and K such that for any $\varepsilon \leq \varepsilon_0$

$$P(\sup_{t\leq T} |x^\varepsilon(t)| \geq K) \leq \delta\,, \tag{3.1}$$

and
(ii) for any T, any $\delta > 0$, any $\eta > 0$, there exist ε_0 and θ_0 such that for any $\varepsilon \leq \varepsilon_0$

$$P(\sup_{v\leq u\leq T\,;\,u-v\leq\theta_0} |x^\varepsilon(u) - x^\varepsilon(v)| > \eta) \leq \delta\,. \tag{3.2}$$

Choose ξ as in Theorem 2.2. The sequence $X^{\xi,\varepsilon}$ converges; therefore it is tight and satisfies (3.1) and (3.2). By definition,

$$X^{\xi,\varepsilon}(t) = X^{\varepsilon}(\varepsilon^2 (A^{\xi})^{-1}(\frac{t}{\varepsilon^2}))\,.$$

Proof of condition (i): let us first check that X^{ε} satisfies (3.1).
Assume that $\sup_{t\leq T}|X^{\xi,\varepsilon}(t)|\leq K$. Given $t_0\leq T$, let $x_0 = X^{\varepsilon}(t_0)$ i.e. $X(\frac{t_0}{\varepsilon^2}) = \frac{1}{\varepsilon}x_0$ and define $s_0 = \varepsilon^2 A^{\xi}(\frac{t_0}{\varepsilon^2})$. Since $A^{\xi}(t)\leq t$, we have $s_0\leq t_0$.
If $\frac{1}{\varepsilon}x_0$ belongs to $\mathcal{C}^{\xi}(\omega)$, then $t_0 = \varepsilon^2(A^{\xi})^{-1}(\frac{s_0}{\varepsilon^2})$ and $X^{\xi,\varepsilon}(s_0) = X^{\varepsilon}(t_0) = x_0$ and therefore $|x_0|\leq K$.
Now suppose that $\frac{1}{\varepsilon}x_0$ does not belong to $\mathcal{C}^{\xi}(\omega)$ and let $t_1 = \varepsilon^2(A^{\xi})^{-1}(\frac{s_0}{\varepsilon^2})$ and $x_1 = X^{\varepsilon}(t_1)$. Then $t_1\leq t_0$ and $\frac{1}{\varepsilon}x_1$ belongs to $\mathcal{C}^{\xi}(\omega)$. The same argument as before shows that $|x_1|\leq K$. On the other hand, by definition of the time changed process X^{ξ}, $\frac{1}{\varepsilon}x_1$ is the last point in $\mathcal{C}^{\xi}(\omega)$ visited by X before time t_0. Thus $\frac{1}{\varepsilon}x_0$ belongs to a hole on the boundary of which sits $\frac{1}{\varepsilon}x_1$. It then follows from Lemma 3.1 that

$$|\frac{1}{\varepsilon}x_1 - \frac{1}{\varepsilon}x_0| \leq (\log\frac{K}{\varepsilon})^a\,.$$

Thus we have proved that

$$|x_0|\leq K + \varepsilon(\log\frac{K}{\varepsilon})^a\,.$$

We can choose ε_0 small enough so that $\varepsilon(\log\frac{K}{\varepsilon})^a\leq K$ and therefore we have

$$\sup_{t\leq T}|X^{\xi,\varepsilon}(t)|\leq K \implies \sup_{t\leq T}|X^{\varepsilon}(t)|\leq 2K\,.$$

Since the sequence $X^{\xi,\varepsilon}$ satisfies (3.1), the event '$\sup_{t\leq T}|X^{\xi,\varepsilon}(t)|\leq K$' has a large probability; therefore $\sup_{t\leq T}|X^{\varepsilon}(t)|\leq 2K$ has a large probability and the sequence X^{ε} satisfies (3.1).

Proof of condition (ii): as before, we will deduce that the sequence X^{ε} satisfies (3.2) from the fact that the sequence $X^{\xi,\varepsilon}$ satisfies (3.1) and (3.2). Assume that

$$\sup_{v\leq u\leq T\,;\,u-v\leq\theta_0}|X^{\xi,\varepsilon}(u) - X^{\xi,\varepsilon}(v)|\leq\eta\,.$$

We further assume that $\sup_{t\leq T}|X^{\xi,\varepsilon}(t)|\leq K$.
Given $v_0\leq u_0\leq T$ such that $u_0 - v_0\leq\theta_0$, let $x_0 = X^{\varepsilon}(u_0)$, $y_0 = X^{\varepsilon}(v_0)$ and define $s_0 = \varepsilon^2 A^{\xi}(\frac{u_0}{\varepsilon^2})$, $t_0 = \varepsilon^2 A^{\xi}(\frac{v_0}{\varepsilon^2})$, $u_1 = \varepsilon^2(A^{\xi})^{-1}(\frac{s_0}{\varepsilon^2})$ and $v_1 = \varepsilon^2(A^{\xi})^{-1}(\frac{t_0}{\varepsilon^2})$. Also let $x_1 = X^{\varepsilon}(u_1)$, $y_1 = X^{\varepsilon}(v_1)$.
Since $A^{\xi}(t) - A^{\xi}(s)\leq t-s$ whenever $s\leq t$, we have $t_0\leq s_0\leq T$ and $s_0 - t_0\leq\theta_0$. Besides, by definition of A^{ξ}, we have $x_1 = X^{\xi,\varepsilon}(s_0)$ and $y_1 = X^{\xi,\varepsilon}(t_0)$. We conclude that

$$|x_1 - y_1|\leq\eta\,.$$

On the other hand, the same argument as in the proof of condition (i) based on Lemma 3.1 shows that

$$|x_1 - x_0| + |y_1 - y_0|\leq 2\varepsilon(\log\frac{K}{\varepsilon})^a\,.$$

We have proved that

$$\sup_{v\le u\le T\,;\,u-v\le\theta_0}|X^\varepsilon(u)-X^\varepsilon(v)|\le\eta+2\varepsilon(\log\frac{K}{\varepsilon})^a\,.$$

Since both events '$\sup_{v\le u\le T\,;\,u-v\le\theta_0}|X^{\xi,\varepsilon}(u)-X^{\xi,\varepsilon}(v)|\le\eta$' and '$\sup_{t\le T}|X^{\xi,\varepsilon}(t)|\le K$' have large probabilities, we deduce that the processes X^ε satisfy condition (ii). ■

3.2 Convergence

To conclude the derivation of Theorem 1.3 from Theorem 2.2, it only remains to argue that, for any given time t, the two random variables $X^\varepsilon(t)$ and $X^{\xi,\varepsilon}(t)$ are close to each other in probability.

Lemma 3.3 *Under the assumptions of Theorem 1.3, Q_0 almost surely, for any t, any $\delta>0$, any $\eta>0$, then, for small enough ξ,*

$$\limsup_{\varepsilon\to0}P_0^\omega(|X^\varepsilon(t)-X^{\xi,\varepsilon}(t)|>\eta)\le\delta\,.$$

Proof: we shall rely on Lemma 2.4. If $|X^\varepsilon(t)-X^{\xi,\varepsilon}(t)|>\eta$, then one of the following two events must hold:

$$(I)=\{\sup_{\theta c(\xi)t\le s\le t}|X^{\xi,\varepsilon}(s)-X^{\xi,\varepsilon}(t)|>\frac{\eta}{2}\}\,,$$

$$(II)=\{\inf_{\theta c(\xi)t\le s\le t}|X^{\xi,\varepsilon}(s)-X^{\varepsilon}(t)|>\frac{\eta}{2}\}\,.$$

Here θ is a parameter in $]0,1[$.
The invariance principle for $X^{\xi,\varepsilon}$, see Theorem 2.2, implies that the probability of (I) converges as ε tends to 0 to the probability $P(\sup_{\theta c(\xi)t\le s\le t}\sigma(\xi)|B(s)-B(t)|>\frac{\eta}{2})$, where B is a Brownian motion. Since $\sigma(\xi)$ is bounded away from 0, see Lemma 2.5, and since $c(\xi)\to1$ as $\xi\to0$, we deduce that there exists a value for θ such that

$$\limsup_{\xi\to0}\limsup_{\varepsilon\to0}P_0^\omega(I)\le\delta\,. \tag{3.3}$$

We now assume that θ has been chosen so that (3.3) holds. We shall end the proof of the Lemma by showing that

$$\limsup_{\varepsilon\to0}P_0^\omega(II)=0\,. \tag{3.4}$$

Since, from the tightness of the processes X^ε, see Lemma 3.2, we have

$$\limsup_{\varepsilon\to0}P_0^\omega(\sup_{s\le t}|X^\varepsilon(s)|\ge\varepsilon^{-1})=0\,,$$

we will estimate the probability that both events (II) and '$\sup_{s\le t}|X^\varepsilon(s)|\le\varepsilon^{-1}$' hold.
Let $u=\varepsilon^2A^\xi(\frac{t}{\varepsilon^2})$ and note that $u\le t$. From Lemma 2.4, we know that $u\ge\theta c(\xi)t$ for small enough ε depending on ω.

If $X^\varepsilon(t)$ belongs to $\mathcal{C}^\xi(\omega)$, then $X^\varepsilon(t) = X^{\xi,\varepsilon}(u)$ and therefore (II) does not hold.
Otherwise $X^\varepsilon(t)$ belongs to a hole on the boundary of which sits $X^{\xi,\varepsilon}(u)$. Using the condition $\sup_{s\le t}|X^\varepsilon(s)| \le \varepsilon^{-1}$ and Lemma 3.1, we get that

$$|X^\varepsilon(t) - X^{\xi,\varepsilon}(u)| \le \varepsilon(\log\frac{1}{\varepsilon})^a\,.$$

For sufficiently small ε we have $\varepsilon(\log 1/\varepsilon)^a < \frac{\eta}{2}$ and therefore (II) fails. The proof of (3.4) is complete. ■

End of the proof of Theorem 1.3: choose times $0 < t_1 < ... < t_k$. Use Lemma 3.3, to deduce that for small enough ξ, as ε tends to 0, the law of $(X^\varepsilon(t_1), ..., X^\varepsilon(t_k))$ comes close to the law of $(X^{\xi,\varepsilon}(t_1), ..., X^{\xi,\varepsilon}(t_k))$, which in turn, according to Theorem 2.2, converges to the law of $(\sigma(\xi)B(t_1), ..., \sigma(\xi)B(t_k))$, where B is a Brownian motion. We now let ξ tend to 0: since $\sigma(\xi)$ converges to σ, see Lemma 2.5, the limiting law of $(X^\varepsilon(t_1), ..., X^\varepsilon(t_k))$ is the law of $(\sigma B(t_1), ..., \sigma B(t_k))$ i.e. we have proved that X^ε converges in law to a Brownian motion with variance σ^2 in the sense of finite dimensional marginals. The tightness Lemma 3.2 implies that the convergence in fact holds in the Skorokhod topology. ■

4 Proof of Theorem 2.2

We will outline here a proof of Theorem 2.2. Our strategy is quite similar to the one recently used in [16], [2] and [15] to study the simple symmetric random walk on a percolation cluster. No new idea is required.

Step 0: notation

As before, we use the notation ω to denote a typical environment under the measure Q. For a given edge $e \in \mathbb{E}_d$ (and a given choice of ω), we define

$$\alpha(e) = \mathbf{1}_{\omega(e)>0}\,;\ \alpha'(e) = \mathbf{1}_{\omega(e)\ge\xi}\,.$$

As in part 2, let $\mathcal{C}^\xi(\omega)$ be the infinite cluster of the percolation graph α'. For $x, y \in \mathcal{C}^\xi(\omega)$, we define the *chemical distance* $d^\xi_\omega(x, y)$ as the minimal number of jumps required for the process X^ξ to go from x to y, see part 5.3.

We recall the definition of the generator $\mathcal{L}^{\xi,\omega}$ from formula (2.1). Since the function ω^ξ is symmetric, the operator $\mathcal{L}^{\xi,\omega}$ is reversible with respect to the measure $\mu_\omega = \sum_{z\in\mathcal{C}^\xi(\omega)} n^\omega(z)\delta_z$.

Let $\mathcal{C}^n(\omega)$ be the connected component of $\mathcal{C}^\xi(\omega) \cap [-n, n]^d$ that contains the origin. Let $(X^{\xi,n}(t),\ t \ge 0)$ be the random walk X^ξ restricted to the set $\mathcal{C}^n(\omega)$. The definition of $X^{\xi,n}$ is the same as for X^ξ except that jumps outside $\mathcal{C}^n$ are now forbidden. Its Dirichlet form is

$$\mathcal{E}^{\xi,\omega,n}(f, f) = \frac{1}{2}\sum_{x\sim y\in\mathcal{C}^n(\omega)} \omega^\xi(x, y)(f(x) - f(y))^2$$

We use the notation τ^n for the exit time of the process X^ξ from the box $[-2n + 1, 2n - 1]^d$ i.e. $\tau^n = \inf\{t\,;\ X^\xi(t) \notin [-2n + 1, 2n - 1]^d\}$.

Step 1: Carne-Varopoulos bound

The measure μ_ω being reversible for the process X^ξ, the transition probabilities satisfy a Carne-Varopoulos bound:

$$P^\omega_x(X^\xi(t)=y)\le Ce^{-d^\xi_\omega(x,y)^2/(4t)}+e^{-ct}\,,$$

where $c=\log 4-1$ and C is some constant that depends on ξ and ω. (See [16], appendix C.)

By Lemma 5.4, we can replace the chemical distance $d^\xi_\omega(x,y)$ by the Euclidean distance $|x-y|$, provided that $x\in[-n,n]^d$ and n is large enough. We get that, Q^ξ_0 almost surely, for large enough n, for any $x\in[-n,n]^d$ and any $y\in\mathbb{Z}^d$ such that $|x-y|\ge(\log n)^2$, then

$$P^\omega_x(X^\xi(t)=y)\le Ce^{-\frac{|x-y|^2}{Ct}}+e^{-ct}\,. \tag{4.1}$$

The same reasoning as in [16], appendix C (using Lemma 5.4 again) then leads to upper bounds for the exit time τ^n: Q^ξ_0 almost surely, for large enough n, for any $x\in[-n,n]^d$ and any t, we have

$$P^\omega_x[\tau^n\le t]\le Ctn^de^{-\frac{n^2}{Ct}}+e^{-ct}\,. \tag{4.2}$$

Indeed, let $N(t)$ be the number of jumps the random walk performs until time t and let σ^n be the number of jumps of the walk until it exits the box $[-2n+1,2n-1]^d$, so that $\sigma^n=N(\tau^n)$. Note that the process $(N(t)\,,\,t\in\mathbb{R}_+)$ is a Poisson process of rate 1. With probability larger than $1-e^{-ct}$, we have $N(t)\le 2t$. If $N(t)\le 2t$ and $\tau^n\le t$, then $\sigma^n\le 2t$ and there are at most $2t$ choices for the value of σ^n. Let y be the position of the walk at the exit time and let z be the last point visited before exiting. Note that $d^\xi_\omega(z,y)=1$. Due to Lemma 5.4, we have

$$|x-y|\le\frac{1}{c^-}d^\xi_\omega(x,y)\le\frac{1}{c^-}(d^\xi_\omega(x,z)+1)\le\frac{c^+}{c^-}(1+|x-z|)\le\frac{c^+}{c^-}(1+2n)\,.$$

Note that our use of Lemma 5.4 here is legitimate. Indeed $|x-y|$ is of order n and, since $d^\xi_\omega(z,y)=1$, Lemma 3.1 implies that $|y-z|$ is at most of order $(\log n)^7$. Therefore $|x-z|$ is of order n and thus certainly larger that $(\log n)^2$.

Thus we see that there are at most of order n^d possible choices for y. Finally, due to (4.1),

$$P^\omega_x(X^\xi(s)=y)\le Ce^{-\frac{n^2}{Ct}}\,,$$

for any $s\le 2t$, $x\in[-n,n]^d$ and $y\notin[-2n+1,2n-1]^d$. Putting everything together, we get (4.2).

Step 2: Nash inequalities and on-diagonal decay

Lemma 4.1 *For any $\theta>0$, there exists a constant $c_u(\theta)$ such that, Q^ξ_0 a.s. for large enough t, we have*

$$P^\omega_x[X^\xi(t)=y]\le\frac{c_u(\theta)}{t^{d/2}}\,, \tag{4.3}$$

for any $x\in\mathcal{C}^\xi(\omega)$ and $y\in\mathbb{Z}^d$ such that $|x|\le t^\theta$.

Proof:

We use the notation $\alpha'(e) = \mathbf{1}_{\omega(e)\geq\xi}$. Note that the random variables $(\alpha'(e)\,;\,e\in E_d)$ are independent Bernoulli variables with common parameter $Q(\alpha'(e)>0) = Q(\omega(e)\geq\xi)$. Since we have assumed that $Q(\omega(e)\geq\xi) > p_c$, the environment α' is a typical realization of super-critical bond percolation.

The following Nash inequality is proved in [16], equation (5): there exists a constant β such that Q_0^ξ a.s. for large enough n, for any function $f:\mathcal{C}^n(\omega)\to\mathbb{R}$ one has

$$\mathrm{Var}(f)^{1+\frac{2}{\varepsilon(n)}} \leq \beta\, n^{2(1-\frac{d}{\varepsilon(n)})}\,\mathcal{E}^{\alpha',n}(f,\,f)\,\|f\|_1^{4/\varepsilon(n)}\,,$$

where

$$\mathcal{E}^{\alpha',n}(f,f) = \frac{1}{2}\sum_{x\sim y\in\mathcal{C}^n(\omega)} \alpha'(x,y)(f(x)-f(y))^2\,.$$

The variance and the L_1 norms are computed with respect to the counting measure on $\mathcal{C}^n(\omega)$ and $\varepsilon(n) = d + 2d\frac{\log\log n}{\log n}$. (Note that there is a typo in [16] where it is claimed that (5) holds for the uniform probability on $\mathcal{C}^n(\omega)$ instead of the counting measure.)

Inequality (2.3) implies that $\alpha'(x,y)\leq\xi^{-1}\omega^\xi(x,y)$. Therefore $\mathcal{E}^{\alpha',n}$ and $\mathcal{E}^{\xi,\omega,n}$ satisfy the inequality

$$\mathcal{E}^{\alpha',n}(f,f) \leq \frac{1}{\xi}\,\mathcal{E}^{\xi,\omega,n}(f,f)\,. \tag{4.4}$$

Using inequality (4.4) in the previous Nash inequality, we deduce that there exists a constant β (that depends on ξ) such that Q_0^ξ a.s. for large enough n, for any function $f:\mathcal{C}^n(\omega)\to\mathbb{R}$ one has

$$\mathrm{Var}(f)^{1+\frac{2}{\varepsilon(n)}} \leq \beta\, n^{2(1-\frac{d}{\varepsilon(n)})}\,\mathcal{E}^{\xi,\omega,n}(f,\,f)\,\|f\|_1^{4/\varepsilon(n)}\,. \tag{4.5}$$

As shown in [16] part 4, the Carne-Varopoulos inequality (4.1), inequality (4.2) and the Nash inequality (4.5) can be combined to prove upper bounds on the transition probabilities. We thus obtain that: there exists a constant c_u such that, Q_0^ξ a.s. for large enough t, we have

$$P_0^\omega[X^\xi(t)=y] \leq \frac{c_u}{t^{d/2}}\,, \tag{4.6}$$

for any $y\in\mathbb{Z}^d$.

Using the translation invariance of Q, it is clear that estimate (4.6) in fact holds if we choose another point $x\in\mathbb{Z}^d$ to play the role of the origin. Thus, for any $x\in\mathbb{Z}^d$, Q a.s. on the set $x\in\mathcal{C}^\xi(\omega)$, for t larger than some random value $t_0(x)$, we have

$$P_x^\omega[X^\xi(t)=y] \leq \frac{c_u}{t^{d/2}}\,, \tag{4.7}$$

for any $y\in\mathbb{Z}^d$.

In order to deduce the Lemma from the upper bound (4.7), one needs control the tail of the law of $t_0(0)$.

Looking at the proofs in [16], one sees that all the error probabilities decay faster than any

polynomial. More precisely, the Q_0^ξ probability that inequality (4.5) fails for some $n \geq n_0$ decays faster than any polynomial in n_0. From the proof of Lemma 5.4, we also know that the Q_0^ξ probability that inequality (4.1) fails for some $n \geq n_0$ decays faster than any polynomial in n_0. As a consequence, a similar bound holds for inequality (4.2).
To deduce error bounds for (4.6), one then needs to go to part 4 of [16]. Since the proof of the upper bound (4.6) is deduced from (4.1), (4.2) and (4.5) by choosing $t \log t = bn^2$ for an appropriate constant b, we get that Q_0^ξ(inequality (4.6) fails for some $t \geq t_0$) decays faster than any polynomial in t_0. By translation invariance, the same holds for (4.7) i.e. for any $A > 0$, there exists T such that

$$Q(x \in \mathcal{C}^\xi(\omega) \,\mathrm{and}\, t_0(x) \geq t_0) \leq t_0^{-A} ,$$

for any $t_0 > T$. Therefore,

$$Q(\exists x \in \mathcal{C}^\xi(\omega)\,;\, |x| \leq t_0^\theta \,\mathrm{and}\, t_0(x) \geq t_0) \leq t_0^{d\theta - A} .$$

One then chooses A larger than $d\theta + 1$ and the Borel-Cantelli lemma gives the end of the proof of (4.3). ∎

Step 3: exit times estimates and tightness

We denote with $\tau(x, r)$ the exit time of the random walk from the ball of center x and Euclidean radius r.

Lemma 4.2 *For any $\theta > 0$, there exists a constant c_e such that, Q_0^ξ a.s. for large enough t, we have*

$$P_x^\omega[\tau(x, r) < t] \leq c_e \frac{\sqrt{t}}{r} , \tag{4.8}$$

for any $x \in \mathbb{Z}^d$ and r such that $|x| \leq t^\theta$ and $r \leq t^\theta$.

Proof: the argument is the same as in [2], part 3. We define

$$M_x(t) = E_x^\omega[d_\omega^\xi(x, X^\xi(t))]$$

and

$$Q_x(t) = -E_x^\omega[\log q_t^\omega(x, X^\xi(t))] ,$$

where $q_t^\omega(x, y) = P_x^\omega(X^\xi(t) = y)/\mu_\omega(x)$. Then, for large enough t and for $|x| \leq t^\theta$, one has:

$$\begin{aligned} Q_x(t) &\geq -\log c_u + \frac{d}{2} \log t , \\ M_x(t) &\geq c_2 \exp(Q_x(t)/d) , \\ Q_x'(t) &\geq \frac{1}{2}(M_x'(t))^2 . \end{aligned}$$

The first inequality is obtained as an immediate consequence of Lemma 4.1. The second one is proved as in [2], Lemma 3.3 and the third one as in [2], equation (3.10), using ideas from [3] and [17]. Note that, in the proof of the second inequality, we used Lemma 5.4 to control the volume growth in the chemical distance d^ξ_ω. One now integrates these inequalities to deduce that

$$c_1\sqrt{t} \le M_x(t) \le c_2\sqrt{t}\,. \tag{4.9}$$

Once again the proof is the same as in [2], Proposition 3.4. Note that, in the notation of [2], $T_B = |x|^{1/\theta}$ so that equation (4.9) holds for $t \ge \frac{1}{\theta}|x|^{1/\theta}\log|x|$. The end of the proof is identical to the proof of Equation (3.13) in [2]. ■

Lemma 4.3 *Q^ξ_0 a.s. for large enough t, we have*

$$P^\omega_x[\tau(x,r) < t] \le 27(c_e)^3(\frac{\sqrt{t}}{r})^3\,, \tag{4.10}$$

for any $x \in \mathbb{Z}^d$ and r such that $|x| \le t^\theta$ and $r \le t^\theta$.

Proof: let $x' = X^\xi(\tau(x,r/3))$, $x'' = X^\xi(\tau'(x',r/3))$ where $\tau'(x',r/3)$ is the exit time from the ball of center x' and radius $r/3$ after time $\tau(x,r/3)$ and let $\tau''(x'',r/3)$ be the exit time from the ball of center x'' and radius $r/3$ after time $\tau'(x,r/3)$. In order that $\tau(x,r) < t$ under P^ω_x we must have $\tau(x,r/3) < t$ and $\tau'(x',r/3) < t$ and $\tau''(x'',r/3) < t$. We can then use Lemma 4.2 to estimate the probabilities of these 3 events and conclude that (4.10) holds. ■

Lemma 4.4 *For small enough ξ, Q_0 almost surely, under P^ω_0, the family of processes $(X^{\xi,\varepsilon}(t) = \varepsilon X^\xi(\frac{t}{\varepsilon^2}), t \in \mathbb{R}_+)$ is tight in the Skorokhod topology (as ε goes to 0).*

Proof: we shall prove that, for any $T > 0$, for any $\eta > 0$ and for small enough θ_0 then

$$\limsup_\varepsilon \sup_{v\le T} P^\omega_0(\sup_{u\le T\,;\, v\le u\le v+\theta_0} |X^{\xi,\varepsilon}(u) - X^{\xi,\varepsilon}(v)| > \eta) \le 27(c_e)^3(\frac{\sqrt{\theta_0}}{\eta})^3\,. \tag{4.11}$$

Indeed inequality (4.11) implies that

$$\limsup_{\theta_0} \frac{1}{\theta_0} \limsup_\varepsilon \sup_{v\le T} P^\omega_0(\sup_{u\le T\,;\, v\le u\le v+\theta_0} |X^{\xi,\varepsilon}(u) - X^{\xi,\varepsilon}(v)| > \eta) = 0\,. \tag{4.12}$$

According to Theorem 8.3 in Billingsley's book [6], this last inequality is sufficient to ensure the tightness.

We use Lemma 4.2 with $\theta = 1$ to check that

$$P^\omega_0(\sup_{t\le T} |X^{\xi,\varepsilon}(t)| \ge K) = P^\omega_0(\tau(0,\frac{K}{\varepsilon}) \le \frac{T}{\varepsilon^2}) \le c_e\frac{\sqrt{T}}{K}\,.$$

(We could use Lemma 4.2 since $\frac{K}{\varepsilon} \le \frac{T}{\varepsilon^2}$ for small ε.)

Next choose $\eta > 0$ and use Lemma 4.3 with $\theta = 3$ and the Markov property to get that

$$P^\omega_0(\sup_{v\le u\le T\,;\, u-v\le\theta_0} |X^{\xi,\varepsilon}(u) - X^{\xi,\varepsilon}(v)| > \eta) \le P^\omega_0(\sup_{t\le T} |X^{\xi,\varepsilon}(t)| \ge K) + \sup_{y\,;\,|y|\le K/\varepsilon} P^\omega_y(\tau(y,\frac{\eta}{\varepsilon}) \le \frac{\theta_0}{\varepsilon^2})\,.$$

If we choose K of order $1/\varepsilon$ and pass to the limit as ε tends to 0, then, due to the previous inequality, the contribution of the first term vanishes. As for the second term, by Lemma 4.3, it is bounded by $27(c_e)^3(\frac{\sqrt{\theta_0}}{\eta})^3$. Note that we could use Lemma 4.3 since $\frac{K}{\varepsilon} \leq (\frac{\theta_0}{\varepsilon^2})^3$ and $\frac{\eta}{\varepsilon} \leq (\frac{\theta_0}{\varepsilon^2})^3$ for small ε. Thus the proof of (4.11) is complete. ∎

Step 4: Poincaré inequalities and end of the proof of Theorem 2.2
Applied to a centered function f, Nash inequality (4.5) reads:

$$\|f\|_2^{2+\frac{4}{\varepsilon(n)}} \leq \beta\, n^{2(1-\frac{d}{\varepsilon(n)})}\, \mathcal{E}^{\xi,\,\omega,\,n}(f,\, f)\, \|f\|_1^{4/\varepsilon(n)}\,.$$

Holder's inequality implies that

$$\|f\|_1 \leq \|f\|_2(2n+1)^{d/2}$$

since $\# C^n(\omega) \leq (2n+1)^d$. We deduce that any centered function on $\mathcal{C}^n(\omega)$ satisfies

$$\|f\|_2^2 \leq \beta n^2\, \mathcal{E}^{\xi,\,\omega,\,n}(f,\, f)\,,$$

for some constant β. Equivalently, any (not necessarily centered) function on $\mathcal{C}^n(\omega)$ satisfies

$$\mathrm{Var}(f) \leq \beta n^2\, \mathcal{E}^{\xi,\,\omega,\,n}(f,\, f)\,.$$

Thus we have proved the following Poincaré inequality on $C^n(\omega)$: there is a constant β such that, Q_0^ξ.a.s. for large enough n, for any function $f : \mathcal{C}^n(\omega) \to \mathbb{R}$ then

$$\sum_{x\in\mathcal{C}^n(\omega)} f(x)^2 \leq \beta n^2 \sum_{x\sim y\in\mathcal{C}^n(\omega)} \omega^\xi(x,y)(f(x)-f(y))^2 \tag{4.13}$$

Our second Poincaré inequality is derived from [2], see Definition 1.7, Theorem 2.18, Lemma 2.13 part a) and Proposition 2.17 part b): there exist constants $M < 1$ and β such that Q_0^ξ.a.s. for any $\delta > 0$, for large enough n, for any $z \in \mathbb{Z}^d$ s.t. $|z| \leq n$ and for any function $f : \mathbb{Z}^d \to \mathbb{R}$ then

$$\sum_{x\in\mathcal{C}^\xi(\omega)\cap(z+[-M\delta n, M\delta n]^d)} f(x)^2 \leq \beta\delta^2 n^2 \sum_{x\sim y\in\mathcal{C}^\xi(\omega)\cap(z+[-\delta n,\delta n]^d)} \omega^\xi(x,y)(f(x)-f(y))^2 \tag{4.14}$$

In [2], inequality (4.14) is in fact proved for the Dirichlet form $\mathcal{E}^{\alpha',n}$ but the comparison inequality (4.4) implies that it also holds for the Dirichlet form $\mathcal{E}^{\xi,\,\omega,\,n}$.

One can now conclude the proof of the Theorem following the argument in [15] line by line starting from paragraph 2.2. ∎

5 Percolation results

5.1 Prerequisites on site percolation

We shall use some properties of site percolation that we state below.

By site percolation of parameter r on $\mathbb{Z}^d$, we mean the product Bernoulli measure of parameter r on the set of applications $\zeta : \mathbb{Z}^d \to \{0, 1\}$. We identify any such application with the sub-graph of the grid whose vertices are the points $x \in \mathbb{Z}^d$ such that $\zeta(x) = 1$ and equipped with the edges of the grid linking two points x, y such that $\zeta(x) = \zeta(y) = 1$.

Let $l > 1$. Call a sub-set of $\mathbb{Z}^d$ *l-connected* if it is connected for the graph structure defined by: two points are neighbors when the Euclidean distance between them is less than l.

We recall our notation $|x - y|$ for the Euclidean distance between x and y.

A *path* is a sequence of vertices of $\mathbb{Z}^d$ such that two successive vertices in π are neighbors. We mostly consider injective paths. With some abuse of vocabulary, a sequence of vertices of $\mathbb{Z}^d$ in which two successive vertices are at distance not more than l will be called a *l-nearest-neighbor path*. Let $\pi = (x_0, ..., x_k)$ be a sequence of vertices. We define its length

$$|\pi| = \sum_{j=1}^{k} |x_{j-1} - x_j| ,$$

and its cardinality $\#\pi = \#\{x_0, ..., x_k\}$. ($\#\pi = k + 1$ for an injective path.) When convenient, we identify an injective path with a set (its range).

Lemma 5.1 *Let $l > 1$. There exists $p_1 > 0$ such that for $r < p_1$, almost any realization of site percolation of parameter r has only finite l-connected components and, for large enough n, any l-connected component that intersects the box $[-n, n]^d$ has volume smaller than $(\log n)^{6/5}$.*

Proof: the number of l-connected sets that contain a fixed vertex and of volume m is smaller than $e^{a(l)m}$ for some constant $a(l)$, see [12]. Thus the number of l-connected sets of volume m that intersect the box $[-n, n]^d$ is smaller than $(2n+1)^d e^{a(l)m}$. But the probability that a given set of volume m contains only opened sites is $r^m \le p_1^m$. We now choose p_1 small enough so that $\sum_n \sum_{m \ge (\log n)^{6/5}} (2n + 1)^d e^{a(l)m} p_1^m < \infty$ and the Borel-Cantelli lemma yields the conclusion of Lemma 5.1. ∎

As in the case of bond percolation discussed in the introduction, it is well known that for r larger than some critical value then almost any realization of site percolation of parameter r has a unique infinite connected component - the *infinite cluster* - that we will denote with $\mathcal{C}$.

Lemma 5.2 *There exists $p_2 < 1$ such that for $r > p_2$, for almost any realization of site percolation of parameter r and for large enough n, any connected component of the complement of the infinite cluster $\mathcal{C}$ that intersects the box $[-n, n]^d$ has volume smaller than $(\log n)^{5/2}$.*

Proof: let ζ be a typical realization of site percolation of parameter r. We assume that r is above the critical value so that there is a unique infinite cluster, $\mathcal{C}$. We also assume that $1 - r < p_1$ where p_1 is the value provided by Lemma 5.1 for $l = d$.

Let A be a connected component of the complement of $\mathcal{C}$. Define the *interior boundary of A*: $\partial_{int} A = \{x \in A\, ; \exists y\, s.t.\, (x, y) \in \mathbb{E}_d \text{ and } y \notin A\}$. It is known that $\partial_{int} A$ is d-connected, see [9], Lemma 2.1. By construction any $x \in \partial_{int} A$ satisfies $\zeta(x) = 0$. Since the application $x \to 1 - \zeta(x)$ is a typical realization of site percolation of parameter $1 - r$ and $1 - r < p_1$, as an application of Lemma 5.1 we get that $\partial_{int} A$ is finite. Because we already know that the complement of A is infinite (since it contains $\mathcal{C}$), it implies that A itself is finite.

We now assume that A intersects the box $[-n,n]^d$. Choose n large enough so that $\mathcal{C} \cap [-n,n]^d \neq \emptyset$ so that $[-n,n]^d$ is not a sub-set of A. Then it must be that $\partial_{int} A$ intersects $[-n,n]^d$. Applying Lemma 5.1 again, we get that, for large n, the volume of $\partial_{int} A$ is smaller than $(\log n)^{6/5}$. The classical isoperimetric inequality in $\mathbb{Z}^d$ implies that, for any finite connected set B, one has $(\#\partial_{int} B)^{d/(d-1)} \geq \mathcal{I}\#B$ for some constant $\mathcal{I}$. Therefore $\#A \leq \mathcal{I}^{-1}(\log n)^{6d/5(d-1)}$. Since $6d/5(d-1) < 5/2$, the proof is complete. ■

Lemma 5.3 *There exists $p_3 < 1$ and a constant c_3 such that for $r > p_3$, for almost any realization of site percolation of parameter r and for large enough n, for any two points x, y in the box $[-n,n]^d$ such that $|x-y| \geq (\log n)^{3/2}$ we have*
(i) for any injective d-nearest-neighbor path π from x to y then

$$\#\{z \in \pi\,;\, \zeta(z) = 1\} \geq c_3|x-y|\,.$$

(ii) for any injective (1-nearest-neighbor) path π from x to y then

$$\#(\mathcal{C} \cap \pi) \geq c_3|x-y|\,.$$

Proof: we assume that r is close enough to 1 so that there is a unique infinite cluster $\mathcal{C}$. We also assume that $1-r < p_1$, where p_1 is the constant appearing in Lemma 5.1 for $l = 1$. Then the complement of $\mathcal{C}$ only has finite connected components.

Part (i) of the Lemma is proved by a classical Borel-Cantelli argument based on the following simple observations: the number of injective d-nearest-neighbor paths π from x of length L is bounded by $(c_d)^L$ for some constant c_d that depends on the dimension d only; the probability that a given set of cardinality L contains less than dc_3L sites where $\zeta = 1$ is bounded by $exp(\lambda dc_3 L)(re^{-\lambda} + 1 - r)^L$ for all $\lambda > 0$. We choose $c_3 < \frac{1}{d}$ and λ such that $c_d e^{-(1-dc_3)\lambda} < 1$ and p_3 such that $\gamma = c_d e^{\lambda dc_3}(p_3 e^{-\lambda} + 1 - p_3) < 1$. Let now x and y be as in the Lemma. Note that any injective d-nearest-neighbor path π from x to y satisfies $\#\pi \geq \frac{1}{d}|x-y| \geq \frac{1}{d}(\log n)^{3/2}$. Therefore the probability that there is an injective d-nearest-neighbor path π from x to y such that $\#\{z \in \pi\,;\, \zeta(z) = 1\} < c_3|x-y|$ is smaller than $\sum_{L \geq \frac{1}{d}(\log n)^{3/2}} \gamma^L$ and the probability that (i) fails for some x and y is smaller than $(2n+1)^{2d} \sum_{L \geq \frac{1}{d}(\log n)^{3/2}} \gamma^L$. Since $\sum_n (2n+1)^{2d} \sum_{L \geq \frac{1}{d}(\log n)^{3/2}} \gamma^L < \infty$, the Borel-Cantelli lemma then yields that, for large enough n, part (i) of Lemma 5.3 holds.

We prove part (ii) by reducing it to an application of part (i). Assume that, for some points x and y as in the Lemma, there exists an injective nearest-neighbor path π from x to y such that $\#(\mathcal{C} \cap \pi) < c_3|x-y|$. We first modify the path π into a d-nearest-neighbor path from x to y, say π', in the following way: the parts of π that lie in $\mathcal{C}$ remain unchanged but the parts of π that visit the complement of $\mathcal{C}$ are modified so that they only visit points where $\zeta = 0$. Such a modified path π' exists because the interior boundary of a connected component of the complement of $\mathcal{C}$ is d connected (as we already mentioned in the proof of Lemma 5.2) and only contains points where $\zeta = 0$.

Observe that $\mathcal{C} \cap \pi' = \mathcal{C} \cap \pi$ and that $\mathcal{C} \cap \pi' = \{z \in \pi'\,;\, \zeta(z) = 1\}$ so that

$$\#\{z \in \pi'\,;\, \zeta(z) = 1\} < c_3|x-y|\,.$$

Next turn π' into an injective d-nearest-neighbor path, say π'', by suppressing loops in π'. Clearly $\{z \in \pi''\,;\, \zeta(z) = 1\} \subset \{z \in \pi'\,;\, \zeta(z) = 1\}$ and therefore

$$\#\{z \in \pi''\,;\, \zeta(z) = 1\} < c_3|x-y|\,,$$

a contradiction with part (i) of the Lemma. ■

5.2 Proof of Lemma 3.1

Lemma 3.1 only deals with the geometry of percolation clusters, with no reference to random walks. We will restate it as a percolation lemma at the cost of changing a little our notation. In order to make a distinction with a typical realization of an environment for which we used the notation ω, we will use the letters α or α' to denote typical realizations of a percolation graphs. Thus one switches from the notation of the following proof back to the notation of part 3 using the following dictionary:

$$\alpha(e) = \mathbf{1}_{\omega(e)>0} \quad ; \quad \alpha'(e) = \mathbf{1}_{\omega(e)\geq\xi}$$
$$q = Q(\omega(e) > 0) \quad ; \quad p = Q(\omega(e) \geq \xi \,|\, \omega(e) > 0)\,.$$

This way taking ξ close to 0 is equivalent to taking p close to 1.

We very much rely on renormalization technics, see Proposition 2.1. in [1].

As in the introduction, we identify a sub-graph of $\mathbb{Z}^d$ with an application $\alpha : \mathbb{E}_d \to \{0,1\}$, writing $\alpha(x,y) = 1$ if the edge (x,y) is present in α and $\alpha(x,y) = 0$ otherwise. Thus $\mathcal{A} = \{0,1\}^{\mathbb{E}_d}$ is identified with the set of sub-graphs of $\mathbb{Z}^d$. Edges pertaining to α are then called *open*. Connected components of such a sub-graph will be called *clusters*.

Define now Q to be the probability measure on $\{0,1\}^{\mathbb{E}_d}$ under which the random variables $(\alpha(e),\, e \in \mathbb{E}_d)$ are Bernoulli(q) independent variables with

$$q > p_c.$$

Then, Q almost surely, the graph α has a unique infinite cluster denoted with $\mathcal{C}(\alpha)$.

For a typical realization of the percolation graph under Q, say α, let Q^α be the law of bond percolation on $\mathcal{C}(\alpha)$ with parameter p. We shall denote α' a typical realization under Q^α i.e. α' is a random subgraph of $\mathcal{C}(\alpha)$ obtained by keeping (resp. deleting) edges with probability p independently of each other. We always assume that p is close enough to 1 so that Q^α almost surely there is a unique infinite cluster in α' that we denote $\mathcal{C}^\alpha(\alpha')$. By construction $\mathcal{C}^\alpha(\alpha') \subset \mathcal{C}(\alpha)$. Connected components of the complement of $\mathcal{C}^\alpha(\alpha')$ in $\mathcal{C}(\alpha)$ are called *holes*.

We now restate Lemma 3.1:

there exists $p_0 < 1$ such that for $p > p_0$, for Q almost any α, for Q^α almost any α', for large enough n, then any hole intersecting the box $[-n,n]^d$ has volume smaller than $(\log n)^a$.

Renormalization: let α be a typical realization of percolation under Q.

Let N be an integer. We chop $\mathbb{Z}^d$ in a disjoint union of boxes of side length $2N+1$. Say $\mathbb{Z}^d = \cup_{\mathbf{i}\in\mathbb{Z}^d} B_{\mathbf{i}}$, where $B_{\mathbf{i}}$ is the box of center $(2N+1)\mathbf{i}$. Following [1], let $B'_{\mathbf{i}}$ be the box of center $(2N+1)\mathbf{i}$ and side length $\frac{5}{2}N + 1$. From now on, the word *box* will mean one of the boxes $B_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d$.

We say that a box $B_{\mathbf{i}}$ is *white* if $B_{\mathbf{i}}$ contains at least one edge from α and the event $R_{\mathbf{i}}^{(N)}$ in equation (2.9) of [1] is satisfied. Otherwise, $B_{\mathbf{i}}$ is a *black* box. We recall that the event $R_{\mathbf{i}}^{(N)}$ is defined by: there is a unique cluster of α in $B'_{\mathbf{i}}$, say $K_{\mathbf{i}}$; all open paths contained in $B'_{\mathbf{i}}$ and of radius larger than $\frac{1}{10}N$ intersect $K_{\mathbf{i}}$ within $B'_{\mathbf{i}}$; $K_{\mathbf{i}}$ is crossing for each subbox $B \subset B'_{\mathbf{i}}$ of side larger than $\frac{1}{10}N$. See [1] for details. We call $K_{\mathbf{i}}$ the *crossing cluster* of α in the box $B_{\mathbf{i}}$. Note the following consequences of this definition.

(Fact i) If x and y belong to the same white box $B_{\mathbf{i}}$ and both x and y belong to the infinite cluster of α, then there is a path in $\mathcal{C}(\alpha)$ connecting x and y within $B'_{\mathbf{i}}$.

(Fact ii) Choose two neighboring indices $\mathbf{i}$ and $\mathbf{j}$ with $|\mathbf{i}-\mathbf{j}| = 1$ and such that both boxes $B_{\mathbf{i}}$ and $B_{\mathbf{j}}$ are white. As before, let $K_{\mathbf{i}}$ and $K_{\mathbf{j}}$ be the crossing clusters in $B_{\mathbf{i}}$ and $B_{\mathbf{j}}$ respectively. Let $x \in K_{\mathbf{i}}$ and $y \in K_{\mathbf{j}}$. Then there exists a path in α connecting x and y within $B'_{\mathbf{i}} \cup B'_{\mathbf{j}}$.

We call *renormalized* process the random subsets of $\mathbb{Z}^d$ obtained by taking the image of the initial percolation model by the application ϕ_N, see equation (2.11) in [1]. A site $\mathbf{i} \in \mathbb{Z}^d$ is thus declared *white* if the box $B_{\mathbf{i}}$ is white.

Let $\mathbf{Q}$ be the law of the renormalized process. The comparison result of Proposition 2.1 in [1] states that $\mathbf{Q}$ stochastically dominates the law of site percolation with parameter $p(N)$ with $p(N) \to 1$ as N tends to ∞.

We now introduce the extra percolation Q^α. Let us call *grey* a white box $B_{\mathbf{i}}$ that contains an edge $e \in \mathcal{C}(\alpha)$ such that $\alpha'(e) = 0$. We call *pure white* white boxes that are not grey.

Let $\mathbf{Q}'$ be the law on subsets of the renormalized grid obtained by keeping pure white boxes, and deleting both black and grey boxes. We claim that $\mathbf{Q}'$ dominates the law of site percolation with parameter $p'(N) = p(N)p^{e_N(d)}$ where $e_N(d)$ is the number of edges in a box of side length $2N+1$. (Remember that p is the parameter of Q^α.) This claim is a consequence of the three following facts. We already indicated that $\mathbf{Q}$ stochastically dominates the law of site percolation with parameter $p(N)$. The conditional probability that a box $B_{\mathbf{i}}$ is pure white given it is white is larger or equal than $p^{e_N(d)}$. Besides, still under the condition that $B_{\mathbf{i}}$ is white, the event '$B_{\mathbf{i}}$ is pure white' is independent of the colors of the other boxes.

We further call *immaculate* a pure white box $B_{\mathbf{i}}$ such that any box $B_{\mathbf{j}}$ intersecting $B'_{\mathbf{i}}$ is also pure white. Call $\mathbf{Q}''$ the law on subsets of the renormalized grid obtained by keeping only immaculate boxes. Since the event '$B_{\mathbf{i}}$ is immaculate' is an increasing function with respect to the percolation process of pure white boxes, we get that $\mathbf{Q}''$ stochastically dominates the law of site percolation with parameter $p''(N) = p'(N)^{3^d}$.

End of the proof of Lemma 3.1: choose p_0 and N such that $p''(N)$ is close enough to 1 so that, $\mathbf{Q}''$ almost surely, there is an infinite cluster of immaculate boxes that we call $\mathbb{C}$.

For $\mathbf{i} \in \mathbb{C}$, let $K_{\mathbf{i}}$ be the crossing cluster in the box $B_{\mathbf{i}}$ and let $K = \cup_{\mathbf{i}\in\mathbb{C}} K_{\mathbf{i}}$. Then K is connected (This follows from the definition of white boxes, see (Fact i) and (Fact ii) above.) and infinite (Because $\mathbb{C}$ is infinite.). Thus we have $K \subset \mathcal{C}^\alpha(\alpha')$.

Let A be a hole and let $\mathbf{A}$ be the set of indices $\mathbf{i}$ such that $B_{\mathbf{i}}$ intersects A. Observe that $\mathbf{A}$ is connected. We claim that

$$\mathbf{A} \cap \mathbb{C} = \emptyset\,.$$

Indeed, assume there exists $x \in B_{\mathbf{i}}$ such that $\mathbf{i} \in \mathbb{C}$ and $x \in A$. By definition A is a subset of $\mathcal{C}(\alpha)$ and therefore $x \in \mathcal{C}(\alpha)$. Let $y \in K_{\mathbf{i}}$, $y \neq x$. As we already noted $y \in \mathcal{C}^\alpha(\alpha')$. Since $x \in \mathcal{C}(\alpha)$ and $y \in \mathcal{C}(\alpha)$ there is a path, π, connecting x and y within $B'_{\mathbf{i}}$, see (Fact i) above. But $B_{\mathbf{i}}$ is immaculate and therefore $B'_{\mathbf{i}}$ only contains edges e with $\alpha'(e) = 1$. Therefore all edges along the path π belong to α' which imply that $x \in \mathcal{C}^\alpha(\alpha')$. This is in contradiction with the assumptions that $x \in A$. We have proved that $\mathbf{A} \cap \mathbb{C} = \emptyset$.

To conclude the proof of Lemma 3.1, it only remains to choose p_0 and N such that $p''(N) \geq p_2$ and apply Lemma 5.2. We deduce that the volume of $\mathbf{A}$ is bounded by $(\log n)^{5/2}$ and therefore the volume of A is smaller than $(2N+1)^d(\log n)^{5/2}$. ■

5.3 Deviation of the chemical distance

We use the same notation as in the preceeding section. For given realizations of the percolations α and α', we define the corresponding *chemical distance* $d^{\alpha}_{\alpha'}$ on $\mathcal{C}^{\alpha}(\alpha')$: two points $x \neq y$ in $\mathcal{C}^{\alpha}(\alpha')$ satisfy $d^{\alpha}_{\alpha'}(x,y) = 1$ if and only if one (at least) of the following two conditions is satisfied: either x and y are neighbors in $\mathbb{Z}^d$ and $\alpha'(x,y) = 1$ or both x and y are at the boundary of a hole h i.e. there is a hole h and $x', y' \in h$ such that x' is a neighbor of x and y' is a neighbor of y. In general, $d^{\alpha}_{\alpha'}(x,y)$ is defined as the smaller integer k such that there exists a sequence of points $x_0, ..., x_k$ in $\mathcal{C}^{\alpha}(\alpha')$ with $x_0 = x$, $x_k = y$ and such that $d^{\alpha}_{\alpha'}(x_j, x_{j+1}) = 1$ for all j.

Lemma 5.4 *There exists $p_4 < 1$ such that for $p > p_4$, there exist constants c^+ and c^- such that for Q almost any α, for Q^{α} almost any α', for large enough n, then*

$$c^-|x-y| \leq d^{\alpha}_{\alpha'}(x,y) \leq c^+|x-y|\,, \tag{5.1}$$

for any $x, y \in \mathcal{C}^{\alpha}(\alpha')$ such that $x \in [-n,n]^d$ and $|x-y| \geq (\log n)^2$.

Proof: let $d^{\alpha}(x,y)$ be the chemical distance between x and y within $\mathcal{C}(\alpha)$ i.e. $d^{\alpha}(x,y)$ is the minimal length of a path from x to y, say π, such that any edge $e \in \pi$ satisfies $\alpha(e) = 1$. Applying Theorem 1.1 in [1] together with the Borel-Cantelli Lemma, we deduce that there exists a constant c^+ such that $d^{\alpha}(x,y) \leq c^+|x-y|$ for any $x, y \in \mathcal{C}(\alpha)$ such that $x \in [-n,n]^d$ and $|x-y| \geq (\log n)^2$. Since $d^{\alpha}_{\alpha'}(x,y) \leq d^{\alpha}(x,y)$, it gives the upper bound in (5.1).

We now give a proof of the lower bound. As for Lemma 3.1, we use a renormalization argument. The notation used below is borrowed from the proof of Lemma 3.1 except that the role of p_0 is now played by p_4. .

We wish to be able to apply Lemma 5.3 (ii) to the renormalized site percolation model with law $\mathbf{Q}''$ (i.e. the percolation model of immaculate boxes): therefore we choose p_4 and N such that $p''(N) \geq p_3$ and observe that the event considered in Lemma 5.3 (ii) is increasing.

Consider two points x and y as in Lemma 5.4 and let π be an injective path from x to y within $\mathcal{C}(\alpha)$. We shall prove that

$$\#\mathcal{E}_{\pi} \geq c_5|x-y|\,, \tag{5.2}$$

where $\mathcal{E}_{\pi} = \{z, z' \in \pi \cap \mathcal{C}^{\alpha}(\alpha')\,;\, \alpha'(z,z') = 1\}$. By construction of the chemical distance $d^{\alpha}_{\alpha'}$, (5.2) implies the lower bound in (5.1) with $c^- = c_5$.

Let Π' be the sequence of the indices of the boxes $B_{\mathbf{i}}$ that π intersects. At the level of the renormalized grid, Π' is a nearest-neighbor path from $\mathbf{i}_0$ to $\mathbf{i}_k$ with $x \in B_{\mathbf{i}_0}$ and $y \in B_{\mathbf{i}_k}$. Let $\Pi = (\mathbf{i}_0, ..., \mathbf{i}_k)$ be the injective path obtained by suppressing loops in Π'. We may, and will, assume that n is large enough so that $i_0 \neq i_k$ so that $|\mathbf{i}_0 - \mathbf{i}_k|$ and $|x-y|$ are comparable. Applying Lemma 5.3 (ii) to $\mathbf{Q}''$, we get that

$$\#(\mathbb{C} \cap \Pi) \geq c_3|\mathbf{i}_0 - \mathbf{i}_k| \geq c_3'|x-y|\,, \tag{5.3}$$

for some constant c_3'.

Let $\mathbf{i} \in \mathbb{C} \cap \Pi$ and choose $z \in B_{\mathbf{i}} \cap \pi$. Since the path π is not entirely contained in one box, it must be that π connects z to some point $z' \notin B_{\mathbf{i}}$. Since $z' \in \pi$, we also have $z' \in \mathcal{C}(\alpha)$. By definition of a white box, it implies that $z \in K_{\mathbf{i}}$. Since $\mathbf{i} \in \mathbb{C}$, it implies that actually $z \in K$ and therefore $z \in \mathcal{C}^\alpha(\alpha')$. As a matter of fact, since the box $B_{\mathbf{i}}$ is pure white, we must have $\alpha' = 1$ on all the edges of π from z to z'. In particular z has a neighbor in $\mathcal{C}(\alpha)$, say z'', such that $\alpha'(z, z'') = 1$. Therefore $(z, z'') \in \mathcal{E}_\pi$. We conclude that any indice in $\mathbb{C} \cap \Pi$ gives a contribution of at least 1 to $\#\mathcal{E}_\pi$. Therefore (5.3) implies that

$$\#\mathcal{E}_\pi \geq c_3' |x - y| \,.$$

■

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