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QUENCHING CRITERIA FOR A PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE IN AN INFINITE STRIP

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Abstract. This article studies a semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source in an N-dimensional infinite strip. Criteria for the solution to quench are given.

1. Introduction. A first initial-boundary value parabolic quenching problem due to a concentrated nonlinear source in one spatial dimension was studied by Chan and Jiang [1]. It was shown that the larger the domain, the larger the solution. This implies that quenching occurs if the domain is sufficiently large. Chan and Tragoonsirisak [3] investigated a multi-dimensional version in the N-dimensional Euclidean space $\mathbb{R}^N$. They showed that for this unbounded spatial domain, the solution always quenches in a finite time for $N \leq 2$; for $N \geq 3$, they proved that there exists a unique critical number α^* (corresponding physically to the strength of the source) such that the solution exists globally for $\alpha \leq \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$. Instead of $\mathbb{R}^N$, here we would like to consider the spatial domain to be an infinite strip, which is unbounded. Since we show later that such a problem is equivalent to a one-dimensional mixed-boundary value problem, our results for an infinite strip with $N \geq 2$ are also valid for the corresponding problem with $N = 1$. We prove that for any dimension $N\,(\geq 1)$, the solution can still exist globally. Quenching criteria, involving the strength of the source, the width of the strip and the location of the concentrated nonlinear source, are given. Our result that for $N \geq 1$, u exists globally for $\alpha \leq \alpha^*$ is different from that when the domain is $\mathbb{R}^N$, since in the latter case for $N \leq 2$, u always quenches for any α.

Let a point $(x_1, x_2, \ldots, x_{N-1}, x_N)$ in the N-dimensional Euclidean space $\mathbb{R}^N$ be denoted by $(x, \tilde{x})$, where x stands for x_1. Let L and b be positive numbers such that $b < L$,

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$S = (-L, L) \times \mathbb{R}^{N-1}$, $s = (-b, b) \times \mathbb{R}^{N-1}$, $\partial S = \left\{(x, \tilde{x}) : x \in \{-L, L\}, \text{ and } \tilde{x} \in \mathbb{R}^{N-1}\right\}$, $\partial s = \left\{(x, \tilde{x}) : x \in \{-b, b\}, \text{ and } \tilde{x} \in \mathbb{R}^{N-1}\right\}$, $\nu(x, \tilde{x})$ denote the unit outward normal at $(x, \tilde{x}) \in \partial s$, and $\chi_s(x, \tilde{x})$ denote a function which is 1 for $|x| > b$ and 0 for $|x| < b$. Since the Dirac delta function is the derivative of the Heaviside function, it follows that $\partial \chi_s(x, \tilde{x}) / \partial \nu$ gives a Dirac delta function at each point on $x = |b|$ and is zero everywhere else (cf. Chan and Tragoonsirisak [3]). Hence, we have a concentrated source on ∂s. Recently, Chan and Tragoonsirisak [5] studied the following problem with a concentrated nonlinear source on ∂s:

$$\left.\begin{array}{c} u_t - \triangle u = \alpha \dfrac{\partial \chi_s(x, \tilde{x})}{\partial \nu} f(u) \text{ in } S \times (0, T], \\ u(x, 0) = 0 \text{ on } \bar{S},\ u(x, t) = 0 \text{ on } \partial S \times (0, T], \end{array}\right\} \tag{1.1}$$

where α and T are positive numbers, $\bar{S}$ is the closure of S, f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \le u < c$. Let $H = \partial/\partial t - \partial^2/\partial x^2$, $D = (0, L)$, $\bar{D} = [0, L]$, and $\Omega = D \times (0, T]$. Using symmetry, problem (1.1) is equivalent to the one-dimensional problem

$$\left.\begin{array}{c} Hu = \alpha \delta(x - b) f(u) \text{ in } \Omega, \\ u(x, 0) = 0 \text{ on } \bar{D},\ u_x(0, t) = u(L, t) = 0 \text{ for } 0 < t \le T], \end{array}\right\} \tag{1.2}$$

where $\delta(x - b)$ is the Dirac delta function. A solution u is said to quench if there exists an extended real number $t_q \in (0, \infty]$ such that

$$\sup\left\{u(x, t) : x \in \bar{D}\right\} \to c^- \text{ as } t \to t_q.$$

If $t_q < \infty$, then u is said to quench in a finite time. If $t_q = \infty$, then u quenches in infinite time. They proved existence, uniqueness, and locations where quenching occurs for the solution. For ease of reference, let us summarize their main results in the following theorem. Green's function $g(x, t; \xi, \tau)$ corresponding to problem (1.2) is given by

$$g(x, t; \xi, \tau) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\cos \frac{(2n-1)\pi x}{2L}\right) \left(\cos \frac{(2n-1)\pi \xi}{2L}\right) \exp\left(-\frac{(2n-1)^2 \pi^2 (t - \tau)}{4L^2}\right)$$

(cf. Chan and Tragoonsirisak [5]).

THEOREM 1.1. (a) For $(x, t; \xi, \tau) \in \left(\bar{D} \times (\tau, T]\right) \times \left(\bar{D} \times [0, T)\right)$, $g(x, t; \xi, \tau)$ is continuous.

(b) For $x, \xi \in D$ and $0 \le \tau < t \le T$, $g(x, t; \xi, \tau)$ is positive.
(c) If $r(t) \in C([0, T])$, then $\int_0^t g(x, t; b, \tau) r(\tau) d\tau$ is continuous for $x \in \bar{D}$ and $t \in [0, T]$.
(d) There exists some t_q such that for $0 \le t < t_q$, the nonlinear integral equation

$$u(x, t) = \alpha \int_0^t g(x, t; b, \tau) f(u(b, \tau)) d\tau \tag{1.3}$$

has a unique continuous nonnegative solution u. This solution u is the unique solution of problem (1.2) and is a strictly increasing function of t in D. For any $t \in (0, t_q)$, $u(x, t)$

attains its absolute maximum at (b,t) on the region $\bar{D}\times[0,t]$. If t_q is finite, then at t_q, u quenches only at $x=b$.

We remark that the above results are valid not only for an infinite strip with $N \geq 2$, but also for $N=1$ (which is problem (1.2)). In Section 2, we show that there exists a unique positive number α^* such that u exists globally for $\alpha \leq \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$. We also derive a formula for computing α^*. In Section 3, we study the effects of L and b respectively on quenching.

2. Critical α. Let us study the effect of α on quenching. We fix L and b.

LEMMA 2.1. For each $b \in (0,L)$,

$$\lim_{t\to\infty}\int_0^t g\left(b,t;b,\tau\right)d\tau = L-b. \tag{2.1}$$

Proof. Let

$$g_i\left(b,t;b,\tau\right)=\frac{2}{L}\sum_{n=1}^{i}\left(\cos^2\frac{(2n-1)\,\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\,\pi^2\,(t-\tau)}{4L^2}\right).$$

For $i=1,2,3,\ldots$, the sequence

$$\int_0^t g_i\left(b,t;b,\tau\right)d\tau=\frac{2}{L}\sum_{n=1}^{i}\int_0^t\left(\cos^2\frac{(2n-1)\,\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\,\pi^2\,(t-\tau)}{4L^2}\right)d\tau$$

is monotone increasing. As i tends to ∞, the sequence

$$\lim_{i\to\infty}\int_0^t g_i\left(b,t;b,\tau\right)d\tau=\int_0^t g\left(b,t;b,\tau\right)d\tau$$

exists by Theorem 1.1(c). Thus,

$$\begin{aligned}
&\int_0^t g\left(b,t;b,\tau\right)d\tau\\
&=\frac{2}{L}\sum_{n=1}^{\infty}\int_0^t\left(\cos^2\frac{(2n-1)\,\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\,\pi^2\,(t-\tau)}{4L^2}\right)d\tau\\
&=\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos^2\frac{(2n-1)\pi b}{2L}}{(2n-1)^2}-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}\\
&=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1+\cos\frac{(2n-1)\pi b}{L}}{(2n-1)^2}-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}\\
&=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\frac{(2n-1)\pi b}{L}}{(2n-1)^2}\\
&\quad-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}.
\end{aligned}\tag{2.2}$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8},$$
$$\sum_{n=1}^{\infty}\frac{\cos\frac{(2n-1)\pi b}{L}}{(2n-1)^2}=\frac{\pi^2}{8}-\frac{\pi^2 b}{4L}$$

(cf. Stromberg [6, p. 518]), it follows from (2.2) that

$$\int_0^t g(b,t;b,\tau)\,d\tau=L-b-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}. \tag{2.3}$$

We have

$$\begin{aligned}0&<\sum_{n=1}^{\infty}\frac{\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}\\&\le\sum_{n=1}^{\infty}\frac{\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}\\&\le\sum_{n=1}^{\infty}\frac{\exp\left(-\frac{n^2\pi^2 t}{4L^2}\right)}{n^2}\\&\le\sum_{n=1}^{\infty}\exp\left(-\frac{n\pi^2 t}{4L^2}\right),\end{aligned}$$

which is a geometric series with the common ratio $\exp\left(-\pi^2 t/\left(4L^2\right)\right)$. Hence,

$$0\le\lim_{t\to\infty}\sum_{n=1}^{\infty}\frac{\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)}{(2n-1)^2}\le\lim_{t\to\infty}\frac{1}{\exp\left(\frac{\pi^2 t}{4L^2}\right)-1}=0. \tag{2.4}$$

From (2.3) and (2.4), we have (2.1). □

LEMMA 2.2. For α sufficiently small, u exists globally.

Proof. For any $t\in(0,t_q)$, it follows from Theorem 1.1(d) that on the region $\bar{D}\times[0,t]$, u attains its absolute maximum at the point (b,t). Thus for any $u(x,t)\le c/2$, it follows from $f'(u)>0$ that

$$u(x,t)\le u(b,t)\le\alpha f\left(\frac{c}{2}\right)\int_0^t g(b,t;b,\tau)\,d\tau.$$

Differentiating the right-hand side of (2.3) term by term with respect to t, we obtain

$$\frac{2}{L}\sum_{n=1}^{\infty}\left(\cos^2\frac{(2n-1)\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\pi^2 t}{4L^2}\right)\le\frac{2}{L}\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2 t}{4L^2}\right),$$

which converges uniformly for t in any compact subset of $(0, t_q)$. Thus, we can differentiate the right-hand side of (2.3) term by term (cf. Wade [7, pp. 190-191]) to obtain

$$\frac{\partial}{\partial t}\int_0^t g\left(b,t,b,\tau\right)d\tau = \frac{2}{L}\sum_{n=1}^{\infty}\left(\cos^2\frac{(2n-1)\,\pi b}{2L}\right)\exp\left(-\frac{(2n-1)^2\,\pi^2 t}{4L^2}\right) > 0. \qquad (2.5)$$

Using Lemma 2.1, we have

$$\begin{aligned} u\left(b,t\right) &\le \alpha f\left(\frac{c}{2}\right)\lim_{t\to\infty}\int_0^t g\left(b,t;b,\tau\right)d\tau \\ &= \alpha\left(L-b\right)f\left(\frac{c}{2}\right). \end{aligned}$$

By choosing α sufficiently small, namely

$$\alpha \le \frac{c}{2\left(L-b\right)f\left(\frac{c}{2}\right)},$$

we have $u(x,t) \le u\left(b,t\right) \le c/2$ for $t \in (0,\infty)$. This proves the lemma. □

Lemma 2.3. For α sufficiently large, u quenches in a finite time.

Proof. From (1.3),

$$u(b,t) \ge \alpha f(0)\int_0^t g(b,t;b,\tau)d\tau.$$

From (2.1) and (2.5), there exists some $\tilde{t} \in (0,\infty)$ such that for $t \ge \tilde{t}$, $\int_0^t g(b,t;b,\tau)d\tau \ge \left(L-b\right)/2$. Thus for $t \ge \tilde{t}$,

$$u(b,t) \ge \frac{\alpha\left(L-b\right)f(0)}{2}.$$

By choosing

$$\alpha \ge \frac{2c}{\left(L-b\right)f\left(0\right)},$$

we have $u(b,t) \ge c$, which implies that u quenches in a finite time. This proves the lemma. □

The proof of the following result is similar to that of Theorem 4 of Chan and Jiang [1] for a first initial-boundary value problem (cf. Chan and Kaper [2], and Chan and Tragoonsirisak [4]).

Theorem 2.4. If $u(x,t) \le C$ for some constant $C \in (0,c)$, then $u(x,t)$ converges from below to a solution $U(x) = \lim_{t\to\infty} u(x,t)$ of the nonlinear two-point boundary value problem

$$\left.\begin{array}{c} -U''\left(x\right) = \alpha\delta\left(x-b\right)f\left(U\right) \text{ in } D, \\ U'\left(0\right) = 0,\, U\left(L\right) = 0. \end{array}\right\} \qquad (2.6)$$

Furthermore,

$$U(x) = \alpha G(x;b)f(U(b)), \qquad (2.7)$$

where

$$G(x;\xi) = \begin{cases} L-\xi, & x \le \xi, \\ L-x, & x > \xi \end{cases} \qquad (2.8)$$

is Green's function corresponding to problem (2.6).

Proof. The homogeneous problem corresponding to (2.6) has only the trivial solution. Its Green's function $G(x;\xi)$ exists and is given by (2.8). Let

$$F(x,t)=\int_D G(x;\xi)\,u(\xi,t)\,d\xi,$$

which is bounded since the integrand is bounded. Because u is the solution of problem (1.2), $F(x,t)$ may be regarded as a distribution. Using Green's second identity, we have

$$\begin{aligned}F_t(x,t)&=\int_D G(x;\xi)\left(u_{\xi\xi}(\xi,t)+\alpha\delta(\xi-b)\,f(u(\xi,t))\right)d\xi\\&=-u(x,t)+\alpha G(x;b)\,f(u(b,t)).\end{aligned}$$

By Theorem 1.1(d), u is a strictly increasing function of t in D. The term $\alpha G(x;b)\,f(u(b,t))$ is monotone increasing with respect to t since $f'(u)>0$. Thus,

$$\lim_{t\to\infty}F_t(x,t)=-\lim_{t\to\infty}u(x,t)+\alpha G(x;b)\,f\left(\lim_{t\to\infty}u(b,t)\right),$$

which exists, because $u\le C$. It follows from u being a strictly increasing function of t in D that F is also a strictly increasing function of t in D. If $\lim_{t\to\infty}F_t(x,t)$ were strictly positive at some point x, then $F(x,t)$ would increase without bound as t tends to infinity. Hence, $\lim_{t\to\infty}F_t(x,t)=0$. This gives

$$\tilde U(x)=\alpha G(x;b)\,f\left(\tilde U(b)\right),$$

where $\tilde U(x)$ denotes $\lim_{t\to\infty}u(x,t)$. It follows from u being a strictly increasing function of t in D that $u(x,t)<\tilde U(x)$ in $D\times(0,\infty)$. Differentiating $\tilde U(x)$ twice, we have

$$\tilde U''(x)=-\alpha\delta(x-b)\,f\left(\tilde U(b)\right)=-\alpha\delta(x-b)\,f\left(\tilde U(x)\right).$$

Also, $\tilde U'(0)=0$ and $\tilde U(L)=0$. Thus, $\tilde U(x)$ is a solution of the problem (2.6), and we have (2.7). □

The next result shows that there exists a critical value α^* for α. The proof for showing that the solution exists globally when $\alpha=\alpha^*$ is the same as that for Theorem 7 of Chan and Jiang [1]

THEOREM 2.5. There exists a unique

$$\alpha^*=\frac{1}{L-b}\sup_{0<U(b)<c}\left(\frac{U(b)}{f(U(b))}\right)\tag{2.9}$$

such that u exists globally for $\alpha\le\alpha^*$ and quenches in a finite time for $\alpha>\alpha^*$. Furthermore, u does not quench in infinite time.

Proof. Let us construct the sequence $\{u_n\}$ by $u_0(x,t)=0$, and for $n=0,1,2,\ldots$,

$$u_{n+1}(x,t)=\alpha\int_0^t g(x,t;b,\tau)f(u_n(b,\tau))d\tau.$$

Using mathematical induction, we have $0<u_1<u_2<u_3<\cdots<u_n<u_{n+1}$ in Ω. Since u_n is an increasing sequence as n increases, it follows from the Monotone Convergence

Theorem (cf. Stromberg [6, pp. 266-268]) that

$$u(x,t) = \alpha \int_0^t g(x,t;b,\tau) f(u(b,\tau)) d\tau,$$

where $\lim_{n\to\infty} u_n(x,t) = u(x,t)$. To show that the larger the α, the larger the solution, let β be a positive number such that $\beta < \alpha$. We construct the sequence $\{v_n\}$ by $v_0\,(x,t) = 0$, and for $n = 0, 1, 2, \ldots$,

$$v_{n+1}(x,t) = \beta \int_0^t g(x,t;b,\tau) f(v_n(b,\tau)) d\tau.$$

Similarly, $0 < v_1 < v_2 < v_3 < \cdots < v_n < v_{n+1}$ in Ω, and

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \beta \int_0^t g(x,t;b,\tau) f(v(b,\tau)) d\tau.$$

Because $u_n > v_n$ for $n = 1, 2, 3, \ldots$, we have $u \geq v$. Hence, the solution u is a nondecreasing function of α. Since the solution u of problem (1.2) is unique, it follows from Lemmas 2.2 and 2.3 that there exists a unique α^* such that u exists globally for $\alpha < \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$.

The critical value α^* is determined as the supremum of all positive values α for which a solution U of (2.6) exists. From Theorem 1.1(d), $U(x)$ attains its maximum at $x = b$. From (2.7) and (2.8),

$$U(b) = \alpha\,(L-b)\, f\,(U\,(b))\,.$$

Thus, we have (2.9).

To show that u exists globally when $\alpha = \alpha^*$, let us consider the function $\psi(s) = s/f(s)$. Since $\psi(s) > 0$ for $0 < s < c$, and $\psi(0) = 0 = \lim_{s\to c^-} \psi(s)$, a direct computation shows that $\psi(s)$ attains its maximum when $\psi(s) = 1/f'(s)$, where $s \in (0, c)$ by the Rolle Theorem. Thus, $\sup_{0<U(b)<c} (U(b)/f(U(b)))$ occurs with $U\,(b) \in (0, c)$. This implies that when $\alpha = \alpha^*$, $U(x)$ exists and is bounded away from c. Thus, u exists globally when $\alpha = \alpha^*$. Since u quenches in a finite time for $\alpha > \alpha^*$, it does not quench in infinite time. $\square$

3. Critical domain and location of the source. In this section, we fix α and study the effects of L and b on quenching.

Theorem 3.1. (i) If

$$L - b \leq \frac{1}{\alpha} \sup_{0<U(b)<c} \left(\frac{U(b)}{f(U(b))} \right), \tag{3.1}$$

then u exists globally.

(ii) If

$$L - b > \frac{1}{\alpha} \sup_{0<U(b)<c} \left(\frac{U(b)}{f(U(b))} \right), \tag{3.2}$$

then u quenches in a finite time.

Proof. (i) Using (3.1), we have

$$\alpha \leq \frac{1}{L-b} \sup_{0<U(b)<c} \left(\frac{U(b)}{f(U(b))} \right) = \alpha^*.$$

It follows from Theorem 2.5 that u exists globally.

(ii) From (3.2),

$$\alpha > \frac{1}{L-b} \sup_{0<U(b)<c} \left(\frac{U(b)}{f(U(b))} \right) = \alpha^*.$$

By Theorem 2.5, u quenches in a finite time. □

The following result follows from Theorem 3.1(i).

COROLLARY 3.2. If $L \leq L^*$, where

$$L^* = \frac{1}{\alpha} \sup_{0<U(b)<c} \left(\frac{U(b)}{f(U(b))} \right),$$

then u exists globally for any location b.

With Corollary 3.2, we obtain the following results respectively from Theorem 3.1(i) and (ii).

COROLLARY 3.3. For $L > L^*$, there exists a unique number

$$b^* = L - \frac{1}{\alpha} \sup_{0<U(b)<c} \left(\frac{U(b)}{f(U(b))} \right)$$

such that

(i) if $b \in [b^*, L)$, then u exists globally;

(ii) if $b \in (0, b^*)$, then u quenches in a finite time.

For illustration, let $\alpha = 1$ and $f(u) = 1/(1-u)$. A direct computation shows that $U(b)\,(1-U(b))$ attains its supremum when $U(b) = 0.5$. Thus, $L^* = 0.25$. For any $L \leq 0.25$, it follows from Corollary 3.2 that independent of where b is, u exists globally. If $L > 0.25$, then it follows from Corollary 3.3 that u exists globally when $b \in [L-0.25, L)$ and quenches in a finite time when $b \in (0, L-0.25)$.

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