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QUENCHING FOR A DEGENERATE PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE

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Abstract. Let q, a, T, and b be any real numbers such that $q \geq 0$, $a > 0$, $T > 0$, and $0 < b < 1$. This article studies the following degenerate semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source situated at b:

$$x^q u_t - u_{xx} = a^2 \delta(x-b) f(u(x,t)) \text{ in } (0,1) \times (0,T],$$
$$u(x,0) = 0 \text{ on } [0,1], \, u(0,t) = u(1,t) = 0 \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function, f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and $f'(u)$ are positive for $0 \leq u < c$. It is shown that the problem has a unique continuous solution u before $\max\{u(x,t) : 0 \leq x \leq 1\}$ reaches c^-, u is a strictly increasing function of t for $0 < x < 1$, and if $\max\{u(x,t) : 0 \leq x \leq 1\}$ reaches c^-, then u attains the value c only at the point b. The problem is shown to have a unique a^* such that a unique global solution u exists for $a \leq a^*$, and $\max\{u(x,t) : 0 \leq x \leq 1\}$ reaches c^- in a finite time for $a > a^*$; this a^* is the same as that for $q = 0$. A formula for computing a^* is given, and no quenching in infinite time is deduced.

1. Introduction. Let q, β, a, and ρ be any real numbers with $q \geq 0$, $0 < \beta < a$, and $\rho > 0$. Let us consider the following degenerate semilinear parabolic first initial-boundary value problem,

$$\left.\begin{aligned} \varsigma^q u_\gamma - u_{\varsigma\varsigma} = \delta(\varsigma - \beta) F(u(\varsigma,\gamma)) \text{ in } (0,a) \times (0,\rho], \\ u(\varsigma,0) = 0 \text{ on } [0,a], \, u(0,\gamma) = u(a,\gamma) = 0 \text{ for } 0 < \gamma \leq \rho, \end{aligned}\right\} \tag{1}$$

where $\delta(x)$ is the Dirac delta function and F is a given function. These types of problems are motivated by applications in which the ignition of a combustible medium is accomplished through the use of either a heated wire or a pair of small electrodes to supply a large amount of energy to a very confined area. When $q = 0$, the problem

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(1) can be used to describe the temperature of a one-dimensional rod having a length a and a concentrated nonlinear source at β. When $q = 1$, it may also be used to describe the temperature u of the channel flow of a fluid with temperature-dependent viscosity in the boundary layer (cf. Chan and Kong [4]) with a concentrated nonlinear source at β; here, ς and γ denote the coordinates perpendicular and parallel to the channel wall, respectively.

Let $\varsigma = ax$, $\gamma = a^{q+2}t$, $\beta = ab$, $Lu = x^q u_t - u_{xx}$, $F(u(\varsigma, \gamma)) = f(u(x,t))$, $D = (0,1)$, $\bar{D} = [0,1]$, and $\Omega = D \times (0,T]$. Then (1) is transformed into the following problem:

$$\left.\begin{aligned} Lu = a^2\delta(x-b)f(u(x,t)) \text{ in } \Omega, \\ u(x,0) = 0 \text{ on } \bar{D},\ u(0,t) = u(1,t) = 0 \text{ for } 0 < t \leq T, \end{aligned}\right\} \tag{2}$$

with $0 < b < 1$ and $T = \rho/a^{q+2}$. We assume that $\lim_{u\to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and $f'(u)$ are positive for $0 \leq u < c$.

The case $q = 0$ was studied by Deng and Roberts [7] by analyzing its corresponding nonlinear Volterra equation at the site b of the concentrated source:

$$u(b,t) = a^2 \int_0^t g(b,t;b,\tau) f(u(b,\tau))\, d\tau,$$

where $g(x,t;\xi,\tau)$ denotes Green's function corresponding to (2) with $q = 0$. By also assuming that $f''(u) > 0$ for $u \geq 0$, they showed that there exists a length a^* such that for $a \leq a^*$, the solution $u(b,t)$ of the integral equation exists for all time and is uniformly bounded away from c while for $a > a^*$, there exists some finite t_q such that $\lim_{t\to t_q} u(b,t) = c$ and $\lim_{t\to t_q} u_t(b,t) = \infty$.

Instead of studying a solution $u(b,t)$ of the nonlinear Volterra equation, we would like to investigate a solution $u(x,t)$ of the degenerate problem (2). Since $u(x,t)$ need not be differentiable at b, we say that a solution of the problem (2) is a continuous function satisfying (2). In Sec. 2, we show that the problem (2) has a unique solution u, and $u_{xx} \geq 0$ for $x \in (0,b)$ and $x \in (b,1)$. It follows from $x^q u_t(x,t) = u_{xx}(x,t) + a^2\delta(x-b)f(u(x,t))$ that $u_t(b,t) = \infty$ for each $t > 0$. Hence, we say that a solution u of the problem (2) is said to quench if there exists some t_q such that $\max\{u(x,t) : x \in \bar{D}\} \to c^-$ as $t \to t_q$ (cf. Chan and Liu [5]). If t_q is finite, then u is said to quench in a finite time. On the other hand, if $t_q = \infty$, then u is said to quench in infinite time. We also show that u is a strictly increasing function of t in D, and if u quenches, then b is the single quenching point.

The length a^* is called the critical length (cf. Chan and Kong [3]) if a unique global solution u exists for $a < a^*$, and if the solution u quenches in a finite time for $a > a^*$. In Sec. 3, we show existence of a unique critical length, and that it is the same as that for $q = 0$. By making use of $\lim_{u\to c^-} f(u) = \infty$, we show that for $a = a^*$, u exists for $0 < t < \infty$, and is uniformly bounded away from c. This shows that quenching does not occur in infinite time. We also derive a formula for calculating a^*.

2. Existence, uniqueness, and single-point quenching. Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (2) is determined by the following system:

for x and ξ in D, and t and τ in $(-\infty, \infty)$,

$$LG = \delta(x-\xi)\delta(t-\tau),$$
$$G(x,t;\xi,\tau) = 0 \text{ for } t < \tau,\ G(0,t;\xi,\tau) = G(1,t;\xi,\tau) = 0.$$

By Chan and Chan [1], we have

$$G(x,t;\xi,\tau) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(\xi)e^{-\lambda_i(t-\tau)},$$

where λ_i $(i = 1, 2, 3, \dots)$ are the eigenvalues of the problem

$$\phi'' + \lambda x^q \phi = 0,\ \phi(0) = \phi(1) = 0,$$

and their corresponding eigenfunctions are given by

$$\phi_i(x) = (q+2)^{1/2} x^{1/2} \frac{J_{\frac{1}{q+2}}\left(\frac{2\lambda_i^{1/2}}{q+2} x^{(q+2)/2}\right)}{\left| J_{1+\frac{1}{q+2}}\left(\frac{2\lambda_i^{1/2}}{q+2}\right)\right|}$$

with $J_{1/(q+2)}$ denoting the Bessel function of the first kind of order $1/(q+2)$. From Chan and Chan [1], $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_i < \lambda_{i+1} < \cdots$. The set $\{\phi_i(x)\}$ is a maximal (that is, complete) orthonormal set with the weight function x^q (cf. Gustafson [9, p. 176]).

To derive the integral equation from the problem (2), let us consider the adjoint operator L^*, which is given by $L^*u = -x^q u_t - u_{xx}$. Using Green's second identity, we obtain

$$u(x,t) = a^2 \int_0^t G(x,t;b,\tau) f(u(b,\tau))d\tau. \tag{3}$$

For ease of reference, let us state below Lemmas 1(a), 1(b), 1(d), and 4 of Chan and Chan [1] as Lemma 1(a), 1(b), 1(c), and 1(d), respectively; we also state below Lemma 2.2(a), 2.2(b), 2.2(c), and 2.2(d) of Chan and Tian [6] as Lemma 1(e), 1(f), 1(g), and 1(h), respectively.

LEMMA 1. (a). For some positive constant k_1, $|\phi_i(x)| \le k_1 x^{-q/4}$ for $x \in (0,1]$.

(b). For some positive constant k_2, $|\phi_i(x)| \le k_2 x^{1/2}\lambda_i^{1/4}$ for $x \in \bar{D}$.

(c). For any $x_0 > 0$, and $x \in [x_0, 1]$, there exists some positive constant k_3 depending on x_0 such that $|\phi_i'(x)| \le k_3 \lambda_i^{1/2}$.

(d). In $\{(x,t;\xi,\tau) : x \text{ and } \xi \text{ are in } D,\ T \ge t > \tau \ge 0\}$, $G(x,t;\xi,\tau)$ is positive.

(e). For $(x,t;\xi,\tau) \in (\bar{D} \times (\tau, T]) \times (\bar{D} \times [0,T))$, $G(x,t;\xi,\tau)$ is continuous.

(f). For each fixed $(\xi,\tau) \in \bar{D} \times [0,T)$, $G_t(x,t;\xi,\tau) \in C(\bar{D} \times (\tau,T])$.

(g). For each fixed $(\xi,\tau) \in \bar{D} \times [0,T)$, $G_x(x,t;\xi,\tau)$ and $G_{xx}(x,t;\xi,\tau)$ are in $C((0,1] \times (\tau,T])$.

(h). If $r \in C([0,T])$, then $\int_0^t G(x,t;b,\tau)r(\tau)d\tau$ is continuous for $x \in \bar{D}$ and $t \in [0,T]$.

We modify the techniques in proving Lemma 2.3 and Theorems 2.4 and 2.6 of Chan and Tian [6] to show that the integral equation (3) has a unique nonnegative continuous solution. Unlike theirs, we achieve this without using the contraction mapping and without considering the integral equation (3) at $x = b$; also by making use of (3), we prove further that u is a strictly increasing function of t in D.

THEOREM 1. There exists some t_q ($\leq \infty$) such that for $0 \leq t < t_q$, the integral equation (3) has a unique nonnegative continuous solution u, and u is a strictly increasing function of t in D. If t_q is finite, then u reaches c^- at t_q.

Proof. Let us construct a sequence $\{u_i\}$ in Ω by $u_0 \equiv 0$, and for $i = 0, 1, 2, \ldots,$

$$Lu_{i+1} = a^2\delta(x-b)f(u_i(x,t)) \text{ in } \Omega,$$
$$u_{i+1}(x,0) = 0 \text{ on } \bar{D},\ u_{i+1}(0,t) = u_{i+1}(1,t) = 0 \text{ for } 0 < t \leq T.$$

Let $\partial\Omega$ denote the parabolic boundary $(\bar{D} \times \{0\}) \cup (\{0,1\} \times (0,T])$ of Ω. We have

$$L(u_1 - u_0) = a^2\delta(x-b)f(0) \text{ in } \Omega,\ u_1 - u_0 = 0 \text{ on } \partial\Omega.$$

By Lemma 1(d) and 1(e), $G(x,t;\xi,\tau)$ is positive and continuous. From (3), $u_1 > u_0$ in Ω. Let us assume that for some positive integer j,

$$0 < u_1 < u_2 < \cdots < u_{j-1} < u_j \text{ in } \Omega.$$

We have

$$L(u_{j+1} - u_j) = a^2\delta(x-b)(f(u_j) - f(u_{j-1})) \text{ in } \Omega,\ u_{j+1} - u_j = 0 \text{ on } \partial\Omega.$$

Since f is a strictly increasing function and $u_j > u_{j-1}$, it follows from (3) that $u_{j+1} > u_j$. By the principle of mathematical induction,

$$0 < u_1 < u_2 < \cdots < u_{n-1} < u_n \text{ in } \Omega$$

for any positive integer n.

To show that each u_n is an increasing function of t, let us construct a sequence $\{w_n\}$ such that for $n = 0, 1, 2, \ldots, w_n(x,t) = u_n(x,t+h) - u_n(x,t)$, where h is any positive number less than T. Then, $w_0(x,t) = 0$. By (3), we have

$$\begin{aligned} w_1(x,t) &= u_1(x,t+h) - u_1(x,t) \\ &= a^2 f(0)\left(\int_0^{t+h} G(x,t+h;b,\tau)d\tau - \int_0^t G(x,t;b,\tau)d\tau\right). \end{aligned}$$

We note that $G(x,t+h;b,\tau) = G(x,t+h-\tau;b,0)$. Let $\sigma = \tau - h$. Then,

$$\int_0^{t+h} G(x,t+h;b,\tau)d\tau = \int_0^h G(x,t+h;b,\tau)d\tau + \int_0^t G(x,t;b,\sigma)d\sigma.$$

We have

$$w_1(x,t) = a^2 f(0)\int_0^h G(x,t+h;b,\tau)d\tau,$$

which is positive for $0 < t \leq T - h$. Let us assume that for some positive integer j, $w_j > 0$ for $0 < t \leq T - h$. Let $\sigma = \tau - h$. We have

$$\begin{aligned}
& a^2 \int_0^{t+h} G(x, t+h; b, \tau) f(u_j(b, \tau)) d\tau \\
&= a^2 \left(\int_0^h G(x, t+h; b, \tau) f(u_j(b, \tau)) d\tau + \int_0^t G(x, t; b, \sigma) f(u_j(b, \sigma + h)) d\sigma \right) \\
&> a^2 \left(\int_0^h G(x, t+h; b, \tau) f(u_j(b, \tau)) d\tau + \int_0^t G(x, t; b, \sigma) f(u_j(b, \sigma)) d\sigma \right).
\end{aligned}$$

In $D \times (0, T - h]$,

$$\begin{aligned}
w_{j+1}(x, t) &= u_{j+1}(x, t+h) - u_{j+1}(x, t) \\
&> a^2 \int_0^h G(x, t+h; b, \tau) f(u_j(b, \tau)) d\tau > 0.
\end{aligned}$$

By the principle of mathematical induction, $w_n > 0$ for $0 < t \leq T - h$ and all positive integers n. Thus, each u_n is an increasing function of t.

By Lemma 1(h), $G(x, t; b, \tau)$ is integrable. Thus for any given positive constant M $(< c)$, it follows from

$$u_n(x, t) = a^2 \int_0^t G(x, t; b, \tau) f(u_{n-1}(b, \tau)) d\tau \tag{4}$$

and u_n being an increasing function of t that there exists some t_1 such that $u_n \leq M$ for $0 \leq t \leq t_1$ and $n = 0, 1, 2, \ldots$. In fact, t_1 satisfies

$$a^2 f(M) \int_0^{t_1} G(x, t_1; b, \tau) d\tau \leq M.$$

Let u denote $\lim_{n \to \infty} u_n$. From (4) and the Monotone Convergence Theorem (cf. Royden [10, p. 87]), we have (3) for $0 \leq t \leq t_1$. Thus, u is a nonnegative solution of the integral equation (3) for $0 \leq t \leq t_1$.

To prove that u is unique, let us assume that the integral equation (3) has two distinct solutions u and $\tilde{u}$ on the interval $[0, t_1]$. Let $\Theta = \max_{\bar{D} \times [0, t_1]} |u - \tilde{u}|$. From (3),

$$u(x, t) - \tilde{u}(x, t) = a^2 \int_0^t G(x, t; b, \tau) \left(f(u(b, \tau)) - f(\tilde{u}(b, \tau)) \right) d\tau. \tag{5}$$

Using the Mean Value Theorem, we have

$$|f(u(b, \tau)) - f(\tilde{u}(b, \tau))| \leq f'(M) \Theta.$$

Let ϵ be some positive number. It follows from Lemma 1(a), 1(b), and (5) that

$$\begin{aligned}\Theta &\leq a^2 f'(M)\Theta \int_0^t G(x,t;b,\tau)d\tau \\ &\leq a^2 f'(M)\Theta k_1 k_2 b^{-q/4} \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\sum_{i=1}^{\infty}\lambda_i^{1/4}e^{-\lambda_i(t-\tau)}d\tau \\ &\leq a^2 f'(M)\Theta k_1 k_2 b^{-q/4} \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\sum_{i=1}^{\infty}\lambda_i^{1/4}e^{-\lambda_i\epsilon}d\tau.\end{aligned}$$

Since $\sum_{i=1}^{\infty}\lambda_i^{1/4}e^{-\lambda_i\epsilon}$ converges uniformly with respect to $\tau \in [0, t-\epsilon]$, it follows from the Weierstrass M-Test that

$$\begin{aligned}\Theta &\leq a^2 f'(M)\Theta k_1 k_2 b^{-q/4} \lim_{\epsilon\to 0}\sum_{i=1}^{\infty}\int_0^{t-\epsilon}\lambda_i^{1/4}e^{-\lambda_i(t-\tau)}d\tau \\ &\leq a^2 f'(M)\Theta k_1 k_2 b^{-q/4} \lim_{\epsilon\to 0}\sum_{i=1}^{\infty}\lambda_i^{-3/4}(e^{-\lambda_i\epsilon}-e^{-\lambda_i t}) \\ &\leq a^2 f'(M)\Theta k_1 k_2 b^{-q/4} \lim_{\epsilon\to 0}\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-e^{-\lambda_i t}) \\ &= a^2 f'(M) k_1 k_2 b^{-q/4}\left[\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-e^{-\lambda_i t})\right]\Theta, \end{aligned} \tag{6}$$

which converges since $O(\lambda_i) = O(i^2)$ (cf. Watson [12, p. 506]) for large i. Let us choose some positive number σ_1 $(\leq t_1)$ such that for $t \in [0, \sigma_1]$,

$$a^2 f'(M) k_1 k_2 b^{-q/4}\left[\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-e^{-\lambda_i t})\right] < 1. \tag{7}$$

This gives a contradiction. Thus, u is unique for $0 \leq t \leq \sigma_1$. If $\sigma_1 < t_1$, then it follows from (3) that for $\sigma_1 \leq t \leq t_1$,

$$u(x,t) = a^2\int_0^{\sigma_1} G(x,t;b,\tau)f(u(b,\tau))d\tau + a^2\int_{\sigma_1}^{t} G(x,t;b,\tau)f(u(b,\tau))d\tau. \tag{8}$$

Since $u = \tilde{u}$ for $0 \leq t \leq \sigma_1$, we have for $\sigma_1 \leq t \leq t_1$,

$$u(x,t) - \tilde{u}(x,t) = a^2\int_{\sigma_1}^{t} G(x,t;b,\tau)\left(f(u(b,\tau)) - f(\tilde{u}(b,\tau))\right)d\tau,$$

from which we obtain

$$\Theta \leq a^2 f'(M) k_1 k_2 b^{-q/4}\left[\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-e^{-\lambda_i(t-\sigma_1)})\right]\Theta. \tag{9}$$

It follows from (7) that for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$,

$$a^2 f'(M) k_1 k_2 b^{-q/4}\left[\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-e^{-\lambda_i(t-\sigma_1)})\right] < 1. \tag{10}$$

This gives a contradiction. Thus, we have uniqueness of a solution for $0 \leq t \leq \min\{2\sigma_1, t_1\}$. By proceeding in this way, the integral equation (3) has a unique nonnegative solution u for $0 \leq t \leq t_1$.

From Lemma 1(h) and (4), each u_n is continuous. To show that u is continuous, we note that

$$u_{n+1}(x,t) - u_n(x,t) = a^2 \int_0^t G(x,t;b,\tau)\left[f(u_n(b,\tau)) - f(u_{n-1}(b,\tau))\right] d\tau.$$

Let $S_n = \max_{\bar{D}\times[0,t_1]}(u_n - u_{n-1})$. Using the Mean Value Theorem, we have

$$f(u_n(b,\tau)) - f(u_{n-1}(b,\tau)) \leq f'(M) S_n.$$

As in the derivation of (6), we obtain for $0 \leq t \leq \sigma_1$,

$$S_{n+1} \leq a^2 f'(M) k_1 k_2 b^{-q/4}\left[\sum_{i=1}^{\infty} \lambda_i^{-3/4}(1 - e^{-\lambda_i t})\right] S_n.$$

By (7), the sequence $\{u_n\}$ converges uniformly to u with respect to $x \in \bar{D}$ and $t \in [0, \sigma_1]$. Thus, the integral equation (3) has a unique nonnegative continuous solution u for $0 \leq t \leq \sigma_1$. From Lemma 1(a) and 1(b), the integrand of the first term on the right-hand side of (8) is bounded by $f(M)\, k_1 k_2 b^{-q/4} \sum_{i=1}^{\infty} \lambda_i^{1/4} e^{-\lambda_i (t-\tau)}$, which is integrable for $t > \sigma_1$. Thus, if $\sigma_1 < t_1$, then for $\sigma_1 \leq t \leq t_1$, the first term on the right-hand side of (8) is continuous. Let

$$z(x,t) = a^2 \int_{\sigma_1}^t G(x,t;b,\tau) f(z(b,\tau)) d\tau.$$

From (8), $z < M$. For $\sigma_1 \leq t \leq t_1$, let us construct a sequence $\{z_i\}$ by $z_0(x,t) \equiv 0$, and for $i = 0, 1, 2, \ldots$,

$$z_{i+1}(x,t) = a^2 \int_{\sigma_1}^t G(x,t;b,\tau) f(z_i(b,\tau)) d\tau.$$

By Lemma 1(h), each z_n is continuous. We have

$$z_{n+1}(x,t) - z_n(x,t) = a^2 \int_{\sigma_1}^t G(x,t;b,\tau)\left(f(z_n(b,\tau)) - f(z_{n-1}(b,\tau))\right) d\tau.$$

Let $Z_n = \max_{\bar{D}\times[\sigma_1, \min\{2\sigma_1, t_1\}]} |z_n - z_{n-1}|$. As in the derivation of (9),

$$Z_{n+1} \leq a^2 f'(M) k_1 k_2 b^{-q/4}\left[\sum_{i=1}^{\infty} \lambda_i^{-3/4}(1 - e^{-\lambda_i (t-\sigma_1)})\right] Z_n.$$

From (10), the sequence $\{z_n\}$ converges uniformly to z, and hence, z is continuous for $\sigma_1 \leq t \leq \min\{2\sigma_1, t_1\}$. From (8), u is continuous for $\sigma_1 \leq t \leq \min\{2\sigma_1, t_1\}$. If $2\sigma_1 < t_1$, then for $2\sigma_1 \leq t \leq t_1$,

$$u(x,t) = a^2 \int_0^{2\sigma_1} G(x,t;b,\tau) f(u(b,\tau)) d\tau + a^2 \int_{2\sigma_1}^t G(x,t;b,\tau) f(u(b,\tau)) d\tau.$$

Since the first term on the right-hand side is continuous, we consider the second term. An argument analogous to the above shows that u is continuous for $0 \leq t \leq \min\{3\sigma_1, t_1\}$. By proceeding in this way, the integral equation (3) has a unique nonnegative continuous solution u for $0 \leq t \leq t_1$.

Let t_q be the supremum of all t_1, where $[0, t_1]$ is the interval for which the integral equation (3) has a unique nonnegative continuous solution u. If t_q is finite and u does not reach c^- at t_q, then given any number between $\max_{\bar{D}} u(x, t_q)$ and c, a proof similar to the above shows that there exists some $t_2 > t_q$ such that the integral equation (3) has a unique nonnegative continuous solution u for $0 \leq t \leq t_2$. This contradicts the definition of t_q. Hence, if t_q is finite, then u reaches c^- at t_q.

It follows from u_n being an increasing function of t in D that u is a nondecreasing function of t. Let $\sigma = \tau - \epsilon$. We have

$$\begin{aligned}
&a^2 \int_0^{t+\epsilon} G(x, t+\epsilon; b, \tau) f(u(b, \tau)) d\tau \\
&= a^2 \left(\int_0^{\epsilon} G(x, t+\epsilon; b, \tau) f(u(b, \tau)) d\tau + \int_0^t G(x, t; b, \sigma) f(u(b, \sigma+\epsilon)) d\sigma \right) \\
&\geq a^2 \left(\int_0^{\epsilon} G(x, t+\epsilon; b, \tau) f(u(b, \tau)) d\tau + \int_0^t G(x, t; b, \sigma) f(u(b, \sigma)) d\sigma \right)
\end{aligned}$$

since f is an increasing function. Thus,

$$u(x, t+\epsilon) - u(x, t) \geq a^2 \int_0^{\epsilon} G(x, t+\epsilon; b, \tau) f(u(b, \tau)) d\tau > 0.$$

Hence, u is a strictly increasing function of t in D. □

We modify the method of proving Theorem 2.5 of Chan and Tian [6] to show that the solution of the integral equation (3) is actually the solution of the problem (2).

THEOREM 2. Before quenching occurs, the problem (2) has a unique nonnegative solution u. If t_q is finite, then u quenches at t_q.

Proof. It follows from Lemma 1(h) that for any $t_3 \in (0, t)$,

$$\begin{aligned}
&\int_0^t G(x, t; b, \tau) f(u(b, \tau)) d\tau \\
&= \lim_{n\to\infty} \int_0^{t-1/n} G(x, t; b, \tau) f(u(b, \tau)) d\tau \\
&= \lim_{n\to\infty} \int_{t_3}^t \frac{\partial}{\partial \zeta} \left(\int_0^{\zeta - 1/n} G(x, \zeta; b, \tau) f(u(b, \tau)) d\tau \right) d\zeta \\
&\quad + \lim_{n\to\infty} \int_0^{t_3 - 1/n} G(x, t_3; b, \tau) f(u(b, \tau)) d\tau.
\end{aligned}$$

For $\zeta - \tau \geq 1/n$, it follows from Lemma 1(f) and 1(b) that for any $x \in \bar{D}$,

$$\begin{aligned}
G_\zeta(x, \zeta; b, \tau) f(u(b, \tau)) &\leq f(u(b, \tau)) \left(\sum_{i=1}^{\infty} |\phi_i(x)|\, |\phi_i(b)|\, \lambda_i e^{-\lambda_i(\zeta - \tau)} \right) \\
&\leq k_2^2 f(u(b, \tau)) \sum_{i=1}^{\infty} \lambda_i^{3/2} e^{-\lambda_i / n},
\end{aligned}$$

which is integrable with respect to τ over $(0, \zeta - 1/n)$. Using the Leibnitz rule (cf. Stromberg [11, p. 380]), we have

$$\frac{\partial}{\partial \zeta}\left(\int_0^{\zeta-1/n} G(x,\zeta;b,\tau)f(u(b,\tau))d\tau\right)$$
$$= G\left(x,\zeta;b,\zeta-\frac{1}{n}\right) f\left(u\left(b,\zeta-\frac{1}{n}\right)\right) + \int_0^{\zeta-1/n} G_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau.$$

Let us consider the problem

$$Lw = 0 \text{ for } x \text{ and } \xi \text{ in } D,\ 0 < \tau < t,$$
$$w(0,t;\xi,\tau) = w(1,t;\xi,\tau) = 0 \text{ for } 0 < \tau < t,$$
$$\lim_{t\to\tau^+} x^q w(x,t;\xi,\tau) = \delta(x-\xi).$$

Using Green's second identity, we obtain for $t \geq \tau$,

$$w(x,t;\xi,\tau) = \int_D y^q G(x,t;y,\tau)y^{-q}\delta(y-\xi)dy$$
$$= G(x,t;\xi,\tau).$$

It follows that $\lim_{t\to\tau^+} x^q G(x,t;\xi,\tau) = \delta(x-\xi)$.

Since f is an increasing function, and $G(x,\zeta;b,\zeta-1/n) = G(x,1/n;b,0)$, which is independent of ζ, we have

$$\int_0^t x^q G(x,t;b,\tau)f(u(b,\tau))d\tau$$
$$= \delta(x-b)\int_{t_3}^t f(u(b,\zeta))d\zeta + \lim_{n\to\infty}\int_{t_3}^t\int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau d\zeta$$
$$+ \int_0^{t_3} x^q G(x,t_3;b,\tau)f(u(b,\tau))d\tau. \tag{11}$$

Let

$$g_n(x,\zeta) = \int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau.$$

Without loss of generality, let $n > l$. We have

$$g_n(x,\zeta) - g_l(x,\zeta) = \int_{\zeta-1/l}^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau)f(u(b,\tau))d\tau.$$

Since $x^q G_\zeta(x,\zeta;b,\tau) \in C(\bar{D}\times(\tau,T])$ and $f(u(b,\tau))$ is an increasing function of τ, it follows from the Second Mean Value Theorem for Integrals (cf. Stromberg [11, p. 328]) that for any $x \neq b$ and any ζ in any compact subset of $(0,t_q)$, there exists some real number ν such that $\zeta - \nu \in (\zeta - 1/l, \zeta - 1/n)$ and

$$g_n(x,\zeta) - g_l(x,\zeta) = f\left(u\left(b,\zeta-\frac{1}{l}\right)\right)\int_{\zeta-1/l}^{\zeta-\nu} x^q G_\zeta(x,\zeta;b,\tau)d\tau$$
$$+ f\left(u\left(b,\zeta-\frac{1}{n}\right)\right)\int_{\zeta-\nu}^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau)d\tau.$$

From $G_\zeta(x,\zeta;b,\tau) = -G_\tau(x,\zeta;b,\tau)$, we have

$$g_n(x,\zeta) - g_l(x,\zeta) = \left[f\left(u\left(b,\zeta-\frac{1}{n}\right)\right) - f\left(u\left(b,\zeta-\frac{1}{l}\right)\right)\right] x^q G(x,\zeta;b,\zeta-\nu)$$
$$+ f\left(u\left(b,\zeta-\frac{1}{l}\right)\right) x^q G\left(x,\zeta;b,\zeta-\frac{1}{l}\right)$$
$$- f\left(u\left(b,\zeta-\frac{1}{n}\right)\right) x^q G\left(x,\zeta;b,\zeta-\frac{1}{n}\right).$$

Since for $x \neq b$, $x^q G(x,\zeta;b,\zeta-\epsilon) = x^q G(x,\epsilon;b,0)$, which converges to 0 uniformly with respect to ζ as $\epsilon \to 0$, it follows that for $x \neq b$, $\{g_n\}$ is a Cauchy sequence, and hence $\{g_n\}$ converges uniformly with respect to ζ in any compact subset of $(0,t_q)$. Hence for $x \neq b$,

$$\lim_{n\to\infty}\int_{t_3}^{t}\int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau) f(u(b,\tau))\,d\tau\,d\zeta$$
$$= \int_{t_3}^{t}\lim_{n\to\infty}\int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau) f(u(b,\tau))\,d\tau\,d\zeta$$
$$= \int_{t_3}^{t}\int_0^{\zeta} x^q G_\zeta(x,\zeta;b,\tau) f(u(b,\tau))\,d\tau\,d\zeta.$$

For $x = b$,

$$-G_\zeta(b,\zeta;b,\tau) f(u(b,\tau)) = \sum_{i=1}^{\infty} \phi_i^2(b)\lambda_i e^{-\lambda_i(\zeta-\tau)} f(u(b,\tau)),$$

which is positive. Thus, $\{-g_n\}$ is a nondecreasing sequence of nonnegative functions with respect to ζ. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty}\int_{t_3}^{t}\int_0^{\zeta-1/n} b^q G_\zeta(b,\zeta;b,\tau) f(u(b,\tau))\,d\tau\,d\zeta = \int_{t_3}^{t}\int_0^{\zeta} b^q G_\zeta(b,\zeta;b,\tau) f(u(b,\tau))\,d\tau\,d\zeta.$$

Thus from (11),

$$\frac{\partial}{\partial t}\int_0^t x^q G(x,t;b,\tau) f(u(b,\tau))\,d\tau = \delta(x-b) f(u(b,t)) + \int_0^t x^q G_t(x,t;b,\tau) f(u(b,\tau))\,d\tau.$$

By Lemma 1(b) and 1(c),

$$\left|\sum_{i=1}^{\infty}\frac{\partial}{\partial x}\phi_i(x)\,\phi_i(\xi)\,e^{-\lambda_i(t-\tau)}\right| \le \sum_{i=1}^{\infty}|\phi_i'(x)|\,|\phi_i(\xi)|\,e^{-\lambda_i(t-\tau)}$$
$$\le k_2 k_3 \sum_{i=1}^{\infty}\lambda_i^{3/4} e^{-\lambda_i(t-\tau)},$$

which converges uniformly with respect to x in any compact subset of $(0,1]$ with $t-\tau \ge \epsilon$. By Lemma 1(g) and the Leibnitz rule, we have for any x in any compact subset of $(0,1]$ and t in any compact subset of $(0,t_q)$,

$$\frac{\partial}{\partial x}\int_0^{t-\epsilon} G(x,t;b,\tau) f(u(b,\tau))\,d\tau = \int_0^{t-\epsilon} G_x(x,t;b,\tau) f(u(b,\tau))\,d\tau.$$

Since ϕ_i is an eigenfunction, it follows from Lemma 1(b) that

$$\begin{aligned}\left|\sum_{i=1}^{\infty}\frac{\partial^2}{\partial x^2}\phi_i(x)\,\phi_i(\xi)\,e^{-\lambda_i(t-\tau)}\right| &\le \sum_{i=1}^{\infty}|\phi_i''(x)|\,|\phi_i(\xi)|\,e^{-\lambda_i(t-\tau)}\\ &= \sum_{i=1}^{\infty}\lambda_i x^q\,|\phi_i(x)|\,|\phi_i(\xi)|\,e^{-\lambda_i(t-\tau)}\\ &\le k_2^2\sum_{i=1}^{\infty}\lambda_i^{3/2}e^{-\lambda_i(t-\tau)},\end{aligned}$$

which converges uniformly with respect to x in any compact subset of $(0,1]$ with $t-\tau\ge\epsilon$. By Lemma 1(g) and the Leibnitz rule, we have for any x in any compact subset of $(0,1]$ and t in any compact subset of $(0,t_q)$,

$$\frac{\partial}{\partial x}\int_0^{t-\epsilon}G_x(x,t;b,\tau)f(u(b,\tau))d\tau=\int_0^{t-\epsilon}G_{xx}(x,t;b,\tau)f(u(b,\tau))d\tau.$$

For any $x_1\in D$,

$$\begin{aligned}&\lim_{\epsilon\to0}\int_0^{t-\epsilon}G(x,t;b,\tau)f(u(b,\tau))d\tau\\ &=\lim_{\epsilon\to0}\int_{x_1}^{x}\left(\frac{\partial}{\partial\eta}\int_0^{t-\epsilon}G(\eta,t;b,\tau)f(u(b,\tau))d\tau\right)d\eta+\lim_{\epsilon\to0}\int_0^{t-\epsilon}G(x_1,t;b,\tau)f(u(b,\tau))d\tau\\ &=\lim_{\epsilon\to0}\int_{x_1}^{x}\int_0^{t-\epsilon}G_\eta(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta+\int_0^{t}G(x_1,t;b,\tau)f(u(b,\tau))d\tau.\end{aligned}\tag{12}$$

By the Fubini Theorem (cf. Stromberg [11, p. 352]),

$$\begin{aligned}\lim_{\epsilon\to0}\int_{x_1}^{x}\int_0^{t-\epsilon}G_\eta(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta&=\lim_{\epsilon\to0}\int_0^{t-\epsilon}(f(u(b,\tau))\int_{x_1}^{x}G_\eta(\eta,t;b,\tau)d\eta)d\tau\\ &=\lim_{\epsilon\to0}\int_0^{t-\epsilon}f(u(b,\tau))(G(x,t;b,\tau)-G(x_1,t;b,\tau))d\tau\\ &=\int_0^{t}f(u(b,\tau))(G(x,t;b,\tau)-G(x_1,t;b,\tau))d\tau,\end{aligned}$$

which exists by Lemma 1(h). Therefore,

$$\begin{aligned}\int_0^{t}f(u(b,\tau))(G(x,t;b,\tau)-G(x_1,t;b,\tau))d\tau&=\int_0^{t}f(u(b,\tau))\left(\int_{x_1}^{x}G_\eta(\eta,t;b,\tau)d\eta\right)d\tau\\ &=\int_{x_1}^{x}\int_0^{t}G_\eta(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta.\end{aligned}$$

From (12),

$$\begin{aligned}&\int_0^{t}G(x,t;b,\tau)f(u(b,\tau))d\tau\\ &=\int_{x_1}^{x}\int_0^{t}G_\eta(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta+\int_0^{t}G(x_1,t;b,\tau)f(u(b,\tau))d\tau.\end{aligned}$$

Thus,

$$\frac{\partial}{\partial x}\int_0^t G(x,t;b,\tau)f(u(b,\tau))d\tau = \int_0^t G_x(x,t;b,\tau)f(u(b,\tau))d\tau. \tag{13}$$

For any $x_2 \in D$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^{t-\epsilon} G_x(x,t;b,\tau)f(u(b,\tau))d\tau \\
&= \lim_{\epsilon\to 0}\int_{x_2}^{x}\left(\frac{\partial}{\partial\eta}\int_0^{t-\epsilon} G_\eta(\eta,t;b,\tau)f(u(b,\tau))d\tau\right)d\eta + \lim_{\epsilon\to 0}\int_0^{t-\epsilon} G_\eta(x_2,t;b,\tau)f(u(b,\tau))d\tau \\
&= \lim_{\epsilon\to 0}\int_{x_2}^{x}\int_0^{t-\epsilon} G_{\eta\eta}(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta + \int_0^t G_\eta(x_2,t;b,\tau)f(u(b,\tau))d\tau. \qquad (14)
\end{aligned}$$

By the Fubini Theorem,

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{x_2}^{x}\int_0^{t-\epsilon} G_{\eta\eta}(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta &= \lim_{\epsilon\to 0}\int_0^{t-\epsilon}(f(u(b,\tau))\int_{x_2}^{x} G_{\eta\eta}(\eta,t;b,\tau)d\eta)d\tau \\
&= \lim_{\epsilon\to 0}\int_0^{t-\epsilon} f(u(b,\tau))(G_\eta(x,t;b,\tau) - G_\eta(x_2,t;b,\tau))d\tau \\
&= \int_0^t f(u(b,\tau))(G_\eta(x,t;b,\tau) - G_\eta(x_2,t;b,\tau))d\tau,
\end{aligned}$$

which exists by (13). Therefore,

$$\int_0^t f(u(b,\tau))(G_\eta(x,t;b,\tau) - G_\eta(x_2,t;b,\tau))d\tau = \int_{x_2}^{x}\int_0^t G_{\eta\eta}(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta.$$

From (14),

$$\begin{aligned}
&\int_0^t G_x(x,t;b,\tau)f(u(b,\tau))d\tau \\
&= \int_{x_2}^{x}\int_0^t G_{\eta\eta}(\eta,t;b,\tau)f(u(b,\tau))d\tau d\eta + \int_0^t G_\eta(x_2,t;b,\tau)f(u(b,\tau))d\tau.
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial x}\int_0^t G_x(x,t;b,\tau)f(u(b,\tau))d\tau = \int_0^t G_{xx}(x,t;b,\tau)f(u(b,\tau))d\tau.$$

It follows from (13) that for any x in any compact subset of D and any t in any compact subset of $(0,t_q)$,

$$\frac{\partial^2}{\partial x^2}\int_0^t G(x,t;b,\tau)f(u(b,\tau))d\tau = \int_0^t G_{xx}(x,t;b,\tau)f(u(b,\tau))d\tau.$$

Before quenching occurs, it follows from the integral equation (3) that for $x \in D$ and $t > 0$,

$$\begin{aligned}
Lu &= a^2\delta(x-b)f(u(b,t)) + a^2\int_0^t LG(x,t;b,\tau)f(u(b,\tau))d\tau \\
&= a^2\delta(x-b)f(u(b,t)) + a^2\delta(x-b)\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\delta(t-\tau)f(u(b,\tau))d\tau \\
&= a^2\delta(x-b)f(u(x,t)).
\end{aligned}$$

From (3), $\lim_{t\to 0} u(x,t) = 0$ for $x \in \bar{D}$. Since $G(0,t;\xi,\tau) = G(1,t;\xi,\tau) = 0$, we have $u(0,t) = u(1,t) = 0$. Thus, the nonnegative continuous solution of the integral equation (3) is a solution of the problem (2). Since a solution of the problem (2) is a solution of the integral equation (3), which has a unique solution before quenching occurs, u is the unique solution of the problem (2).

If t_q is finite, then by Theorem 1, u reaches c^- at t_q. Thus, u quenches at t_q. □

Our next result shows that b is the single quenching point.

THEOREM 3. The solution $u(x,t)$ of the problem (2) attains its maximum at (b,t). If u quenches, then b is the single quenching point.

Proof. Since $u(b,t)$ is known, let us consider the following problems:

$$\left.\begin{array}{c} Lu = 0 \text{ in } (0,b)\times(0,t_q), \\ u(x,0) = 0 \text{ for } 0 \le x \le b,\ u(0,t) = 0 \text{ and } u(b,t) = u(b,t) \text{ for } 0 < t < t_q, \end{array}\right\} \tag{15}$$

$$\left.\begin{array}{c} Lu = 0 \text{ in } (b,1)\times(0,t_q), \\ u(x,0) = 0 \text{ for } b \le x \le 1,\ u(b,t) = u(b,t) \text{ and } u(1,t) = 0 \text{ for } 0 < t < t_q. \end{array}\right\} \tag{16}$$

By the weak maximum principle (cf. Friedman [8, p. 39]), u attains its maximum at b for each of the problems (15) and (16). Since u is a strictly increasing function of t in D, $u(x,t)$ attains its maximum at (b,t).

If u quenches, then it quenches at b. For the problem (15), it follows from the parabolic version of Hopf's Lemma (cf. Friedman [8, p. 49]) that for any fixed $t \in (0,t_q)$, $u_x(0,t) > 0$. For any $x \in (0,b)$, $u_{xx} = x^q u_t$, which is nonnegative by Theorem 1. Hence, u is concave up. Similarly for any fixed $t \in (0,t_q)$, $u_x(1,t) < 0$. For any $x \in (b,1)$, $u_{xx} = x^q u_t \ge 0$, and hence, u is concave up. Thus if u quenches, then b is the single quenching point. □

3. Critical length and no quenching in infinite time. We modify the proof of Theorem 3 of Chan and Kaper [2] to obtain the following result.

THEOREM 4. If there exists a constant $C \in (0,c)$ such that $\lim_{t\to\infty} u(x,t) \le C$, then $u(x,t)$ converges uniformly on $\bar{D}$ from below to a solution $U(x)$ of the nonlinear two-point boundary value problem,

$$-U''(x) = a^2\delta(x-b)f(U(x)) \text{ in } D,\ U(0) = U(1) = 0. \tag{17}$$

Furthermore, $u(x,t) < U(x)$ in $D \times (0,\infty)$.

Proof. Since the homogeneous problem corresponding to (17) has only the trivial solution, its Green's function $g(x;\xi)$ exists. A direct computation gives

$$g(x;\xi) = \begin{cases} \xi(1-x), & 0 \le \xi \le x, \\ x(1-\xi), & x < \xi \le 1. \end{cases}$$

Let

$$F(x,t) = \int_D g(x;\xi)\xi^q u(\xi,t)d\xi,$$

which is bounded, because the integrand is bounded. Since u is the solution of the problem (2), $F(x,t)$ may be regarded as a distribution. By Green's second identity,

$$\begin{aligned} F_t(x,t) &= \int_D g(x;\xi)(u_{\xi\xi}(\xi,t) + a^2\delta(\xi-b)f(u(\xi,t)))d\xi \\ &= -u(x,t) + a^2 g(x;b)f(u(b,t)). \end{aligned} \tag{18}$$

Since u is a strictly increasing function of t in D and $f(u)$ is strictly increasing, the term $a^2g(x;b)f(u(b,t))$ in (18) is monotone increasing with respect to t. It follows from the continuity of f that

$$\lim_{t\to\infty} F_t(x,t) = -\lim_{t\to\infty} u(x,t) + a^2 g(x;b) f(\lim_{t\to\infty} u(b,t)), \tag{19}$$

which exists since $u \le C$. We note that F is a strictly increasing function of t in D. If $\lim_{t\to\infty} F_t(x,t)$ were strictly positive at some point x, then $F(x,t)$ would increase without bound as t tends to infinity. Thus, $\lim_{t\to\infty} F_t(x,t) = 0$. It follows from (19) that

$$\tilde{U}(x) = a^2 g(x;b) f(\tilde{U}(b)), \tag{20}$$

where $\tilde{U}(x) = \lim_{t\to\infty} u(x,t)$. Since u is a strictly increasing function of t in D, we have $u(x,t) < \tilde{U}(x)$ in $D \times (0,\infty)$. By a direct differentiation,

$$\tilde{U}''(x) = -a^2\delta(x-b)f(\tilde{U}(b)) = -a^2\delta(x-b)f(\tilde{U}(x)).$$

From (20), $\tilde{U}(0) = \tilde{U}(1) = 0$. Thus, $\tilde{U}(x)$ is a solution of the problem (17). Since $\tilde{U}$ is continuous, the uniform convergence of u to $\tilde{U}$ on $\bar{D}$ follows from the Dini Theorem. The theorem is then proved. □

Let v denote the solution of the problem (1) with a and β being replaced by $a+\alpha$ and $\beta(a+\alpha)/a$ respectively for some constant $\alpha > 0$. Let $\varsigma = (a+\alpha)x$, $\gamma = (a+\alpha)^{q+2}t$, $\beta = (a+\alpha)b$, $F(u(\varsigma,\gamma)) = f(v(x,t))$, $T_\alpha = \rho/(a+\alpha)^{q+2}$, and $\Omega_\alpha = D \times (0, T_\alpha]$. Then,

$$\left.\begin{aligned} Lv &= (a+\alpha)^2\delta(x-b)f(v(x,t)) \text{ in } \Omega_\alpha, \\ v(x,0) &= 0 \text{ on } \bar{D},\ v(0,t) = v(1,t) = 0 \text{ for } 0 < t \le T_\alpha. \end{aligned}\right\} \tag{21}$$

THEOREM 5. In Ω_α, $v(x,t) > u(x,t)$.

Proof. Let us construct two sequences $\{v_i(x,t)\}$ and $\{u_i(x,t)\}$ in Ω_α by $v_0(x,t) = u_0(x,t) = 0$, and for $i = 0,1,2,\ldots$,

$$\begin{aligned} Lv_{i+1} &= (a+\alpha)^2\delta(x-b)f(v_i(x,t)) \text{ in } \Omega_\alpha, \\ v_{i+1}(x,0) &= 0 \text{ on } \bar{D},\ v_{i+1}(0,t) = v_{i+1}(1,t) = 0 \text{ for } 0 < t \le T_\alpha; \end{aligned}$$

$$\begin{aligned} Lu_{i+1} &= a^2\delta(x-b)f(u_i(x,t)) \text{ in } \Omega_\alpha, \\ u_{i+1}(x,0) &= 0 \text{ on } \bar{D},\ u_{i+1}(0,t) = u_{i+1}(1,t) = 0 \text{ for } 0 < t \le T_\alpha. \end{aligned}$$

By (3),

$$v_1(x,t) - u_1(x,t) = \left[(a+\alpha)^2 - a^2\right] f(0) \int_0^t G(x,t;b,\tau)d\tau > 0.$$

Let us assume that for some positive integer j, $v_j(x,t) > u_j(x,t)$. Then,

$$v_{j+1}(x,t) - u_{j+1}(x,t) > a^2 \int_0^t G(x,t;b,\tau)\left(f(v_j(b,\tau)) - f(u_j(b,\tau))\right)d\tau > 0.$$

By the principle of mathematical induction, $v_n(x,t) > u_n(x,t)$ for all positive integers n. A proof similar to those of Theorems 1 and 2 shows that $v(x,t) = \lim_{n\to\infty} v_n(x,t)$ is the solution of the problem (21). Hence, $v(x,t) \geq u(x,t)$ in Ω_α.

By (3),

$$v(x,t) - u(x,t) > a^2 \int_0^t G(x,t;b,\tau)\left(f(v(b,\tau)) - f(u(b,\tau))\right) d\tau \geq 0.$$

The theorem is then proved. □

THEOREM 6. If $\lim_{t\to s} u(b,t) = c^-$, where $s \leq \infty$, then $v(b,t)$ quenches in a finite time.

Proof. From (3),

$$\lim_{t\to s} \int_0^t G(b,t;b,\tau) f(u(b,\tau)) d\tau = \frac{c}{a^2}.$$

Since in D, u and v are strictly increasing functions of t and $v > u$, there exists some t_q ($< s$) such that

$$\lim_{t\to t_q} \int_0^t G(b,t;b,\tau) f(v(b,\tau)) d\tau = \frac{c}{(a+\alpha)^2}.$$

Hence, $v(b,t)$ quenches in a finite time. □

Theorems 4 and 5 imply that there exists a critical length a^* such that u exists on $\bar{D}$ for all $t > 0$ if $a < a^*$. The critical length a^* is determined as the supremum of all positive values a for which a solution U of (17) exists. Hence, it is the same as that for $q = 0$. According to Theorem 6, u quenches in a finite time if $a > a^*$.

THEOREM 7. The solution u does not quench in infinite time.

Proof. Since $u(x,t)$ attains its maximum at (b,t), $U(x) = \lim_{t\to\infty} u(x,t)$ attains its maximum at b. From (20),

$$\begin{aligned} a^* &= \max\left(\frac{U(b)}{g(b,b) f(U(b))}\right)^{1/2} \\ &= \frac{1}{[b(1-b)]^{1/2}} \max\left(\frac{U(b)}{f(U(b))}\right)^{1/2}. \end{aligned}$$

Let us consider the function

$$\psi(s) = \frac{s}{f(s)}.$$

Since $\psi(s) > 0$ for $0 < s < c$, and $\psi(0) = 0 = \lim_{s\to c^-} \psi(s)$, a direct computation shows that $\psi(s)$ attains its maximum when $\psi(s) = 1/f'(s)$, where $s \in (0,c)$ by Rolle's Theorem. Thus, $\max\left(U(b)/f(U(b))\right)^{1/2}$ occurs when

$$\frac{U(b)}{f(U(b))} = \frac{1}{f'(U(b))},$$

where $0 < U(b) < c$. This implies that $U(x)$ exists when $a = a^*$. Hence for $a \leq a^*$, u exists globally and is uniformly bounded away from c. Since u quenches in a finite time for $a > a^*$, u does not quench in infinite time. □

For illustration, let $f(u) = (1-u)^{-p}$. A direct computation shows that $U(b)\,[1 - U(b)]^p$ attains its maximum when $U(b) = 1/(1+p)$. Therefore,

$$a^* = \left(\frac{p^p}{b(1-b)(1+p)^{1+p}} \right)^{1/2}.$$

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