QUEUES FOR A FIXED-CYCLE TRAFFIC LIGHT

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1. Summary. In their book *Studies in the Economics of Transportation*, Beckmann, McGuire and Winsten (BMW) ([2], pp. 11–13, 40–42) proposed a simple queuing model for traffic flow through a fixed-cycle traffic light. Although they derived a relation between the average delay per car and the average length of the queue at the beginning of a red phase of the light, they only indicated some possible numerical schemes for evaluating the latter. Here we shall derive analytic expressions for the average queue length and consequently also the average delay under equilibrium conditions for the BMW model.

2. Introduction. Several papers have been written on the subject of queuing at a fixed-cycle traffic light. Wardrop [7] and Webster [8] describe very extensive studies based upon experimental observation, computer simulation and semi-empirical theory with the theory based upon the assumption that the arrivals of cars at the light form a Poisson process. Uematu [6] investigated the queues for a model quite similar to that of BMW but was mainly concerned with the question of how long it takes an empty queue to reach some preassigned length for the first time. The present author also made a previous study [5] of delays but only considered arrival rates which were not too close to the critical value and used a more elaborate model than that considered here.

In the model proposed by BMW, it is assumed that events such as the arrival or departure of a car at the traffic light may occur only on a set of discrete and equally spaced time points. The traffic light pattern is periodic in time with each cycle represented by a sequence of r consecutive time points designated as red points followed by a sequence of g points designated as green. At either a red or green point there is a probability α that one new car will arrive and a probability $1 - \alpha$ that no new cars arrive, these probabilities being independent of the number of arrivals at any other time points. No cars are allowed to leave the light at red points but one car leaves at any green point provided that either a new car also arrives at that time or the queue just prior to this time point is non-empty.

From these rules it follows that the lengths of the queue immediately before time points define a non-stationary Markov chain in which at any red point there is a probability α that the queue increases by one car and a probability $1 - \alpha$ that it remains unchanged, whereas at any green point there is a probability α that a non-empty queue remains unchanged and a probability $1 - \alpha$ that it decreases by one. The lengths of the queue before corresponding time points of successive cycles of the light, however, form a stationary Markov

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chain. If we let q_x denote the length of queue just before the first red point of the xth cycle, the q_x satisfy the recursion relation,

$$q_{x+1} = \max\{q_x + u_x - g, 0\}, \tag{2.1}$$

in which u_x represents the total number of arrivals during the xth cycle. The u_x are independent random variables having a binomial distribution

$$\Pr\{u_x = m\} = \binom{r+g}{m}(1-\alpha)^{r+g-m}\alpha^m. \tag{2.2}$$

Our problem here is to find the equilibrium distribution for q_x. Once this has been found and $E(q_x)$ evaluated, the average waiting time per car measured in units of the time interval between consecutive time points can be evaluated from the formula derived by BMW,

$$w = r(1-\alpha)^{-1}(g+r)^{-1}[E(q_x)/\alpha + (r+1)/2]. \tag{2.3}$$

Relation (2.1) is equivalent to the recursion formula for a queue with bulk service of g customers at a time. It has been studied previously by Bailey [1] and Downton [3], [4] when arrivals have a Poisson distribution and service time a χ^2 distribution (a special case of which is service at constant time intervals). Some of the analysis here for a binomial distribution of arrivals, particularly Section 4, closely parallels the analysis described by Bailey.

3. Low Rates of Arrival. One method of determining the equilibrium distribution of q_x is to take any initial distribution, for example $q_1 = 0$ with probability one, and evaluate the distribution for q_2, q_3, etc. from (2.1) and (2.2). If the average rate of arrivals per cycle is less than the maximum rate of departure, i.e.,

$$\alpha(r+g) < g \tag{3.1}$$

then this sequence of distributions will always converge to the equilibrium distribution.

If, in addition to (3.1), the difference between these rates is larger than the dispersion of u_x, i.e.,

$$g - \alpha(r+g) > [\alpha(1-\alpha)(r+g)]^{\frac{1}{2}}, \tag{3.2}$$

then $\Pr\{q_2 > 0\}$ will be small compared with $\Pr\{q_2 = 0\}$, $\Pr\{q_2 > 0 \text{ and } q_3 > 0\}$ will be relatively much smaller yet, and the sequence of distributions for q_2, q_3, etc. will converge rapidly to the equilibrium distribution, the more rapidly the larger the difference in the two sides of (3.2).

If we take $\Pr\{q_1 = 0\} = 1$, the next approximation to the equilibrium distribution is

$$\begin{aligned} \Pr\{q_2 = j\} &= \binom{r+g}{g+j}(1-\alpha)^{r-j}\alpha^{g+j}, \qquad j > 0, \\ \Pr\{q_2 = 0\} &= 1 - \sum_{j=1}^{r} \Pr\{q_2 = j\}. \end{aligned} \tag{3.3}$$

The evaluation of the distributions for q_3, q_4, etc. is straightforward but becomes quite tedious.

Since, in most practical applications, r and g are in the range of 10 to 20, we expect that estimations of $E(q_x)$ in the limit r and $g \to \infty$ with r/g fixed will be of some value. In this limit, (3.3) can be used to approximate $E(q_x)$ whenever

$$\mu \equiv [g - \alpha(r + g)][rg/(r + g)]^{-\frac{1}{2}} > 1, \tag{3.4}$$

a condition which excludes only a range of α in which the difference between α and the critical value, $g/(r + g)$, is of order $r^{-\frac{1}{2}}$. For r sufficiently large, this excluded range can be made arbitrarily small but if $r = g = 10$, for example, it is from $\alpha \sim 0.38$ to 0.5 and for $r = g = 20$ from $\alpha \sim 0.42$ to 0.5.

From (3.3) we obtain for $0 < j \ll r$

$$\Pr\{q_2 = j\} = \left\{\frac{(r + g)!(1 - \alpha)^r\alpha^g}{g!r!}\right\} \left\{\frac{r\alpha}{g(1 - \alpha)}\right\}^j \exp\left\{\frac{-j^2(r + g)}{2rg} + O\left(\frac{j^3}{r}\right)\right\} \tag{3.5}$$

and

$$E(q_2) = \frac{(r + g + 1)!(1 - \alpha)^{r+1}\alpha^{g+1}}{g!r!\mu^2}[1 + O(\mu^{-2})]. \tag{3.6}$$

If we disregard the smaller values of α and assume that $\mu \ll r^{1/6}$, then (3.6) can be simplified further by using Stirling's formula and expansions of log α in powers of μ to give

$$E(q_2) = \left[\frac{gr}{2\pi(r + g)}\right]^{\frac{1}{2}} \frac{\exp(-\mu^2/2)}{\mu^2}\left[1 + O\left(\frac{\mu^3}{r^{\frac{1}{2}}}\right) + O(\mu^{-2})\right]. \tag{3.7}$$

For $\mu^2 \gg 1$, we can also estimate that $E(q_3)$ will differ from $E(q_2)$ only by an additional term that is smaller than $E(q_2)$ by a factor proportional to $\exp(-\mu^2/2)$.

Whereas in practical applications, the error terms in (3.6) or (3.7) may be quite significant, these equations at least give an accurate description of what happens for sufficiently large r and μ and a qualitative description even for moderately large r. In the range $\mu > 1$, $E(q_2)$ is a monotone increasing function of α and is of order $r^{\frac{1}{2}}$ for $\mu = O(1)$. For $r \gg 1$, μ is a rapidly varying function of α and as α decreases $E(q_2)$ also decreases very rapidly. Even for the largest α at which we may apply these formulas, however, where $E(q_x)$ is of order $r^{\frac{1}{2}}$, the effect of the queue on w is small because in (2.3) $E(q_x)$ must be added to another term that is of order r. For $x = g = 10$, $E(q_x)$ causes only about a 20% increase in w even when $\mu = 1$.

To investigate what happens for $\mu < 1$, we consider below a different method of evaluating $E(q_x)$

4. Use of Generating Functions. Let

$$G_x(z) = \sum_{j=0}^{\infty} z^j \Pr\{q_x = j\} \tag{4.1}$$

denote the probability generating function (p.g.f.) for q_x. From (2.2) u_x has the p.g.f. $(1 - \alpha + \alpha z)^{r+g}$ and, since u_x and q_x are independent, $u_x + q_x - g$ has the p.g.f. $(1 - \alpha + \alpha z)^{r+g}z^{-g}G_x(z)$. If we subtract from this the probabilities for negative values of $u_x + q_x - g$ and reassign them to the event $q_{x+1} = 0$, we obtain from (2.1) the p.g.f. for q_{x+1},

$$G_{x+1}(z) = z^{-g}\left[(1 - \alpha + \alpha z)^{r+g}G_x(z) - \sum_{k=0}^{g-1} a_k z^k\right] + \sum_{k=0}^{g-1} a_k, \tag{4.2}$$

in which the a_k are the Taylor expansion coefficients of $(1 - \alpha + \alpha z)^{r+g}G_x(z)$.

If there is an equilibrium distribution for the queue length with $G_{x+1}(z) = G_x(z) \equiv G(z)$ then (4.2) gives

$$G(z) = Q^{-1}(z)\left[z^g \sum_{k=0}^{g-1} a_k - \sum_{k=0}^{g-1} a_k z^k\right] \tag{4.3}$$

with

$$Q(z) = z^g - (1 - \alpha + \alpha z)^{r+g}. \tag{4.4}$$

We do not know the a_k unless we know $G(z)$, but (4.3) and (4.4) at least describe the form of $G(z)$, a polynomial of degree g divided by another polynomial of degree $r + g$. We also know that, if $G(z)$ is a p.g.f., it must be analytic in the unit circle $| z | \leqq 1$ of the complex plane, and in particular at any points in this circle where $Q(z)$ has a zero.

Since $Q(z)$ is analytic, the number of zeros of $Q(z)$ inside or on the circle $| z | = 1$ is equal to g plus the number of cycles through which the complex phase of $z^{-g}Q(z)$ changes when z traverses a path just outside the unit circle, or equivalently g plus the number of times the image of this path under the transformation $z^{-g}Q(z)$ encircles the origin. Since for $| z | = 1$ and $0 < \alpha < 1$

$$| z^{-g}Q(z) - 1 | = | 1 - \alpha + \alpha z |^{r+g} \leqq 1,$$

with the last equality sign valid only at $z = 1$, the image of the unit circle itself passes through the origin once as z passes through $z = 1$ but otherwise lies to the right of the origin. Whether or not $z^{-g}Q(z)$ encircles the origin as z traverses a path just outside the unit circle is, therefore determined by what happens to $z^{-g}Q(z)$ for z in the neighborhood of $z = 1$. By expanding $z^{-g}Q(z)$ in a Taylor series about $z = 1$, one can easily show that as z passes to the right of $z = 1$, $z^{-g}Q(z)$ passes to the right of the origin if $\alpha(r + g) < g$ and so fails to encircle the origin but passes to the left of the origin thereby encircling it once if $\alpha(r + g) \geqq g$. We conclude from this that $Q(z)$ has g zeros inside or on the unit circle if $\alpha(r + g) < g$ but $g + 1$ zeros if $\alpha(r + g) \geqq g$. Since $\alpha(r + g) < g$

is also the condition for existence of an equilibrium distribution of q_x, only this case is of interest here.

If $G(z)$ is to be analytic for $|z| \leqq 1$, each of the g factors $(z - z_l)$ of $Q(z)$ with $|z_l| \leqq 1$ must cancel a corresponding factor of the gth degree polynomial in the numerator of $G(z)$ and $G(z)$ will reduce to the form

$$G(z) = A \prod_{l=1}^{r} (z - z_l)^{-1},$$

in which z_l, $l = 1, 2, \cdots, r$ are the r zeros of $Q(z)$ with $|z_l| > 1$. Since, in addition, any p.g.f. must satisfy the condition $G(1) = 1$, we finally obtain

$$G(z) = \prod_{l=1}^{r} (1 - z_l)(z - z_l)^{-1} \tag{4.5}$$

and

$$E(q_x) = dG(z)/dz \,|_{z=1} = \sum_{l=1}^{r} (z_l - 1)^{-1}. \tag{4.6}$$

The study of the q_x distribution is thus reduced to a study of the roots z_l of $Q(z)$ with $|z_l| > 1$.

It is not generally possible to obtain explicit expressions for the roots z_l, but they must all lie on a curve of the complex plane defined by the equation

$$|z| = |1 - \alpha + \alpha z|^{(r+g)/g}. \tag{4.7}$$

For any specified direction of z in the complex plane, one can sketch the graphs of the two sides of (4.7) as a function of $|z|$ and show that for $\alpha(r + g) < g$, the two graphs always intersect twice, once for $|z| \leqq 1$ and once for $|z| > 1$. The curve of (4.7), therefore, consists of two closed paths C' and C such as shown in Fig. 1, one lying inside the unit circle and the other outside.

The roots z_l must also satisfy the equation

$$z_l^{g/r} = \gamma_l(1 - \alpha + \alpha z_l)^{(r+g)/r}, \tag{4.8}$$

with $\gamma_l^r = 1$, and one can show that there is one and only one root of (4.8) on the curve C of Fig. 1 corresponding to each of the r distinct values of γ_l with $\gamma_l^r = 1$. By suitable numbering of the roots z_l we can choose γ_l so that

$$\gamma_l = \exp[2\pi i(l - 1)/r]. \tag{4.9}$$

We can also interpret (4.8) as a one-to-one mapping of the r roots on C into the r values of γ_l equally spaced around the unit circle. If we let r and $g \to \infty$ keeping r/g and α fixed, the curve C also stays fixed but the values of γ_l become densely and uniformly distributed on the unit circle. At the same time, the roots z_l become dense on C.

We know already from Section 3 that for the above limiting process $E(q_x) \to 0$. This can also be derived from (4.6) by observing that for $r \to \infty$ the sum in (4.6) becomes the Riemann sum for an integral which we may interpret as

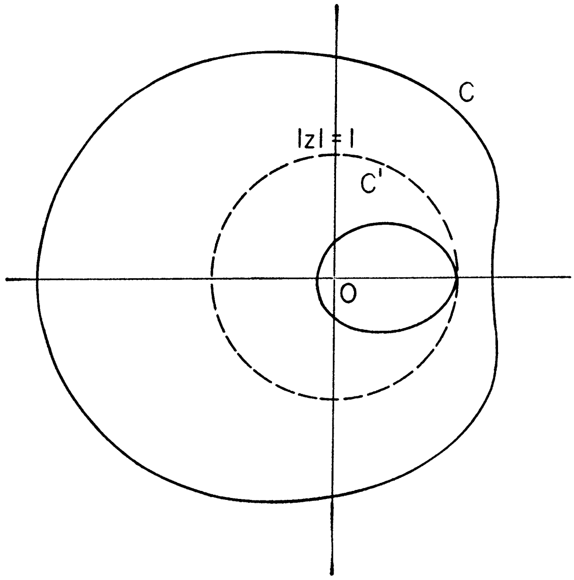

FIG. 1

either an integral with respect to the continuous real variable l or with respect to the complex variable γ around the unit circle. If we choose the last form we find

$$r^{-1}E(q_x) \to 1/2\pi i \int \{\gamma[z(\gamma) - 1]\}^{-1}\, d\gamma. \tag{4.10}$$

The function $z(\gamma)$ defined by (4.8) for $|\gamma| = 1$ is also defined for $|\gamma| > 1$. For $|\gamma| \geqq 1$, $z(\gamma)$ is analytic, $|z(\gamma)| > 1$, and is of order γ for $\gamma \to \infty$. The contour integral in (4.10) therefore vanishes by virtue of Cauchy's theorem. In addition, the difference between the Riemann sum of an analytic function and the integral over any closed path is asymptotically smaller than any finite power of the spacing between points. $E(q_x)$ is, therefore, smaller than any finite power of r^{-1} for $r \to \infty$.

If we define z_l for non-integer real l through (4.8) and (4.9), it follows that

$$\int_{\frac{1}{2}}^{r+\frac{1}{2}} (z_l - 1)^{-1}\, dl = 0.$$

By dividing this integral into r parts and subtracting it from (4.6), we can also write $E(q_x)$ as the difference between a Riemann sum and its limiting integral, i.e.

$$E(q_x) = \sum_{l=1}^{r} \left\{ (z_l - 1)^{-1} - \int_{l-\frac{1}{2}}^{l+\frac{1}{2}} (z_l - 1)^{-1}\, dl \right\}. \tag{4.11}$$

5. Nearly Critical Arrival Rate. Equations (4.6) and (4.11) are particularly well suited to the evaluation of $E(q_x)$ when $\alpha \to g/(r+g)$ because in this case we find that $z_1 \to 1$ and the one term of (4.6) for $l = 1$ becomes infinite. If, however, we let $r \to \infty$ and $\alpha \to g/(r+g)$ simultaneously then some of the neighboring roots to z_1, for example z_2 and z_r also approach 1.

Since z_l is defined by (4.8) and (4.9) also for negative values of l and is periodic in l with period r, we may consider l in the range $-r/2 < l \leqq r/2$, for example, so that the roots nearest to z_1 are those with small $|l|$, $l = \cdots -1, 0, 2, \cdots$. To locate these roots we take the logarithm of both sides of (4.8) and expand in powers of $(z_l - 1)$ and μ to obtain

$$-4\pi i(l-1)(r+g)r^{-1}g^{-1} - 2(z_l - 1)(r+g)^{\frac{1}{2}}(rg)^{-\frac{1}{2}}\mu + (z_l-1)^2 + O[(z_l-1)^2\mu r^{-\frac{1}{2}}, \quad (z_l-1)^3] = 0.$$

The roots of this approximately quadratic equation with $|z_l| > 1$ are

$$(5.1) \qquad z_l - 1 = (r+g)^{\frac{1}{2}}(rg)^{-\frac{1}{2}}\{\mu + [\mu^2 + 4\pi i(l-1)]^{\frac{1}{2}}\} + O[(z_l-1)^2]$$

and in particular

$$(5.2) \qquad z_1 - 1 = 2(r+g)^{\frac{1}{2}}(rg)^{-\frac{1}{2}}\mu + O[(r+g)r^{-1}g^{-1}\mu^2].$$

We conclude immediately from this that, if r and g are finite,

$$(5.3) \qquad \begin{aligned} E(q_x) &= (z_1 - 1)^{-1} + O(1), && \text{for } \mu \to 0, \\ &= rg\{2(r+g)[g - \alpha(r+g)]\}^{-1} + O(1), && \text{for } \alpha \to g/(r+g), \end{aligned}$$

in which $O(1)$ here means order relative to μ as $\mu \to 0$ but not relative to r and g.

Suppose we now let $r \to \infty$ with μ fixed, particularly with $\mu \lesssim 1$ since this is the only case that could not be handled satisfactorily in Section 3. Except when $l \ll r$ and $z_l \sim 1$, the difference between $(z_l - 1)^{-1}$ and its integral between $l - \frac{1}{2}$ and $l + \frac{1}{2}$ is of the order of magnitude of the second derivative of $(z_l - 1)^{-1}$ with respect to l, which in turn is of order r^{-2} according to (4.8) and (4.9). The sum of all such terms in (4.11) is at most of order r^{-1} and so any significant contribution to (4.11) can come only for the small values of $|l|$ where (5.1) is applicable. From (4.11) and (5.1), we obtain

$$(5.4) \qquad E(q_x) = \left(\frac{rg}{r+g}\right)^{\frac{1}{2}} \sum_{l=-\infty}^{l+\infty} \left\{(\mu + [\mu^2 + 4\pi i(l-1)]^{\frac{1}{2}})^{-1} - \int_{l-\frac{1}{2}}^{l+\frac{1}{2}} (\mu + [\mu^2 + 4\pi i(l-1)]^{\frac{1}{2}})^{-1}\, dl\right\} + O(1)$$

with the dominant error team of $O(1)$ relative to r coming from the error term of (5.1), particularly for $l = 1$ and to a lesser extent from the other l with $|l| \ll r$.

The terms in the sum (5.4) are $O(l^{-\frac{3}{2}})$ for $|l| \gg \mu$, so the series converges rapidly enough to be of practical use even for $\mu \sim (4\pi)^{\frac{1}{2}} \sim 3$. For small μ, the main contribution, however, comes from $l = 1$ where the first term in the bracket of (5.4) is $(2\mu)^{-1}$ while all other contributions to the series are at most of order

1 even for $\mu \to 0$. Generally we obtain for μ of order 1 or less

$$(5.5) \qquad E(q_x) = [rg/(r+g)]^{\frac{1}{2}}[(2\mu)^{-1} + O(1)]$$

and for $\mu \lesssim 1$ we can expand (5.4) in powers of μ to obtain

$$(5.6) \qquad E(q_x) = \left(\frac{rg}{r+g}\right)^{\frac{1}{2}}\left[\frac{1}{2\mu} - A + \frac{\mu}{4} + O(\mu^2)\right]$$

with

$$(5.7) \qquad A = (2\pi)^{-\frac{1}{2}} \lim_{R\to\infty} [2(R+\tfrac{1}{2})^{\frac{1}{2}} - \sum_{l=1}^{R} l^{-\frac{1}{2}}] \sim 0.582.$$

One can estimate that $O(\mu^2)$ is roughly $-\mu^2/20$ and in succeeding terms the important parameter is $\mu/(4\pi)^{\frac{1}{2}}$, so that (5.6) will be correct to within about 30% even for $\mu = 1$. The error in (3.7) for $\mu = 1$ should be of comparable size and if one compares (5.6) with (3.7) one finds that they agree to within a factor of about $\frac{3}{2}$ for $r \to \infty$ and $\mu = 1$.

Since for $r \to \infty$, the effect of the queue on w will not be significant unless $E(q_x)$ is of order r, this will not occur unless μ is $O(r^{-\frac{1}{2}})$ and $g - \alpha(r+g) = O(1)$. If, in fact, $g - \alpha(r+g) = 1$ the queue causes w to increase by a factor of 2.

We note finally that for certain values of $(r+g)/r$, namely 2, $\frac{3}{2}$, 3, $\frac{4}{3}$ and 4, one can obtain exact expressions for the roots z_l by virtue of the fact that (4.8) gives a set of quadratic, cubic or quartic equations. One can, therefore, also obtain exact explicit formulas for $E(q_x)$ and w. If, for example, $r = g$, then

$$(5.8) \qquad z_l - 1 = \tfrac{1}{2}\gamma_l^{-1}\alpha^{-2}\{1 - 2\alpha\gamma_l + [1 - 4\alpha(1-\alpha)\gamma_l]^{\frac{1}{2}}\}$$

in which the square root must be chosen in the right half of the complex plane to give $|z_l| > 1$. In particular

$$(5.9) \qquad z_1 - 1 = \alpha^{-2}(1 - 2\alpha).$$

6. Comparison with Webster's Formula. The only formula with which we can compare the above results is Webster's semi-empirical formula [8] for delays which is based upon the assumption that the arrivals form a Poisson distribution rather than a binomial distribution as assumed here. Webster's formula consists of three terms; the first is essentially the same as (2.3) with $E(q_x) = 0$ and represents the delay for regularly spaced arrivals; the second term is the delay that results from a queue when arrivals have a Poisson distribution but the service time is a constant equal to $(r+g)/g$ time intervals; and the third term is an empirical correction obtained by fitting curves to values calculated by computer simulation.

Since a Poisson distribution allows arbitrary small time intervals between arrivals, fluctuations may cause more cars to arrive in some green period than can leave. Because of this one finds for a Poisson distribution of arrivals that even when $r \to 0$ (no traffic light) one still has a queue and furthermore the average length of the queue becomes infinite as the arrival rate approaches the critical value. As pointed out by BMW, the binomial distribution has the advantage of forcing a minimum spacing between cars and so we avoid this unfortunate limiting behavior, even though this is accomplished in a somewhat artificial way wherein the spacings are confined to be integer multiples of the minimum spacing.

By using methods very similar to those described in Sections 1 to 5, it is possible also to compute the queue lengths and delays when the arrivals have a Poisson distribution, provided we assume that (2.1) still holds. We need only replace the p.g.f. for the binomial distribution of u_x by the corresponding expression for the Poisson distribution. By doing this one finds as the analogue of (5.5)

$$E(q_x) = \tfrac{1}{2}g[g - \alpha(r + g)]^{-1} + O(r^{\frac{1}{2}}), \tag{6.1}$$

the leading term of which is $(g + r)/r$ times as large as in (5.5). The average waiting time for nearly critical arrival rate is then given by

$$w = r^2/\{2(1 - \alpha)(g + r)\} + (g + r)/\{2[g - \alpha(r + g)]\} + O(r^{\frac{1}{2}}). \tag{6.2}$$

Furthermore, in (6.1) and (6.2), the $O(r^{\frac{1}{2}})$ are asymptotically proportional to $r^{\frac{1}{2}}$.

The first term of (6.2) is essentially the same as the first term of Webster's formula and the coefficient of $[g - \alpha(r + g)]^{-1}$ in the second term has the same value as in Webster's formula for $\alpha \to g/(r + g)$. The third term of Webster's formula, however, is not asymptotically proportional to $r^{\frac{1}{2}}$, nor does it indicate in any obvious way the importance of the magnitude of $g - \alpha(r + g)$ as compared with $[rg/(r + g)]^{\frac{1}{2}}$.

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