QUEUES SUBJECT TO SERVICE INTERRUPTION

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Summary. A treatment is given of the M/G/1 queue with interruptions of Poisson incidence occasioned either by server breakdown or the arrival of customers with higher priority. Interruption times and priority service times have arbitrary distribution. After pre-emptive interruption, ordinary service is either repeated or resumed. The time dependent behavior of the system is discussed in a complete state space and the joint density in all system variables of this space is constructed systematically from the densities associated with a set of simpler first-passage problems. Equilibrium distributions are available as limiting forms and server busy period distributions obtained.

1. Introduction. Queues subject to interruption due to service breakdown or the arrival of priority customers have been extensively treated in the literature, and bibliographies covering all but the most recent work may be found in the papers of Miller [9] and Heathcote [3]. Three types of service discipline governing the disposition of the ordinary customer in service when an interruption occurs have been discussed, postponeable or "head-of-the-line" discipline, "pre-emptive resume" discipline, and "pre-emptive repeat" discipline. A detailed description of these may be found in Section 2.

Until recently, the only service time and interruption time distributions considered were exponential, limiting somewhat severely the usefulness of the results. This paper was motivated at its inception by the need for a treatment considering more general distributions. Since the paper was first presented,[1] however, two such treatments by Gaver [2] and Jaiswal [4, 5] have been published which duplicate many of the results obtained and make a lengthy exposition undesirable. A condensed version of the original paper [6] is here presented emphasizing those elements of the methodology and those results which are believed to be new.

As in the phase space procedure previously developed [7, 8] for the discussion of multidimensional systems, the supplementary variables required for a Markovian characterization of the queue length processes are explicitly carried. A system of equations describing the motion in this phase space may be written down but is of considerable complexity. The disadvantages associated with a formal analysis [4, 5] of the system may be avoided by a constructive description of the phase space motion compounded successively from the distributions associated with a set of simpler first passage problems. The procedure enables one to treat all three disciplines simultaneously and to generate the joint time dependent distribution of all system variables, from which the joint or individual

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distribution of any subset of variables may be deduced. Equilibrium distributions may be examined as limiting forms and server busy period distributions extracted from the time-dependent solution. A similar structure may be seen in the extended chain method treatment of Gaver [2], who does not, however, deal with the joint queue length distributions.

The basic first passage process, given for each of the three service disciplines in Section 2, is the server sojourn process describing the state of the server from the time an ordinary customer enters service to the time the server is available for a next ordinary customer. The description of the full process (Section 3) follows from standard results for the M/G/1 queuing system.

Throughout the paper, discussion will be conducted in terms of probability densities rather than distributions. These densities and the operations of integration and differentiation applied to them may either be understood in a generalized sense, such as that of the theory of distributions of L. Schwartz, or regarded as limiting forms of functions on which the operations may be unambiguously performed. Effort will be made wherever possible to present results in real time. The paper however is principally intended as methodological.

2. The basic server sojourn process. The process commences at time $\theta = 0$ with the entry of an ordinary customer having a service time requirement s_0 of probability $D_0(s_0)$. For the preemptive disciplines, service of the customer is subject to interruptions with Poisson incidence and frequency λ_1, the interruptions having duration s_1 of probability density $D_1(s_1)$. These interruptions may be due to breakdown or to a sequence of priority customers the last of whom must be served before service can be resumed. After an interruption, service may continue from its phase at interruption (resume policy) or from the starting phase (repeat policy). When service is completed, the sojourn process ends in the completion state R. The server sojourn problem for head-of-the-line discipline describes the completion of service of the customer without interruption and the subsequent removal of all priority demands on the server.

(a) *Pre-emptive resume discipline.* We first consider the process for the resume policy. At time θ, the server will be in one of the following states:

Interruption state $I:(x_0, x_1)$ Service has been interrupted. Time x_1 has elapsed since interruption, at which time a total service time x_0 had been received by the customer.

Free state $F:(x_0)$ Customer is being served, and has now received a total x_0 of service time.

Rest state R The customer has completed service.

The states F, I, and R are mutually exclusive and exhaustive and provide a Markovian characterization of the server sojourn process. A typical trajectory in the phase space is shown in Figure 1. We are interested in the distribution on this phase space at time θ. Sojourn (first passage) probability densities will be denoted here and henceforth by script and Greek letters. State densities for the full process will be denoted by Roman letters. Let the probability densities

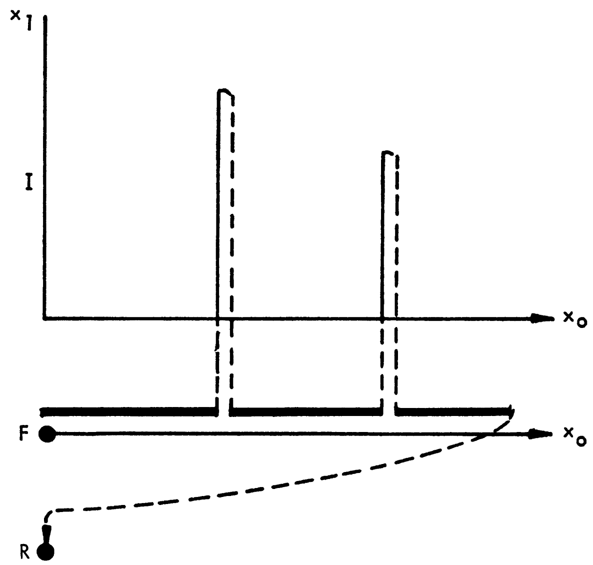

FIGURE 1

for states F and I be denoted by $\mathcal{W}_0(x_0, \theta)$ and $\mathcal{W}_1(x_0, x_1, \theta)$. Let $\eta_0(x_0)\, dx_0$ be the probability of service completion in the interval $(x_0, x_0 + dx_0)$ of a customer who has been in service for time x_0, so that $D_0(x_0) = \eta_0(x_0) \exp[-N_0(x_0)]$ where $N_0(x_0) = \int_0^{x_0} \eta_0(x_0)\, dx_0$. Similarly let $\eta_1(x_1)$ be the conditional interruption termination rate and $D_1(x_1) = \eta_1 \exp[-N_1]$. For motion in the subspace $I:(x_0, x_1)$, x_0 remains fixed and $\mathcal{W}_1(x_0, x_1, \theta)$ obeys $\partial\mathcal{W}_1/\partial\theta + \partial\mathcal{W}_1/\partial x_1 = -\eta_1(x_1)\mathcal{W}_1$ so that

$$\mathcal{W}_1(x_0, x_1, \theta) = \mathcal{W}_1(x_0, 0, \theta - x_1) \exp[-N_1(x_1)].$$

For all x_0 and θ, moreover, continuity of probability requires that $\mathcal{W}_1(x_0, 0, \theta) = \lambda_1\mathcal{W}_0(x_0, \theta)$ so that the density $\mathcal{W}_1(x_0, x_1, \theta)$ is related to $\mathcal{W}_0(x_0, \theta)$ by

$$\mathcal{W}_1(x_0, x_1, \theta) = \lambda_1\mathcal{W}_0(x_0, \theta - x_1) \exp[-N_1(x_1)]. \tag{2.1}$$

Another simple continuity argument leads to the equation

$$\left[\frac{\partial}{\partial x_0} + \frac{\partial}{\partial\theta} + \lambda_1 + \eta_0(x_0)\right]\mathcal{W}_0(x_0, \theta) = \int \mathcal{W}_1(x_0, x_1, \theta)\eta_1(x_1)\, dx_1 \tag{2.2}$$
$$= \lambda_1 \mathcal{W}_0(x_0, \theta) * D_1(\theta)$$

the asterisk denoting convolution in θ. Boundary conditions on (2.2) at $\theta = 0$ and $x_0 = 0$ are given by

$$\mathcal{W}_0(x_0, 0) = 0; \qquad \mathcal{W}_0(0, \theta) = \delta(\theta - 0+) \tag{2.3}$$

where $\delta(\theta)$ is the Dirac delta function. Laplace transformation of (2.2) with respect to θ and solution of the elementary differential equation resulting yields

$$\omega_0(x_0, p) = \exp\{-N_0(x_0) - \lambda_1[1 - d_1(p)]x_0 - px_0\}. \tag{2.4a}$$

Also from (2.1)

$$\omega_1(x_0, x_1, p) = \lambda_1\omega_0(x_0, p) \exp[-N_1(x_1) - px_1]. \tag{2.4b}$$

ω_0, ω_1 and d_1 denote the Laplace transforms of $\mathcal{W}_0$, $\mathcal{W}_1$, and D_1 respectively.

The probability of service completion $R(\theta)$ is found from $R(0) = 0$ and

$$\frac{dR}{d\theta} = \int_0^\infty \mathcal{W}_0(x_0, \theta)\eta_0(x_0)\, dx_0 = D_e(\theta). \tag{2.5}$$

Equation (2.5) defines $D_e(s_e)$, the density of effective service times s_e for resume discipline. From (2.4a), $D_e(s_e)$ is given immediately in transformed form by

$$d_e(p) = d_0(p + \lambda_1[1 - d_1(p)]) \tag{2.6}$$

where d_e and d_0 correspond to D_e and D_0. We observe from (2.6) that when the rate of interruption or the duration of interruptions goes to zero, $d_e(p)$ becomes simply $d_0(p)$ as it must. It is also clear that if interruptions always terminate, i.e., if $d_1(0) = 1$, $d_e(0) = 1$ and a customer will always complete service. The state densities and effective service time density are readily exhibited in real time. Thus from (2.4a) we have

$$\mathcal{W}_0(x_0, \theta) = \sum_0^\infty [(\lambda_1 x_0)^n/n!]\, D_1^{(n)}(\theta - x_0) \exp\{-N_0(x_0) - \lambda_1 x_0\} \tag{2.7}$$

where $D_1^{(n)}(\theta)$ is the n-fold convolution of $D_1(\theta)$ with itself. $D_1^{(0)}(\theta)$ is the delta function $\delta(\theta)$. $\mathcal{W}_1(x_0, x_1, \theta)$ is then given by (2.1). For $D_e(\theta)$ we find from (2.6)

$$D_e(\theta) = \sum_0^\infty \{[(\lambda_1\theta)^n/n!]e^{-\lambda_1\theta}D_0(\theta)\} * D_1^{(n)}(\theta). \tag{2.8}$$

In (2.8) the asterisk denotes convolution. When $D_0(\theta)$ and $D_1(\theta)$ are both Erlangian functions, $D_e(\theta)$ may be expressed via (2.8) as a series of Erlangian functions.

When interruptions are due to the arrival of priority customers at rate λ_P with service time density $D_P(s_P) = \eta_P(s_P)e^{-N_P(s_P)}$ the state of the server may be described by the states $F:(x_0)$ and $I:(x_0\,;\, m_P, x_P)$ in place of $F:(x_0)$ and $I:(x_0, x_1)$. Here m_P is the number of priority customers in line, and x_P is the length of time the priority customer in service has been in service. If $\mathcal{W}_P^{[k]}(m_P, x_P\,;\,\theta)$ is employed to designate the density of states in the space $\{(m_P, x_P)\}$ at time θ for a tour initiated in that space at $(k, 0)$, and terminated by departure of the last priority customer, and if $\mathcal{W}_0(x_0, \theta)$ and $\mathcal{W}_1(x_0\,;\, m_P, x_P\,;\,\theta)$ denote the densities for the server sojourn tour initiated by the entry of an ordinary customer into service and terminated by his departure from service, then in parallel with (2.1) we have

$$\mathcal{W}_1(x_0\,;\, m_P, x_P\,;\,\theta) = \lambda_P\mathcal{W}_0(x_0, \theta) * \mathcal{W}_P^{[0]}(m_P, x_P\,;\,\theta), \tag{2.9}$$

where the asterisk denotes convolution in θ. For the Laplace transforms of the generating functions, (2.9) becomes

$$\gamma_1(x_0\,;\,u_P\,,\,x_P\,;\,p) = \lambda_P\omega_0(x_0\,,\,p)\gamma_P^{[0]}(u_P\,,\,x_P\,;\,p). \tag{2.10}$$

The transformed generating function of $\mathcal{W}_P^{[k]}$ may be obtained quickly from the M/G/1 state space discussion of Keilson and Kooharian [7] or Gaver [1] to be

$$\begin{aligned}\gamma_P^{[k]}&(u_P\,,\,x_P\,;\,p)\\ &= \{u_P^{k+1} - r_{Pp}^{k+1}/[u_P - d_P(p + \lambda_P[1 - u_P])]\}e^{-N_P(x_P)}e^{-\lambda_P(1-u_P)x_P}.\end{aligned} \tag{2.11}$$

In (2.10) and (2.11) r_{Pp} is the unique root of the denominator inside the unit circle. $D_1(s_1)$ is now the classical M/G/1 server busy period density $S_P(s_1)$ for the service of the priority customers, with transform $s_P(p) = r_{Pp}$. $D_e(s_e)$, the effective service time density, continues to be given by (2.8). $S_P(s_1)$ is given in real time [10] by

$$S_P(s_1) = \sum_0 e^{-\lambda_P s_1}(\lambda_P s_1)^n D_P^{(n+1)}(s_1)/(n+1)!$$

(b) *Pre-emptive repeat discipline.* In the foregoing process our system resumes its sojourn in the state $F:(x_0)$ at the point x_0 at which it was interrupted. Next we shall briefly examine the process for repeat discipline in which service must start over at $x_0 = 0$ after an interruption. Again we will have state $F:(x_0)$ and state R. For the interruption state I, indexing with x_0 is now unnecessary since the server forgets where he was at interruption, but for convenience of notation we will retain x_0 in the state designation and the corresponding density. The more important subcase of repeat discipline is that where the customer diverted is returned to service and his service time requirement is unchanged. We then find by an argument similar to that for resume discipline

$$\omega_0(x_0\,,\,p) = \int_0^\infty \left\{\frac{ds D_0(s)U(s - x_0)e^{-\lambda_1 x_0 - p x_0}}{1 - \lambda_1 d_1(p)[1 - e^{-(p+\lambda_1)s}](p + \lambda_1)^{-1}}\right\} \tag{2.4'a}$$

where $U(s)$ is the unit step function, vanishing for negative s,

$$\omega_1(x_0\,,\,x_1\,,\,p) = \int_0^\infty \left\{\frac{ds D_0(s)[1 - e^{-(\lambda_1+p)s}]\lambda_1 \exp\{-N_1(x_1) - p x_1\}}{\lambda_1 + p - \lambda_1 d_1(p)[1 - e^{-(\lambda_1+p)s}]}\right\} \tag{2.4'b}$$

and

$$d_e(p) = \int_0^\infty \frac{ds D_0(s)e^{-(\lambda_1+p)s}}{1 - \lambda_1 d_1(p)[1 - e^{-(p+\lambda_1)s}](p + \lambda_1)^{-1}}. \tag{2.4'c}$$

When the state of the server is described in terms of the variables m_P and x_P, the sojourn tour densities $\omega_0(x_0\,,\,p)$, $\omega_1(x_0\,;\,m_P\,,\,x_P\,;\,p)$, γ_1 $(x_0\,;\,u_P\,,\,x_P\,;\,p)$ and $d_e(p)$ are obtained from (2.4′a), (2.4′b) and (2.4′c) by the substitutions $\lambda_1 = \lambda_P$, $d_1(p) = s_P(p)$, and replacement of $e^{-N_1(x_1)-p x_1}$ by $\gamma_P^{[0]}(u_P\,,\,x_P\,;\,p)$.

(c) *Head-of-the-line discipline.* The server sojourn process commences with the entry of an ordinary customer. After he has completed service, any priority demands that have appeared during the service period are given attention. The server sojourn process ends when all priority demands on hand have been accommodated. The number of priority customers appearing will depend on the service period of the ordinary customer and the interruption duration will vary accordingly. If time s_0 has been spent in service, m_P priority customers will be present at the end of the service period with probability $(\lambda_P s_0)^{m_P} e^{-\lambda_P s_0}/m_P!$. A period s_1 with density $S_P^{(m_P)}(s_1)$ must be expended subsequently to eliminate all priority demands that will have presented themselves. The interruption duration subsequent to a service period s_0 has density

$$D_{1s_0}(s_1) = \sum_0 [(\lambda_P s_0)^m/m!]e^{-\lambda_P s_0} S_P^{(m)}(s_1)$$

so that the density $D_e(s_e)$ of the clearance time $s_e = s_0 + s_1$ is again given by (2.8). The density $\mathcal{W}_0(x_0, \theta)$ is just $e^{-N_0(x_0)}\delta(x_0 - \theta)$ and $\mathcal{W}_1(x_0 ; m_P, x_P ; \theta)$, where x_0 is again retained for uniformity of notation, is given by

$$\sum_{m=0}^{\infty} \{D_0(\theta)(\lambda_P\theta)^{m+1}[(m+1)!]^{-1}e^{-\lambda_P\theta}\} * \mathcal{W}_P^{(m)}(m_P, x_P ; \theta).$$

For the transformed generating functions we then have from (2.11)

$$\gamma_0(x_0, p) = e^{-N_0(x_0)}e^{-px_0} \tag{2.12}$$

and

$$\gamma_1(x_0; u_P, x_P; p) = \left\{\frac{d_0(p + \lambda_P[1 - u_P]) - d_0(p + \lambda_P[1 - r_{Pp}])}{u_P - d_P(p + \lambda_P[1 - u_P])}\right\} e^{-N_P(x_P)}e^{-\lambda_P(1-u_P)x_P}. \tag{2.13}$$

3. Associated queuing process. The server sojourn processes of Section 2 may be combined with elementary results for the M/G/1 queue to give a complete discussion of the queue length process for a queue with general interruptions, and of the joint queue length process for priority queues.

Consider such a system from the point of view of the ordinary customers. We assume that (a) such customers arrive at rate λ_0 with Poisson interarrival time density; (b) if the server is attending an ordinary customer there is probability per unit time λ_1 of interruption; (c) if the server is idle there is probability per unit time λ_2 of interruption. The use of distinct λ_1 and λ_2 enables us to treat simultaneously the cases $\lambda_2 = \lambda_1$ appropriate for priority queues and $\lambda_2 = 0$ appropriate for some breakdown situations. When the system is in the idle state, an ordinary customer will arrive with probability $\lambda_0/(\lambda_0 + \lambda_2)$ and enter service. The system will then embark on a sequence $\mathfrak{J}_0$ of sojourn tours each having a duration s_e of density $D_e(s_e)$ appropriate to the nature of the interruptions and the service discipline prevalent. If the queue is stable the sequence of sojourn tours will terminate at the idle state when a tour ends and no ordinary customers are in queue. With probability $\lambda_2/(\lambda_0 + \lambda_2)$, the system will leave the empty state and be unavailable to ordinary customers for a period s_2 with density $D_1(s_2)$. For interruptions due to priority customers $D_1(s_2) = S_P(s_2)$. At the end of the interruption, m_0 ordinary customers will have accumulated with probability $(\lambda_0 s_2)^{m_0} e^{-\lambda_0 s_2}/m_0!$. A sequence $\mathfrak{I}_{m_0-1}$ of sojourn tours starting with $(m_0 - 1)$ customers in line will then ensue, if $m_0 \neq 0$, ending at the idle state for the stable case. We refer to the first passage process commencing with departure from the idle state and terminating at the idle state as the regeneration tour. The time elapsed since the initiation of this tour will be denoted by τ.

Let $\mathcal{I}_{m_0}(\tau)$ be the probability per unit time that a sequence $\mathfrak{I}_{m_0}$ of sojourn tours will commence at time τ during the regeneration tour. Then

$$\mathcal{I}_{m_0}(\tau) = \frac{\lambda_0}{\lambda_0 + \lambda_2} \delta(\tau)\delta_{m_0,0} + \frac{\lambda_2}{\lambda_0 + \lambda_2} D_1(\tau) \frac{(\lambda_0 \tau)^{m_0+1}}{(m_0 + 1)!} e^{-\lambda_0 \tau} \qquad m_0 = 0, 1, 2 \cdots \tag{3.1}$$

In (3.1) $\delta(\tau)$ is the Dirac delta function. Each sojourn tour has duration s_e with density $D_e(s_e)$. The sequence $\mathfrak{I}_{m_0}$ of tours has duration v of density $S_e^{(m_0+1)}(v)$, where $S_e^{(1)}(v)$ is the M/G/1 server busy period density for arrival rate λ_0 and service time density $D_e(s_e)$. Consequently from (3.1) we have for the density of regeneration times v of the system

$$S(v) = \frac{\lambda_0}{\lambda_0 + \lambda_2} S_e^{(1)}(v) + \frac{\lambda_2}{\lambda_0 + \lambda_2} \sum_{m_0} [(\lambda_0 v)^{m_0} e^{-\lambda_0 v} D_1(v)/m_0!] * S_e^{(m_0)}(v) \tag{3.2}$$

with Laplace transform

$$s(p) = \frac{\lambda_0}{\lambda_0 + \lambda_2} s_e(p) + \frac{\lambda_2}{\lambda_0 + \lambda_2} d_1(p + \lambda_0[1 - s_e(p)]). \tag{3.3}$$

From $\mathcal{I}_{m_0}(\tau)$ and the same result in M/G/1 theory called upon for (2.11) we may obtain the probability per unit time $\mathfrak{A}_m(\tau)$ that a sojourn tour is initiated at time τ during the regeneration tour leaving m ordinary customers in line. The component $\mathfrak{A}_m^{[m_0]}(\theta)$ for a tour $\mathfrak{I}_{m_0}$ commenced at $\theta = 0$ has g.f. transform

$$\alpha^{[m_0]}(u, p) = \mathfrak{L}\left\{\sum_0^\infty u^m \mathfrak{A}_m^{[m_0]}(\theta)\right\} = [u^{m_0+1} - s_e(p)^{m_0+1}]/[u - d_e(p + \lambda_0[1 - u])].$$

Since $\mathfrak{A}_m(\tau) = \sum_{m_0} \mathcal{I}_{m_0}(\tau) * \mathfrak{A}_m^{[m_0]}(\tau)$, we have for $\alpha(u, p) = \mathfrak{L}\{\sum u^m \mathfrak{A}_m(\tau)\}$

$$\alpha(u, p) = \frac{\lambda_0[u - s_e(p)] + \lambda_2\{d_1(p + \lambda_0[1 - u]) - d_1(p + \lambda_0[1 - s_e(p)])\}}{(\lambda_0 + \lambda_2)\{u - d_e(p + \lambda_0[1 - u])\}}. \tag{3.4}$$

Consider now the full system motion. If the system is known to be in the idle state $\mathcal{E}$ at $t = 0$, and $E(t)$ denotes the probability that the system is idle at time t, the probability $A_m(t)$ per unit time of the entry of an ordinary customer into service leaving m in line is given by $A_m(t) = (\lambda_0 + \lambda_2)E(t) * \mathcal{A}_m(t)$, so that $a(u, p) = \mathcal{L}\{\sum_0 u^m A_m(t)\} = (\lambda_0 + \lambda_2)e(p)\alpha(u, p)$. Between ordinary customer entries, the number of ordinary customers arriving is independent of the coordinates describing the server sojourn state. If we denote by $W_0(m, x_0, t)$, and $W_1(m, x_0, x_1, t)$ the densities associated with the states $F:(x_0)$ and $I:(x_0, x_1)$, and their generating functions by $G_0(u, x_0, t)$ and $G_1(u; x_0, x_1, t)$, we have at once for the pre-emptive disciplines and general interruption

$$G_0(u, x_0, t) = [\sum u^m A_m(t)] * [e^{-\lambda_0(1-u)t}\mathcal{W}_0(x_0, t)]$$

$$G_1(u, x_0, x_1, t) = [\sum u^m A_m(t)] * [e^{-\lambda_0(1-u)t}\mathcal{W}_1(x_0, x_1, t)]$$

and for the Laplace transforms

$$g_0(u, x_0, p) = (\lambda_0 + \lambda_2)e(p)\alpha(u, p)\omega_0(x_0, p + \lambda_0(1 - u)) \tag{3.5}$$

and

$$g_1(u, x_0, x_1, p) = (\lambda_0 + \lambda_2)e(p)\alpha(u, p)\omega_1(x_0, x_1, p + \lambda_0(1 - u)) \tag{3.6}$$

where $\alpha(u, p)$ is given by (3.4). For the states (m, x_1) corresponding to in-interruptions that took place when the server was idle there is a density $W_2(m, x_1, t) = \lambda_2 E(t - x_1)(\lambda_0 x_1)^m e^{-\lambda_0 x_1} e^{-N_1(x_1)}/m!$ whose transformed generating function is given by

$$g_2(u, x_1, p) = \lambda_2 e(p) \exp[-N_1(x_1) - px_1 - \lambda_0(1 - u)x_1]. \tag{3.7}$$

For $E(t)$ we observe that $dE/dt = -(\lambda_0 + \lambda_2)E(t) + (\lambda_0 + \lambda_2)E(t) * S(t)$ where $S(t)$ is given by (3.2). We thus have

$$e(p) = \{p + (\lambda_0 + \lambda_2)[1 - s(p)]\}^{-1} \tag{3.8}$$

For a priority queue with any of the three basic disciplines, (3.4), (3.5), (3.7) and (3.8) retain their utility and (3.6) is replaced by

$$g_1(u, x_0; u_P, x_P; p) = (\lambda_0 + \lambda_2)e(p)\alpha(u, p)\gamma_1(x_0; u_P, x_P; p + \lambda_0[1 - u]) \tag{3.9}$$

where γ_1 is given by (2.10), (2.13) and in the paragraph below (2.4′c). The steady state distributions are obtained by the customary Tauberian procedure.

For all of the many results checked against those of Miller [9] Heathcote [3] Gaver [2] and Jaiswal [4, 5], agreement has been found.

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