Quiver varieties and cluster algebras

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To the memory of the late Professor Masayoshi Nagata

Abstract Motivated by a recent conjecture by Hernandez and Leclerc, we embed a Fomin-Zelevinsky cluster algebra into the Grothendieck ring $\mathbf{R}$ of the category of representations of quantum loop algebras $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ of a symmetric Kac-Moody Lie algebra, studied earlier by the author via perverse sheaves on graded quiver varieties. Graded quiver varieties controlling the image can be identified with varieties which Lusztig used to define the canonical base. The cluster monomials form a subset of the base given by the classes of simple modules in $\mathbf{R}$, or Lusztig's dual canonical base. The conjectures that cluster monomials are positive and linearly independent (and probably many other conjectures) of Fomin and Zelevinsky follow as consequences when there is a seed with a bipartite quiver. Simple modules corresponding to cluster monomials factorize into tensor products of "prime" simple ones according to the cluster expansion.

1. Introduction

1.1. Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky [21]. A cluster algebra $\mathscr{A}$ is a subalgebra of the rational function field $\mathbb{Q}(x_1, \dots, x_n)$ of n indeterminates equipped with a distinguished set of variables (cluster variables) grouped into overlapping subsets (clusters) consisting of n elements, defined by a recursive procedure (mutation) on quivers. Let us quote the motivation from the original text [21, p. 498, second paragraph]:

> This structure should serve as an algebraic framework for the study of "dual canonical bases" in these coordinate rings and their q-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the cluster *monomials*) belong to this dual canonical basis.

Here dual canonical base means a conjectural analogue of the dual of Lusztig's [43] canonical base of $\mathbf{U}_q^-$, the $-$ part of the quantized enveloping algebra. One of the deepest properties of the dual canonical base is positivity: the structure constants are in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$. But the existence and positivity are not known for cluster algebras except some examples.

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The theory of cluster algebras has been developed in various directions different from the original motivation (see the list of references in a recent survey [37]).

One of the most active directions is the theory of the cluster category (see [6]). It is defined as the orbit category of the derived category $\mathscr{D}(\operatorname{rep}\mathcal{Q})$ of finite-dimensional representations of a quiver $\mathcal{Q}$ under the action of an automorphism. This theory is quite useful to understand combinatorics of the cluster algebra: clusters are identified with tilting objects, and mutations are interpreted as exchange triangles (see the survey [37] for more detail).

However, the cluster category *does not* have enough structures, compared with the cluster algebra. For example, multiplication of the cluster algebra roughly corresponds to the direct sum of the cluster category, but addition remains obscure. So the cluster category is called *additive categorification* of the cluster algebra. The cluster algebra is recovered from the cluster category by the so-called cluster character. (Some call it the Caldero-Chapoton map.) But it is not clear how to obtain all the dual canonical base elements from this method.

Very recently, Hernandez and Leclerc [31] proposed another categorical approach. They conjecture that there exists a monoidal abelian category $\mathscr{M}$ whose Grothendieck ring is the cluster algebra. All of the structures of the cluster algebra can be conjecturally lifted to the monoidal category. For example, the dual canonical base is given by simple objects, the combinatorics of mutation is explained by decomposing tensor products into simple objects, and so on. Here we give the table of structures:

cluster algebra	*additive categorification*	*monoidal categorification*
$+$	?	$\oplus$
$\times$	$\oplus$	$\otimes$
clusters	cluster tilting objects	real simple objects
mutation	exchange triangle	$0 \to S \to X_i \otimes X_i^* \to S' \to 0$
cluster variables	rigid indecomposables	real prime simple objects
dual canonical base	?	simple objects
?	?	prime simple objects

In the bottom line, we have a definition of *prime simple* objects, those that cannot be factored into smaller simple objects. There is no counterpart in the theory of the cluster algebra, so it is a completely new notion.

However, the monoidal categorification seems to have a drawback. We do not have many tools to study the tensor product factorization in an abstract setting. We need additional input from other sources. Therefore it is natural to demand functors connecting two categorifications exchanging $\oplus$ and $\otimes$, and hopefully ? and $\oplus$. We call them *tropicalization* and detropicalization functors[†]

[†]Leclerc himself already had hoped to make a connection between two categorifications (see [40]). He calls them exponential and log.

expecting that the top ? in the additive categorification column is something like min:

$$
\begin{tikzcd}
& \text{cluster algebras} & \\
\text{additive categorifications} \arrow[ur, "\text{cluster character}"] \arrow[rr, dotted, shift right, "\text{detropicalization functors}"'] & & \text{monoidal categorifications} \arrow[ul, "\text{Grothendieck group}"'] \arrow[ll, dotted, shift right, "\text{tropicalization functors}"']
\end{tikzcd}
$$

The author believes that this is an interesting idea to pursue, but it is so far just a *slogan*; it seems difficult even to make definitions of (de)tropicalization functors precise. Therefore we set aside categorical approaches, return to the origin of the cluster algebra, that is, the construction of the canonical base, and ask why it has many structures.

The answer is simple. Lusztig's construction of the canonical base is based on the category of perverse sheaves on the space of representations $\mathbf{E}_W$ of the quiver. Therefore

(a) it has the structure of the monoidal abelian category, where the tensor product is given by the convolution diagram coming from exact sequences of quiver representations;

(b) it inherits various combinatorial structure from the module category $\operatorname{rep} \mathcal{Q}$ and probably also from the cluster category.

In this sense, we already have (de)tropicalization functors!

Thus we are led to ask a naive question sounding much more elementary compared with categorical approaches:

> Is it possible to realize a cluster algebra entirely in Lusztig's framework, that is, via a certain category of perverse sheaves on the space $\mathbf{E}_W$ of representations of a quiver?

If the answer is affirmative, the positivity conjecture is a direct consequence of that of the canonical base.

As far as the author has searched the literature on the subject, there is no explicit mention of this conjecture, although many examples of cluster algebras arise really as subalgebras of $\mathbf{U}_q^-$. Usually Lusztig's perverse sheaves appear only as a motivation and are not used in a fundamental way. A closest result is the work of Geiss, Leclerc, and Schröer [25], [26], where the cluster algebra is realized as a space of constructible functions on Λ_W, the space of nilpotent representations of the preprojective algebra. This Λ_W is a Lagrangian in the cotangent space $T^*\mathbf{E}_W$ of the space $\mathbf{E}_W$ of representations. The space of constructible functions was used also by Lusztig [45] to construct the *semicanonical* base. Constructible functions are vaguely related to perverse sheaves (or D-modules) via characteristic cycle construction, although nobody makes the relation precise. And it was proved that cluster monomials are indeed elements of the dual semicanonical base (see [25], [26]). But constructible functions have fewer structures than perverse sheaves; in particular, the positivity of the multiplication is unknown.

Before explaining our framework for the cluster algebra via perverse sheaves, we need to explain the author's earlier work [49]. It is another child of Lusztig's work.

1.2. Graded quiver varieties and quantum loop algebras

In [49] we studied the category $\mathscr{R}$ of l-integrable representations of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ via perverse sheaves on graded quiver varieties $\mathfrak{M}_0^\bullet(W)$ (denoted by $\mathfrak{M}_0(\infty,\mathbf{w})^A$ in [49]). If $\mathfrak{g}$ is a simple Lie algebra of type ADE, $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is a subquotient of Drinfeld-Jimbo quantized enveloping algebra of affine type ADE (usually called the quantum affine algebra), and $\mathscr{R}$ is nothing but the category of finite-dimensional representations of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. The graded quiver varieties are fixed point sets of the quiver varieties $\mathfrak{M}_0(W)$ introduced in [47] and [48] with respect to torus actions. The main result says that the Grothendieck group $\mathbf{R}$ of $\mathscr{R}$ has a natural t-deformation $\mathbf{R}_t$ which can be constructed from a category $\mathscr{P}_W$ of perverse sheaves on $\mathfrak{M}_0^\bullet(W)$ so that simple (resp., standard) modules correspond to dual of intersection cohomology (IC) complexes (resp., constant sheaves) of natural strata of $\mathfrak{M}_0^\bullet(W)$. Here, the parameter t comes from the cohomological grading. Furthermore, the transition matrix of two bases of simple and standard modules (i.e., dimensions of stalks of IC complexes) is given by analogue of Kazhdan-Lusztig polynomials, which can be computed* using purely combinatorial objects $\chi_{q,t}$ called t-analogues of q-characters (see [51], [54]). If we set $t=1$, we get the q-character defined in [38] and [24] as the generating function of the dimensions of l-weight spaces, simultaneous generalized eigenspaces with respect to a commutative subalgebra of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. For the simple module corresponding to an IC complex L, $\chi_{q,t}$ is the generating function of multiplicities of L in direct images of constant sheaves on various nonsingular graded quiver varieties $\mathfrak{M}^\bullet(V,W)$ under morphisms $\pi\colon \mathfrak{M}^\bullet(V,W)\to\mathfrak{M}_0^\bullet(W)$.

We have a noncommutative multiplication on $\mathbf{R}_t$, which is a t-deformation of a commutative multiplication on $\mathbf{R}$. When $\mathfrak{g}$ is of type ADE, the commutative multiplication on $\mathbf{R}$ comes from the tensor product $\otimes$ on the category $\mathscr{R}$ as $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is a Hopf algebra. (It is not known whether the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ can be equipped with the structure of a Hopf algebra in general.) The t-deformed multiplication was originally given in terms of t-analogue of q-characters, but Varagnolo and Vasserot [59] later introduced a convolution diagram on $\mathfrak{M}_0^\bullet(W)$ which gives the multiplication in a more direct and geometric way.

These geometric structures are similar to ones used to define the canonical base of $\mathbf{U}_q^-$ by Lusztig [43]. We have the following table of analogy:

$\mathbf{R}_t$	*geometry*	*dual of* $\mathbf{U}_q^-$
standard modules $M(W)$	constant sheaves	dual PBW base elements
simple modules $L(W)$	IC complexes	dual canonical base elements
t-deformed $\otimes$	convolution diagram	multiplication

*The meaning of the word *compute* is explained in Remark 6.4.

Note that $\mathbf{U}_q^-$ is not commutative even at $q=1$, while its dual $(\mathbf{U}_q^-)^*|_{q=1}$ is the coordinate ring $\mathbb{C}[\mathfrak{n}^-]$ and thus is commutative. Hence, we should compare $\mathbf{R}_t$ with $(\mathbf{U}_q^-)^*$, not with $\mathbf{U}_q^-$. Also, the convolution diagram looks similar to one for comultiplication, not to one for multiplication. The only difference is relevant varieties: Lusztig used the vector spaces $\mathbf{E}_W$ of representations of the quiver with group actions (or the moduli *stacks* of representations of the quiver), while the author used graded quiver varieties, which are framed moduli *spaces* of graded representations of the preprojective algebra associated with the underlying graph.

The computation of the transition matrix is hard to use in practice, like the Kazhdan-Lusztig polynomials. On the other hand, many people have been studying special modules (say, tame modules, Kirillov-Reshetikhin modules, minimal affinization, etc.) by purely algebraic approaches, at least when $\mathfrak{g}$ is of finite type (see [11] and the references therein). Their structure is different from that of general modules. Thus it is natural to look for a special geometric property which holds *only* for graded quiver varieties corresponding to these classes of modules. In [49, Section 10] the author introduced two candidates for such properties. We name the corresponding modules *special* and *small*, respectively. These properties are easy to state both in geometric and algebraic terms, but it is difficult to check whether a given module is special or small. Since [49], we have gradually understood that smallness is not a right concept, as there are only a very few examples (see [30]), but the speciality is a useful concept and there are many special modules, say, Kirillov-Reshetikhin modules.† One of the applications of this study was a proof of the T-system, which was conjectured in 1994 by Kuniba, Nakanishi, and Suzuki (see [53] and the references therein). Several steps in the proof of the main result in [53] depended on the geometry, but they were replaced by purely algebraic arguments and generalized to nonsymmetric quantum loop algebras cases later by Hernandez [29]. It was a fruitful interplay between geometric and algebraic approaches.

1.3. Realization of cluster algebras via perverse sheaves

Hernandez and Leclerc [31] give not only an abstract definition of a monoidal categorification but also its candidate for a certain cluster algebra. It is a monoidal (i.e., closed under the tensor product) subcategory $\mathscr{C}_1$ of $\mathscr{R}$ when $\mathfrak{g}$ is of type ADE. They indeed show that $\mathscr{C}_1$ is a monoidal categorification for types A and D_4. Therefore we have strong evidence that it is a right candidate. From what we have reviewed just above, if it indeed is a monoidal categorification, the cluster algebra is a subalgebra of $\mathbf{R}$, constructed via perverse sheaves on graded quiver varieties! Moreover, from the philosophy explained above, we could expect that graded quiver varieties corresponding to $\mathscr{C}_1$ have very *special* features compared with general ones.

†Special modules form a *special* class of modules. Small modules form a *small* class of modules. But the name *small* originally comes from the smallness of a morphism.

In this article, we show that it turns out to be true. The first main observation (see Proposition 4.6) is that the graded quiver varieties $\mathfrak{M}_0^\bullet(W)$ become just the vector spaces $\mathbf{E}_W$ of representations of the *decorated quiver*. Here, the decorated quiver† is constructed from a given finite graph with a bipartite orientation by adding a new (frozen) vertex i' and an arrow $i' \to i$ (resp., $i \to i'$) if i is a sink (resp., source) for each vertex i (see Definition 4.3). Therefore the underlying variety is nothing but what Lusztig used. Also, the convolution diagram turns out to be the same as Lusztig's. Thus the Grothendieck group $\mathcal{K}(\mathscr{C}_1)$ is also a subquotient of the dual of $\mathbf{U}_q^-$, associated with the Kac-Moody Lie algebra corresponding to the decorated quiver.

To define a cluster algebra with frozen variables (or with coefficients in the terminology of [21]), we choose a quiver with choices of frozen vertices. We warn the reader that this quiver for the cluster algebra (we call it the $\mathbf{x}$-quiver; see Definition 5.4) is slightly different from the decorated quiver: the principal part has the opposite orientation, while the frozen part is the same.

1.4. Second key observation

Once we get a correct candidate for the class of perverse sheaves, we next study structures of the dual canonical base and try to pull out the cluster algebra structure from it. We hope to see a shadow of the structure of a cluster category.

As we mentioned above, our $\mathcal{K}(\mathscr{C}_1)$ is a subquotient of the dual of $\mathbf{U}_q^-$. In particular, we introduce an equivalence relation on the canonical base. The second key observation is that each equivalence class contains exactly one skyscraper sheaf $1_{\{0\}}$ of the origin 0 of $\mathbf{E}_W$ (the simplest perverse sheaf!). This equivalence relation is built on the theory of graded quiver varieties. From this observation together with the first observation that the graded quiver varieties are vector spaces, we can apply the Fourier-Sato-Deligne transform (see [36], [39]) to make a reduction to a study of constant sheaves $1_{\mathbf{E}_W^*}$ on the whole space.

There is a certain natural family of projective morphisms $\pi^\perp\colon \widetilde{\mathcal{F}}(\nu, W)^\perp \to \mathbf{E}_W^*$ from nonsingular varieties $\widetilde{\mathcal{F}}(\nu, W)^\perp$. This family appears as monomials in Lusztig's context and as q-characters in the theory reviewed in Section 1.2. Using these morphisms, we define a homomorphism from $\mathbf{R}$ to the cluster algebra. Fibers of these morphisms are what are called *quiver Grassmannian* varieties. People study their Euler characters and define the cluster character as their generating function. This is clearly related to the study of the pushforward

$$\pi_!^\perp\bigl(1_{\widetilde{\mathcal{F}}(\nu,W)^\perp}[\dim \widetilde{\mathcal{F}}(\nu, W)^\perp]\bigr).$$

If $\mathbf{E}_W^*$ contains an open orbit, then the Euler number of the fiber over a point in the orbit is nothing but the coefficient of $1_{\mathbf{E}_W^*}[\dim \mathbf{E}_W]$ in the above pushforward. When the dual canonical base element is a cluster monomial, $\mathbf{E}_W^*$ indeed contains

†The decorated quiver is different from one in [46], where there are no arrows between i and i'.

an open orbit. Therefore we immediately see that all cluster monomials are dual canonical base elements. This very simple observation between the cluster character and the pushforward appeared in the work of Caldero and Reineke [9].†

To be more precise, we need to apply reflection functors at all sink vertices in the decorated quiver with opposite orientations to identify fibers of $\widetilde{F}(\nu, W)^\perp$ with quiver Grassmannian varieties. The resulting quiver corresponds to the cluster algebra with principal coefficients (see [23]).

An appearance of the cluster character formula in the category $\mathscr{C}_1$ was already pointed out in [31, Section 12] as it is nothing but a leading part of the q-character mentioned above. (We call the leading part the *truncated* q-character.)

From a result on graded quiver varieties, it also follows that quiver Grassmannian varieties have vanishing odd cohomology groups under the above assumption. The generating function of all Betti numbers is nothing but the truncated t-analogue of q-character of a simple module. In particular, it was computed in [54].

We have assumed that $\mathbf{E}_W^*$ contains an open orbit. But the only necessary assumption we need is that perverse sheaves corresponding to canonical base elements have strictly smaller supports than $\mathbf{E}_W^*$, except $1_{\mathbf{E}_W^*}[\dim \mathbf{E}_W^*]$. Even if this condition is not satisfied, we can consider the *almost simple module* $\mathbb{L}(W)$ corresponding to the sum of perverse sheaves whose supports are the whole $\mathbf{E}_W^*$. Then the total sum of Betti numbers (the Euler number is not natural in this wider context) of the quiver Grassmannian give the truncated q-character of the almost simple module. An almost simple module $\mathbb{L}(W)$ is not necessarily simple in general.

It is rather simple to study tensor product factorization of $\mathbb{L}(W)$ since we computed their truncated q-characters. First we observe that Kirillov-Reshetikhin modules simply factor out. Then we may assume that W has zero entries on frozen vertices. Thus W is supported on the first given vertices. We next observe that $\mathbb{L}(W)$ factors as

$$\mathbb{L}(W) \cong \mathbb{L}(W^1) \otimes \cdots \otimes \mathbb{L}(W^s)$$

according to the *canonical decomposition* $W = W^1 \oplus \cdots \oplus W^s$ of W. Recall that the canonical decomposition is the decomposition of a general representation of $\mathbf{E}_W$ first introduced by Kac [34], [35] and studied further by Schofield [57]. It is known that each W^k is a Schur root (i.e., a general representation is indecomposable) and Ext^1 between general representations from two different factors W^k, W^l vanishes.

We prove that a simple module $L(W)$ corresponds to a cluster monomial if and only if the canonical decomposition contains only real Schur roots. In this case, $\mathbf{E}_W^*$ contains an open orbit. Then we have $\mathbb{L}(W) = L(W)$, $\mathbb{L}(W^k) =$

†There is a gap in the proof of [9, Theorem 1] since Lusztig's v is identified with q. The correct identification is $v = -\sqrt{q}$. We give a corrected proof in the appendix.

$L(W^k)$, and each $L(W^k)$ corresponds to a cluster variable, and the above tensor factorization corresponds to the cluster expansion.

1.5. To do list

In this article, basically due to laziness of the author, at least four natural topics are not discussed

(1) Our Grothendieck ring $\mathbf{R}$ has a natural noncommutative deformation $\mathbf{R}_t$. It should contain the quantum cluster algebra in [4]. In fact, we already give our main formula (in Theorem 6.3) in Poincaré polynomials of quiver Grassmannian varieties. Therefore the only remaining thing is to prove the quantum version of the cluster character formula. Any proof in the literature should be modified to the quantum version naturally, as it is based on the counting of rational points.

After an earlier version of this article was posted on the arXiv, Qin [56] proved the quantum version of the cluster character formula for an acyclic cluster algebra. This is the most essential part for this problem, but we still need to check that the multiplication $\mathbf{R}_t$ is the same as that of the quantum cluster algebra. This will be checked elsewhere.

(2) We treat only the case when the underlying quiver is bipartite. Since the choice of the quiver orientation is not essential in Lusztig's construction (in fact, the Fourier transform provides a technique to change orientations), this assumption probably can be removed.

(3) We treat only the symmetric cases. Symmetrizable cases can be studied by considering quiver automorphisms as in Lusztig's work. Though the corresponding theory was not studied in the author's theory, it should correspond to the representations of twisted quantum affine algebras.

(4) In [25] and [26] it was proved that cluster monomials are semicanonical base elements. It was conjectured that they are also canonical base elements. It is desirable to study the precise relation of this work to ours.

The author or his colleagues will hopefully come back to these problems in the near future.

In [31] a further conjecture is proposed for the monoidal subcategory $\mathscr{C}_\ell$, where $\mathscr{C}_1$ is the special case $\ell = 1$. Since the graded quiver varieties are no longer vector spaces for $\ell > 1$, the method of this article does not work. But it is certainly an interesting direction to pursue. We also remark that other connections between the cluster algebra theory and the representation theory of quantum affine algebras have been found by Di Francesco and Kedem [18] and Inoue and colleagues [33]. It is interesting to make a connection to their works.

This article is organized as follows. Sections 2 and 3 are preliminaries for cluster algebras and graded quiver varieties, respectively. In Section 4 we introduce the category $\mathscr{C}_1$ following [31] and study the corresponding graded quiver varieties. In Section 5 we define a homomorphism from the Grothendieck group $\mathbf{R}_{\ell=1}$ of $\mathscr{C}_1$ to a rational function field which is endowed with a cluster algebra structure. In Section 6 we explain the relation between the cluster character and the pushforward and derive several consequences on factorizations of simple modules. In Section 7 we prove that cluster monomials are dual canonical base elements. In the appendix we prove that the quiver Grassmannian of a rigid module of an acyclic quiver has no odd cohomology. It implies the positive conjecture for an acyclic cluster algebra for the special case of an *initial* seed.

2. Preliminaries, I: Cluster algebras

We review the definition and properties of cluster algebras.

2.1. Definition

Let $\mathcal{G} = (I, E)$ be a finite graph, where I is the set of vertices and E is the set of edges. Let H be the set of pairs consisting of an edge together with its orientation. For $h \in H$, we denote by $i(h)$ (resp., $o(h)$) the incoming (resp., outgoing) vertex of h. For $h \in H$, we denote by $\overline{h}$ the same edge as h with the reverse orientation. A *quiver* $\mathcal{Q} = (I, \Omega)$ is the finite graph $\mathcal{G}$ together with a choice of an *orientation* $\Omega \subset H$ such that $\Omega \cap \overline{\Omega} = \emptyset$, $\Omega \cup \overline{\Omega} = H$.

We consider a pair of a quiver $\mathcal{Q} = (I, \Omega)$ and a larger quiver $\widetilde{\mathcal{Q}} = (\widetilde{I}, \widetilde{\Omega})$ containing $\mathcal{Q}$, where I is a subset of $\widetilde{I}$ and Ω is obtained from $\widetilde{\Omega}$ by removing arrows incident to a point in $\widetilde{I} \setminus I$. Set $I_{\mathrm{fr}} = \widetilde{I} \setminus I$. We call $i \in I_{\mathrm{fr}}$ (resp., $i \in I$) a *frozen* (resp., *principal*) vertex.

We assume that $\widetilde{\mathcal{Q}}$ has no loops or 2-cycles and that there are no edges connecting points in I_{fr}. We define a matrix $\widetilde{\mathbf{B}} = (b_{ij})_{i \in \widetilde{I}, j \in I}$ by

$$\begin{aligned} b_{ij} &:= (\text{the number of oriented edges from } j \text{ to } i) \\ &\qquad \text{or } -(\text{the number of oriented edges from } i \text{ to } j). \end{aligned}$$

Since we have assumed that $\widetilde{\mathcal{Q}}$ contains no 2-cycles, this is well defined. Moreover, giving $\widetilde{\mathbf{B}}$ is equivalent to a quiver $\widetilde{\mathcal{Q}}$ with the decomposition $\widetilde{I} = I \sqcup I_{\mathrm{fr}}$ as above. The *principal part* $\mathbf{B}$ of $\widetilde{\mathbf{B}}$ is the matrix obtained from $\widetilde{\mathbf{B}}$ by taking entries for $I \times I$. From the definition, $\mathbf{B}$ is skew-symmetric.

For a vertex $k \in I$, we define the *matrix mutation* $\mu_k(\widetilde{\mathbf{B}})$ of $\widetilde{\mathbf{B}}$ in direction k as the new matrix (b'_{ij}) indexed by $(i, j) \in \widetilde{I} \times I$ given by the formula

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(b_{ik} b_{kj}, 0) & \text{otherwise.} \end{cases} \tag{2.1}$$

If $\widetilde{\Omega}^*$ denotes the corresponding quiver, it is obtained from $\widetilde{\Omega}$ by the following rules.

(1) For each $i \to k$, $k \to j \in \widetilde{\Omega}$, create a new arrow $i \to j$ if either i or $j \in I$.
(2) Reverse all arrows incident to k.
(3) Remove 2-cycles between i and j of the resulting quiver after (1) and (2).

Graphically it is given by

$$\widetilde{\Omega}: \begin{tikzcd} i \arrow[rr, "r"] \arrow[dr, "s"'] & & j \\ & k \arrow[ur, "t"'] & \end{tikzcd} \quad \Longrightarrow \quad \widetilde{\Omega}^*: \begin{tikzcd} i \arrow[rr, "r+st"] & & j, \arrow[dl, "t"] \\ & k \arrow[ul, "s"] & \end{tikzcd}$$

where s, t are nonnegative integers and $i \xrightarrow{l} j$ means that there are l arrows from i to j if $l \geq 0$ and $(-l)$ arrows from j to i if $l \leq 0$. The new quiver $\widetilde{\Omega}^*$ has no loops or 2-cycles.

Let $\mathscr{F} = \mathbb{Q}(x_i)_{i\in\widetilde{I}}$ be the field of rational functions in commuting indeterminates $\mathbf{x} = (x_i)_{i\in\widetilde{I}}$ indexed by $\widetilde{I}$. For $k \in I$ we define a new variable x_k^* by the *exchange relation*:

$$x_k^* = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k}. \tag{2.2}$$

Let $\mu_k(\mathbf{x})$ be the set of variables obtained from $\mathbf{x}$ by replacing x_k by x_k^*. The pair $(\mu_k(\mathbf{x}), \mu_k(\widetilde{\mathbf{B}}))$ is called the *mutation* of $(\mathbf{x}, \widetilde{\mathbf{B}})$ in direction k. We can iterate this procedure and obtain new pairs by mutating $(\mu_k(\mathbf{x}), \mu_k(\widetilde{\mathbf{B}}))$ in any direction $l \in I$. We do not make mutations in the direction of a frozen vertex $k \in I_{\mathrm{fr}}$. Variables x_i for $i \in I_{\mathrm{fr}}$ are always in $\mu_k(\mathbf{x})$; they are called *frozen variables* (or *coefficients* in [21]).

Now a *seed* is a pair $(\mathbf{y}, \widetilde{\mathbf{C}})$ of $\mathbf{y} = (y_i)_{i\in\widetilde{I}} \in \mathscr{F}^{\widetilde{I}}$ and a matrix $\widetilde{\mathbf{C}} = (c_{ij})_{i\in\widetilde{I}, j\in I}$ obtained from the *initial seed* $(\mathbf{x}, \widetilde{\mathbf{B}})$ by a successive application of mutations in various directions $k \in I$. The set of seeds is denoted by $\mathscr{S}$. A *cluster* is $\{y_i \mid i \in \widetilde{I}\}$ of a seed $(\mathbf{y}, \widetilde{\mathbf{C}})$, considered as a subset of $\mathscr{F}$ by forgetting the $\widetilde{I}$-index. A *cluster variable* is an element of the union of all clusters. Note that clusters may overlap: a cluster variable may belong to another cluster. Also, the $\widetilde{I}$-index may be different from the original one. The *cluster algebra* $\mathscr{A}(\widetilde{\mathbf{B}})$ is the subalgebra of $\mathscr{F}$ generated by all the cluster variables. The integer $\#I$ is called the *rank* of $\mathscr{A}(\widetilde{\mathbf{B}})$. A *cluster monomial* is a monomial in the cluster variables of a *single* cluster. The exchange relation (2.2) is of the form

$$x_k x_k^* = m_+ + m_-, \tag{2.3}$$

where $m_\pm = \prod_{\pm b_{ik}>0} x_i^{\pm b_{ik}}$ are cluster monomials.

When we say a *cluster algebra*, it may mean the subalgebra $\mathscr{A}(\widetilde{\mathbf{B}})$ or all the above structures.

One of the important results in the cluster algebra theory is the Laurent phenomenon: every cluster variable z in $\mathscr{A}(\widetilde{\mathbf{B}})$ is a Laurent polynomial in any given cluster $\mathbf{y}$ with coefficients in $\mathbb{Z}$. It is conjectured that the coefficients are nonnegative. A cluster monomial is a subtraction-free rational expression in $\mathbf{x}$, but this is not enough to ensure the positivity of its Laurent expansion, as an example $x^2 - x + 1 = (x+1)^3/(x+1)$ shows.

2.2. F-polynomial

It is known that cluster variables of $\mathscr{A}(\widetilde{\mathbf{B}})$ are expressed by the $\mathbf{g}$-vectors and F-polynomials (see [23]), which are constructed from another cluster algebra with the same principal part but a simpler frozen part. We recall their definition in this subsection.

We first prepare some notation. We consider the multiplicative group $\mathbb{P}$ of all Laurent monomials in $(x_i)_{\in I}$. We introduce the addition $\oplus$ by

$$\prod_i x_i^{a_i} \oplus \prod_i x_i^{b_i} = \prod_i x_i^{\min(a_i,b_i)}.$$

In this operation together with the ordinary multiplication and division, $\mathbb{P}$ becomes a semifield, called the *tropical semifield*. Let F be a subtraction-free rational expression with integer coefficients in variables y_i. Then we evaluate it in $\mathbb{P}$ by specializing the y_i to some elements p_i of $\mathbb{P}$. We denote it by $F|_{\mathbb{P}}(\mathbf{p})$, where $\mathbf{p} = (p_i)_{i\in I}$.

Let $\mathscr{A}_{\mathrm{pr}}$ be the cluster algebra with *principal coefficients*. It is given by the initial seed $((\mathbf{u},\mathbf{f}),\widetilde{\mathbf{B}}_{\mathrm{pr}})$ with $(\mathbf{u},\mathbf{f}) = (u_i, f_i)_{i\in I}$, and $\widetilde{\mathbf{B}}_{\mathrm{pr}}$ is the matrix indexed by $(I \sqcup I) \times I$ with the same principal $\mathbf{B}$ as $\widetilde{\mathbf{B}}$ and the identity matrix in the frozen part. Here I_{fr} is a copy of I and $\widetilde{I} = I \sqcup I$. We write a cluster variable α as

$$\alpha = X_\alpha(\mathbf{u},\mathbf{f}),$$

a subtraction-free rational expression in $\mathbf{u}$, $\mathbf{f}$. We then specialize all the u_i to 1:

$$F_\alpha(\mathbf{f}) = X_\alpha(\mathbf{u},\mathbf{f})|_{u_i=1}.$$

It becomes a polynomial in f_i and is called the F-polynomial (see [23], Section 3). It is also known (see [23], Section 6) that X_α is homogeneous with respect to the $\mathbb{Z}^I$-grading given by

$$\deg u_i = i, \qquad \deg f_j = -\sum_i b_{ij} i,$$

where b_{ij} is the matrix entry for the principal part $\mathbf{B}$, and the vertex i is identified with the coordinate vector in $\mathbb{Z}^I$. We then define $\mathbf{g}$-vector by

$$\mathbf{g}_\alpha \overset{\mathrm{def}}{=} \deg X_\alpha \in \mathbb{Z}^I.$$

We now return back to the original cluster algebra $\mathscr{A}(\widetilde{\mathbf{B}}) \subset \mathbb{Q}(x_i)_{i\in\widetilde{I}}$. We introduce the following variables:

$$y_j = \prod_{i\in I_{\mathrm{fr}}} x_i^{b_{ij}}, \qquad \widehat{y}_j = y_j \prod_{i\in I} x_i^{b_{ij}} \quad (j \in I).$$

We write $\mathbf{y} = (y_i)_{i\in I}$, $\widehat{\mathbf{y}} = (\widehat{y}_i)_{i\in I}$.

We consider the corresponding cluster variable $x[\alpha]$ in the seed of the original cluster algebra $\mathscr{A}(\widetilde{\mathbf{B}})$ obtained by the same mutation processes as we obtained α in the cluster algebra with principal coefficients. We then have [23, Corollary 6.5]:

$$x[\alpha] = \frac{F_\alpha(\widehat{\mathbf{y}})}{F_\alpha|_{\mathbb{P}}(\mathbf{y})}\mathbf{x}^{\mathbf{g}_\alpha},$$

where $\mathbf{x}^{\mathbf{g}_\alpha} = \prod_{i\in I} x_i^{(\mathbf{g}_\alpha)_i}$ if $(\mathbf{g}_\alpha)_i$ is the ith entry of $\mathbf{g}_\alpha$.

2.3. Hernandez-Leclerc monoidal categorification conjecture

We recall Hernandez and Leclerc's monoidal categorification conjecture in this subsection.

Let $\mathscr{A}$ be a cluster algebra, and let $\mathscr{M}$ be an abelian monoidal category. A simple object $L \in \mathscr{M}$ is *prime* if there exists no nontrivial factorization $L \cong L_1 \otimes L_2$. We say that L is *real* if $L \otimes L$ is simple.

DEFINITION 2.4 ([31, DEFINITION 2.1])

Let $\mathscr{A}$ and $\mathscr{M}$ be as above. We say that $\mathscr{M}$ is a *monoidal categorification* of $\mathscr{A}$ if the Grothendieck ring of $\mathscr{M}$ is isomorphic to $\mathscr{A}$ and if

(1) the cluster monomials m of $\mathscr{A}$ are the classes of all the real simple objects $L(m)$ of $\mathscr{M}$;

(2) the cluster variables of $\mathscr{A}$ (including the frozen ones) are the classes of all the real prime simple objects of $\mathscr{M}$.

If two cluster variables x, y belong to the common cluster, then xy is a cluster monomial. Therefore the corresponding simple objects $L(x)$, $L(y)$ satisfy $L(x) \otimes L(y) \cong L(y) \otimes L(x) \cong L(xy)$.

PROPOSITION 2.5 ([31, SECTION 2])

Suppose that a cluster algebra $\mathscr{A}$ has a monoidal categorification $\mathscr{M}$.

(1) *Every cluster monomial has a Laurent expansion with positive coefficients with respect to any cluster $\mathbf{y} = (y_i)_{i \in \widetilde{I}} \in \mathscr{S}$:*

$$m = \frac{N_m(\mathbf{y})}{\prod_i y_i^{d_i}}, \quad d_i \in \mathbb{Z}_{\geq 0},\ N(y_i) \in \mathbb{Z}_{\geq 0}[y_i^{\pm}].$$

In fact, the coefficient of $\prod y_i^{k_i}$ in $N_m(\mathbf{y})$ is equal to the multiplicity of $L(\prod y_i^{k_i}) = \bigotimes L(y_i)^{\otimes k_i}$ in $L(m) \otimes L(\prod_i y_i^{d_i}) = L(m) \otimes \bigotimes L(y_i)^{\otimes d_i}$.

(2) *The cluster monomials of $\mathscr{A}$ are linearly independent.*

CONJECTURE 2.6 ([31, CONJECTURE 4.6])

The cluster algebra for the quiver defined in Section 5 has a monoidal categorification when the underlying graph is of type ADE. More precisely, it is given by a certain explicitly defined monoidal subcategory $\mathscr{C}_1$ of the category of finite-dimensional representations of the quantum affine algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$.

The monoidal subcategory is defined in Section 4.1 in terms of graded quiver varieties for arbitrary symmetric Kac-Moody cases. And we prove the conjecture for type ADE. This is new for D_n for $n \geq 5$ and E_6, E_7, E_8 since the conjecture was already proved in [31] for types A and D_4.

However, we cannot control the prime factorization of arbitrary simple modules except in the ADE cases. We can just prove that cluster monomials are real simple objects. And there are *imaginary* simple objects for types other than ADE. So it is still not clear that our monoidal subcategory is a monoidal categorification in the above sense in general (see the paragraph at the end of Section 6 for a partial result). Nonetheless, the statement that cluster monomials are classes of simple objects is enough to derive the conclusions (1) and (2) of Proposition 2.5.

3. Preliminaries, II: Graded quiver varieties

We review the definition of graded quiver varieties and the convolution diagram for the tensor product in this section. Our notation mainly follows [54]. Some materials are borrowed from [59].

We do not explain anything about representations of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ except in Theorem 3.17. This is because we can work directly in the category of perverse sheaves on graded quiver varieties. Another reason is that it is not known whether the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ can be equipped with the structure of a Hopf algebra in general. Therefore tensor products of modules do not make sense. On the other hand, the category of perverse sheaves *has* the coproduct induced from the convolution diagram.

3.1. Definition of graded quiver varieties

Let q be a nonzero complex number. We assume that it is not a root of unity later, but it can be at the beginning.

Suppose that a finite graph $\mathcal{G} = (I, E)$ is given. We assume that the graph $\mathcal{G}$ contains no edge loops. Let $\mathbf{A} = (a_{ij})$ be the adjacency matrix of the graph, namely,

$$a_{ij} = (\text{the number of edges joining } i \text{ to } j).$$

Let $\mathbf{C} = 2\mathbf{I} - \mathbf{A} = (c_{ij})$ be the Cartan matrix.

Let H be the set of pairs consisting of an edge together with its orientation as in Section 2. We choose and fix an orientation Ω of $\mathcal{G}$ and define $\varepsilon(h) = 1$ if $h \in \Omega$ and -1 otherwise.

Let V, W be $(I \times \mathbb{C}^*)$-graded vector spaces such that its $(i \times a)$-component, denoted by $V_i(a)$, is finite-dimensional and zero for all but finitely many $i \times a$. In what follows we consider only $(I \times \mathbb{C}^*)$-graded vector spaces with this condition. We say that the pair (V, W) of $(I \times \mathbb{C}^*)$-graded vector spaces is *l-dominant* if

$$\dim W_i(a) - \dim V_i(aq) - \dim V_i(aq^{-1}) - \sum_{j : j \neq i} c_{ij} \dim V_j(a) \geq 0 \tag{3.1}$$

for any i, a.

Let $\mathbf{C}_q$ (q-analogue of the Cartan matrix) be an endomorphism of $\mathbb{Z}^{I \times \mathbb{C}^*}$ given by

$$\big(v_i(a)\big) \mapsto \big(v_i'(a)\big), \qquad v_i'(a) = v_i(aq) + v_i(aq^{-1}) + \sum_{j : j \neq i} c_{ij} v_j(a). \tag{3.2}$$

Considering $\dim V$, $\dim W$ as vectors in $\mathbb{Z}_{\geq 0}^{I \times \mathbb{C}^*}$, we view the left-hand side of (3.1) as the (i, a)-component of $(\dim W - \mathbf{C}_q \dim V)$. This is an analogue of a weight.

We say that $V \leq V'$ if

$$\dim V_i(a) \geq \dim V'_i(a)$$

for any i, a. We say that $V < V'$ if $V \leq V'$ and $V \neq V'$. This is analogue of the dominance order. We say $(V,W) \leq (V',W')$ if there exists $\mathbf{v}'' \in \mathbb{Z}_{\geq 0}^{I\times\mathbb{C}^*}$ whose entries are zero for all but finitely many (i,a) such that

$$\dim W - \mathbf{C}_q \dim V = \dim W' - \mathbf{C}_q(\dim V' + \mathbf{v}'').$$

When $W = W'$, $(V,W) \leq (V',W')$ if and only if $V \leq V'$.

These conditions originally come from the representation theory of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$.

For an integer n, we define vector spaces by

$$\begin{aligned} \mathrm{L}^\bullet(V,W)^{[n]} &\overset{\text{def}}{=} \bigoplus_{i\in I,\ a\in\mathbb{C}^*} \operatorname{Hom}\big(V_i(a), W_i(aq^n)\big), \\ \mathrm{E}^\bullet(V,W)^{[n]} &\overset{\text{def}}{=} \bigoplus_{h\in H,\ a\in\mathbb{C}^*} \operatorname{Hom}\big(V_{o(h)}(a), W_{i(h)}(aq^n)\big). \end{aligned} \tag{3.3}$$

If V and W are $(I\times\mathbb{C}^*)$-graded vector spaces as above, we consider the vector spaces

$$\mathbf{M}^\bullet \equiv \mathbf{M}^\bullet(V,W) \overset{\text{def}}{=} \mathrm{E}^\bullet(V,V)^{[-1]} \oplus \mathrm{L}^\bullet(W,V)^{[-1]} \oplus \mathrm{L}^\bullet(V,W)^{[-1]}, \tag{3.4}$$

where we use the notation $\mathbf{M}^\bullet$ unless we want to specify V, W. The above three components for an element of $\mathbf{M}^\bullet$ are denoted by B, α, β, respectively. (In [49], α and β were denoted by i, j, respectively.) The $\operatorname{Hom}(V_{o(h)}(a), V_{i(h)}(aq^{-1}))$-component of B is denoted by $B_{h,a}$. Similarly, we denote by $\alpha_{i,a}$, $\beta_{i,a}$ the components of α, β.

We define a map $\mu\colon \mathbf{M}^\bullet \to \mathrm{L}^\bullet(V,V)^{[-2]}$ by

$$\mu_{i,a}(B,\alpha,\beta) = \sum_{i(h)=i} \varepsilon(h) B_{h,aq^{-1}} B_{\overline{h},a} + \alpha_{i,aq^{-1}}\beta_{i,a},$$

where $\mu_{i,a}$ is the (i,a)-component of μ.

Let $G_V \overset{\text{def}}{=} \prod_{i,a} \mathrm{GL}(V_i(a))$. It acts on $\mathbf{M}^\bullet$ by

$$(B,\alpha,\beta) \mapsto g\cdot(B,\alpha,\beta) \overset{\text{def}}{=} (g_{i(h),aq^{-1}} B_{h,a} g_{o(h),a}^{-1}, g_{i,aq^{-1}}\alpha_{i,a}, \beta_{i,a} g_{i,a}^{-1}).$$

The action preserves the subvariety $\mu^{-1}(0)$ in $\mathbf{M}^\bullet$.

DEFINITION 3.5

A point $(B,\alpha,\beta) \in \mu^{-1}(0)$ is said to be *stable* if the following condition holds:

If an $I\times\mathbb{C}^*$-graded subspace V' of V is B-invariant and contained in $\operatorname{Ker}\beta$, then $V' = 0$.

Let us denote by $\mu^{-1}(0)^{\mathrm{s}}$ the set of stable points.

Clearly, the stability condition is invariant under the action of G_V. Hence we may say whether an orbit is stable or not.

We consider two kinds of quotient spaces of $\mu^{-1}(0)$:

$$\mathfrak{M}_0^\bullet(V,W) \overset{\text{def}}{=} \mu^{-1}(0)/\!\!/G_V, \qquad \mathfrak{M}^\bullet(V,W) \overset{\text{def}}{=} \mu^{-1}(0)^{\mathrm{s}}/G_V.$$

Here $/\!\!/$ is the affine algebro-geometric quotient; that is, the coordinate ring of $\mathfrak{M}_0^\bullet(V,W)$ is the ring of G_V-invariant functions on $\mu^{-1}(0)$. In particular, it is an affine variety. It is the set of closed G_V-orbits. The second one is the set-theoretical quotient but coincides with a quotient in the geometric invariant theory (see [48, Section 3]). The action of G_V on $\mu^{-1}(0)^{\mathrm{s}}$ is free thanks to the stability condition (see [48, Lemma 3.10]). By the general theory, there exists a natural projective morphism

$$\pi\colon \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W)$$

(see [48, (3.18)]). The inverse image of zero under π is denoted by $\mathfrak{L}^\bullet(V,W)$. We call these varieties *cyclic quiver varieties* or *graded quiver varieties*, according as q is a root of unity or not. In this article we consider only the case when q is not a root of unity hereafter. When we want to distinguish $\mathfrak{M}^\bullet(V,W)$ and $\mathfrak{M}_0^\bullet(V,W)$, we call the former (resp., latter) the *nonsingular* (resp., *affine*) *graded quiver variety*. But it does not mean that $\mathfrak{M}_0^\bullet(V,W)$ is actually singular. As we see later, it is possible that $\mathfrak{M}_0^\bullet(V,W)$ happens to be nonsingular.

We have

$$\dim \mathfrak{M}^\bullet(V,W) = \langle \dim V, (q+q^{-1})\dim W - q^{-1}\mathbf{C}_q \dim V\rangle,$$

where $q^{\pm}\cdot$ is an automorphism of $\mathbb{Z}^{I\times\mathbb{C}^*}$ given by $(v_i(a)) \mapsto (v_i'(a))$; $v_i'(a) = v_i(aq^{\pm})$, and $\langle\ ,\ \rangle$ is the natural pairing on $\mathbb{Z}^{I\times\mathbb{C}^*}$ (see [54, (4.11)]).

The original quiver varieties (see [47], [48]) are the special case when $q=1$ and $V_i(a) = W_i(a) = 0$ except when $a=1$. On the other hand, the above varieties $\mathfrak{M}^\bullet(W)$, $\mathfrak{M}_0^\bullet(W)$ are a fixed point set of the original quiver varieties with respect to a semisimple element in a product of general linear groups (see [49, Section 4]). In particular, it follows that $\mathfrak{M}^\bullet(V,W)$ is nonsingular since the corresponding original quiver variety is. This can also be checked directly.

It is known that the coordinate ring of $\mathfrak{M}_0^\bullet(V,W)$ is generated by the following type of elements:

$$(B,\alpha,\beta) \mapsto \langle \chi, \beta_{j,aq^{-n-1}} B_{h_n,aq^{-n}} \dots B_{h_1,aq^{-1}} \alpha_{i,a}\rangle, \tag{3.6}$$

where χ is a linear form on $\operatorname{Hom}(W_i(a), W_j(aq^{-n-2}))$ (see [44]). Here we do not need to consider generators of a form $\operatorname{tr}(B_{h_N,aq^{N-1}} B_{h_{N-1},aq^{N-2}} \cdots B_{h_1,a})$ corresponding to an oriented cycle $h_1, \dots, h_N$ as they automatically vanish as q is not a root of unity. (Our definition of the graded quiver variety is different from one in [49] when there are multiple edges joining two vertices. See Remark 3.13 for more detail. The above generators may not vanish in the original definition, but *do* vanish in our definition.)

Let $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W) \subset \mathfrak{M}_0^\bullet(V,W)$ be a possibly empty open subset of $\mathfrak{M}_0^\bullet(V,W)$ consisting of closed free G_V-orbits. It is known that π is an isomorphism on $\pi^{-1}(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W))$ (see [48, Proposition 3.24]). In particular, $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ is nonsingular and pure-dimensional.

The G_V-orbit through (B, α, β), considered as a point of $\mathfrak{M}^\bullet(V, W)$, is denoted by $[B, \alpha, \beta]$.

Suppose that we have two $I \times \mathbb{C}^*$-graded vector spaces V, V' such that $V_i(a) \subset V'_i(a)$ for all i, a. Then $\mathfrak{M}_0^\bullet(V, W)$ can be identified with a closed subvariety of $\mathfrak{M}_0^\bullet(V', W)$ by the extension by zero to the complementary subspace (see [49, Lemma 2.5.3]). We consider the limit

$$\mathfrak{M}_0^\bullet(W) \stackrel{\text{def}}{=} \bigcup_V \mathfrak{M}_0^\bullet(V, W).$$

(It was denoted by $\mathfrak{M}_0(\infty, \mathbf{w})^A$ in [49] and by $\mathfrak{M}_0^\bullet(\infty, W)$ in [54].)

We have $\mathfrak{M}_0^\bullet(V, 0) = \{0\}$ for $W = 0$ since generators (3.6) vanish. Then [47, Lemma 6.5] or [48, Lemma 3.27] implies that

$$\mathfrak{M}_0^\bullet(W) = \bigsqcup_{[V]} \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V, W), \tag{3.7}$$

where $[V]$ denotes the isomorphism class of V. It is known that

(3.8) $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V, W) \neq \emptyset$ if and only if $\mathfrak{M}^\bullet(V, W) \neq \emptyset$ and (V, W) is l-dominant (see [49, Theorem 14.3.2(2)]).

(3.9) If $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V, W) \subset \overline{\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V', W)}$, then $V' \leq V$. (This follows from [49, Section 3.3].)

It is also easy to show that

$$\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V, W) = \emptyset \quad \text{if } V \text{ is sufficiently large} \tag{3.10}$$

(see the argument in the proof of Proposition 4.6(1)). Thus $\mathfrak{M}_0^\bullet(W) \stackrel{\text{def}}{=} \bigcup_V \mathfrak{M}_0^\bullet(V, W)$ stabilizes at some V.

On the other hand, we consider the disjoint union for $\mathfrak{M}^\bullet(V, W)$:

$$\mathfrak{M}^\bullet(W) \stackrel{\text{def}}{=} \bigsqcup_{[V]} \mathfrak{M}^\bullet(V, W).$$

Note that there are no obvious morphisms between $\mathfrak{M}^\bullet(V, W)$ and $\mathfrak{M}^\bullet(V', W)$ since the stability condition is not preserved under the extension. We have a morphism $\mathfrak{M}^\bullet(W) \to \mathfrak{M}_0^\bullet(W)$, still denoted by π.

It is known that $\mathfrak{M}^\bullet(V, W)$ becomes empty if V is sufficiently large when $\mathfrak{g}$ is of type ADE (since the usual quiver variety $\mathfrak{M}(V, W)$ is nonempty if and only if $(\dim W - \mathbf{C} \dim V)$ is a weight of the irreducible representation with the highest weight $\dim W$; see [48, Theorem 10.2]). But it is not true in general, and dimensions of $\mathfrak{M}^\bullet(V, W)$ may go to ∞ when V becomes large. In the following, we use $\mathfrak{M}^\bullet(W)$ for brevity of notation and consider its geometric structure on each $\mathfrak{M}^\bullet(V, W)$ individually. We never consider it as an infinite-dimensional variety. Furthermore, we only need $\mathfrak{M}^\bullet(V, W)$ such that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V, W) \neq \emptyset$ in practice. From the above remark, we can stay in finite V's.

The following three-term complex plays an important role:

$$C^\bullet_{i,a}(V,W): V_i(aq) \xrightarrow{\sigma_{i,a}} \bigoplus_{h:i(h)=i} V_{o(h)}(a) \oplus W_i(a) \xrightarrow{\tau_{i,a}} V_i(aq^{-1}), \tag{3.11}$$

where

$$\sigma_{i,a} = \bigoplus_{i(h)=i} B_{\overline{h},aq} \oplus \beta_{i,aq}, \qquad \tau_{i,a} = \sum_{i(h)=i} \varepsilon(h) B_{h,a} + \alpha_{i,a}.$$

This is a complex thanks to the equation $\mu(B,\alpha,\beta)=0$. If (B,α,β) is stable, $\sigma_{i,a}$ is injective as the $(I\times\mathbb{C}^*)$-graded vector space V' given by $V'_i(aq) := \operatorname{Ker}\sigma_{i,a}$, $V'_j(b) := 0$ (otherwise) is B-invariant and contained in $\operatorname{Ker}\beta$ and hence must be zero.

We assign the degree zero to the middle term. We define the rank of complex $C^\bullet$ by $\sum_p (-1)^p \operatorname{rank} C^p$. It is exactly the left-hand side of (3.1). Therefore (V,W) is l-dominant if and only if

$$\operatorname{rank} C^\bullet_{i,a}(V,W) \geq 0$$

for any i,a. From this observation the "only-if" part of (3.8) is clear. If we consider the complex at a point $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$, it is easy to see that $\tau_{i,a}$ is surjective. Therefore $\operatorname{rank} C^\bullet_{i,a}(V,W)$ is the dimension of the middle cohomology group. When (V,W) is l-dominant, we define an $I\times\mathbb{C}^*$-graded vector space $C^\bullet(V,W)$ by

$$\dim\bigl(C^\bullet(V,W)\bigr)_i(a) = \operatorname{rank} C^\bullet_{i,a}(V,W). \tag{3.12}$$

REMARK 3.13

Since we treat only graded quiver varieties of type ADE in [54], we explain what must be modified for general types.

In [49] the graded quiver varieties are the $\mathbb{C}^*$-fixed points of the ordinary quiver varieties. When there are multiple edges joining two vertices, there are several choices of the $\mathbb{C}^*$-action. A choice corresponds to a choice of the q-analogue $\mathbf{C}_q$ of the Cartan matrix $\mathbf{C}$ which implicitly appears in the defining relation of the quantum loop algebras (see [49, (1.2.9)] for the defining relation, and see [49, (2.9.1)] or (3.11) for its relation to the $\mathbb{C}^*$-action). For example, consider type $A_1^{(1)}$. In [49] the q-analogue of the Cartan matrix was

$$\begin{pmatrix} [2]_q & -[2]_q \\ -[2]_q & [2]_q \end{pmatrix} = \begin{pmatrix} q+q^{-1} & -(q+q^{-1}) \\ -(q+q^{-1}) & q+q^{-1} \end{pmatrix},$$

while it is

$$\begin{pmatrix} [2]_q & -2 \\ -2 & [2]_q \end{pmatrix} = \begin{pmatrix} q+q^{-1} & -2 \\ -2 & q+q^{-1} \end{pmatrix}$$

in this article. When there is at most one edge joining two vertices, we do not have this choice as $[1]_q = 1$. The theory developed in [49] works for any choice of the $\mathbb{C}^*$-action.

For the results in [54], we need a little care. First of all, [54, Corollary 3.7] does not make sense since it is not known whether we have tensor products in general, as we already mentioned. For the choice of the $\mathbb{C}^*$-action in this article, all other results hold without any essential changes, except assertions when ε is a root of unity or ± 1. (In these cases, we get new types of strata, so the assertion must be modified. For the affine type, they can be understood from [52].) If we take the $\mathbb{C}^*$-action in [49], the recursion used to prove Axiom 2 does not work. So we first take the $\mathbb{C}^*$-action in this article, and then apply the same trick used to deal with cyclic quiver varieties. In particular, we need to include an analogue of Axiom 4. Details are left as an exercise for the reader of [54].

3.2. Transversal slice

Take a point $x \in \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V^0, W)$. Let T be the tangent space of $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V^0, W)$ at x. Since $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V^0, W)$ is nonempty, (V^0, W) is l-dominant; that is, (3.1) holds by (3.8). Let $W^\perp = C^\bullet(V^0, W)$ as in (3.12).

We consider another graded quiver variety $\mathfrak{M}_0^\bullet(V, W)$ which contains x in its closure. By (3.9), we have $V \le V^0$. Therefore we can consider $V^\perp$, an $(I \times \mathbb{C}^*)$-graded vector space whose (i,a)-component has the dimension $\dim V_i(a) - \dim V_i^0(a)$. We have $\dim W - \mathbf{C}_q \dim V = \dim W^\perp - \mathbf{C}_q \dim V^\perp$, which means the weight is unchanged under this procedure.

THEOREM 3.14 ([49, SECTION 3.3])

We work in the complex analytic topology. There exist neighborhoods U, U_T, $U_{\mathfrak{S}}$ of $x \in \mathfrak{M}_0^\bullet(V, W)$, $0 \in T$, $0 \in \mathfrak{M}_0^\bullet(V^\perp, W^\perp)$, respectively, and biholomorphic maps $U \to U_T \times U_{\mathfrak{S}}$, $\pi^{-1}(U) \to U_T \times \pi^{-1}(U_{\mathfrak{S}})$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathfrak{M}^\bullet(V,W) \supset \pi^{-1}(U) & \xrightarrow[\cong]{} & U_T \times \pi^{-1}(U_{\mathfrak{S}}) \subset T \times \mathfrak{M}^\bullet(V^\perp, W^\perp) \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\times\pi} \\
\mathfrak{M}_0^\bullet(V,W) \supset \quad U & \xrightarrow[\cong]{} & U_T \times U_{\mathfrak{S}} \quad \subset T \times \mathfrak{M}_0^\bullet(V^\perp, W^\perp)
\end{array}$$

Furthermore, a stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V', W)$ of $\mathfrak{M}_0^\bullet(V, W)$ is mapped to a product of U_T and the stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V'^\perp, W^\perp)$ of $\mathfrak{M}_0^\bullet(V^\perp, W^\perp)$.

Here $V'^\perp$ is defined exactly as $V^\perp$ replacing V by V'; that is, $\dim V'^\perp = \dim V' - \dim V^0$.

Note that $V'' \le V' \Leftrightarrow V''^\perp \le V'^\perp$ if we define $V''^\perp$ for V'' in the same way.

See also [14] for the same result in the étale topology.

3.3. The additive category $\mathscr{Q}_W$ and the Grothendieck ring

Let X be a complex algebraic variety. Let $\mathscr{D}(X)$ be the bounded derived category of constructible sheaves of $\mathbb{C}$-vector spaces on X. For $j \in \mathbb{Z}$, the shift functor is denoted by $L \mapsto L[j]$. The Verdier duality is denoted by D. For a locally closed subvariety $Y \subset X$, we denote by 1_Y the constant sheaf on Y. We denote by $IC(Y)$ the intersection cohomology complex associated with the trivial local system 1_Y on Y. Our degree convention is such that $IC(Y)|_Y = 1_Y[\dim Y]$.

Since $\pi\colon \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W)$ is proper and $\mathfrak{M}^\bullet(V,W)$ is smooth, $\pi_!(1_{\mathfrak{M}^\bullet(V,W)})$ is a direct sum of shifts of simple perverse sheaves on $\mathfrak{M}_0^\bullet(V,W)$ by the decomposition theorem (see [2]). We denote by $\mathscr{P}_W$ the set of isomorphism classes of simple perverse sheaves obtained in this manner, considered as a complex on $\mathfrak{M}_0^\bullet(W)$ by extension by zero to the complement of $\mathfrak{M}_0^\bullet(V,W)$. By [49, Section 14], $\mathscr{P}_W = \{IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)) \mid \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W) \neq \emptyset\}$. By (3.10), $\#\mathscr{P}_W < \infty$. Set $IC_W(V) \overset{\mathrm{def}}{=} IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W))$. Let $\mathscr{Q}_W$ be the full subcategory of $\mathscr{D}(\mathfrak{M}_0^\bullet(W))$ whose objects are the complexes isomorphic to finite direct sums of $IC_W(V)[k]$ for various $IC_W(V) \in \mathscr{P}_W$, $k \in \mathbb{Z}$. Let $\pi_W(V) \overset{\mathrm{def}}{=} \pi_!(1_{\mathfrak{M}^\bullet(V,W)} \times [\dim \mathfrak{M}^\bullet(V,W)])$. By the definition, we have $\pi_W(V) \in \mathscr{Q}_W$. The subcategory $\mathscr{Q}_W$ is preserved under D, and elements in $\mathscr{P}_W$ are fixed by D.

Let $\mathcal{K}(\mathscr{Q}_W)$ be the abelian group with one generator (L) for each isomorphism class of objects of $\mathscr{Q}_W$ and with relations $(L) + (L') = (L'')$ whenever L'' is isomorphic to $L \oplus L'$. It is a module over $\mathcal{A} = \mathbb{Z}[t,t^{-1}]$ by $t(L) = (L[1])$, $t^{-1}(L) = (L[-1])$. It is a free $\mathcal{A}$-module with base $\{(IC_W(V)) \mid IC_W(V) \in \mathscr{P}_W\}$. The duality D defines the *bar involution* $\overline{}$ on $\mathcal{K}(\mathscr{Q}_W)$ fixing $(IC_W(V))$ and satisfying $\overline{t(L)} = t^{-1}\overline{(L)}$. Since π is proper and $\mathfrak{M}^\bullet(V,W)$ is smooth, we also have $\overline{(\pi_W(V))} = (\pi_W(V))$. We do not write ( ) hereafter.

There is another base

$$\{\pi_W(V) \mid (V,W) \text{ is } l\text{-dominant}, \mathfrak{M}^\bullet(V,W) \neq \emptyset\}.$$

Note that $\pi_W(V)$ make sense for any V without the l-dominance condition, but we need to take only l-dominant ones to get a base. Let us define $a_{V,V';W}(t) \in \mathcal{A}$ by

$$\pi_W(V) = \sum_{V'} a_{V,V';W}(t) IC_W(V'). \tag{3.15}$$

Then we have $a_{V,V';W}(t) \in \mathbb{Z}_{\geq 0}[t,t^{-1}]$, $a_{V,V;W}(t) = 1$ and $a_{V,V';W} = 0$ unless $V' \leq V$. Since both $\pi_W(V)$ and $IC_W(V')$ are fixed by the bar involution, we have $a_{V,V';W}(t) = a_{V,V';W}(t^{-1})$. It also follows that we only need to consider $\pi_!(1_{\mathfrak{M}^\bullet(V,W)})$ for which (V,W) is *l-dominant* in the definition of $\mathscr{P}_W$.

Take V^0 such that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V^0,W) \neq \emptyset$. Taking into account the transversal slice in Section 3.2, we define a surjective homomorphism $p_{W^\perp,W}\colon \mathcal{K}(\mathscr{Q}_W) \to \mathcal{K}(\mathscr{Q}_{W^\perp})$ by

$$IC_W(V) \mapsto \begin{cases} IC_{W^\perp}(V^\perp) & \text{if } \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V^0,W) \subset \overline{\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)}, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.14, this homomorphism is also compatible with $\pi_W(V)$. Taking various V's, the $\mathcal{K}(\mathscr{Q}_W)$'s form a projective system.

We consider the dual $\mathcal{K}(\mathscr{Q}_W)^* = \mathrm{Hom}_{\mathcal{A}}(\mathcal{K}(\mathscr{Q}_W), \mathcal{A})$. Let $\{L_W(V)\}$, $\{\chi_W(V)\}$ be the bases of $\mathcal{K}(\mathscr{Q}_W)^*$ dual to $\{IC_W(V)\}$, $\{\pi_W(V)\}$, respectively. Here, V runs over the set of isomorphism classes of $(I \times \mathbb{C}^*)$-graded vector spaces such that (V, W) is l-dominant. We consider yet another base $\{M_W(V)\}$ of $\mathcal{K}(\mathscr{Q}_W)^*$ given by

$$\mathcal{K}(\mathscr{Q}_W) \ni (L) \mapsto \sum_k t^{\dim \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)-k} \dim H^k(i^!_{x_{V,W}} L) \in \mathcal{A},$$

where $x_{V,W}$ is a point in $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ and $i_{x_{V,W}}$ is the inclusion of the point $x_{V,W}$ in $\mathfrak{M}_0^\bullet(W)$. By Theorem 3.14, it is independent of the choice of $x_{V,W}$. Also, it is compatible with the projective system: if $V^0 \geq V' \geq V$, $\langle M_W(V'), IC_W(V)\rangle = \langle M_{W^\perp}(V'^\perp), IC_{W^\perp}(V^\perp)\rangle$.

By the defining property of perverse sheaves, we have

$$L_W(V) \in M_W(V) + \sum_{V' : V' > V} t^{-1}\mathbb{Z}[t^{-1}] M_W(V'). \tag{3.16}$$

Since there are only finitely many V' with $V' > V$, this is a finite sum. This shows that $\{M_W(V)\}_V$ is a base. Recall also that the canonical base $L_W(V)$ is characterized by this property together with $\overline{L_W(V)} = L_W(V)$. It is the analogue of the characterization of the Kazhdan-Lusztig base. This is not relevant in this article, but it was important to compute $L_W(V)$ explicitly in [54].

Let

$$\mathbf{R}_t \overset{\text{def}}{=} \left\{ (f_W) \in \prod_W \operatorname{Hom}_{\mathcal{A}}\big(\mathcal{K}(\mathscr{Q}_W), \mathcal{A}\big) \;\middle|\; \begin{array}{c} \langle f_W, IC_W(V)\rangle = \langle f_{W^\perp}, IC_{W^\perp}(V^\perp)\rangle \\ \text{for any } W,\, W^\perp \text{ as above} \end{array} \right\}.$$

A functional $(f_W) \in \mathbf{R}_t$ is determined when all values $\langle f_{W^\perp}, IC_{W^\perp}(0)\rangle$ are given for any $W^\perp$. Let $L(W)$, $\chi(W)$, $M(W)$ be the functional determined from $L_W(0)$, $\chi_W(0)$, $M_W(0)$, respectively. For example, $\langle L(W), IC_{W'}(V')\rangle = \delta_{\dim W, \dim W' - \mathbf{C}_q \dim V'}$. They form an analogue of *canonical*, *monomial*, and *PBW* bases of $\mathbf{R}_t$, respectively. From (3.16) the transition matrix between the canonical and monomial bases are upper triangular with respect to the ordering $(0, W) \leq (0, W')$.

The following is the main result in [49].

THEOREM 3.17 ([49, THEOREM 14.3.10])

As an abelian group, $\mathbf{R}_t|_{t=1}$ is isomorphic to the Grothendieck group of the category $\mathscr{R}$ of l-integrable representations of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ of the symmetric Kac-Moody Lie algebra $\mathfrak{g}$ given by the Cartan matrix $\mathbf{C}$, so that

(1) *$L(W)$ corresponds to the class of the simple modules whose Drinfeld polynomial is given by*

$$P_i(u) = \prod_{a \in \mathbb{C}^*} (1 - au)^{\dim W_i(a)} \quad (i \in I).$$

(2) *$M(W)$ corresponds to the class of the standard modules whose Drinfeld polynomial is given by the same formula.*

Since we do not need this result in this article, except for an explanation of our approach to one in [31], we do not explain terminologies and concepts in the statement (see [49]).

From a general theory of the convolution algebra (see [12]), $\mathcal{K}(\mathscr{Q}_W)$ is the Grothendieck group of the category of *graded* representations of the convolution algebra $H_*(\mathfrak{M}^\bullet(W)\times_{\mathfrak{M}_0^\bullet(W)}\mathfrak{M}^\bullet(W))\cong\bigoplus_{V^1,V^2}\operatorname{Ext}^*_{\mathscr{D}(\mathfrak{M}_0^\bullet(W))}(\pi_W(V^1),\pi_W(V^2))$, where the grading is for $\operatorname{Ext}^\bullet$-group. And $\{L_W(V)\}$ is the base given by classes of simple modules.

Let us briefly explain how we glue the abelian categories for various W to get a single abelian category. A family of graded module structures $\{\rho_W : H_*(\mathfrak{M}^\bullet(W)\times_{\mathfrak{M}_0^\bullet(W)}\mathfrak{M}^\bullet(W))\to\operatorname{End}_{\mathbb{C}}(M)\}_W$ on a single vector space M is said to be *compatible* if ρ_W factors through various restrictions to open subsets in Theorem 3.14 and the restrictions are compatible with the restriction of $\rho_{W^\perp}$ under the local isomorphisms in Theorem 3.14. For example, we fix W^0 and choose various points $x_{V,W}\in\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ with $\dim W-\mathbf{C}_q\dim V=\dim W^0$. We identify $H_*(\pi^{-1}(x_{V,W}))$ with a single vector space M, say, $H_*(\pi^{-1}(x_{0,W^0}))$, by the local isomorphisms. It is a compatible family of module structures. Compatible families form an abelian category. Let us denote it by $\mathscr{R}_{\mathrm{conv}}$. Then we have $\mathcal{K}(\mathscr{R}_{\mathrm{conv}})\cong\mathbf{R}_t$. In Theorem 3.17, we have families of homomorphisms $\mathbf{U}_q(\mathbf{L}\mathfrak{g})\to H_*(\mathfrak{M}^\bullet(W)\times_{\mathfrak{M}_0^\bullet(W)}\mathfrak{M}^\bullet(W))$ compatible with the local isomorphisms. Therefore we have a functor from $\mathscr{R}_{\mathrm{conv}}$ to the category $\mathscr{R}$ of l-integrable representations of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. It sends a simple object to a simple module. We do not know whether it is an equivalence (after forgetting the grading on $\mathscr{R}_{\mathrm{conv}}$), but we can get enough information practically.

3.4. t-Analogue of q-characters

For each $(i,a)\in I\times\mathbb{C}^*$, we introduce an indeterminate $Y_{i,a}$. Let

$$\mathscr{Y}_t\overset{\mathrm{def}}{=}\mathcal{A}[Y_{i,a},Y_{i,a}^{-1}]_{i\in I,a\in\mathbb{C}^*}.$$

We associate polynomials e^W, $e^V\in\mathscr{Y}_t$ to graded vector spaces V, W by

$$e^W=\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{\dim W_i(a)},\qquad e^V=\prod_{i\in I,a\in\mathbb{C}^*}V_{i,a}^{\dim V_i(a)},$$

$$\text{where } V_{i,a}=Y_{i,aq^{-1}}^{-1}Y_{i,aq}^{-1}\prod_{\substack{h\in H\\ o(h)=i}}Y_{i(h),a}.$$

We define the *t-analogue* of the *q-character* for $M(W)$ by

$$\chi_{q,t}\big(M(W)\big)\overset{\mathrm{def}}{=}\sum_V\sum_k t^{-k}\dim H^k\big(i_0^!\pi_W(V)\big)e^We^V,$$

where zero is the unique point of $\mathfrak{M}_0^\bullet(0,W)$. From the definition in Section 3.3, this is nothing but the generating function of pairings $\langle M_W(0),\pi_W(V)\rangle$ for various V. If $\mathfrak{g}$ is of type ADE, $\mathfrak{M}^\bullet(V,W)$ becomes empty for large V, as we mentioned in Section 3.1. Therefore this is a finite sum. If $\mathfrak{g}$ is not of type ADE, this becomes an infinite series, so it lives in a completion of $\mathscr{Y}_t$. Since the difference is not essential, we keep the notation $\mathscr{Y}_t$. Anyway we use only the truncated q-character, which is in $\mathscr{Y}_t$, in this article.

Suppose that (V^0, W) is l-dominant and we define $V^\perp$, $W^\perp$ as in Section 3.2. Then

$$\sum_V \langle M_W(V^0), \pi_W(V)\rangle e^W e^V = \sum_V \langle M_{W^\perp}(0), \pi_{W^\perp}(V^\perp)\rangle e^{W^\perp} e^{V^\perp} = \chi_{q,t}\bigl(M(W^\perp)\bigr)$$

as $e^W e^V = e^{W^\perp} e^{V^\perp}$.

Since $\{M(W)\}$ is a base of $\mathbf{R}_t$, we can extend $\chi_{q,t}$ to $\mathbf{R}_t$ linearly. We have

$$\chi_{q,t}\bigl(L(W)\bigr) = \sum_V \langle L_W(0), \pi_W(V)\rangle e^W e^V = \sum_V a_{V,0;W}(t) e^W e^V, \tag{3.18}$$

where $a_{V,0;W}$ is the coefficient of $IC_W(0) = 1_{\{0\}}$ in $\pi_W(V)$ (in $\mathcal{K}(\mathscr{Q}_W)$) as in (3.15).

Since $\{M_W(V)\}_{(V,W):l\text{-dominant}}$ forms a base of $\mathbf{R}_t$, we have the following.

THEOREM 3.19

The q-character homomorphism $\chi_{q,t}\colon \mathbf{R}_t \to \mathscr{Y}_t$ is injective.

But $\chi_{q,t}$ also contains terms from $\pi_W(V)$ with (V,W) not necessarily l-dominant. This is redundant information.

REMARK 3.20

By [54, Theorem 3.5], the coefficient of $e^W e^V$ in the t-analogue of q-characters for standard modules $M(W)$ is in $t^{\dim \mathfrak{M}^\bullet(V,W)}\mathbb{Z}_{\ge 0}[t^{-2}]$. This was a consequence of vanishing of odd cohomology groups of $\mathfrak{L}^\bullet(V,W)$. From the proof of [12, Lemma 8.7.8] together with the above vanishing result, we have

$$a_{V,0;W}(t) \in t^{\dim \mathfrak{M}^\bullet(V,W)}\mathbb{Z}_{\ge 0}[t^{-2}].$$

3.5. A convolution diagram

Let us take a 2-step flag $0 \subset W^2 \subset W$ of $(I \times \mathbb{C}^*)$-graded vector spaces. We put $W/W^2 = W^1$. Following [50], we introduce closed subvarieties in $\mathfrak{M}_0^\bullet(W)$ and $\mathfrak{M}^\bullet(W)$:

$$\begin{aligned}\mathfrak{Z}_0^\bullet(W^1; W^2) = \bigl\{[B,\alpha,\beta] \in \mathfrak{M}_0^\bullet(W) \bigm| W^2 \\ \text{is invariant under } \beta B^k \alpha \text{ for any } k \in \mathbb{Z}_{\ge 0}\bigr\},\\ \mathfrak{Z}^\bullet(W^1; W^2) = \pi^{-1}\bigl(\mathfrak{Z}_0^\bullet(W^1; W^2)\bigr).\end{aligned}$$

This definition is different from the original one but equivalence was proved by [50, Lemma 3.6, Remark 3.7]. The latter has an α-partition

$$\mathfrak{Z}^\bullet(W^1; W^2) = \bigsqcup \mathfrak{Z}^\bullet(V^1, W^1; V^2, W^2)$$

such that $\mathfrak{Z}^\bullet(V^1, W^1; V^2, W^2)$ is a vector bundle over $\mathfrak{M}^\bullet(V^1, W^1) \times \mathfrak{M}^\bullet(V^2, W^2)$ of rank

$$\langle \dim V^1, q^{-1}(\dim W^2 - \mathbf{C}_q \dim V^2)\rangle + \langle \dim V^2, q \dim W^1\rangle$$

(see [50, Proposition 3.8]). Let us denote this rank by

$$d(V^1, W^1; V^2, W^2).$$

(It was denoted by $d(e^{V^1}e^{W^1}, e^{V^2}e^{W^2})$ in [54].)

Following [59], we consider the diagram

$$\mathfrak{M}_0^\bullet(W^1) \times \mathfrak{M}_0^\bullet(W^2) \xleftarrow{\kappa} \mathfrak{Z}_0^\bullet(W^1; W^2) \xrightarrow{\iota} \mathfrak{M}_0^\bullet(W),$$

where ι is the inclusion and κ is given by the induced maps from $\beta B^k\alpha$ to $W^1 = W/W^2$, W^2. Then we define a functor

$$\widetilde{\mathrm{Res}}_{W^1,W^2} \stackrel{\text{def}}{=} \kappa_! \iota^* \colon \mathscr{D}\big(\mathfrak{M}_0^\bullet(W)\big) \to \mathscr{D}\big(\mathfrak{M}_0^\bullet(W^1) \times \mathfrak{M}_0^\bullet(W^2)\big).$$

We have

$$\begin{aligned}&\widetilde{\mathrm{Res}}_{W^1,W^2}(\pi_W(V))\\ &\quad = \bigoplus_{V^1+V^2=V} \pi_{W^1}(V^1) \boxtimes \pi_{W^2}(V^2)[d(V^2,W^2;V^1,W^1) - d(V^1,W^1;V^2,W^2)]\end{aligned}$$

(see [59, Lemma 4.1]; a weaker statement was given in [54, Proposition 6.2(3)]). From this observation, objects in $\mathscr{Q}_W$ are sent to $\mathscr{Q}_{W^1\times W^2}$, the full subcategory of $\mathscr{D}(\mathfrak{M}_0^\bullet(W^1) \times \mathfrak{M}_0^\bullet(W^2))$ whose objects are complexes isomorphic to finite direct sums of $IC_{W^1}(V^1) \boxtimes IC_{W^2}(V^2)[k]$ for various $IC_{W^1}(V^1) \in \mathscr{P}_{W^1}$, $IC_{W^2}(V^2) \in \mathscr{P}_{W^2}$, $k \in \mathbb{Z}$ (see [59, Lemma 4.1]). Therefore this functor induces a homomorphism $\mathcal{K}(\mathscr{Q}_W) \to \mathcal{K}(\mathscr{Q}_{W^1}) \otimes_{\mathcal{A}} \mathcal{K}(\mathscr{Q}_{W^2})$. It is coassociative, as $\mathcal{K}(\mathscr{Q}_W)$ is spanned by classes $\pi_W(V)$ and they satisfy the coassociativity from the above formula. We denote it also by $\widetilde{\mathrm{Res}}_{W^1,W^2}$.

Let $\mathbf{C}_q^{-1}$ be the inverse of $\mathbf{C}_q$. We define it by solving the equation $(u_i(a)) = \mathbf{C}_q(x_i(a))$ recursively starting from $x_i(aq^s) = 0$ for sufficiently small s. Note that $x_i(a)$ may be nonzero for infinitely many a. We then observe that

$$d(V^1, W^1; V^2, W^2) - \langle \mathbf{C}_q^{-1} \dim W^1, q^{-1} \dim W^2 \rangle$$

is preserved under the replacement $\mathfrak{M}^\bullet(V^1, W^1) \times \mathfrak{M}^\bullet(V^2, W^2) \rightsquigarrow \mathfrak{M}^\bullet(V^{1\perp}, W^{1\perp}) \times \mathfrak{M}^\bullet(V^{2\perp}, W^{2\perp})$ by the transversal slice (see 59, Lemma 3.2]). Therefore we define

$$\varepsilon(W^1, W^2) \stackrel{\text{def}}{=} \langle \mathbf{C}_q^{-1} \dim W^1, q^{-1} \dim W^2 \rangle - \langle \mathbf{C}_q^{-1} \dim W^2, q^{-1} \dim W^1 \rangle,$$

$$\mathrm{Res} \stackrel{\text{def}}{=} \sum_{W=W^1\oplus W^2} \widetilde{\mathrm{Res}}[\varepsilon(W^1, W^2)].$$

Then its transpose defines a multiplication on $\mathbf{R}_t$, which is denoted by $\otimes$.

We also define the twisted multiplication on $\mathscr{Y}_t$ given by

$$m_1 * m_2 = t^{\varepsilon(\vec{m}_1, \vec{m}_2)} m_1 m_2, \tag{3.21}$$

where m_1, m_2 are monomials in $Y_{i,a}^{\pm}$ and $\vec{m}_\alpha = (m_i^\alpha(a))$ is given by $m_\alpha = \prod Y_{i,a}^{m_i^\alpha(a)}$.

The following is the main result of [59].

THEOREM 3.22

(1) *The structure constant of the product with respect to the base* $\{L(W)\}$ *is positive:*

$$L(W^1)\otimes L(W^2)\in\sum_W a^W_{W^1,W^2}(t)L(W)$$

with $a^W_{W^1,W^2}(t)\in\mathbb{Z}_{\geq 0}[t,t^{-1}]$.

(2) $\chi_{q,t}\colon \mathbf{R}_t\to\mathscr{Y}_t$ *is an algebra homomorphism with respect to* $\otimes$ *and the twisted product* $*$.

The following corollary of the positivity is also due to [59].

COROLLARY 3.23

The following are equivalent:

(1) $L(W^1)\otimes L(W^2)=L(W^1\oplus W^2)$ *holds at* $t=1$*;*
(2) $L(W^1)\otimes L(W^2)=t^{\varepsilon(W^1,W^2)}L(W^1\oplus W^2)$.

It is tiresome to keep powers of t when tensor products of simple modules are simple. From this corollary, there is no loss of information even if we forget powers. Therefore we do not write $t^{\varepsilon(W^1,W^2)}$ hereafter.

The restriction functor defines an algebra homomorphism

$$\begin{aligned}&H_*\big(\mathfrak{M}^\bullet(W)\times_{\mathfrak{M}_0^\bullet(W)}\mathfrak{M}^\bullet(W)\big)\\&\quad\to H_*\big(\mathfrak{M}^\bullet(W^1)\times_{\mathfrak{M}_0^\bullet(W^1)}\mathfrak{M}^\bullet(W^1)\big)\otimes H_*\big(\mathfrak{M}^\bullet(W^2)\times_{\mathfrak{M}_0^\bullet(W^2)}\mathfrak{M}^\bullet(W^2)\big).\end{aligned}$$

It gives us a monoidal structure on the ungraded version of $\mathscr{R}_{\mathrm{conv}}$.

4. Graded quiver varieties for the monoidal subcategory $\mathscr{C}_1$

4.1. Graded quiver varieties and the decorated quiver

The monoidal subcategory $\mathscr{C}_1$ introduced in [31] is, in fact, the first (or second) of a series of subcategories $\mathscr{C}_\ell$ indexed by $\ell\in\mathbb{Z}_{\geq 0}$. Let us describe all of them in terms of the category $\mathscr{R}_{\mathrm{conv}}$.

We suppose that (I,E) contains no odd cycles and take a *bipartite* partition $I=I_0\sqcup I_1$; that is, every edge connects a vertex in I_0 with one in I_1. We set

$$\xi_i=\begin{cases}0 & \text{if } i\in I_0,\\ 1 & \text{if } i\in I_1.\end{cases}$$

Fix a nonnegative integer ℓ. We consider the graded quiver varieties $\mathfrak{M}^\bullet(V,W)$, $\mathfrak{M}_0^\bullet(V,W)$ under the condition

$$(*_\ell)\qquad W_i(a)=0\quad\text{unless } a=q^{\xi_i},q^{\xi_i+2},\dots,q^{\xi_i+2\ell}.$$

It is clear that if W satisfies $(*_\ell)$, both W^1 and W^2 satisfy $(*_\ell)$ in the convolution product $\mathrm{Res}\colon\mathscr{Q}_W\to\mathscr{Q}_{W^1}\times\mathscr{Q}_{W^2}$. Also, from the proof of Proposition 4.6(1), it is clear that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)\neq\emptyset$ implies $V_i(a)=0$ unless $a=q^{\xi_i+1},\dots,q^{\xi_i+2\ell-1}$.

Since $W_i^{\perp}(a)$ in Section 3.2 is the middle cohomology of the complex (3.11), $W_i^{\perp}(a)$ also satisfies $(*_\ell)$. Therefore the condition $(*_\ell)$ is also compatible with the projective system $\mathcal{K}(\mathscr{Q}_W) \to \mathcal{K}(\mathscr{Q}_{W^\perp})$. Therefore we have the subring $\mathbf{R}_{t,\ell}$ of $\mathbf{R}_t$. We set $\mathbf{R}_\ell = \mathbf{R}_{t,\ell}|_{t=1}$. It is also clear that the definition in [31] in terms of roots of Drinfeld polynomials corresponds to our definition when $\mathfrak{g}$ is of type ADE from the theory developed in [49].

EXAMPLE 4.1

Consider the simplest case $\ell = 0$. By [49, Proposition 4.2.2] or the argument below we have $\mathfrak{M}_0^\bullet(V,W) = \{0\}$ if W satisfies $(*_0)$. Therefore $\mathscr{Q}_W$ consists of finite direct sums of shifts of a single object $1_{\mathfrak{M}_0^\bullet(0,W)}$. We have $\operatorname{Res}(1_{\mathfrak{M}_0^\bullet(0,W)}) = 1_{\mathfrak{M}_0^\bullet(0,W^1)} \boxtimes 1_{\mathfrak{M}_0^\bullet(0,W^2)}$. This corresponds to the fact that any tensor product of simple modules in $\mathscr{C}_0$ remains simple (see [31, Example 3.3]).

We now start to analyze the condition $(*_{\ell=1})$. Let

$$\begin{aligned}\mathbf{E}_W \overset{\text{def}}{=} &\bigoplus_i \operatorname{Hom}\bigl(W_i(q^{\xi_i+2}), W_i(q^{\xi_i})\bigr) \\ &\oplus \bigoplus_{h:o(h)\in I_1, i(h)\in I_0} \operatorname{Hom}\bigl(W_{o(h)}(q^3), W_{i(h)}(1)\bigr)\end{aligned} \tag{4.2}$$

This vector space $\mathbf{E}_W$ is the space of representations of the *decorated quiver*.

DEFINITION 4.3

Suppose that a finite graph $\mathcal{G} = (I,E)$ together with a bipartite partition $I = I_0 \sqcup I_1$ is given. We define the *decorated quiver* $\widetilde{\mathcal{Q}} = (\widetilde{I}, \widetilde{\Omega})$ by the following two steps.

(1) We put an orientation to each edge in E so that vertices in I_0 (resp., I_1) are sinks (resp., sources). Let Ω be the set of all oriented edges, and let $\mathcal{Q} = (I,\Omega)$ be the corresponding quiver.

(2) Let I_{fr} be a copy of I. For $i \in I$, we denote by i' the corresponding vertex in I_{fr}. Then we add a new vertex i' and an arrow $i' \to i$ (resp., $i \to i'$) if $i \in I_0$ (resp., $i \in I_1$) for each $i \in I$. Let Ω_{dec} be the set of these arrows. The decorated quiver is $\widetilde{\mathcal{Q}} = (\widetilde{I}, \widetilde{\Omega}_{\mathrm{dec}}) = (I \sqcup I_{\mathrm{fr}}, \Omega \sqcup \Omega_{\mathrm{dec}})$.

We call $\mathcal{Q} = (I,\Omega)$ the *principal part* of the decorated quiver.

For example, for type A_3 with $I_0 = \{1,3\}$, we get the following quiver:

$$\begin{tikzcd} W_1(1) & W_2(q^3) \arrow[l, "\mathbf{y}_{1,2}=\beta_{1,q}B_{1,2,q^2}\alpha_{2,q^3}"'] \arrow[r, "\mathbf{y}_{3,2}=\beta_{3,q}B_{3,2,q^2}\alpha_{2,q^3}"] \arrow[d, "\mathbf{x}_2=\beta_{2,q^2}\alpha_{2,q^3}"] & W_3(1) \\ W_1(q^2) \arrow[u, "\mathbf{x}_1=\beta_{1,q}\alpha_{1,q^2}"] & W_2(q) & W_3(q^2) \arrow[u, "\mathbf{x}_3=\beta_{3,q}\alpha_{3,q^2}"'] \end{tikzcd} \tag{4.4}$$

The maps attached with arrows are explained in the proof of Proposition 4.6.

The following is a variant of a variety corresponding to a monomial in F_i in Lusztig's theory (see [43, Section 9.1.3]).

DEFINITION 4.5

(1) Let $\nu = (\nu_i) \in \mathbb{Z}^I_{\geq 0}$. Let $\mathcal{F}(\nu, W)$ be the variety parameterizing collections of vector spaces $X = (X_i)_{i \in I}$ indexed by I such that $\dim X_i = \nu_i$ and

$$X_i \subset W_i(1) \quad (i \in I_0), \qquad X_i \subset W_i(q) \oplus \bigoplus_{h \in \Omega : o(h) = i} X_{i(h)} \quad (i \in I_1).$$

It is a kind of partial flag variety and nonsingular projective.

(2) Let $\tilde{\mathcal{F}}(\nu, W)$ be the variety of all triples $(\bigoplus \mathbf{x}_i, \bigoplus \mathbf{y}_h, X)$, where $(\bigoplus \mathbf{x}_i, \bigoplus \mathbf{y}_h) \in \mathbf{E}_W$ and $X \in \mathcal{F}(\nu, W)$ such that

$$\operatorname{Im} \mathbf{x}_i \subset X_i \ (i \in I_0), \qquad \operatorname{Im}\Big(\mathbf{x}_i \oplus \bigoplus_{h \in \Omega : o(h) = i} \mathbf{y}_h\Big) \subset X_i \quad (i \in I_1).$$

This is a vector bundle over $\mathcal{F}(\nu, W)$ and hence is nonsingular. Let $\pi_\nu \colon \tilde{\mathcal{F}}(\nu, W) \to \mathbf{E}_W$ be the natural projection. It is a proper morphism.

PROPOSITION 4.6

Suppose that W satisfies $(*_\ell)$ with $\ell = 1$.

(1) *If $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V, W) \neq \emptyset$, we have*

$$V_i(a) = 0 \quad \textit{unless } a = q^{\xi_i + 1}. \tag{4.7}$$

Moreover, we have an isomorphism $\mathfrak{M}_0^\bullet(W) \cong \mathbf{E}_W$ given by

$$[B, \alpha, \beta] \mapsto \Big(\bigoplus_{i \in I} \mathbf{x}_i, \bigoplus_{h \in \Omega} \mathbf{y}_h\Big), \qquad \mathbf{x}_i = \beta_{i, q^{\xi_i + 1}} \alpha_{i, q^{\xi_i + 2}}, \ \mathbf{y}_h = \beta_{i(h), q} B_{h, q^2} \alpha_{o(h), q^3}.$$

(2) *Suppose that V satisfies (4.7). Let us define $\nu \in \mathbb{Z}^I_{\geq 0}$ by $\nu_i = \dim V_i(q^{\xi_i + 1})$. Then $\mathfrak{M}^\bullet(V, W)$ is isomorphic to $\tilde{\mathcal{F}}(\nu, W)$, and the following diagram is commutative:*

$$\begin{tikzcd}
\mathfrak{M}^\bullet(V, W) \arrow[r, "\cong"] \arrow[d, "\pi"'] & \tilde{\mathcal{F}}(\nu, W) \arrow[d, "\pi_\nu"] \\
\mathfrak{M}_0^\bullet(W) \arrow[r, "\cong"] & \mathbf{E}_W
\end{tikzcd}$$

Proof

(1) Recall that the coordinate ring of $\mathfrak{M}_0^\bullet(V, W)$ is generated by functions given by (3.6).

Consider a map

$$\beta_{j, aq^{-n-1}} B_{h_n, q^{-n}} \cdots B_{h_1, aq^{-1}} \alpha_{i, a} \colon W_i(a) \to W_j(aq^{-n-2})$$

with $i(h_a) = o(h_{a+1})$ for $a = 1, \ldots, n-1$. From the assumption $(*_1)$, this is nonzero only when $i = j$, $n = 0$, $a = q^{\xi_i + 2}$ or $n = 1$, $i \in I_1$, $j \in I_0$, $a = q^3$. From this observation we have

$$\mathfrak{M}_0^\bullet(W) = \mathfrak{M}_0^\bullet(V, W)$$

for some V with $V_i(a) = 0$ unless $a = q^{\xi_i+1}$. Thus we obtain the first assertion. Moreover, the equation $\mu(B, \alpha, \beta) = 0$ is automatically satisfied, and the second assertion follows from the standard fact $\mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W')/\!/ \mathrm{GL}(V) \cong \mathrm{Hom}(W, W')$ for V with $\dim V \geq \min(\dim W, \dim W')$.

(2) We first observe the following. $\square$

CLAIM

Under the assumption (B, α, β) is stable if and only if the following linear maps are all injective:

$$\beta_{i,q} \colon V_i(q) \to W_i(1) \ (i \in I_0), \qquad \sigma_{i,q} \colon V_i(q^2) \to \bigoplus_{h:o(h)=i} V_{i(h)}(q) \oplus W_i(q) \ (i \in I_1)$$

(see (3.11) and the subsequent formula for the definition of $\sigma_{i,q}$).

Proof

Consider the $(I \times \mathbb{C}^*)$-graded vector space given by $V_i'(q) = \mathrm{Ker}\, \beta_{i,q}$ and all other $V_j'(a) = 0$. Then the stability condition implies $V_i'(q) = 0$. Therefore $\beta_{i,q}$ is injective. The same argument shows the injectivity of $\sigma_{i,q}$. Conversely, suppose that all the above maps are injective. Take an $(I \times \mathbb{C}^*)$-graded subspace V' of V as in Definition 3.5. First, consider $V_i'(q)$ for $i \in I_0$. We have $\beta_{i,q}|V_i'(q) = 0$. Therefore the injectivity of $\beta_{i,q}$ implies $V_i'(q) = 0$. Next, consider $V_j'(q^2) \subset V_j(q^2)$ for $j \in I_0$. We have $\beta_{j,q^2}|V_j'(q^2) = 0$ from the assumption. We also have $B_{\overline{h},q^2}(V_j'(q^2)) \subset V_i'(q) = 0$ from what we have just proved. Therefore the injectivity of $\sigma_{j,q}$ implies that $V_j'(q^2) = 0$. This completes the proof of the claim.

Suppose that $[B, \alpha, \beta] \in \mathfrak{M}^\bullet(V, W)$ is given. We set

$$\widetilde{\sigma}_{i,q} := \Big(\bigoplus_{h:o(h)=i} \beta_{i(h),q} \oplus \mathrm{id}_{W_i(\mathbf{q})} \Big) \circ \sigma_{i,q} \colon V_i(q^2) \to \bigoplus_{h:o(h)=i} W_{i(h)}(1) \oplus W_i(q),$$

$$X_i := \mathrm{Im}\, \beta_{i,q} \quad (i \in I_0), \qquad X_i := \mathrm{Im}\, \widetilde{\sigma}_{i,q} \quad (i \in I_1).$$

The spaces X_i are independent of the choice of a representative (B, α, β) of $[B, \alpha, \beta]$. From the above claim, we have $\dim X_i = \dim V_i(q)$ $(i \in I_0)$ and $\dim X_i = \dim V_i(q^2)$ $(i \in I_1)$. The remaining properties are automatically satisfied by the construction.

Conversely, suppose that $(\bigoplus \mathbf{x}_i, \bigoplus \mathbf{y}_h, X)$ is given. We set $V_i(q) := X_i$ $(i \in I_0)$, $V_i(q^2) := X_i$ $(i \in I_1)$ and define linear maps (B, α, β) by

$$\beta_{i,q} := \big(\text{the inclusion } X_i \subset W_i(1)\big), \qquad \alpha_{i,q^2} := \mathbf{x}_i \quad (i \in I_0),$$

$$\beta_{i,q^2} \oplus \bigoplus_{h:o(h)=i} B_{h,q^2} := \big(\text{the inclusion } X_i \subset W_i(q) \oplus \bigoplus X_{i(h)}\big),$$

$$\alpha_{i,q^3} := \mathbf{x}_i \quad (i \in I_1).$$

From the claim, the data (B,α,β) are stable and define a point in $\mathfrak{M}^\bullet(V,W)$. These two assignments are inverse to each other; hence, they are isomorphisms. $\square$

4.2. A contravariant functor σ

For a later application we study further the description in Proposition 4.6(2). By (2), $\mathfrak{M}^\bullet(V,W) \cong \tilde{\mathcal{F}}(\nu,W)$ can be considered a vector bundle over $\mathcal{F}(\nu,W)$. It is naturally a subbundle of the trivial bundle $\mathcal{F}(\nu,W)\times \mathbf{E}_W$. Let $\tilde{\mathcal{F}}(\nu,W)^\perp$ be its annihilator in the dual trivial bundle $\mathcal{F}(\nu,W)\times\mathbf{E}_W^*$, and let $\pi^\perp\colon \tilde{\mathcal{F}}(\nu,W)^\perp \to \mathbf{E}_W^*$ be the natural projection. We denote the dual variables of $\mathbf{x}_i$, $\mathbf{y}_h$ by $\mathbf{x}_i^*$, $\mathbf{y}_{\overline{h}}^*$, respectively, that is,

$$\mathbf{x}_i^* \in \operatorname{Hom}\bigl(W_i(q^{\xi_i}), W_i(q^{\xi_i+2})\bigr), \qquad \mathbf{y}_{\overline{h}}^* \in \operatorname{Hom}\bigl(W_{i(h)}(1), W_{o(h)}(q^3)\bigr).$$

By (2), $((\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\overline{h}}^*), X)$ is contained in $\tilde{\mathcal{F}}(\nu,W)^\perp$ if and only if

$$\mathbf{x}_i^*(X_i) = 0 \quad (i\in I_0), \qquad \Bigl(\mathbf{x}_i^* + \sum_{h:o(h)=i} \mathbf{y}_{\overline{h}}^*\Bigr)(X_i) = 0 \quad (i \in I_1). \tag{4.8}$$

It is important to understand a fiber of $\pi^\perp$ on a general point $(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\overline{h}}^*)$ in $\mathbf{E}_W^*$. Since considering a subspace X_i in $W_i(q)\oplus W_{i(h)}(1)$ looks slightly strange, let us apply the Bernstein-Gelfand-Ponomarev reflection functors from [5] (see [1, Section VII.5]) to $(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\overline{h}}^*)$ at all the vertices $i\in I_1$ (where $W_i(q^3)$ is put). First, observe that $(\pi^\perp)^{-1}(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\overline{h}}^*)$ is unchanged even if we replace $W_i(q^3)$ by the image of the map

$$\mathbf{x}_i^* + \sum_{h:o(h)=i} \mathbf{y}_{\overline{h}}^* \colon W_i(q) \oplus \bigoplus_{h:o(h)=i} W_{i(h)}(1) \to W_i(q^3) \tag{4.9}$$

for all $i\in I_1$. Then we may assume that $\mathbf{x}_i^* + \sum_{h:o(h)=i}\mathbf{y}_{\overline{h}}^*$ is surjective. Then we can go back to $(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\overline{h}}^*)$ by the inverse reflection functor. Hence the following operation gives an isomorphism between the relevant varieties.

We set

$$^\sigma W_i(q^3) \overset{\text{def}}{=} \operatorname{Ker}\Bigl(\mathbf{x}_i^* + \sum_{h:o(h)=i} \mathbf{y}_{\overline{h}}^*\Bigr)$$

and define linear maps $^\sigma\mathbf{x}_i\colon {}^\sigma W_i(q^3) \to W_i(q)$ $(i\in I_1)$, $^\sigma\mathbf{y}_h\colon {}^\sigma W_i(q^3)\to W_{i(h)}(1)$ $(h\in H$ with $o(h)=i\in I_1)$ as the compositions of the inclusion $^\sigma W_i(q^3) \to W_i(q)\oplus\bigoplus_{h:o(h)=i} W_{i(h)}(1)$ and the projections to factors. We have

$$\dim {}^\sigma W_i(q^3) = \max\Bigl(\dim W_i(q) + \sum_{h:o(h)=i} \dim W_{i(h)}(1) - \dim W_i(q^3), 0\Bigr). \tag{4.10}$$

We denote by $^\sigma W$ the new $(I\times\mathbb{C}^*)$-graded vector space given obtained from W by replacing $W_i(q^3)$ by $^\sigma W_i(q^3)$ for all $i\in I_1$. We also set $^\sigma\mathbf{x}_i = \mathbf{x}_i^*$ for $i\in I_0$. We do not change $W_i(1)$, $W_i(q^2)$ for $i\in I_0$, and $W_i(q)$ for $i \in I_1$.

We consider X_i $(i\in I_1)$ as a subspace of $^\sigma W_i(q^3)$ thanks to the second equation of (4.8). Since X_i was originally a subspace of $W_i(q)\oplus\bigoplus_{h:o(h)=i} X_{i(h)}$, the above definition implies $^\sigma\mathbf{y}_h(X_{o(h)}) \subset X_{i(h)}$.

For convenience we change the notation for a subspace from X_i to $X_i(1)$ ($i \in I_0$) or $X_i(q^3)$ ($i \in I_1$) to indicate the $\mathbb{C}^*$-grading. We also set $X_i(q^2) = 0$ for $i \in I_0$ and $X_i(q) = {}^\sigma W_i(q)$ for $i \in I_1$. Under these definitions, ${}^\sigma\mathbf{x}_i(X_i(1)) \subset X_i(q^2)$ is nothing but the first equation in (4.8), and ${}^\sigma\mathbf{x}_i(X_i(q^3)) \subset X_i(q)$ is automatically true. Thus the conditions can be phrased simply as "X is invariant under $(\bigoplus_{i\in I} {}^\sigma\mathbf{x}_i, \bigoplus_{h\in\Omega} {}^\sigma\mathbf{y}_h)$."

LEMMA 4.11
Let ${}^\sigma\mathbf{x}_i$, ${}^\sigma\mathbf{y}_h$ be as above. Then $(\pi^\perp)^{-1}(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\bar{h}}^*)$ is isomorphic to the variety of $(I \times \mathbb{C}^*)$-graded subspaces X of ${}^\sigma W$ satisfying

$$X_i(q^2) = 0 \ (i \in I_0), \qquad X_i(q) = {}^\sigma W_i(q) \ (i \in I_1),$$
$$\dim X_i(1) = \dim V_i(q) \ (i \in I_0), \qquad \dim X_i(q^3) = \dim V_i(q^2) \ (i \in I_1),$$
$$X \quad \textit{is invariant under } \Big(\bigoplus_{i\in I} {}^\sigma\mathbf{x}_i, \bigoplus_{h\in\Omega} {}^\sigma\mathbf{y}_h\Big).$$

This variety is what people call the *quiver Grassmannian* associated with the quiver representation $(\bigoplus_i {}^\sigma\mathbf{x}_i, \bigoplus_{h\in\Omega} {}^\sigma\mathbf{y}_h)$. Its importance in cluster algebra theory was first noticed in [7]. We are interested only in its Poincaré polynomial, which is independent of the choice of a general point. We denote this variety simply by $\mathrm{Gr}_V({}^\sigma W)$, suppressing the choice $(\bigoplus_i {}^\sigma\mathbf{x}_i, \bigoplus_{h\in\Omega} {}^\sigma\mathbf{y}_h)$. Note also that the I-grading is only relevant in $\mathrm{Gr}_V({}^\sigma W)$. Therefore we use this notation also for an I-graded vector space V.

Note that the orientation is different from the decorated quiver (4.4). This corresponds to the cluster algebra with principal coefficients considered in Section 2.2. Therefore we call it the quiver with *principal decoration*. For example, in type A_3 with $I_0 = \{1, 3\}$, we get the following quiver:

$$\begin{tikzcd}
W_1(1) \arrow[d, "{}^\sigma\mathbf{x}_1 = \mathbf{x}_1^*"'] & {}^\sigma W_2(q^3) \arrow[l, "{}^\sigma\mathbf{y}_{1,2}"'] \arrow[r, "{}^\sigma\mathbf{y}_{3,2}"] \arrow[d, "{}^\sigma\mathbf{x}_2"] & W_3(1) \arrow[d, "{}^\sigma\mathbf{x}_3 = \mathbf{x}_3^*"] \\
W_1(q^2) & W_2(q) & W_3(q^2)
\end{tikzcd} \tag{4.12}$$

REMARK 4.13
The quiver Grassmannian is a fiber of a projective morphism which played a fundamental role in Lusztig's construction of the canonical base. It is denoted by $\pi_{\boldsymbol{\nu}} : \tilde{\mathcal{F}}_{\boldsymbol{\nu}} \to \mathbf{E}_{\mathbf{V}}$ in [43, Part II]. But note that Lusztig considered more generally various spaces of *flags*, not only subspaces.

Later it is useful to view σ as a functor between categories of representations of quivers. Let $\mathrm{rep}\widetilde{\mathcal{Q}}$ be the category of finite-dimensional representations of the decorated quiver $\widetilde{\mathcal{Q}}$. Let ${}^\sigma\widetilde{\mathcal{Q}}$ be the quiver with the principal decoration obtained by reversing the arrows between i and i' for $i \in I_0$ as above. Let $\mathrm{rep}^\sigma\widetilde{\mathcal{Q}}$ be the corresponding category, and let $\mathrm{rep}^\sigma\widetilde{\mathcal{Q}}^{\mathrm{op}}$ be its opposite category. Then σ is the functor

$$^{\sigma}(\bullet) = \prod_{i \in I_1} \Phi_i^- \circ D(\bullet) \colon \operatorname{rep}\widetilde{\mathcal{Q}} \to \operatorname{rep}^{\sigma}\widetilde{\mathcal{Q}}^{\mathrm{op}},$$

where Φ_i^- is the reflection functor at the vertex for $W_i(q^3)$ and D is the duality operator

$$D(\bullet) = \operatorname{Hom}_{\mathbb{C}}(\bullet, \mathbb{C}).$$

To make an identification with the above picture, we fix an isomorphism $W \cong W^*$ of $(I \sqcup I_{\mathrm{fr}})$-graded vector spaces.

Let $\operatorname{rep}^-\widetilde{\mathcal{Q}}$ be the full subcategory of $\operatorname{rep}\widetilde{\mathcal{Q}}$ consisting of representations having no direct summands isomorphic to simple modules corresponding to vertices $i \in I_1$. Similarly, we define $\operatorname{rep}^{-\sigma}\widetilde{\mathcal{Q}}^{\mathrm{op}}$. Then σ defines an equivalence between $\operatorname{rep}^-\widetilde{\mathcal{Q}}$ and $\operatorname{rep}^{-\sigma}\widetilde{\mathcal{Q}}^{\mathrm{op}}$. We write the quasi-inverse functor $\sigma_- = D \circ \prod_{i \in I_1} \Phi_i^+$.

In fact, it is more elegant to consider σ as a functor between derived categories of $\operatorname{rep}\widetilde{\mathcal{Q}}$ and $\operatorname{rep}^{\sigma}\widetilde{\mathcal{Q}}^{\mathrm{op}}$ as in [27, IV.4, Example 6] (see also Remark 7.7).

5. From Grothendieck rings to cluster algebras

Since W always satisfies $(*_{\ell=1})$ hereafter, we denote $W_i(q^{3\xi_i})$ and $W_i(q^{2-\xi_i})$ by W_i and $W_{i'}$, respectively. This is compatible with the notation in Definition 4.3 as $W_i(q^{2-\xi_i})$ is on the new vertex i'.

We denote the simple modules of the decorated quiver by S_i, $S_{i'}$ corresponding to vertices $i \in I$, $i' \in I_{\mathrm{fr}}$. We consider modules of two completely different algebras,

(a) modules in $\mathscr{R}_{\mathrm{conv}}$ (or of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$) and
(b) modules of the decorated quiver.

Simple modules for the former are denoted by $L(W)$, while S_i, $S_{i'}$ denote the latter. We hope there will be no confusion. We denote the underlying $\widetilde{I} = (I \sqcup I_{\mathrm{fr}})$-graded vector space of S_i, $S_{i'}$ also by the same letter.

The Grothendieck ring $\mathbf{R}_\ell$ is a polynomial ring in the classes $L(W)$ with $\dim W = 1$ satisfying $(*_\ell)$ (l-fundamental representations in $\mathscr{C}_\ell$ when $\mathfrak{g}$ is of type ADE). This result was proved as a consequence of the theory of q-characters in [31, Proposition 3.2] for $\mathfrak{g}$ of type ADE. Since q-characters make sense for arbitrary $\mathfrak{g}$, the same argument works. The corresponding result for the whole category $\mathscr{R}$ is well known.

For $\mathbf{R}_{\ell=1}$, we have $2\#I$ variables corresponding to l-fundamental representations. We denote them by x_i and x_i', exchanging i and i' from the index of the decorated quiver (see Definition 4.3):

$$x_i = L(W) \longleftrightarrow W = S_{i'}, \qquad x_i' = L(W) \longleftrightarrow W = S_i. \tag{5.1}$$

This is confusing, but we cannot avoid it to get a correct statement.

We denote the class of the Kirillov-Reshetikhin module in $\mathscr{C}_1$ by f_i. It corresponds to the class $L(W)$, where W is a 2-dimensional $\widetilde{I}$-graded vector space with $\dim W_i = \dim W_{i'} = 1$, and 0 at other gradings. We have

$$f_i = x_i x_i' - \prod_{h \in H : o(h) = i} x_{i(h)}. \tag{5.2}$$

This is an example of the T-system proved in [53], but in fact, easy to check by studying the convolution diagram as $\mathbf{E}_W \cong \mathbb{C}$ has only two strata, the origin and the complement. It is also a simple consequence of Theorem 6.3 below. It is a good exercise for the reader.

REMARK 5.3

In [53] a more precise relation at the level of modules, not only in the Grothendieck group, was shown: for $i \in I_0$, there exists a short exact sequence

$$0 \to \bigotimes_{h \in H : o(h) = i} x_{i(h)} \to x_i' \otimes x_i \to f_i \to 0,$$

and we replace the middle term by $x_i \otimes x_i'$ if $i \in I_1$.

We have an algebra embedding

$$\mathbf{R}_{\ell=1} = \mathbb{Z}[x_i, x_i']_{i \in I} \to \mathscr{F} = \mathbb{Q}(x_i, f_i)_{i \in I}.$$

We now put the cluster algebra structure on the right-hand side. It is enough to specify the initial seed. We take x_i, f_i as cluster variables of the initial seed. We make f_i a frozen variable. We call the quiver for the initial seed the $\mathbf{x}$-*quiver*. It looks almost the same as the decorated quiver in Definition 4.3 but is a little different and is given as follows.

DEFINITION 5.4

Suppose that a finite graph $\mathcal{G} = (I, E)$ together with a bipartite partition $I = I_0 \sqcup I_1$ is given. We define the $\mathbf{x}$-*quiver* $\widetilde{\mathcal{Q}}_{\mathbf{x}} = (\widetilde{I}, \widetilde{\Omega}_{\mathbf{x}})$ by the following two steps.

(1) The underlying graph is the same as one of the decorated quiver: $\mathcal{G} = (I \sqcup I_{\mathrm{fr}}, E \sqcup \bigcup \{i - i'\})$. The variable x_i corresponds to the vertex i in the original quiver, while f_i corresponds to the new vertex i'.

(2) The rule for drawing arrows is

$$\begin{gathered} f_i \to x_i \quad (i \in I_0), \qquad x_i \to f_i \quad (i \in I_1), \\ x_{o(h)} \xrightarrow{h} x_{i(h)} \quad (\text{if } o(h) \in I_0,\ i(h) \in I_1). \end{gathered} \tag{5.5}$$

For our favorite example, A_3 with $I_0 = \{1, 3\}$, we get the following quiver:

$$\begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[d] & x_3 \arrow[l] \\ f_1 \arrow[u] & f_2 & f_3 \arrow[u] \end{tikzcd}$$

Note that the orientation differs from the decorated quiver (4.4) and the principal decoration (4.12). Also, the vertex f_i corresponds to $W_{i'}$, and x_i corresponds to W_i. This is different from the identification (5.1). If we look at the principal part, the orientation is reversed.

If we make a mutation in direction x_i, the new variable given by the exchange relation (2.2) is nothing but

$$x'_i = \frac{f_i + \prod_{h \in H : o(h) = i} x_{i(h)}}{x_i}$$

from (5.2). Note that the exchange relation is correct for the $\mathbf{x}$-quiver given by our rule (5.5) but *wrong* for the decorated quiver. Thus, this confusion cannot be avoided.

We thus have the following.

PROPOSITION 5.6
The Grothendieck ring $\mathbf{R}_{\ell=1}$ *is a subalgebra of the cluster algebra* $\mathscr{A}(\widetilde{\mathbf{B}})$.

The argument in [31, 4.4] (based on 3, 1.21) implies that $\mathbf{R}_{\ell=1} \cong \mathscr{A}(\widetilde{\mathbf{B}})$, but we will see that all cluster monomials come from simple modules in $\mathbf{R}_{\ell=1}$, so we have a different proof later.

We also need the seed obtained by applying the sequence of mutations $\prod_{i \in I_1} \mu_i$ (see [31, Section 7.1]). Then

(1) x_i $(i \in I_1)$ is replaced by x'_i,
(2) the orientation of arrows are reversed in the principal part and $i \to i'$ $(i \in I_1)$, and
(3) we add a_{ij} arrows from i to j'.

In our A_3-example, we obtain

(5.7)
$$\begin{tikzcd}
x_1 \arrow[dr] & x'_2 \arrow[l] \arrow[r] & x_3 \arrow[dl] \\
f_1 \arrow[u] & f_2 \arrow[u] & f_3 \arrow[u]
\end{tikzcd}$$

We set

(5.8)
$$z_i \overset{\text{def}}{=} \begin{cases} x_i & \text{if } i \in I_0, \\ x'_i & \text{if } i \in I_1. \end{cases}$$

We call the one above the $\mathbf{z}$-*quiver*.

6. Cluster character and prime factorizations of simple modules

6.1. An almost simple module

Fix an $\widetilde{I}$-graded vector space W. Let Ψ be the Fourier-Sato-Deligne functor for the vector space $\mathbf{E}_W \cong \mathfrak{M}_0^\bullet(W)$ (see [36, 39]). We define a subset $\mathscr{L}_W \subset \mathscr{P}_W$

by

$$L \in \mathscr{L}_W \iff \text{the support of } \Psi(L) \text{ is the whole space } \mathbf{E}_W^*.$$

If $L \in \mathscr{L}_W$, $\Psi(L)$ is an IC complex associated with a local system defined over an open set in $\mathbf{E}_W$. We denote its rank by $r_W(L) \in \mathbb{Z}_{>0}$.

Since the Fourier transform of $IC_W(0) = 1_{\{0\}}$ is $1_{\mathbf{E}_W^*}[\dim \mathbf{E}_W^*]$, we always have $IC_W(0) \in \mathscr{L}_W$. We have $r_W(IC_W(0)) = 1$.

We extend this definition for a condition on simple modules $L(W')$. Recall that $IC_W(V)$ is identified with $IC_{W^\perp}(0)$ such that $\dim W^\perp = \dim W - \mathbf{C}_q \dim V$. We say $L(W') \in \mathscr{L}_W$ if $IC_W(V) \in \mathscr{L}_W$ with $W' = W^\perp$. We similarly define $r_W(L(W'))$.

We define the *almost simple module* associated with W by

$$\mathbb{L}(W) = \sum_{L(W') \in \mathscr{L}_W} r_W\big(L(W')\big) L(W').$$

This is an element in $\mathbf{R}_t$.

From the definition of $L(W') \in \mathscr{L}_W$, we have $W' \leq W$. Therefore almost simple modules $\{\mathbb{L}(W)\}$ form a basis of $\mathbf{R}_t$ such that the transition matrix between it and $\{L(W)\}$ is upper triangular with diagonal entries 1.

We will see that an almost simple module is not necessarily simple later. There will be also a simple sufficient condition guaranteeing that an almost simple module is simple.

REMARK 6.1

As we will see soon, almost simple modules are given in terms of quiver Grassmannians for a general representation of $\mathbf{E}_W^*$. This, at first sight, looks similar to the set of generic variables considered by Dupont [20] (see also [19]). But there is a crucial difference. We consider the total sum of Betti numbers of the quiver Grassmannian, while Dupont considers Euler numbers. There is an example with nontrivial odd degree cohomology groups [17, Example 3.5], so this is really different.

Note that from the representation theory of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, it is natural to specialize as $t = 1$ since the t-analogue becomes the ordinary q-character (and the positivity is preserved). This difference cannot be seen for cluster monomials, thanks to Remark 3.20.

In fact, we can also consider a specialization at $t = -1$, but then the positivity is lost and the proof of the factorization (Proposition 6.12) breaks.

6.2. Truncated q-character

In [31, Section 6] Hernandez and Leclerc introduced the *truncated q-character* $\chi_q(M)_{\leq 2}$ from the ordinary q-character $\chi_q(M)$ by setting variables $V_{i,q^r} = 0$ for $r \geq 3$. From the geometric definition of the q-character reviewed in Section 3.4, it just means that we consider only nonsingular quiver varieties $\mathfrak{M}^\bullet(V, W)$ satisfying (4.7), that is, those studied in Proposition 4.6(2). In particular, its t-analogue also makes sense:

$$\begin{aligned}
&\chi_{q,t}\big(M(W)\big)_{\le 2} \overset{\text{def}}{=} \sum_{V \text{ satisfies } (4.7)} \sum_k t^{-k} \dim H^k\big(i_0^! \pi_W(V)\big) e^W e^V, \\
(6.2)\qquad &\chi_{q,t}\big(L(W)\big)_{\le 2} = \sum_{V \text{ satisfies } (4.7)} a_{V,0;W}(t) e^W e^V,
\end{aligned}$$

where $a_{V,0;W}(t)$ is the coefficient of $IC_W(0) = 1_{\{0\}}$ in $\pi_W(V)$ in $\mathcal{K}(\mathscr{Q}_W)$. Since V satisfies (4.7) if (V,W) is l-dominant, the truncated q-character still embeds $\mathbf{R}_{\ell=1}$ to $\mathscr{Y}_t$ (see [31, Proposition 6.1] for an algebraic proof).

The following is one of the main results in this article.

THEOREM 6.3

Suppose that W satisfies $(*_\ell)$ with $\ell = 1$. Then the truncated t-analogue of the q-character of an almost simple module is given by

$$\chi_{q,t}\big(\mathbb{L}(W)\big)_{\le 2} = \sum_V P_t\big(\mathrm{Gr}_V({}^\sigma W)\big) e^W e^V,$$

where the summation runs over all $(I \times \mathbb{C}^*)$-graded vector spaces V with (4.7) and $P_t(\ )$ is the normalized Poincaré polynomial for the Borel-Moore homology group

$$P_t\big(\mathrm{Gr}_V({}^\sigma W)\big) = \sum_i t^{i - \dim \mathfrak{M}^\bullet(V,W)} \dim H_i\big(\mathrm{Gr}_V({}^\sigma W)\big).$$

Since $\mathrm{Gr}_V({}^\sigma W)$ is a fiber of $\pi^\perp \colon \tilde{\mathscr{F}}(\nu, W)^\perp \to \mathbf{E}_W^*$ over a general point in $\mathbf{E}_W^*$ and $\tilde{\mathscr{F}}(\nu, W)^\perp$ is nonsingular, $\mathrm{Gr}_V({}^\sigma W)$ is nonsingular by the generic smoothness theorem. Since $\pi^\perp$ is projective, it is also projective. Therefore the Poincaré polynomial is essentially equal to the virtual one defined by Danilov and Khovanskii [15], using a mixed Hodge structure of Deligne [16]:

$$P_t^{\text{vert}}(X) \overset{\text{def}}{=} \sum_k (-1)^k t^{p+q} h^{p,q}\big(H_c^k(X)\big)$$

(see [15] for the notation $h^{p,q}(H_c^k(X))$). Since our Poincaré polynomial is normalized, we have

$$P_t\big(\mathrm{Gr}_V({}^\sigma W)\big) = t^{-\dim \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)} P_{-t}^{\text{vert}}\big(\mathrm{Gr}_V({}^\sigma W)\big).$$

REMARK 6.4

Recall that $\chi_{q,t}(L(W))$ was computed in [54]. More precisely, a purely combinatorial algorithm to compute $\chi_{q,t}(L(W))$ was given in [54]. If we are interested in simple modules in $\mathscr{C}_1$, the same algorithm works by replacing every '$\chi_{q,t}(\ )$' in [54] by $\chi_{q,t}(\ )_{\le 2}$. Thus the computation is drastically simplified. The algorithm consists of 3 steps. The first step is the computation of $\chi_{q,t}$ for l-fundamental representation. The actual computation of $\chi_{q,t}$ was performed by a supercomputer (see [55]). But this is certainly unnecessary for $\chi_{q,t}(\ )_{\le 2}$. The second step is the computation of $\chi_{q,t}$ for the standard modules. This is just a twisted multiplication of $\chi_{q,t}$'s given in the first step. This step is simple. The third step is analogue of the definition of Kazhdan-Lusztig polynomials. It is still a hard computation if we take large W. It is probably interesting to compare this algorithm with one given by the mutation, for example, for W corresponding to the highest root of E_8. In this case we have $\mathbb{L}(W) = L(W)$ as we see in Proposition 6.9.

In general, if $\mathbb{L}(W) \neq L(W)$, we need to compute $r_W(L(W'))$.

EXAMPLE 6.5

For the Kirillov-Reshetikhin module f_i, we have $\dim W_i = 1 = \dim W_{i'}$. If $i \in I_1$, we have ${}^\sigma W_i = 0$. Therefore $\mathrm{Gr}_V({}^\sigma W)$ is a point if $V = 0$ and $\emptyset$ otherwise. If $i \in I_0$, a general ${}^\sigma\mathbf{x}_i^*\colon W_i \to W_{i'}$ is an isomorphism. Therefore $\mathrm{Gr}_V({}^\sigma W)$ is again a point if $V = 0$ and $\emptyset$ otherwise. Thus we must have $\mathbb{L}(W) = L(W)$ in this case, and $\chi_{q,t}(f_i)_{\le 2}$ contains only the first term:

$$\chi_{q,t}(f_i)_{\le 2} = Y_{i,q^{\xi_i}} Y_{i,q^{\xi_i+2}}.$$

This can be shown in many ways, say, using the main result of [53].

Next, consider x_i. If $i \in I_0$, then ${}^\sigma W$ is 1-dimensional with nonzero entry at ${}^\sigma W_{i'}$. But since we can put only zero-dimensional space $X_{i'}$, we allow only $V = 0$. Thus $\mathbb{L}(W) = L(W)$ and $\chi_{q,t}(x_i)_{\le 2} = Y_{i,q^2}$.

If $i \in I_1$, then ${}^\sigma W$ is 2-dimensional with nonzero entries at ${}^\sigma W_i$ and ${}^\sigma W_{i'}$. Therefore we have either $V = 0$ or 1-dimensional V with nonzero entry at $V_{i'}$. The corresponding varieties are a single point in both cases. Thus $\mathbb{L}(W) = L(W)$ and $\chi_{q,t}(x_i)_{\le 2} = Y_{i,q}(1 + V_{i,q^2})$.

Similarly, we can compute x_i'. We have $\mathbb{L}(W) = L(W)$ always, and the q-character is

$$\chi_{q,t=1}(x_i')_{\le 2} = \begin{cases} Y_{i,1}\bigl(1 + V_{i,q}\prod_j (1 + V_{j,q^2})^{a_{ij}}\bigr) & \text{if } i \in I_0,\\ Y_{i,q^3} & \text{if } i \in I_1.\end{cases}$$

This gives an answer to the exercise we mentioned after (5.2).

Proof of Theorem 6.3

Since $\mathbf{E}_W$ is a vector space by Proposition 4.6 and $IC_W(V)$'s are monodromic (i.e., $H^j(IC_W(V))$ is locally constant on every $\mathbb{C}^*$-orbit of $\mathbf{E}_W$), we can apply the Fourier-Sato-Deligne functor Ψ (see [36, 39]). For example, we have

$$\Psi\bigl(IC_W(0)\bigr) = 1_{\mathbf{E}_W^*}[\dim \mathbf{E}_W].$$

Other $\Psi(IC_W(V))$ are simple perverse sheaves on $\mathbf{E}_W^*$.

Recall that $\tilde{\mathcal{F}}(\nu, W)$ is a vector subbundle of the trivial bundle $\mathcal{F}(\nu, W) \times \mathbf{E}_W$ by Proposition 4.6. Let Ψ' be the Fourier-Sato-Deligne functor for this trivial bundle. We have

$$\Psi'\bigl(1_{\tilde{\mathcal{F}}(\nu,W)}[\dim \tilde{\mathcal{F}}(\nu, W)]\bigr) = 1_{\tilde{\mathcal{F}}(\nu,W)^\perp}[\dim \tilde{\mathcal{F}}(\nu, W)^\perp],$$

where $\tilde{\mathcal{F}}(\nu, W)^\perp$ is the annihilator in the dual trivial bundle $\mathcal{F}(V,W) \times \mathbf{E}_W^*$ as in Section 4.2. Moreover, we have

$$\pi_!^\perp \circ \Psi' = \Psi \circ \pi_!.$$

Therefore if we decompose the pushforward as

$$\pi_!^\perp\big(1_{\tilde{\mathcal{F}}(\nu,W)^\perp}[\dim \tilde{\mathcal{F}}(\nu,W)^\perp]\big) \cong \bigoplus_{V',l} L_{V',l} \otimes \Psi\big(IC_W(V')\big)[l],$$

we have $\sum_l t^l \dim L_{V',l} = a_{V,V';W}(t)$.

Take a general point of $\mathbf{E}_W^*$, and consider the Poincaré polynomial of the stalk of the above. On the left-hand side we get the Poincaré polynomial of $\mathrm{Gr}_V({}^\sigma W)$ by Lemma 4.11. On the other hand, on the right-hand side the factor $\Psi(IC_W(V'))$ with $IC_W(V') \notin \mathscr{L}_W$ disappears as its support is smaller than $\mathbf{E}_W^*$. For $IC_W(V') \in \mathscr{L}_W$, we get $r_W(IC_W(V')) \times a_{V,V';W}(t)$, as $\Psi(IC_W(V'))$ is the IC complex associated with a local system of rank $r_W(IC_W(V'))$ defined over an open subset of $\mathbf{E}_W^*$. Thus we have

$$P_t\big(\mathrm{Gr}_V({}^\sigma W)\big) = \sum_{IC_W(V') \in \mathscr{L}_W} r_W\big(IC_W(V')\big) a_{V,V';W}(t). \tag{6.6}$$

We get the assertion by recalling that $a_{V,V';W}(t)$ is the coefficient of $e^{W^\perp} e^{V^\perp} = e^W e^V$ in the q-character of $L(W^\perp)$, where $\dim W^\perp = \dim W - \mathbf{C}_q \dim V'$, $\dim V^\perp = \dim V - \dim V'$ (Section 3.2). □

6.3. Factorization of KR modules

In the remainder of this section, we give several simple applications of Theorem 6.3.

PROPOSITION 6.7

We have

$$\mathbb{L}(W) \cong \mathbb{L}({}^\varphi W) \otimes \bigotimes_{i \in I} f_i^{\min(\dim W_i, \dim W_{i'})},$$

where ${}^\varphi W$ *is given by*

$$\dim {}^\varphi W_i = \max(\dim W_i - \dim W_{i'}, 0), \qquad \dim {}^\varphi W_{i'} = \max(\dim W_{i'} - \dim W_i, 0).$$

The right-hand side is independent of the order of the tensor product.

From this proposition it becomes enough to understand $\mathbb{L}({}^\varphi W)$. Notice that either ${}^\varphi W_i$ or ${}^\varphi W_{i'}$ is zero for each $i \in I$. If ${}^\varphi W_i = 0$, then ${}^\varphi W_{i'}$ is not connected to any other vertices and is easy to factor out. Thus we eventually reduce to studying the case when all ${}^\varphi W_{i'} = 0$; that is, $\mathbf{E}_{{}^\varphi W}$ is the vector space of representations of the principal part of the decorated quiver obtained by deleting all frozen vertices i'.

Proof
From the definition of ${}^\sigma W$ in the formula (4.10), it is clear that ${}^\sigma W_i$ is unchanged even if we add $\pm(1,1)$ to $(\dim W_i, \dim W_{i'})$ for $i \in I_1$. And the change of $\dim {}^\sigma W_{i'}$ does not affect the quiver Grassmannian. Therefore we can subtract $\min(\dim W_i, \dim W_{i'})$ from both $\dim W_i$ and $\dim W_{i'}$ for each $i \in I_1$. Let $\tilde{W}$ be the resulting $(I \sqcup I_{\mathrm{fr}})$-graded vector space. We have

$$\chi_{q,t}\big(\mathbb{L}(W)\big)_{\le 2} = \chi_{q,t}\big(\mathbb{L}(\tilde{W})\big)_{\le 2} \prod_{i \in I_1} (Y_{i,q}Y_{i,q^3})^{\min(\dim W_i, \dim W_{i'})}.$$

Since the truncated q-character of the Kirillov-Reshetikhin module is equal to $Y_{i,q}Y_{i,q^3}$ by Example 6.5, we have

$$\chi_{q,t}\big(\mathbb{L}(W)\big)_{\le 2} = \chi_{q,t}\big(\mathbb{L}(\tilde{W})\big)_{\le 2} \prod_{i \in I_1} f_i^{\min(\dim W_i, \dim W_{i'})}.$$

Next, we study a similar but slightly different reduction for $i \in I_0$. We consider the variety $(\pi^\perp)^{-1}(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\bar{h}}^*)$ as in the statement of Lemma 4.11. Here $(\bigoplus \mathbf{x}_i^*, \bigoplus \mathbf{y}_{\bar{h}}^*)$ is the representation considered in the proof of Lemma 4.11 before applying the reflection functors. From the condition $X_i \subset \operatorname{Ker} x_i^*$, it is isomorphic to $(\bar{\pi}^\perp)^{-1}(\bigoplus \bar{\mathbf{x}}_i^*, \bigoplus \bar{\mathbf{y}}_{\bar{h}}^*)$, where

(1) $\tilde{W}$ is obtained from W by replacing W_i by $\operatorname{Ker} x_i^*$,
(2) $\bar{\mathbf{y}}_{\bar{h}}^*$ is the restriction of $\mathbf{y}_{\bar{h}}^*$ and other maps are obvious ones.

We have

$$\dim \tilde{W}_i = \max(\dim W_i - \dim W_{i'}, 0).$$

Therefore we have

$$\chi_{q,t}\big(\mathbb{L}(W)\big)_{\le 2} = \chi_{q,t}\big(\mathbb{L}(\tilde{W})\big)_{\le 2} \prod_{i \in I_0} (Y_{i,1}Y_{i,q^2})^{\min(\dim W_i, \dim W_{i'})}.$$

Note again that $Y_{i,1}Y_{i,q^2}$ is the truncated q-character of the Kirillov-Reshetikhin module f_i. Therefore the above equality can be written as

$$\chi_{q,t}\big(\mathbb{L}(W)\big)_{\le 2} = \chi_{q,t}\big(\mathbb{L}(\tilde{W})\big)_{\le 2} \prod_{i \in I_0} f_i^{\min(\dim W_i, \dim W_{i'})}.$$

Combining these two reductions, we obtain the assertion. □

6.4. Factorization and canonical decomposition

Take a general representation $(\bigoplus \mathbf{y}_h)$ of $\mathbf{E}_{{}^\varphi W}$. We decompose it into a sum of indecomposable representations. We have a corresponding decomposition

$${}^\varphi W = W^1 \oplus W^2 \oplus \cdots \oplus W^s$$

of the I-graded graded vector space. It is known (see [34, p. 85]) that $W^1, \dots, W^s$ are independent of a choice of general representation of $\mathbf{E}_{{}^\varphi W}$ up to permutation. This is called the *canonical decomposition* of ${}^\varphi W$ (or $\dim {}^\varphi W$). It is known that all $\dim W^\alpha \in \mathbb{Z}_{\ge 0}^I$ are *Schur roots* and $\operatorname{ext}^1(W^k, W^l) = 0$ for $k \ne l$ (see [35, Proposition 3]). Here $\dim W^k$ is a Schur root if a general representation in $\mathbf{E}_{W^k}$

has only trivial endomorphisms, that is, scalars. It is known that this is equivalent to a general representation and is indecomposable (see [35, Proposition 1]). And $\mathrm{ext}^1(W^k, W^l)$ is the dimension of Ext^1 between general representations in $\mathbf{E}_{W^k}$ and $\mathbf{E}_{W^l}$. Basic results on the canonical decomposition were obtained by Schofield [57], which is used in part below.

Note that the frozen part plays no role in the canonical decomposition, as ${}^{\varphi}W_{i'} \neq 0$ implies ${}^{\varphi}W_i = 0$. Therefore we simply have factors $\underbrace{S_{i'} \oplus \cdots \oplus S_{i'}}_{\dim\ {}^{\varphi}W_{i'}\ \text{factors}}$ in the canonical decomposition. If ${}^{\varphi}W$ contains a factor $S_i^{\oplus m_i}$ for $i \in I_1$, it is killed by ${}^{\sigma}(\ )$. We thus have the following.

PROPOSITION 6.8

Suppose that the canonical decomposition of ${}^{\varphi}W$ *contains factors as*

$$ {}^{\varphi}W = {}^{\psi}W \oplus \bigoplus_{i\in I} S_{i'}^{\oplus \dim {}^{\varphi}W_{i'}} \oplus \bigoplus_{i\in I_1} S_i^{\oplus m_i}. $$

Then we have a factorization

$$ \mathbb{L}({}^{\varphi}W) = \mathbb{L}({}^{\psi}W) \otimes \bigotimes_{i\in I} L(S_{i'})^{\otimes \dim {}^{\varphi}W_{i'}} \otimes \bigotimes_{i\in I_1} L(S_i)^{\otimes m_i}. $$

We consider the following condition.

(C) The canonical decomposition of ${}^{\varphi}W$ contains only *real* Schur roots.

PROPOSITION 6.9

(1) *Assume the condition* (C). *Then* $\mathscr{L}_{{}^{\varphi}W} = \{IC_{{}^{\varphi}W}(0)\}$ *and hence* $\mathbb{L}(W) = L(W)$.

(2) *If* $\mathscr{L}_{{}^{\varphi}W} = \{IC_{{}^{\varphi}W}(0)\}$, $\mathrm{Gr}_V({}^{\sigma}W)$ *has no odd cohomology.*

Proof

(1) From the definition, $\mathbf{E}_{{}^{\varphi}W}$ contains the $(\prod_i \mathrm{GL}(W_i) \times \mathrm{GL}(W_{i'}))$-orbit of a general representation as a Zariski open subset. The same is true for $\mathbf{E}^*_{{}^{\varphi}W}$. Since all $\Psi(IC_W(V))$ are $(\prod_i \mathrm{GL}(W_i) \times \mathrm{GL}(W_{i'}))$-equivariant, we cannot have IC complexes associated with nontrivial local systems as stabilizers are always connected. Therefore we have only $\mathscr{L}_{{}^{\varphi}W} = \{IC_{{}^{\varphi}W}(0)\}$.

(2) Since (6.6) is a single sum, the assertion follows from Remark 3.20.

□

PROPOSITION 6.10

We have

$$ \mathbb{L}({}^{\varphi}W) \cong \mathbb{L}(W^1) \otimes \cdots \otimes \mathbb{L}(W^s). \tag{6.11} $$

Proof

We assume $s = 2$. Since we do not use the assumption that W^1, W^2 are Schur roots, the proof also gives the proof for the general case.

Consider the convolution diagram in Section 3.5. By [43, Section 10.1], the restriction functor commutes with the Fourier-Sato-Deligne functor up to shift. Therefore we consider perverse sheaves defined over $\mathbf{E}^*_{W^1}$, $\mathbf{E}^*_{W^2}$, $\mathbf{E}^*_W$.

We take open subsets U^1, U^2 in $\mathbf{E}^*_{W^1}$, $\mathbf{E}^*_{W^2}$ so that perverse sheaves not in $\mathscr{L}_{W^1}$, $\mathscr{L}_{W^2}$ have support outside of U^1, U^2. Similarly, we take an open subset $U \subset \mathbf{E}^*_W$ consisting of modules isomorphic to the direct sum of modules in U^1 and U^2, and perverse sheaves not in $\mathscr{L}_W$ have support outside of U.

We may assume that Ext-groups between modules in U^1, U^2 vanish. Therefore any module in $\kappa^{-1}(U^1 \times U^2)$ is isomorphic to the direct sum of modules from U^1 and U^2. Therefore $\kappa^{-1}(U^1 \times U^2) \subset U$, and κ is an isomorphism. Therefore for $L \in \mathscr{P}_W \setminus \mathscr{L}_W$, $\operatorname{Res} L$ does not have factors in $IC_{W^1}(V^1) \boxtimes IC_{W^2}(V^2)$ with $IC_{W^\alpha}(V^\alpha) \in \mathscr{L}_{W^\alpha}$ $(\alpha = 1, 2)$. Therefore the product of $L(W'^1) \in \mathscr{L}_{W^1}$ and $L(W'^2) \in \mathscr{L}_{W^2}$ is a linear combination of elements in $\mathscr{L}_W$.

If $IC_W(V) \in \mathscr{L}_W$, the restriction of $\kappa_! \iota^* \Psi(IC_W(V))$ to $U^1 \times U^2$ is a local system of rank $r(IC_W(V))$. Thus if we write

$$\operatorname{Res} IC_W(V) = \sum_{IC_{W^1}(V^1) \in \mathscr{L}_{W^1}, IC_{W^2}(V^2) \in \mathscr{L}_{W^2}} a_V^{V^1, V^2} IC_{W^1}(V^1) \boxtimes IC_{W^2}(V^2)$$
$$+ (\text{linear combination of } L \in \mathscr{P}_W \setminus \mathscr{L}_W),$$

then $a_V^{V^1, V^2}$ is an integer (up to shift). And we have

$$r\big(IC_W(V)\big) = \sum_{V^1, V^2} a_V^{V^1, V^2} r\big(IC_{W^1}(V^1)\big) r\big(IC_{W^2}(V^2)\big).$$

From this we have $\mathbb{L}(W^1) \otimes \mathbb{L}(W^2) = \mathbb{L}(W)$. □

Let us show the converse.

PROPOSITION 6.12

(1) *Suppose that* $\mathbb{L}({}^\varphi W)$ *decomposes as*

$$\mathbb{L}({}^\varphi W) \cong \mathbb{L}(W^1) \otimes \mathbb{L}(W^2).$$

Then we have $\operatorname{ext}^1(W^1, W^2) = 0 = \operatorname{ext}^1(W^2, W^1)$.

(2) *The same assertion is true even if the almost simple modules* $\mathbb{L}(\ )$ *are replaced by simple modules* $L(\ )$.

(3) *The factorization of an almost simple module* $\mathbb{L}(W)$ *is exactly given by the canonical decomposition of* ${}^\varphi W$, *and we have the bijection*

$$\left\{\begin{matrix} \textit{prime almost simple modules} \\ \textit{with } (\mathrm{C}) \end{matrix}\right\} \setminus \{x_i, f_i \mid i \in I\}$$
$$\longleftrightarrow \left\{\begin{matrix} \textit{Schur roots of the principal} \\ \textit{part } \mathcal{Q} \textit{ of the decorated quiver} \end{matrix}\right\}$$

given by $\mathbb{L}(W) \leftrightarrow \dim W$.

Here an almost prime simple module $\mathbb{L}(W)$ means that it does not factor as $\mathbb{L}(W^1)\otimes\mathbb{L}(W^2)$ of almost simple modules.

Proof

(1) Let us first consider the case $x_{i'}=\mathbb{L}(W^2)=L(W^2)$. Taking the truncated q-character, we have

$$\sum_V P_t\big(\mathrm{Gr}_V({}^\sigma W)\big)e^W e^V = Y_{i,q^3} * \Big(\sum_{V^1} P_t\big(\mathrm{Gr}_{V^1}({}^\sigma W^1)\big)e^{W^1}e^{V^1}\Big),$$

where $*$ is the twisted multiplication (3.21).

Since $i\in I_1$ is a source, we have $\mathrm{ext}^1(W^1,S_i)=0$. If we have $\mathrm{ext}^1(S_i,W^1)\neq 0$, then $\dim{}^\sigma W^1=\dim{}^\sigma W+\dim S_i$. Therefore the right-hand side contains the term for V^1 with $\dim V^1=\dim{}^\sigma W+\dim S_i$, as the corresponding quiver Grassmannian $\mathrm{Gr}_{{}^\sigma W^1}({}^\sigma W^1)$ is a single point.

But the left-hand side obviously cannot contain the corresponding term. Therefore we must have $\mathrm{ext}^1(S_i,W^1)=0$.

Now we suppose that general representations of W^1 and W^2 do not contain the direct summand S_i for any $i\in I_1$. Then the vanishing of ext^1 is equivalent to the corresponding statement after applying the functor σ. (Since σ starts with taking the *dual*, we need to exchange the first and the second entries A, B of $\mathrm{ext}^1(A,B)$, but we are studying both $\mathrm{ext}^1(A,B)$ and $\mathrm{ext}^1(B,A)$, so it does not matter.)

We again consider the equality for the truncated q-character:

$$\sum_V P_t\big(\mathrm{Gr}_V({}^\sigma W)\big)e^W e^V = \Big(\sum_{V^1} P_t\big(\mathrm{Gr}_{V^1}({}^\sigma W^1)\big)e^{W^1}e^{V^1}\Big) * \Big(\sum_{V^2} P_t\big(\mathrm{Gr}_{V^2}({}^\sigma W^2)\big)e^{W^2}e^{V^2}\Big).$$

The right-hand side contains the terms with $V^1={}^\sigma W^1$, $V^2=0$ and $V^1=0$, $V^2={}^\sigma W^2$, as both $\mathrm{Gr}_{V^1}({}^\sigma W^1)$ and $\mathrm{Gr}_{V^2}({}^\sigma W^2)$ are points in these cases. These survive thanks to the positivity $P_t(\mathrm{Gr}_{V^1}({}^\sigma W^1))$, $P_t(\mathrm{Gr}_{V^2}({}^\sigma W^2))\in\mathbb{Z}_{\geq 0}[t]$. Therefore the corresponding quiver Grassmannian varieties $\mathrm{Gr}_V({}^\sigma W)$ (two cases) are nonempty in the left-hand side also. Therefore a general representation of $\mathbf{E}_W$ contains two subrepresentations of $\dim{}^\sigma W^1$, $\dim{}^\sigma W^2$, respectively. By [57, Theorem 3.3], it implies that we have both $\mathrm{ext}^1({}^\sigma W^1,{}^\sigma W^2)=0$ and $\mathrm{ext}^1({}^\sigma W^2,{}^\sigma W^1)=0$. This proves the first assertion.

(2) We take a closer look at the above argument. Recall that $P_t(\mathrm{Gr}_V({}^\sigma W))$ is the sum of contributions from $L(W)$ and other perverse sheaves, such as (6.6), and $r_W(IC_W(V'))\in\mathbb{Z}_{>0}$, $a_{V,V';W}(t)\in\mathbb{Z}_{\geq 0}[t]$.

For the case $V^1={}^\sigma W^1$, $V^2=0$ or $V^1=0$, $V^2={}^\sigma W^2$, the corresponding subspace is uniquely determined, and the projection $\pi^\perp\colon\tilde{\mathcal{F}}(\nu,W)^\perp\to\mathbf{E}_W^*$ becomes an isomorphism. Therefore other perverse sheaves do not appear in $\pi^\perp_!(1_{\tilde{\mathcal{F}}(\nu,W)^\perp})$. Therefore we must have $\mathrm{Gr}_V({}^\sigma W)\neq\emptyset$ in the two cases. The remaining argument is the same.

(3) This assertion follows from the first, and the characterization of the canonical decomposition: $\alpha = \sum \beta^i$ is the canonical decomposition if and only if each β^i is a Schur root and $\operatorname{ext}^1(\beta^i, \beta^j) = 0$ for $i \neq j$ (see [35, Proposition 3]). □

COROLLARY 6.13
If $L(W)$ satisfies (C)*, it is real, that is, $L(W) \otimes L(W)$ is simple.*

At this moment, we do not know whether the converse is true or not.

Next, suppose that $\mathcal{G}$ is of type ADE. Then all positive roots are real and Schur. Let Δ_+ be the set of positive roots. Following [22], we introduce the set $\Phi_{\geq -1}$ of *almost positive* roots:

$$\Phi_{\geq -1} = \Delta_+ \sqcup \{-\alpha_i \mid i \in I\},$$

where α_i is the simple root for i.

COROLLARY 6.14
(1) *There are only finitely many prime simple modules in $\mathbf{R}_{\ell=1}$ if and only if the underlying graph $\mathcal{G}$ of the principal part is of type ADE.*

(2) *Suppose that $\mathcal{G}$ is of type ADE. Then all simple modules are real, and there is a bijection*

$$\{\textit{prime simple modules}\} \setminus \{f_i \mid i \in I\} \xrightarrow[1:1]{\dim(\bullet)} \Phi_{\geq -1}.$$

Here the bijection is given by Proposition 6.12(3) together with $x_i \mapsto (-\alpha_i)$.

The first assertion is a simple consequence of the fact that there are infinitely many real Schur roots for non-ADE quivers. This can be shown, for example, by observing that a non-ADE graph always contains an affine graph. Then for an affine graph, real roots α with the defects $\chi(\delta, \alpha) = \dim \operatorname{Hom}(\delta, \alpha) - \dim \operatorname{Ext}^1(\delta, \alpha)$ are nonzero and Schur. Here δ is the generator of positive imaginary roots, and the above is the Euler form for a representation N with $\dim N = \delta$ and M with $\dim M = \alpha$, which is independent of the choice of M, N.

This corollary is nothing but [22] after identifying prime simple modules with cluster variables in Section 7.

Now we consider the affine case.

EXAMPLE 6.15
Suppose that (I, E) is of type $A_1^{(1)}$. The corresponding quiver (I, Ω) is called the Kronecker quiver. Positive roots are $(n \rightrightarrows n+1)$, $(n+1 \rightrightarrows n)$, $(n \rightrightarrows n)$ $(n \in \mathbb{Z}_{\geq 0})$. The vector $(1 \rightrightarrows 1)$ is the generator of positive imaginary roots and is denoted by δ as above.

For $n \in \mathbb{Z}_{>0}$, let nW denote an $(I \sqcup I_{\mathrm{fr}})$-graded vector space with $\mathbb{C}^n$ at the entry i and zero at i' $(i = 0, 1)$: $(nW)_0 = \mathbb{C}^n \rightrightarrows (nW)_1 = \mathbb{C}^n$. Thus $\dim(nW) = n\delta$. Then $nW = W \oplus \cdots \oplus W$ is the canonical decomposition of nW, where W means $1W$. It is well known that a general representation in $\mathbf{E}_W$ corresponds to a point in $\mathbb{P}^1(\mathbb{C})$. And a general representation in $\mathbf{E}_{nW}$ corresponds to distinct n points in $\mathbb{P}^1(\mathbb{C})$.

For a real positive root $(n \rightrightarrows n+1)$ or $(n+1 \rightrightarrows n)$, there is the unique indecomposable module M. It is known that either $\operatorname{Ext}^1(M,W)$ or $\operatorname{Ext}^1(W,M)$ is nonvanishing. Therefore M and W cannot appear in a canonical decomposition simultaneously. It is also known that extensions between $(n \rightrightarrows n+1)$ and $((n+1) \rightrightarrows (n+2))$ vanish. It is also true for $((n+1) \rightrightarrows n)$ and $((n+2) \rightrightarrows (n+1))$. For all other pairs, one of the extensions does not vanish.

From these observations, the canonical decompositions only have real Schur roots, except in the case nW. We consider the case $n=2$. If we consider $\pi^\perp\colon \tilde{\mathcal{F}}(\nu, 2W)^\perp \to \mathbf{E}^*_{2W}$ in Section 4.2, the perverse sheaves appearing (up to shift) in the pushforward $\pi^\perp_!(1_{\tilde{\mathcal{F}}(\nu,2W)^\perp}[\dim \tilde{\mathcal{F}}(\nu,2W)^\perp])$ were studied in [42]. If we take $\nu = (1,1) \in \mathbb{Z}^I_{\geq 0}$, then $\pi^\perp$ is the principal $\{\pm 1\}$-cover over the open set $\mathbf{E}^{*\mathrm{reg}}_{2W}$ corresponding to distinct pairs of points in $\mathbb{P}^1(\mathbb{C})$. Then from [42] we have

$$\mathscr{L}_{2W} = \bigl\{1_{\{0\}}, \Psi^{-1}\bigl(IC(\mathbf{E}^*_{2W},\rho)\bigr)\bigr\},$$

where $IC(\mathbf{E}^*_{2W},\rho)$ is the IC complex associated with the nontrivial local system ρ corresponding to the nontrivial representation of $\{\pm 1\}$. In particular, the almost simple module $\mathbb{L}(2W)$ is not the simple module $L(2W)$. On the other hand, $\mathscr{L}_W = \{1_{\{0\}}\}$.

The coefficient of $\chi_q(L(2W))$ at $Y_{1,1}^2 Y_{2,q^3}^2 \times V_{1,q} V_{2,q^2}$ is 1. The coefficients of $\chi_q(L(W))$ at $Y_{1,1}Y_{2,q^3}V_{1,q}$, $Y_{1,1}Y_{2,q^3}V_{2,q^2}$ are both 1. Therefore $L(2W) \ncong L(W)\otimes L(W)$; that is, $L(W)$ is *not* real. On the other hand, we have $\mathbb{L}(2W) \cong \mathbb{L}(W)\otimes\mathbb{L}(W)$.

There have been many attempts to construct a base for the cluster algebra corresponding to this example in cluster algebra literature (see [58], [10], [20], [19], and [26] in a wider context). The problem is how to understand imaginary root vectors, and the solution is not unique. Relationships between various bases are studied by Leclerc [41].

More generally, if W corresponds to an indivisible isotropic imaginary root (i.e., in the Weyl group orbit of δ of a subdiagram of affine type in $\mathcal{G}$) in an arbitrary $\mathcal{Q}$, we have

$$\mathbb{L}(nW) \cong \mathbb{L}(W)^{\otimes n}.$$

This can be generalized thanks to the results by Schofield [57]. First, we have that if α is a nonisotropic imaginary Schur root, $n\alpha$ is also a Schur root for $n \in \mathbb{Z}_{>0}$ (see [57, Theorem 3.7]). It is also known that an isotropic Schur root must be indivisible (see [57, Theorem 3.8]). Therefore we introduce the following notation. For a W as above and $n \in \mathbb{Z}_{>0}$, let nW be an I-graded vector space with $\dim(nW)_i = n \dim W_i$. For a factor $\mathbb{L}(W^k)$ in (6.11), let $(n\mathbb{L})(W^k)$ be $\mathbb{L}(nW^k)$ if $\dim W^k$ is a nonisotropic Schur imaginary root and $\mathbb{L}(W^k)^{\otimes n}$ otherwise; that is, $\dim W^k$ is a real or indivisible isotropic Schur root.

COROLLARY 6.16

Let W be as above. Let ${}^{\varphi}W = W^1 \oplus W^2 \oplus \cdots \oplus W^s$ be the canonical decomposition. Then we have

$$\mathbb{L}(nW) \cong (n\mathbb{L})(W^1) \otimes \cdots \otimes (n\mathbb{L})(W^s) \otimes \bigotimes_{i\in I} f_i^{n \min(\dim W_i, \dim W_{i'})}.$$

Mimicking the definition in Section 2.3, we say that $\mathbb{L}(W)$ is *real* if $\mathbb{L}(2W) \cong \mathbb{L}(W) \otimes \mathbb{L}(W)$. The above implies that $\mathbb{L}(W)$ is *real* in this sense if and only if there are no nonisotropic imaginary Schur roots in the canonical decomposition of ${}^{\varphi}W$.

If $L(2W) \cong L(W) \otimes L(W)$ (i.e., $L(W)$ is real), we have $\mathrm{ext}^1(W,W) = 0$ by Proposition 6.12(2). By the result of Schofield [57] used above, this can happen only when the canonical decomposition does not contain a nonisotropic imaginary Schur root. This is a step toward proving that $\mathscr{C}_1$ is a monoidal categorification.

7. Cluster algebra structure

In this section we prove, after some preparation, that cluster monomials are dual canonical base elements.

In the previous sections, we use the notation W for an $(I \sqcup I_{\mathrm{fr}})$-graded representation. In this section we also use it for its general representation. Or if we first take a representation, its underlying $(I \sqcup I_{\mathrm{fr}})$-graded vector space is denoted by the same notation.

7.1. Tilting modules

We first review the theory of tilting modules (see [1, Chapter VI], [28]).

Let $\mathcal{Q} = (I, \Omega)$ be a quiver as in Section 2. Let $\mathbb{C}\mathcal{Q}$ be its path algebra defined over $\mathbb{C}$. We consider the category $\mathrm{rep}\,\mathcal{Q}$ of finite-dimensional representations of $\mathcal{Q}$ over $\mathbb{C}$, which is identified with the category of finite-dimensional $\mathbb{C}\mathcal{Q}$-modules.

A module M of the quiver is said to be a *tilting module* if the following two conditions are satisfied.

(1) M is *rigid*, that is, $\mathrm{Ext}^1(M,M) = 0$.

(2) There is an exact sequence $0 \to \mathbb{C}\mathcal{Q} \to M_0 \to M_1 \to 0$ with $M_0,\ M_1 \in \mathrm{add}M$, where $\mathrm{add}M$ denotes the additive category generated by the direct summands of M.

We usually assume that M is multiplicity free.

It is known that the number of indecomposable summands of M equals the number of vertices $\#I$, that is, the rank of $K_0(\mathbb{C}\mathcal{Q})$.

A rigid module M always has a module X so that $M \oplus X$ is a tilting module.

A module M is said to be an *almost complete tilting module* if it is rigid and the number of indecomposable summands of M is $\#I - 1$. We say that an indecomposable module X is the *complement* of M if $M \oplus X$ is a tilting module.

We have the following structure theorem.

THEOREM 7.1 (HAPPEL AND UNGER [28])
Let M be an almost complete tilting module.

(1) *If M is sincere, there exist two nonisomorphic complements X, Y which are related by an exact sequence*

$$0 \to X \to E \to Y \to 0$$

with $E \in \operatorname{add} M$. Moreover, we have $\operatorname{Ext}^1(Y, X) \cong \mathbb{C}$, $\operatorname{Ext}^1(X, Y) = 0$, $\operatorname{Hom}(Y, X) = 0$.

(2) *If M is not sincere, there exists only one complement X up to isomorphism.*

Here a module M is said to be *sincere* if $M_i \neq 0$ for any vertex i.

7.2. Cluster tilting sets

When the quiver $\mathcal{Q}$ does not contain an oriented cycle (i.e., acyclic quiver), combinatorics of the cluster algebra can be understood from cluster category theory. Since we need only the statement, we explain the theory very briefly following [32]. We consider only the case when there are no frozen variables.

Let $n = \#I$. A collection $\mathbf{L} = \{W^1, \dots, W^n\}$ is said to be a *cluster-tilting set* if the following conditions are satisfied.

(0) An element W^i is either an indecomposable representation of the quiver $\mathcal{Q}$ or a vertex. Let $\mathbf{L}_{\text{mod}}$ be the subset of indecomposable representations, $\mathbf{L}_{\text{ver}} = \mathbf{L} \setminus \mathbf{L}_{\text{mod}}$.

(1) The indecomposable modules $W^k \in \mathbf{L}_{\text{mod}}$ are pairwise nonisomorphic. The vertices $W^i \in \mathbf{L}_{\text{ver}}$ are pairwise distinct.

(2) Delete all arrows incident to a vertex $W^i \in \mathbf{L}_{\text{ver}}$. Remove the vertex W^i. Let ${}^{\psi}\mathcal{Q}$ be the resulting quiver.

(3) The entry for $W^k \in \mathbf{L}_{\text{mod}}$ is zero for a vertex $W^j \in \mathbf{L}_{\text{ver}}$. Hence W^k is a representation of ${}^{\psi}\mathcal{Q}$.

(4) The direct sum ${}^{\psi}W \overset{\text{def}}{=} \bigoplus_{W^k \in \mathbf{L}_{\text{mod}}} W^k$ is a tilting module as a representation of ${}^{\psi}\mathcal{Q}$.

Note that $\#\mathbf{L}_{\text{mod}} = \#({}^{\psi}I)$. Therefore ${}^{\psi}W$ is tilting if and only if $\operatorname{ext}^1(W^k, W^l) = 0$ for any k, l (including the case $k = l$). Thus this is stronger than the canonical decomposition and means that $\dim W^k$ is a real Schur root.

The initial cluster-tilting set is the collection $\mathbf{L} = I$ with $\mathbf{L}_{\text{mod}} = \emptyset$. In this case ${}^{\psi}I = \emptyset$, and the condition is trivially satisfied.

If we identify $W^i \in \mathbf{L}_{\text{ver}}$ with $P_{W^i}[1]$ the shift of the indecomposable projective module associated corresponding to the vertex W^i, the above definition is nothing but the definition of a cluster-tilting set for the cluster category (see [6]).

For $k \in \{1, \dots, n\}$, we define the *mutation* $\mu_k(\mathbf{L})$ of $\mathbf{L}$ in direction k as follows.

(1) Suppose that W^k is a vertex. We add it again, together with all arrows incident to it, to the quiver ${}^{\psi}\mathcal{Q}$. Let ${}^{+\psi}\mathcal{Q}$ be the resulting quiver. Since ${}^{\psi}W$ is an almost tilting nonsincere module as a representation of ${}^{+\psi}\mathcal{Q}$, we can add the unique indecomposable ${}^{*}W^k$ to ${}^{\psi}W$ to get a tilting module.

(2) Next, suppose that W^k is a module. We consider an almost tilting module ${}^{-\psi}W$ which is obtained from ${}^{\psi}W$ by subtracting the summand W^k.

(a) If it is sincere, there is another indecomposable module ${}^{*}W^k \neq W^k$ such that ${}^{*}W^k \oplus {}^{-\psi}W$ is a tilting module.

(b) If it is not sincere, there exists the unique simple module S_i, not appearing in the composition factors of ${}^{-\psi}W$. Then we set ${}^{*}W^k = i$.

Let

$$\mu_k(\mathbf{L}) \stackrel{\text{def}}{=} \mathbf{L} \cup \{{}^{*}W^k\} \setminus \{W^k\}.$$

In all cases, $\mu_k(\mathbf{L})$ is again a cluster-tilting set. We can iterate this procedure and obtain new clusters starting from the initial cluster $\mathbf{L} = I$.

7.3. Cluster character

We still continue to assume that the quiver $\mathcal{Q}$ does not contain an oriented cycle. It is known that cluster monomials can be expressed in terms of generating functions of Euler numbers of quiver Grassmannian varieties. This important result was first proved by Caldero and Chapoton [7] in type ADE. Later it was generalized to any acyclic quiver by Caldero and Keller [8] who used various results in the cluster category theory (see [37] for the reference). We recall the formula in this subsection.

Let $(\mathbf{x}, \mathbf{B})$ be the initial seed of the cluster algebra $\mathscr{A}(\mathbf{B})$. We assume that there is no frozen part for simplicity. Let W be a representation of the quiver $\mathcal{Q}$ corresponding to $\mathbf{B}$. Let $\operatorname{Gr}_V(W)$ be the corresponding quiver Grassmannian variety, where V is an I-graded vector space. Although we soon assume that W is a general representation in $\mathbf{E}_W$, it is not necessary for the definition. Let $e(\operatorname{Gr}_V(W))$ be its Euler number. We define

$$X_W \stackrel{\text{def}}{=} \frac{1}{\mathbf{x}^{\dim W}} \sum_V e\big(\operatorname{Gr}_V(W)\big) \mathbf{x}^{\dim V \cdot R} \mathbf{x}^{(\dim W - \dim V)R'},$$

where

$$\mathbf{x}^{\dim W} = \prod_i x_i^{\dim W_i},$$

$$\mathbf{x}^{\dim V \cdot R} = \prod_{h \in \Omega} x_{i(h)}^{\dim V_{o(h)}}, \qquad \mathbf{x}^{(\dim W - \dim V)R'} = \prod_{h \in \Omega} x_{o(h)}^{(\dim W_{i(h)} - \dim V_{i(h)})}.$$

For a vertex i, we set $X_i = x_i$.

Then it is known that the correspondence $W \to X_W$ gives the following:

- the correspondence $W \to X_W$ defines a bijection between the set of isomorphism classes of rigid indecomposable modules with cluster variables minus $\{x_i\}$;

- the correspondence $\mathbb{L} \to \{X_{W^1}, \dots, X_{W^n}\}$ gives a bijection between cluster tilting sets and clusters;
- the mutation on cluster tilting sets corresponds to the cluster mutation.

7.4. Piecewise-linear involution

We give one more preparation before applying results from the cluster category theory to our setting. This last preliminary is not necessary for our argument but helps to make a relation to [31, Section 12.3].

We recall the piecewise-linear involution τ_- on the root lattice considered in [31, Section 7]: for $\gamma = \sum_i \gamma_i i \in \mathbb{Z}^I$, we define $\tau_-(\gamma) = \sum_i \tau_-(\gamma)_i i$ by

$$\tau_-(\gamma)_i = \begin{cases} -\gamma_i - \sum_{j \neq i} c_{ij} \max(0, \gamma_j) & \text{if } i \in I_1, \\ \gamma_i & \text{if } i \in I_0, \end{cases} \tag{7.2}$$

where (c_{ij}) is the Cartan matrix.

Let

$$\gamma = \sum_i (\dim W_i - \dim W_{i'}) i. \tag{7.3}$$

If $i \in I_0$, we have

$$\tau_-(\gamma)_i = \dim W_i - \dim W_{i'} = \dim {}^{\varphi}W_i - \dim {}^{\varphi}W_{i'}.$$

If $i \in I_1$, we have

$$\begin{aligned} \tau_-(\gamma)_i &= \dim W_{i'} - \dim W_i - \sum_{j \neq i} c_{ij} \max(\dim W_j - \dim W_{j'}, 0) \\ &= \dim {}^{\varphi}W_{i'} - \dim {}^{\varphi}W_i - \sum_{j \neq i} c_{ij} \dim {}^{\varphi}W_j. \end{aligned}$$

Therefore we have

$$\dim {}^{\sigma\varphi}W_i = \max\bigl(\tau_-(\gamma)_i, 0\bigr),$$

where ${}^{\sigma\varphi}W = {}^{\sigma}({}^{\varphi}W)$ is obtained by applying σ to ${}^{\varphi}W$.

REMARK 7.4

In [31, Section 12.3], the quiver Grassmannian $\mathrm{Gr}_V(M[\tau_-(\gamma)])$ was considered where $M[\tau_-(\gamma)]$ is a general representation with

$$\dim M[\tau_-(\gamma)]_i = \max\bigl(\tau_-(\gamma)_i, 0\bigr).$$

Here the quiver is the principal part $\mathcal{Q}$ of our decorated quiver. From the above computation, $M[\tau_-(\gamma)]$ is nothing but the principal quiver part of ${}^{\sigma\varphi}W$. The frozen part of ${}^{\sigma\varphi}W$ does not play any role in the quiver Grassmannian by Proposition 6.8. Therefore $\mathrm{Gr}_V(M[\tau_-(\gamma)])$ in [31, Section 12.3] is isomorphic to our $\mathrm{Gr}_V({}^{\sigma\varphi}W)$ under (7.3).

7.5. Cluster monomials

We start to put the cluster algebra structure on $\mathbf{R}$ from this subsection.

PROPOSITION 7.5

(1) *Let W be an I-graded vector space such that* $\dim W$ *is a real Schur root of the principal part of the decorated quiver. Then $L(W)$ is a cluster variable.*

(2) *This correspondence defines a bijection between the set of real Schur roots and the set of cluster variables except variables in the initial seed, that is, x_i, f_i ($i \in I$).*

For type ADE, this together with Corollary 6.14 shows the condition (2) in the monoidal categorification Definition 2.4.

Proof

Roughly this is a consequence of results reviewed in Section 7.3. However, our quiver Grassmannian is for ${}^\sigma W$, not for W. Correspondingly we need to replace the initial seed of $\mathscr{A}(\widetilde{\mathbf{B}})$ by the $\mathbf{z}$-quiver in (5.8). When we mutate from the $\mathbf{x}$-quiver to the $\mathbf{z}$-quiver, the set of cluster variables does not change by definition, but variables in the initial seed change. So let us first consider this effect. The functor ${}^\sigma(\bullet)$ induces an involution on the set

$$\{\text{real Schur roots}\} \setminus \{\alpha_i \mid i \in I_1\}.$$

Therefore we only need to study cluster variables corresponding to α_i in either the $\mathbf{x}$-quiver or the $\mathbf{z}$-quiver.

- In the $\mathbf{x}$-quiver, α_i corresponds to $W = S_i$. We have $L(S_i) = x_i' = z_i$. This is a cluster variable of the seed for the $\mathbf{z}$-quiver but not for the original $\mathbf{x}$-quiver. Note also that ${}^\sigma W = 0$ in this case.
- In the $\mathbf{z}$-quiver, α_i corresponds to the cluster variable obtained as z_i^*. But this is nothing but x_i. The corresponding simple module is $L(S_{i'})$. We do not consider it since it has support in the frozen part.

We now may assume that $\dim {}^\sigma W$ is a real Schur root different from α_i ($i \in I_1$).

We cannot apply the formula in Section 7.3 directly as the $\mathbf{z}$-quiver contains an oriented cycle in general (see (5.7)). We thus first consider the quiver with principal coefficients and write down F-polynomials and $\mathbf{g}$-vectors by using the formula in Section 7.3. Then we apply the result in Section 2.2 to get the formula for cluster variables in the original cluster algebra.

We take $\mathbf{u}$, $\mathbf{f}$ as cluster variables for the initial seed of $\mathscr{A}_{\mathrm{pr}}$ and define

$$X_{{}^\sigma W}(\mathbf{u},\mathbf{f}) \overset{\text{def}}{=} \frac{1}{\prod_{i\in I} u_i^{{}^\sigma w_i}} \sum_V e\bigl(\mathrm{Gr}_V({}^\sigma W)\bigr) \prod_{i\in I_0} u_i^{\sum_j a_{ij} v_j} \prod_{i\in I_1} u_i^{\sum_j a_{ij}({}^\sigma w_j - v_j)} \prod_{i\in I} f_i^{v_i},$$

where $v_i = \dim V_i$, $w_i = \dim W_i$, ${}^\sigma w_i = \dim {}^\sigma W_i$. By Section 7.3, this is a cluster variable α for $\mathscr{A}_{\mathrm{pr}}$, and hence the above gives the Laurent polynomial $X_\alpha(\mathbf{u},\mathbf{f})$ in Section 2.2.

Hence, the F-polynomial is

$$F_{{}^\sigma W}(\mathbf{f}) = \sum_V e\bigl(\mathrm{Gr}_V({}^\sigma W)\bigr) \prod_{i\in I} f_i^{v_i}.$$

And the $\mathbf{g}$-vector is

$$\mathbf{g}_{^\sigma W} = -\sum_{i\in I_0} {}^\sigma w_i i - \sum_{i\in I_1}\Bigl({}^\sigma w_i - \sum_j a_{ij}{}^\sigma w_j\Bigr) i = -\sum_i w_i\varepsilon_i i,$$

where $\varepsilon_i = (-1)^{\xi_i}$.

Now we return to our original cluster algebra. Since our initial seed is given by the $\mathbf{z}$-quiver, we change the notation in Section 2.2 and use z-variables instead of x-variables. We denote the cluster variable corresponding to the above $X_{^\sigma W}$ by $z[{}^\sigma W]$. We have

$$z[{}^\sigma W] = \frac{F_{^\sigma W}(\widehat{\mathbf{y}})}{F_{^\sigma W}|_{\mathbb{P}}(\mathbf{y})}\mathbf{z}^{\mathbf{g}_\alpha},$$

where

$$y_j = \begin{cases} f_j^{-1}\prod_{i\in I} f_i^{a_{ij}} & \text{if } j\in I_0,\\ f_j^{-1} & \text{if } j\in I_1,\end{cases} \qquad \widehat{y}_j = y_j\prod_{i\in I} z_i^{\varepsilon_i a_{ij}} \qquad (j\in I)$$

in this situation. A direct calculation shows (see [31, Lemma 7.2])

$$\chi_q(\widehat{y}_j) = V_{j,q^{\xi_j+1}}.$$

We note that $F_{^\sigma W}$ contains the monomial $\prod_i f_i^{{}^\sigma w_i}$ for $V = {}^\sigma W$ with the coefficient 1, and all other terms are its factor. If we evaluate it at y_j, we have

$$\prod_{i\in I} f_i^{-{}^\sigma w_i}\prod_{i\in I_1} f_i^{\sum a_{ij}{}^\sigma w_j} = \prod_{i\in I_0} f_i^{-{}^\sigma w_i}\prod_{i\in I_1} f_i^{-{}^\sigma w_i + \sum a_{ij}{}^\sigma w_j} = \prod_{i\in I_0} f_i^{-w_i}\prod_{i\in I_1} f_i^{w_i}.$$

We also have the constant term 1 for $V=0$. Therefore

$$F_{^\sigma W}|_{\mathbb{P}}(\mathbf{y}) = \prod_{i\in I_0} f_i^{-w_i}.$$

Thus combining with the above calculation of $\mathbf{g}_{^\sigma W}$, we get (see [31, Lemma 7.3])

$$\frac{\mathbf{z}^{\mathbf{g}_{^\sigma W}}}{F_{^\sigma W}|_{\mathbb{P}}(\mathbf{y})} = \prod_{i\in I_0} f_i^{w_i}\prod_{i\in I} z_i^{-w_i\varepsilon_i}.$$

Its q-character is

$$\chi_q\Bigl(\frac{\mathbf{z}^{\mathbf{g}_{^\sigma W}}}{F_{^\sigma W}|_{\mathbb{P}}(\mathbf{y})}\Bigr) = \prod_{i\in I_0} Y_{i,1}^{w_i}\prod_{i\in I_1} Y_{i,q^3}^{w_i}.$$

We thus get

$$\chi_q(z[{}^\sigma W])_{\le 2} = \sum_V e\bigl(\mathrm{Gr}_V({}^\sigma W)\bigr)e^W e^V.$$

Hence we have $z[{}^\sigma W] = \mathbb{L}(W) = L(W)$, where the first equality follows from Theorem 6.3 and the second equality from Proposition 6.9. □

PROPOSITION 7.6

Let $L(W^1)$, …, $L(W^s)$ be simple modules corresponding to cluster variables w_1, …, w_s (via either Proposition 7.5 or x_i, f_i). Then $L(W^1)\otimes\cdots\otimes L(W^s)$ is simple if and only if all w_1, …, w_s live in a common cluster.

For type ADE, this shows condition (1) in the monoidal categorification Definition 2.4.

Proof
The assertion is trivial for the factor f_i by Proposition 6.7. So we may assume that any $W^1, \dots, W^s$ is not f_i. Therefore we have $W^1 = {}^{\varphi}W^1, \dots, W^s = {}^{\varphi}W^s$.

By Propositions 6.10 and 6.12, $L(W^1) \otimes \cdots \otimes L(W^s)$ is simple if and only if $\operatorname{ext}^1(W^k, W^l) = \operatorname{ext}^1(W^l, W^k) = 0$ for $k \neq l$. Thus we need to show that this is equivalent to the condition that the corresponding w_k and w_l be in a common cluster. Therefore we may assume $k = 1$, $l = 2$.

When $W^1 = W^2$, then $w_1 = w_2$ is in a common cluster. But $\operatorname{ext}^1(W^1, W^2) = 0$ is also true since $\dim W^1 = \dim W^2$ is a real Schur root.

If neither $L(W^1)$ nor $L(W^2)$ is one of x_i and x'_i, then $L(W^1) = z[{}^{\sigma}W^1]$, $L(W^2) = z[{}^{\sigma}W^2]$ as in the proof of Proposition 7.5. We have $\operatorname{ext}^1(W^1, W^2) = \operatorname{ext}^1(W^2, W^1) = 0$ if and only if $\operatorname{ext}^1({}^{\sigma}W^1, {}^{\sigma}W^2) = \operatorname{ext}^1({}^{\sigma}W^2, {}^{\sigma}W^1) = 0$. This happens if and only if ${}^{\sigma}W^1 \oplus {}^{\sigma}W^2$ is rigid, and hence it can be extended to a tilting module. From Sections 7.2 and 7.3, this is equivalent to the fact that the corresponding cluster variables live in a common cluster.

If $L(W^1) = x_i$, $L(W^2) = x'_i$, then $L(W^1) \otimes L(W^2)$ is not simple by the T-system (5.2). They are not in any cluster simultaneously. Any other pairs from x_i, x'_j are always in a common cluster. It is also clear that $L(W^1) \otimes L(W^2)$ is always simple. Therefore we may assume that $L(W^1)$ is one of x_i, x'_i, and $L(W^2)$ is not.

Consider the case $L(W^1) = x_i$ with $i \in I_0$. We have $W^1 = S_{i'}$. From Propositions 6.7 and 6.8, $L(S_{i'}) \otimes L(W^2)$ is simple if and only if $W^2_i = 0$. In this case, $x_i = z_i$ is a cluster variable from the seed for the $\mathbf{z}$-quiver. From Section 7.2 the cluster variable w_2 is in a common cluster with z_i if and only if ${}^{\sigma}W^2_i = 0$. This is equivalent to $W^2_i = 0$ since $i \in I_0$.

The case $L(W^1) = x'_i$ with $i \in I_0$ is not necessary to consider since we have $L(W^1) = L(S_i) = z[S_i]$, which has already been studied.

Next, suppose $L(W^1) = x_i$ with $i \in I_1$. We have $W^1 = S_{i'}$. From Propositions 6.7 and 6.8, $L(S_{i'}) \otimes L(W^2)$ is simple if and only if $W^2_i = 0$ as above. Since i is a source, this is equivalent to $\operatorname{Hom}(W^2, S_i) = 0$. From the definition of the reflection functor, it is equivalent to $\operatorname{Ext}^1(S_i, {}^{\sigma}W^2) = 0$. Since we have $x_i = z_i^*$, the corresponding rigid module for the $\mathbf{z}$-quiver is S_i. Therefore x_i and w are in a common cluster if and only $\operatorname{Ext}^1(S_i, {}^{\sigma}W^2) = 0 = \operatorname{Ext}^1({}^{\sigma}W^2, S_i)$ by Section 7.2. But the latter equality is trivial since i is the source. Thus we have checked the assertion in this case.

Finally, suppose $L(W^1) = x'_i$ for $i \in I_1$. This is z_i and corresponds to a vertex i in the cluster-tilting set for the $\mathbf{z}$-quiver. Therefore w is in the same cluster with z_i if and only if ${}^{\sigma}W^2_i = 0$. By the same argument as above, this is equivalent to $\operatorname{Ext}^1(S_i, W^2) = 0 = \operatorname{Ext}^1(W^2, S_i)$. Thus we have checked the final case. $\square$

REMARK 7.7
As indicated in the proof, it is more natural to define ${}^\sigma S_i$ as $S_i[-1]$, an object in the derived category $\mathscr{D}(\operatorname{rep}^\sigma \widetilde{\mathcal{Q}}^{\mathrm{op}})$. This is also compatible with the cluster category theory, as $S_i[-1] = I_i[-1]$ for $i \in I_1$, where I_i is the indecomposable injective module corresponding to the vertex i.

7.6. Exchange relation

Consider an exchange relation (2.3). Thanks to Propositions 7.5 and 7.6, we have the corresponding equality in $\mathbf{R}_{\ell=1}$:

$$L(x_k) \otimes L(x_k^*) = L(m_+) + L(m_-).$$

Since $L(m_\pm)$ are simple, this inequality in the Grothendieck group implies either of the following:

$$0 \to L(m_+) \to L(x_k) \otimes L(x_k^*) \to L(m_-) \to 0,$$

or

$$0 \to L(m_-) \to L(x_k) \otimes L(x_k^*) \to L(m_+) \to 0$$

in the level of modules. It is natural to conjecture that we always have the above one. For the T-system, this is true thanks to Remark 5.3.

This conjecture follows from a refinement of the exchange relation:

$$\chi_{q,t}\big(L(x_k) \otimes L(x_k^*)\big) = t^{-l+n}\chi_{q,t}\big(L(m_+)\big) + t^n \chi_{q,t}\big(L(m_-)\big)$$

for some $l > 0$, $n \in \mathbb{Z}$. If we write the corresponding perverse sheaves by $P(x_k)$, $P(x_k^*)$, $P(m_+)$, $P(m_-)$, the above means that

$$\begin{aligned}\operatorname{Res}\big(P(m_+)\big) &= P(x_k) \boxtimes P(x_k^*)[l-n] \oplus \cdots,\\ \operatorname{Res}\big(P(m_-)\big) &= P(x_k) \boxtimes P(x_k^*)[-n] \oplus \cdots,\end{aligned}$$

where $\cdots$ means the sum of (shifts of) other perverse sheaves. Since $\operatorname{Hom}(P(x_k) \boxtimes P(x_k^*)[l], P(x_k) \boxtimes P(x_k^*))$ vanishes for $l > 0$ by a property of perverse sheaves (see [12, Corollary 8.4.4], we see that $L(m_+)$ is a submodule of $L(x_k) \otimes L(x_k^*)$.

This refinement of the exchange relation might be proved directly, but it should be proved naturally if we make an isomorphism of the quantum cluster algebra (see [4]) with $\mathbf{R}_{t,\ell=1}$.

Appendix: Odd cohomology vanishing of quiver Grassmannians

In this appendix, we generalize our proof of the odd cohomology group vanishing of the quiver Grassmannian of submodules of a rigid module of a bipartite quiver (see Proposition 6.9(2)) to an acyclic one. Thus we recover the main result of Caldero and Reineke [9]. It implies the positivity conjecture for an acyclic cluster algebra (see Proposition 2.5) for the special case of an *initial* seed.

After an earlier version of this article was posted on the arXiv, Qin [56] proved the quantum version of the cluster character formula for an acyclic cluster algebra. As an application, he observed the odd cohomology group vanishing of quiver Grassmannians. Our proof is different from his.

Let us first fix the notation. Let $\mathcal{Q}=(I,\Omega)$ be a quiver, and let W be an I-graded vector space. We define

$$\mathbf{E}_W=\bigoplus_{h\in\Omega}\operatorname{Hom}(W_{o(h)},W_{i(h)}).$$

Its dual space is

$$\mathbf{E}_W^*=\bigoplus_{h\in\Omega}\operatorname{Hom}(W_{i(h)},W_{o(h)})=\bigoplus_{\overline{h}\in\overline{\Omega}}\operatorname{Hom}(W_{o(\overline{h})},W_{i(\overline{h})}).$$

Those are acted on by $G_W=\prod_i\mathrm{GL}(W_i)$.

Let $\nu\in\mathbb{Z}^I_{\geq 0}$. Let $\mathcal{F}(\nu,W)$ be the product of Grassmanian varieties $\mathrm{Gr}(\nu_i,W_i)$ parameterizing collections of vector subspaces $X_i\subset W_i$ such that $\dim X_i=\nu_i$. Let $\tilde{\mathcal{F}}(\nu,W)$ be the variety of all pairs $(\bigoplus\mathbf{y}_h,X)$, where $\bigoplus\mathbf{y}_h\in\mathbf{E}_W$ and $X\in\mathcal{F}(\nu,W)$ such that

$$\mathbf{y}_h(X_{o(h)})=0,\quad \mathbf{y}_h(W_{o(h)})\subset X_{i(h)},$$

for all $h\in\Omega$. This is a vector bundle over $\mathcal{F}(\nu,W)$. Let $\pi\colon\tilde{\mathcal{F}}(\nu,W)\to\mathbf{E}_W$ be the projection.

Note that $\tilde{\mathcal{F}}(\nu,W)$ is a subbundle of the trivial bundle $\mathcal{F}(\nu,W)\times\mathbf{E}_W$. Let $\tilde{\mathcal{F}}(\nu,W)^\perp$ be its annihilator in the dual trivial bundle $\mathcal{F}(\nu,W)\times\mathbf{E}_W^*$, and let $\pi^\perp\colon\tilde{\mathcal{F}}(\nu,W)^\perp\to\mathbf{E}_W^*$ be the projection. More concretely, $\tilde{\mathcal{F}}(\nu,W)^\perp$ is the variety of all pairs $(\bigoplus\mathbf{y}^*_{\overline{h}},X)$, where $\bigoplus\mathbf{y}^*_{\overline{h}}\in\mathbf{E}_W^*$ and $X\in\mathcal{F}(\nu,W)$ such that

$$\mathbf{y}^*_{\overline{h}}(X_{o(\overline{h})})\subset X_{i(\overline{h})}$$

for all $\overline{h}\in\overline{\Omega}$. Therefore $(\pi^\perp)^{-1}(\bigoplus\mathbf{y}^*_{\overline{h}})$ is the quiver Grassmannian associated with the quiver representation $\bigoplus\mathbf{y}^*_{\overline{h}}$.

THEOREM A.1

Assume that $\mathbf{E}_W^*$ contains an open G_W-orbit, and let $\bigoplus\mathbf{y}^*_{\overline{h}}\in\mathbf{E}_W^*$ be a point in the orbit. Then the quiver Grassmannian $(\pi^\perp)^{-1}(\bigoplus\mathbf{y}^*_{\overline{h}})$ has no odd cohomology.

Proof

Consider the fiber $\pi^{-1}(\bigoplus\mathbf{y}_h)$ of $\pi\colon\tilde{\mathcal{F}}(\nu,W)\to\mathbf{E}_W$. From the definition of $\tilde{\mathcal{F}}(\nu,W)$, it is equal to $X\in\mathcal{F}(\nu,W)$ such that

$$\sum_{h:i(h)=i}\operatorname{Im}\mathbf{y}_h\subset X_i\subset\bigcap_{h:o(h)=i}\operatorname{Ker}\mathbf{y}_h.$$

Thus it is isomorphic to the product of the usual Grassmannian manifolds of subspaces of $\bigcap_{h:o(h)=i}\operatorname{Ker}\mathbf{y}_h/\sum_{h:i(h)=i}\operatorname{Im}\mathbf{y}_h$ of dimension $\nu_i-\dim\sum_{h:i(h)=i}\operatorname{Im}\mathbf{y}_h$. Thus $\pi^{-1}(\bigoplus\mathbf{y}_h)$ has no odd homology.

In the main body of the article, the central fiber was denoted by $\mathfrak{L}^\bullet(V,W)$, and its odd cohomology vanishing was mentioned in Remark 3.20. The remaining part of the proof is the same as in Proposition 6.9(2). Let us sketch it for the sake of the reader.

We consider the pushforward $\pi^{\perp}_{!}(1_{\tilde{\mathcal{F}}(\nu,W)^{\perp}}[\dim \tilde{\mathcal{F}}(\nu,W)^{\perp}])$. By the decomposition theorem, it is isomorphic to a finite direct sum

$$\bigoplus_{P,d} L_{P,d} \otimes P[d]$$

of various simple perverse sheaves P and $d \in \mathbb{Z}$. Here $L_{P,d}$ is a finite-dimensional vector space. Since $\pi^{\perp}$ is G_W-equivariant, all P's appearing above are equivariant. Therefore under our assumption, only the constant sheaf $1_{\mathbf{E}^*_W}[\dim \mathbf{E}^*_W]$ is supported on the whole $\mathbf{E}^*_W$, and all other perverse sheaves P have smaller supports. Taking a fiber at $\bigoplus y^*_{\overline{h}}$, we find that the cohomology group $H_k((\pi^{\perp})^{-1} \times (\bigoplus \mathbf{y}^*_{\overline{h}}))$ is $L_{P,d}$ with $P = 1_{\mathbf{E}^*_W}[\dim \mathbf{E}^*_W]$ and $d = k + \dim \mathbf{E}^*_W - \dim \tilde{\mathcal{F}}^{\perp}(\nu,W)$.

We apply the Fourier-Sato-Deligne functor Ψ for perverse sheaves on $\mathbf{E}_W$. As in the proof of Theorem 6.3, we have

$$\pi_!\big(1_{\tilde{\mathcal{F}}(\nu,W)}[\dim \tilde{\mathcal{F}}(\nu,W)]\big) = \bigoplus_{P,d} L_{P,d} \otimes \Psi(P)[d].$$

We also have that $\Psi(1_{\mathbf{E}^*_W}[\dim \mathbf{E}^*_W])$ is the skyscraper sheaf $1_{\{0\}}$ at the origin of $\mathbf{E}_W$. The fiber of $\pi_!(1_{\tilde{\mathcal{F}}(\nu,W)}[\dim \tilde{\mathcal{F}}(\nu,W)])$ at $0 \in \mathbf{E}_W$ gives the homology group of $\pi^{-1}(0)$, which vanishes in odd degree as we have already observed. Therefore $L_{P,d}$ for $P = 1_{\mathbf{E}^*_W}[\dim \mathbf{E}^*_W]$ vanishes if $d + \dim \tilde{\mathcal{F}}(\nu,W)$ is odd. Thus we have the assertion. □

The odd homology vanishing of the fiber over $\bigoplus \mathbf{y}_h = 0$ is enough for the above argument. We give an analysis of arbitrary cases to show that any fiber is isomorphic to the fiber of zero for a different choice of ν, W. In the main body of this article, this is a consequence of Theorem 3.14.

We also remark that we do not assume that the quiver contains no oriented cycles in the above proof. However, it is implicitly assumed since we consider only the case when $\mathbf{E}^*_W$ contains an open orbit. Thus our result applies only to acyclic cluster algebras.

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