JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 1, Pages 145–238
S 0894-0347(00)00353-2
Article electronically published on October 2, 2000

QUIVER VARIETIES AND FINITE DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

HIRAKU NAKAJIMA

Contents

Introduction

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of type ADE, let $\widehat{\mathfrak{g}}$ be the corresponding (untwisted) affine Lie algebra, and let $\mathbf{U}_q(\widehat{\mathfrak{g}})$ be its quantum enveloping algebra of Drinfel'd-Jimbo, or the quantum affine algebra for short. In this paper we study finite dimensional representations of $\mathbf{U}_q(\widehat{\mathfrak{g}})$, using geometry of quiver varieties which were introduced in [29, 44, 45].

There is a large amount of literature on finite dimensional representations of $\mathbf{U}_q(\widehat{\mathfrak{g}})$; see for example [1, 10, 18, 25, 28] and the references therein. A basic result relevant to us is due to Chari-Pressley [11]: irreducible finite dimensional

Received by the editors December 9, 1999 and, in revised form, July 10, 2000.
2000 *Mathematics Subject Classification.* Primary 17B37; Secondary 14D21, 14L30, 16G20, 33D80.
The author was supported by the Grant-in-aid for Scientific Research (No.11740011), the Ministry of Education, Japan, and National Science Foundation Grant #DMS 97-29992.

representations are classified by an n-tuple of polynomials, where n is the rank of $\mathfrak{g}$. This result was announced for Yangian earlier by Drinfel'd [15]. Hence the polynomials are called Drinfel'd polynomials. However, not much is known about the properties of irreducible finite dimensional representations, say their dimensions, tensor product decomposition, etc.

Quiver varieties are generalizations of moduli spaces of instantons (anti-self-dual connections) on certain classes of real 4-dimensional hyper-Kähler manifolds, called ALE spaces [29]. They can be defined for any finite graph, but we are concerned for the moment with the Dynkin graph of type ADE corresponding to $\mathfrak{g}$. Motivated by results of Ringel [47] and Lusztig [33], the author has been studying their properties [44, 45]. In particular, it was shown that there is a homomorphism

$$\mathbf{U}(\mathfrak{g}) \to H_{\mathrm{top}}(Z(\mathbf{w}), \mathbb{C}),$$

where $\mathbf{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, $Z(\mathbf{w})$ is a certain lagrangian subvariety of the product of quiver varieties (the quiver variety depends on a choice of a dominant weight $\mathbf{w}$), and $H_{\mathrm{top}}(\ ,\mathbb{C})$ denotes the top degree homology group with complex coefficients. The multiplication on the right hand side is defined by the convolution product.

During the study, it became clear that the quiver varieties are analogous to the cotangent bundle $T^*\mathcal{B}$ of the flag variety $\mathcal{B}$. The lagrangian subvariety $Z(\mathbf{w})$ is an analogue of the Steinberg variety $Z = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$, where $\mathcal{N}$ is the nilpotent cone and $T^*\mathcal{B} \to \mathcal{N}$ is the Springer resolution. The above mentioned result is an analogue of Ginzburg's lagrangian construction of the Weyl group W [20]. If we replace homology group by equivariant K-homology group in the case of $T^*\mathcal{B}$, we get the affine Hecke algebra H_q instead of W as was shown by Kazhdan-Lusztig [26] and Ginzburg [13]. Thus it became natural to conjecture that an equivariant K-homology group of the quiver variety gave us the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$. After the author wrote [44], many people suggested this conjecture to him, for example Kashiwara, Ginzburg, Lusztig and Vasserot.

A geometric approach to finite dimensional representations of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ (when $\mathfrak{g} = \mathfrak{sl}_n$) was given by Ginzburg-Vasserot [21, 58]. They used the cotangent bundle of the n-step partial flag variety, which is an example of a quiver variety of type A. Thus their result can be considered as a partial solution to the conjecture.

In [23] Grojnowski constructed the lower-half part $\mathbf{U}_q(\widehat{\mathfrak{g}})^-$ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ on equivariant K-homology of a certain lagrangian subvariety of the cotangent bundle of a variety $\mathbf{E_d}$. This $\mathbf{E_d}$ was used earlier by Lusztig for the construction of canonical bases on the lower-half part $\mathbf{U}_q(\mathfrak{g})^-$ of the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g})$. Grojnowski's construction was motivated in part by Tanisaki's result [52]: a homomorphism from the finite Hecke algebra to the equivariant K-homology of the Steinberg variety is defined by assigning to perverse sheaves (or more precisely Hodge modules) on $\mathcal{B}$ their characteristic cycles. In the same way, he considered characteristic cycles of perverse sheaves on $\mathbf{E_d}$. Thus he obtained a homomorphism from $\mathbf{U}_q(\mathfrak{g})^-$ to K-homology of the lagrangian subvariety. This lagrangian subvariety contains a lagrangian subvariety of the quiver variety as an open subvariety. Thus his construction was a solution to 'half' of the conjecture.

Later Grojnowski wrote an 'advertisement' of his book on the full conjecture [24]. Unfortunately, details were not explained, and his book is not published yet.

The purpose of this paper is to solve the conjecture affirmatively, and to derive results whose analogues are known for H_q. Recall that Kazhdan-Lusztig [26] gave a classification of simple modules of H_q, using the above mentioned K-theoretic construction. Our analogue is the Drinfel'd-Chari-Pressley classification. Also Ginzburg gave a character formula, called a p-adic analogue of the Kazhdan-Lusztig multiplicity formula [13]. (See the introduction in [13] for a more detailed account and historical comments.) We prove a similar formula for $\mathbf{U}_q(\widehat{\mathfrak{g}})$ in this paper.

Let us describe the contents of this paper in more detail. In §1 we recall a new realization of $\mathbf{U}_q(\widehat{\mathfrak{g}})$, called Drinfel'd realization [15]. It is more suitable than the original one for our purpose, or rather, we can consider it as a definition of $\mathbf{U}_q(\widehat{\mathfrak{g}})$. We also introduce the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, which is a subquotient of $\mathbf{U}_q(\widehat{\mathfrak{g}})$, i.e., the quantum affine algebra without central extension and the degree operator. Since the central extension acts trivially on finite dimensional representations, we study $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ rather than $\mathbf{U}_q(\widehat{\mathfrak{g}})$. Introducing a certain $\mathbb{Z}[q,q^{-1}]$-subalgebra $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, we define a specialization $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ at $q=\varepsilon$. This $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ was originally introduced by Chari-Pressley [12] for the study of finite dimensional representations of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ when ε is a root of unity. Then we recall basic results on finite dimensional representations of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. We introduce several concepts, such as l-weights, l-dominant, l-highest weight modules, l-fundamental representation, etc. These are analogues of the same concepts without l for $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules. '$l$' stands for the loop. In the literature, some of these concepts were used without 'l'.

In §2 we introduce two types of quiver varieties, $\mathfrak{M}(\mathbf{w})$, $\mathfrak{M}_0(\infty,\mathbf{w})$ (both depend on a choice of a dominant weight $\mathbf{w}=\sum w_k\Lambda_k$). They are analogues of $T^*\mathcal{B}$ and the nilpotent cone $\mathcal{N}$ respectively, and have the following properties:

(1) $\mathfrak{M}(\mathbf{w})$ is a nonsingular quasi-projective variety, having many components of various dimensions.
(2) $\mathfrak{M}_0(\infty,\mathbf{w})$ is an affine algebraic variety, not necessarily irreducible.
(3) Both $\mathfrak{M}(\mathbf{w})$ and $\mathfrak{M}_0(\infty,\mathbf{w})$ have a $G_{\mathbf{w}}\times\mathbb{C}^*$-action, where $G_{\mathbf{w}}=\prod \mathrm{GL}_{w_k}(\mathbb{C})$.
(4) There is a $G_{\mathbf{w}}\times\mathbb{C}^*$-equivariant projective morphism $\pi\colon \mathfrak{M}(\mathbf{w})\to\mathfrak{M}_0(\infty,\mathbf{w})$.

In §3–§8 we prepare some results on quiver varieties and K-theory which we use in later sections.

In §9–§11 we consider an analogue of the Steinberg variety $Z(\mathbf{w})=\mathfrak{M}(\mathbf{w})\times_{\mathfrak{M}_0(\infty,\mathbf{w})}\mathfrak{M}(\mathbf{w})$ and its equivariant K-homology $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$. We construct an algebra homomorphism

$$\mathbf{U}_q(\mathbf{L}\mathfrak{g})\to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{Q}(q).$$

We first define images of generators in §9, and check the defining relations in §10 and §11. Unlike the case of the affine Hecke algebra, where H_q is isomorphic to $K^{G\times\mathbb{C}^*}(Z)$ ($Z=$ the Steinberg variety), this homomorphism is *not* an isomorphism, neither injective nor surjective.

In §12 we show that the above homomorphism induces a homomorphism

$$\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})\to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))/\mathrm{torsion}\,.$$

(It is natural to expect that $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ is an integral form of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ and that $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ is torsion-free, but we do not have the proofs.)

In §13 we introduce a *standard module* $M_{x,a}$. It depends on the choice of a point $x\in\mathfrak{M}_0(\infty,\mathbf{w})$ and a semisimple element $a=(s,\varepsilon)\in G_{\mathbf{w}}\times\mathbb{C}^*$ such that x is fixed by a. The parameter ε corresponds to the specialization $q=\varepsilon$, while s corresponds to Drinfel'd polynomials. In this paper, we assume ε is *not* a root of unity, although most of our results hold even in that case (see Remark 14.3.9). Let A be the Zariski closure of $a^{\mathbb{Z}}$. We define $M_{x,a}$ as the specialized equivariant K-homology $K^A(\mathfrak{M}(\mathbf{w})_x)\otimes_{R(A)}\mathbb{C}_a$, where $\mathfrak{M}(\mathbf{w})_x$ is a fiber of $\mathfrak{M}(\mathbf{w})\to\mathfrak{M}_0(\infty,\mathbf{w})$ at x, and $\mathbb{C}_a$ is an $R(A)$-algebra structure on $\mathbb{C}$ determined by a. By the convolution product, $M_{x,a}$ has a $K^A(Z(\mathbf{w}))\otimes_{R(A)}\mathbb{C}_a$-module structure. Thus it has a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module structure by the above homomorphism. By the localization theorem of equivariant K-homology due to Thomason [55], $M_{x,a}$ is isomorphic to the complexified (non-equivariant) K-homology $K(\mathfrak{M}(\mathbf{w})_x^A)\otimes\mathbb{C}$ of the fixed point set $\mathfrak{M}(\mathbf{w})_x^A$. Moreover, it is isomorphic to $H_*(\mathfrak{M}(\mathbf{w})_x^A,\mathbb{C})$ via the Chern character homomorphism thanks to a result in §7. We also show that $M_{x,a}$ is a finite dimensional l-highest weight module. As a usual argument for Verma modules, $M_{x,a}$ has the unique (nonzero) simple quotient. The author conjectures that $M_{x,a}$ is a tensor product of l-fundamental representations in some order. This is proved when the parameter is generic in §14.1.

In §14 we show that the standard modules $M_{x,a}$ and $M_{y,a}$ are isomorphic if and only if x and y are contained in the same stratum. Here the fixed point set $\mathfrak{M}_0(\infty,\mathbf{w})^A$ has a stratification $\mathfrak{M}_0(\infty,\mathbf{w})^A=\bigsqcup_\rho\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ defined in §4. Furthermore, we show that the index set $\{\rho\}$ of the stratum coincides with the set $\mathcal{P}=\{P\}$ of l-dominant l-weights of $M_{0,a}$, the standard module corresponding to the central fiber $\pi^{-1}(0)$. Let us denote by ρ_P the index corresponding to P. Thus we may denote $M_{x,a}$ and its unique simple quotient by $M(P)$ and $L(P)$ respectively if x is contained in the stratum $\mathfrak{M}_0^{\mathrm{reg}}(\rho_P)$ corresponding to an l-dominant l-weight P. We prove the multiplicity formula

$$[M(P):L(Q)]=\dim H^*(i_x^!IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q))),$$

where x is a point in $\mathfrak{M}_0^{\mathrm{reg}}(\rho_P)$, $i_x\colon\{x\}\to\mathfrak{M}_0(\infty,\mathbf{w})^A$ is the inclusion, and $IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q))$ is the intersection cohomology complex attached to $\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)$ and the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)}$.

Our result is simpler than the case of the affine Hecke algebra: nonconstant local systems never appear. This phenomenon corresponds to an algebraic result that all modules are l-highest weight. It compensates for the difference of $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ and $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ during the proof of the multiplicity formula.

If $\mathfrak{g}$ is of type A, then $\mathfrak{M}_0(\infty,\mathbf{w})^A$ coincides with a product of varieties $\mathbf{E}_{\mathbf{d}}$ studied by Lusztig [33], where the underlying graph is of type A. In particular, the Poincaré polynomial of $H^*(i_x^!IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)))$ is a Kazhdan-Lusztig polynomial for a Weyl group of type A. We should have a combinatorial algorithm to compute Poincaré polynomials of $H^*(i_x^!IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)))$ for general $\mathfrak{g}$.

Once we know $\dim H^*(i_x^!IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)))$, information about $L(P)$ can be deduced from information about $M(P)$, which is easier to study. For example, consider the following problems:

(1) Compute Frenkel-Reshetikhin's q-characters [18].
(2) Decompose restrictions of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules to $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules (see [28]).

These problems for $M(P)$ are easier than those for $L(P)$, and we have the following answers.

Frenkel-Reshetikhin's q-characters are generating functions of dimensions of l-weight spaces (see §13.5). In §13.5 we show that these dimensions are Euler numbers of connected components of $\mathfrak{M}(\mathbf{w})^A$ for standard modules $M_{0,a}$. As an application, we prove a conjecture in [18] for $\mathfrak{g}$ of type ADE (Proposition 13.5.2). These Euler numbers should be computable.

Let $\operatorname{Res} M(P)$ be the restriction of $M(P)$ to a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module. In §15 we show the multiplicity formula

$$[\operatorname{Res} M(P) : L(\mathbf{w}-\mathbf{v})] = \dim H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}))),$$

where $\mathbf{v}$ is a weight such that $\mathbf{w}-\mathbf{v}$ is dominant, $L(\mathbf{w}-\mathbf{v})$ is the corresponding irreducible finite dimensional module (these are concepts for usual $\mathfrak{g}$ without 'l'), x is a point in $\mathfrak{M}_0^{\mathrm{reg}}(\rho_P)$, $i_x\colon \{x\} \to \mathfrak{M}_0(\infty,\mathbf{w})$ is the inclusion, $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ is a stratum of $\mathfrak{M}(\infty,\mathbf{w})$, and $IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}))$ is the intersection cohomology complex attached to $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ and the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})}$.

If $\mathfrak{g}$ is of type A, then the stratum $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ coincides with a nilpotent orbit cut out by Slodowy's transversal slice [44, 8.4]. The Poincaré polynomials of $H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})))$ were calculated by Lusztig [30] and coincide with Kostka polynomials. This result is compatible with the conjecture that $M(P)$ is a tensor product of l-fundamental representations, for the restriction of an l-fundamental representation is simple for type A, and Kostka polynomials give tensor product decompositions. We should have a combinatorial algorithm to compute Poincaré polynomials of $H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})))$ for general $\mathfrak{g}$.

We give two examples where $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ can be described explicitly.

Consider the case that $\mathbf{w}$ is a fundamental weight of type A, or more generally a fundamental weight such that the label of the corresponding vertex of the Dynkin diagram is 1. Then it is easy to see that the corresponding quiver variety $\mathfrak{M}_0(\infty,\mathbf{w})$ consists of a single point 0. Thus $\operatorname{Res} M(P)$ remains irreducible in this case.

If $\mathbf{w}$ is the highest weight of the adjoint representation, the corresponding $\mathfrak{M}_0(\infty,\mathbf{w})$ is a simple singularity $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of the type corresponding to $\mathfrak{g}$. Then $\mathfrak{M}_0(\infty,\mathbf{w})$ has two strata $\{0\}$ and $(\mathbb{C}^2\setminus\{0\})/\Gamma$. The intersection cohomology complexes are constant sheaves. Hence we have

$$\operatorname{Res} M(P) = L(\mathbf{w}) \oplus L(0).$$

These two results were shown by Chari-Pressley [9] by a totally different method.

As we mentioned, the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ has another realization, called the Drinfel'd new realization. This Drinfel'd construction can be applied to any symmetrizable Kac-Moody algebra $\mathfrak{g}$, not necessarily a finite dimensional one. This generalization also fits our result, since quiver varieties can be defined for arbitrary finite graphs. If we replace finite dimensional representations by l-integrable representations, parts of our result can be generalized to a Kac-Moody algebra $\mathfrak{g}$, at least when it is symmetric. For example, we generalize the Drinfel'd-Chari-Pressley parametrization. A generalization of the multiplicity formula requires further study.

If $\mathfrak{g}$ is an affine Lie algebra, then $\mathbf{U}_q(\widehat{\mathfrak{g}})$ is the quantum affinization of the affine Lie algebra. It is called a *double loop algebra*, or *toroidal algebra*, and has been studied by various people; see for example [22, 48, 49, 56] and the references therein. A first step to a geometric approach to the toroidal algebra using quiver varieties for the affine Dynkin graph of type $\widetilde{A}$ was given by M. Varagnolo and E. Vasserot [57]. In fact, quiver varieties for affine Dynkin graphs are moduli spaces of instantons (or torsion free sheave) on ALE spaces. Thus these cases are relevant to the original motivation, i.e., a study of the relation between 4-dimensional gauge theory and representation theory. In some cases, these quiver varieties coincide with Hilbert schemes of points on ALE spaces, for which many results have been obtained (see [46]). We will return to this in the future.

If we replace equivariant K-homology by equivariant homology, we should get the Yangian $Y(\mathfrak{g})$ instead of $\mathbf{U}_q(\widehat{\mathfrak{g}})$. This conjecture is motivated again by the analogy of quiver varieties with $T^*\mathcal{B}$. The equivariant homology of $T^*\mathcal{B}$ gives the *graded* Hecke algebra [32], which is an analogue of $Y(\mathfrak{g})$ for H_q. As an application, the affirmative solution of the conjecture implies that the representation theory of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ and that of the Yangian are the same. This has been believed by many people, but there is no written proof.

While the author was preparing this paper, he was informed that Frenkel-Mukhin [17] proved the conjecture in [18] (Proposition 13.5.2) for general $\mathfrak{g}$.

Acknowledgement. Part of this work was done while the author enjoyed the hospitality of the Institute for Advanced Study. The author is grateful to G. Lusztig for his interest and encouragement.

1. Quantum affine algebra

In this section, we give a quick review for the definitions of the quantized universal enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ of the Kac-Moody algebra $\mathfrak{g}$ associated with a symmetrizable generalized Cartan matrix, its affinization $\mathbf{U}_q(\widehat{\mathfrak{g}})$, and the associated loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. Although the algebras defined via quiver varieties are automatically symmetric, we treat the nonsymmetric case also for completeness.

1.1. Quantized universal enveloping algebra.

Let q be an indeterminate. For nonnegative integers $n \ge r$, define

$$(1.1.1)\qquad [n]_q \overset{\text{def.}}{=} \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! \overset{\text{def.}}{=} \begin{cases} [n]_q[n-1]_q \cdots [2]_q[1]_q & (n > 0), \\ 1 & (n = 0), \end{cases}$$
$$\begin{bmatrix} n \\ r \end{bmatrix}_q \overset{\text{def.}}{=} \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

Suppose that the following data are given:

(1) P : free $\mathbb{Z}$-module (weight lattice),
(2) $P^* = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ with a natural pairing $\langle\ ,\ \rangle \colon P \otimes P^* \to \mathbb{Z}$,
(3) an index set I of simple roots
(4) $\alpha_k \in P$ $(k \in I)$ (simple root),
(5) $h_k \in P^*$ $(k \in I)$ (simple coroot),
(6) a symmetric bilinear form $(\ ,\ )$ on P.

These are required to satisfy the following:

(a) $\langle h_k, \lambda\rangle = 2(\alpha_k, \lambda)/(\alpha_k, \alpha_k)$ for $k \in I$ and $\lambda \in P$,
(b) $\mathbf{C} \overset{\text{def.}}{=} (\langle h_k, \alpha_l\rangle)_{k,l}$ is a symmetrizable generalized Cartan matrix, i.e., $\langle h_k, \alpha_k\rangle = 2$, and $\langle h_k, \alpha_l\rangle \in \mathbb{Z}_{\le 0}$ and $\langle h_k, \alpha_l\rangle = 0 \iff \langle h_l, \alpha_k\rangle = 0$ for $k \ne l$,
(c) $(\alpha_k, \alpha_k) \in 2\mathbb{Z}_{>0}$,
(d) $\{\alpha_k\}_{k \in I}$ are linearly independent,
(e) there exists $\Lambda_k \in P$ $(k \in I)$ such that $\langle h_l, \Lambda_k\rangle = \delta_{kl}$ (fundamental weight).

The quantized universal enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ of the Kac-Moody algebra is the $\mathbb{Q}(q)$-algebra generated by e_k, f_k $(k \in I)$, q^h $(h \in P^*)$ with relations

$$(1.1.2)\qquad q^0 = 1, \quad q^h q^{h'} = q^{h+h'},$$

$$q^h e_k q^{-h} = q^{\langle h,\alpha_k\rangle} e_k, \quad q^h f_k q^{-h} = q^{-\langle h,\alpha_k\rangle} f_k, \tag{1.1.3}$$

$$e_k f_l - f_l e_k = \delta_{kl} \frac{q^{(\alpha_k,\alpha_k)h_k/2} - q^{-(\alpha_k,\alpha_k)h_k/2}}{q_k - q_k^{-1}}, \tag{1.1.4}$$

$$\sum_{p=0}^{b} (-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_{q_k} e_k^p e_l e_k^{b-p} = \sum_{p=0}^{b} (-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_{q_k} f_k^p f_l f_k^{b-p} = 0 \quad \text{for } k \neq l, \tag{1.1.5}$$

where $q_k = q^{(\alpha_k,\alpha_k)/2}$, $b = 1 - \langle h_k, \alpha_l\rangle$.

Let $\mathbf{U}_q(\mathfrak{g})^+$ (resp. $\mathbf{U}_q(\mathfrak{g})^-$) be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by the elements e_k (resp. f_k). Let $\mathbf{U}_q(\mathfrak{g})^0$ be the $\mathbb{Q}(q)$-subalgebra generated by elements q^h ($h \in P^*$). Then we have the triangle decomposition [36, 3.2.5]:

$$\mathbf{U}_q(\mathfrak{g}) \cong \mathbf{U}_q(\mathfrak{g})^+ \otimes \mathbf{U}_q(\mathfrak{g})^0 \otimes \mathbf{U}_q(\mathfrak{g})^-. \tag{1.1.6}$$

Let $e_k^{(n)} \overset{\text{def.}}{=} e_k^n/[n]_{q_k}!$ and $f_k^{(n)} \overset{\text{def.}}{=} f_k^n/[n]_{q_k}!$. Let $\mathbf{U}_q^{\mathbb{Z}}(\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by elements $e_k^{(n)}$, $f_k^{(n)}$, q^h for $k \in I$, $n \in \mathbb{Z}_{>0}$, $h \in P^*$. It is known that $\mathbf{U}_q^{\mathbb{Z}}(\mathfrak{g})$ is an *integral form* of $\mathbf{U}_q(\mathfrak{g})$, i.e., the natural map $\mathbf{U}_q^{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to \mathbf{U}_q(\mathfrak{g})$ is an isomorphism. (See [10, 9.3.1].) For $\varepsilon \in \mathbb{C}^*$, let us define $\mathbf{U}_\varepsilon(\mathfrak{g})$ as $\mathbf{U}_q^{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ via the algebra homomorphism $\mathbb{Z}[q,q^{-1}] \to \mathbb{C}$ that takes q to ε. It will be called the *specialized* quantized enveloping algebra. We say a $\mathbf{U}_q(\mathfrak{g})$-module M (defined over $\mathbb{Q}(q)$) is a *highest weight module* with *highest weight* $\Lambda \in P$ if there exists a vector $m_0 \in M$ such that

$$e_k * m_0 = 0, \qquad \mathbf{U}_q(\mathfrak{g})^- * m_0 = M, \tag{1.1.7}$$

$$q^h * m_0 = q^{\langle h,\Lambda\rangle} m_0 \qquad \text{for any } h \in P^*. \tag{1.1.8}$$

Then there exists a direct sum decomposition $M = \bigoplus_{\lambda\in P} M_\lambda$ (weight space decomposition) where $M_\lambda \overset{\text{def.}}{=} \{m \mid q^h \cdot v = q^{\langle h,\lambda\rangle} m \text{ for any } h \in P^*\}$. By using the triangular decomposition (1.1.6), one can show that the simple highest weight $\mathbf{U}_q(\mathfrak{g})$-module is determined uniquely by Λ.

We say a $\mathbf{U}_q(\mathfrak{g})$-module M (defined over $\mathbb{Q}(q)$) is *integrable* if M has a weight space decomposition $M = \bigoplus_{\lambda\in P} M_\lambda$ with $\dim M_\lambda < \infty$, and for any $m \in M$, there exists $n_0 \geq 1$ such that $e_k^n * m = f_k^n * m = 0$ for all $k \in I$ and $n \geq n_0$.

The (unique) simple highest weight $\mathbf{U}_q(\mathfrak{g})$-module with highest weight Λ is integrable if and only if Λ is a dominant integral weight Λ, i.e., $\langle \Lambda, h_k\rangle \in \mathbb{Z}_{\geq 0}$ for any $k \in I$ ([36, 3.5.6, 3.5.8]). In this case, the integrable highest weight $\mathbf{U}_q$-module with highest weight Λ is denoted by $L(\Lambda)$.

For a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module M (defined over $\mathbb{C}$), we define highest weight modules, integrable modules, etc. in a similar way.

Suppose Λ is dominant. Let $L(\Lambda)^{\mathbb{Z}} \overset{\text{def.}}{=} \mathbf{U}_q^{\mathbb{Z}}(\mathfrak{g}) * m_0$, where m_0 is the highest weight vector. It is known that the natural map $L(\Lambda)^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to L(\Lambda)$ is an isomorphism and $L(\Lambda)^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ is the simple integrable highest weight module of the corresponding Kac-Moody algebra $\mathfrak{g}$ with highest weight Λ, where $\mathbb{Z}[q,q^{-1}] \to \mathbb{C}$ is the homomorphism that sends q to 1 ([36, Chapter 14 and 33.1.3]). Unless ε is a root of unity, the simple integrable highest weight $\mathbf{U}_\varepsilon(\mathfrak{g})$-module is the specialization of $L(\Lambda)^{\mathbb{Z}}$ ([10, 10.1.14, 10.1.15]).

1.2. **Quantum affine algebra.** The *quantum affinization* $\mathbf{U}_q(\widehat{\mathfrak{g}})$ of $\mathbf{U}_q(\mathfrak{g})$ (or simply *quantum affine algebra*) is an associative algebra over $\mathbb{Q}(q)$ generated by

$e_{k,r}, f_{k,r}$ ($k \in I$, $r \in \mathbb{Z}$), q^h ($h \in P^*$), $q^{\pm c/2}$, $q^{\pm d}$, and $h_{k,m}$ ($k \in I$, $m \in \mathbb{Z} \setminus \{0\}$) with the following defining relations:

$$q^{\pm c/2} \text{ is central,} \tag{1.2.1}$$

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad [q^h, h_{k,m}] = 0, \quad q^d q^{-d} = 1, \quad q^{c/2} q^{-c/2} = 1, \tag{1.2.2}$$

$$\psi_k^{\pm}(z)\psi_l^{\pm}(w) = \psi_l^{\pm}(w)\psi_k^{\pm}(z), \tag{1.2.3}$$

$$\psi_k^-(z)\psi_l^+(w) = \frac{(z - q^{-(\alpha_k,\alpha_l)}q^c w)(z - q^{(\alpha_k,\alpha_l)}q^{-c}w)}{(z - q^{(\alpha_k,\alpha_l)}q^c w)(z - q^{-(\alpha_k,\alpha_l)}q^{-c}w)}\psi_l^+(w)\psi_k^-(z), \tag{1.2.4}$$

$$\begin{gathered} [q^d, q^h] = 0, \quad q^d h_{k,m} q^{-d} = q^m h_{k,m}, \\ q^d e_{k,r} q^{-d} = q^r e_{k,r}, \quad q^d f_{k,r} q^{-d} = q^r f_{k,r}, \end{gathered} \tag{1.2.5}$$

$$q^h e_{k,r} q^{-h} = q^{\langle h,\alpha_k\rangle} e_{k,r}, \quad q^h f_{k,r} q^{-h} = q^{-\langle h,\alpha_k\rangle} f_{k,r}, \tag{1.2.6}$$

$$\begin{aligned} &(q^{\pm sc/2}z - q^{\pm\langle h_k,\alpha_l\rangle}w)\psi_l^s(z)x_k^{\pm}(w) \\ &\qquad = (q^{\pm\langle h_k,\alpha_l\rangle}q^{\pm sc/2}z - w)x_k^{\pm}(w)\psi_l^s(z), \end{aligned} \tag{1.2.7}$$

$$\left[x_k^+(z), x_l^-(w)\right] = \frac{\delta_{kl}}{q_k - q_k^{-1}}\left\{\delta\left(q^c\frac{w}{z}\right)\psi_k^+(q^{c/2}w) - \delta\left(q^c\frac{z}{w}\right)\psi_k^-(q^{c/2}z)\right\}, \tag{1.2.8}$$

$$(z - q^{\pm 2}w)x_k^{\pm}(z)x_k^{\pm}(w) = (q^{\pm 2}z - w)x_k^{\pm}(w)x_k^{\pm}(z), \tag{1.2.9}$$

$$\begin{aligned} &\prod_{p=1}^{-\langle\alpha_k,h_l\rangle}(z - q^{\pm(b'-2p)}w)x_k^{\pm}(z)x_l^{\pm}(w) \\ &\qquad = \prod_{p=1}^{-\langle\alpha_k,h_l\rangle}(q^{\pm(b'-2p)}z - w)x_l^{\pm}(w)x_k^{\pm}(z), \quad \text{if } k \neq l, \end{aligned} \tag{1.2.10}$$

$$\begin{aligned} \sum_{\sigma\in S_b}\sum_{p=0}^{b}(-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_{q_k} & x_k^{\pm}(z_{\sigma(1)})\cdots x_k^{\pm}(z_{\sigma(p)})x_l^{\pm}(w)x_k^{\pm}(z_{\sigma(p+1)}) \\ & \cdots x_k^{\pm}(z_{\sigma(b)}) = 0, \quad \text{if } k \neq l, \end{aligned} \tag{1.2.11}$$

where $q_k = q^{(\alpha_k,\alpha_k)/2}$, $s = \pm$, $b = 1 - \langle h_k, \alpha_l\rangle$, $b' = -(\alpha_k, \alpha_l)$, and S_b is the symmetric group of b letters. Here $\delta(z)$, $x_k^+(z)$, $x_k^-(z)$, $\psi_k^{\pm}(z)$ are generating functions defined by

$$\delta(z) \overset{\text{def.}}{=} \sum_{r=-\infty}^{\infty} z^r, \qquad x_k^+(z) \overset{\text{def.}}{=} \sum_{r=-\infty}^{\infty} e_{k,r}z^{-r}, \qquad x_k^-(z) \overset{\text{def.}}{=} \sum_{r=-\infty}^{\infty} f_{k,r}z^{-r},$$
$$\psi_k^{\pm}(z) \overset{\text{def.}}{=} q^{\pm(\alpha_k,\alpha_k)h_k/2}\exp\left(\pm(q_k - q_k^{-1})\sum_{m=1}^{\infty} h_{k,\pm m}z^{\mp m}\right).$$

We will also need the following generating function later:

$$p_k^{\pm}(z) \overset{\text{def.}}{=} \exp\left(-\sum_{m=1}^{\infty}\frac{h_{k,\pm m}}{[m]_{q_k}}z^{\mp m}\right).$$

We have $\psi_k^{\pm}(z) = q^{\pm(\alpha_k,\alpha_k)h_k/2}p_k^{\pm}(q_k z)/p_k^{\pm}(q_k^{-1}z)$.

Remark 1.2.12. When $\mathfrak{g}$ is finite dimensional, then $\min(\langle\alpha_k, h_l\rangle, \langle\alpha_l, h_k\rangle) = 0$ or 1. Then the relation (1.2.10) reduces to the one in literature. Our generalization seems natural since we will check it later, at least for symmetric $\mathfrak{g}$.

Let $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$ (resp. $\mathbf{U}_q(\widehat{\mathfrak{g}})^-$) be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by the elements $e_{k,r}$ (resp. $f_{k,r}$). Let $\mathbf{U}_q(\widehat{\mathfrak{g}})^0$ be the $\mathbb{Q}(q)$-subalgebra generated by the elements q^h, $h_{k,m}$.

The *quantum loop algebra* $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is the subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})/(q^{\pm c/2}-1)$ generated by $e_{k,r}, f_{k,r}$ ($k \in I$, $r \in \mathbb{Z}$), q^h ($h \in P^*$), and $h_{k,m}$ ($k \in I$, $m \in \mathbb{Z}\setminus\{0\}$), i.e., generators other than $q^{\pm c/2}$, $q^{\pm d}$. We will be concerned only with the quantum loop algebra, and not with the quantum affine algebra in the sequel.

There is a homomorphism $\mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathbf{L}\mathfrak{g})$ defined by

$$q^h \mapsto q^h, \quad e_k \mapsto e_{k,0}, \quad f_k \mapsto f_{k,0}.$$

Let $e_{k,r}^{(n)} \overset{\text{def.}}{=} e_{k,r}^n/[n]_{q_k}!$ and $f_{k,r}^{(n)} \overset{\text{def.}}{=} f_{k,r}^n/[n]_{q_k}!$. Let $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $e_{k,r}^{(n)}$, $f_{k,r}^{(n)}$, q^h and the coefficients of $p_k^{\pm}(z)$ for $k \in I$, $r \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$, $h \in P^*$. (It should be true that $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ is free over $\mathbb{Z}[q,q^{-1}]$ and that the natural map $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to \mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is an isomorphism. But the author does not know how to prove this.) This subalgebra was introduced by Chari-Pressley [12]. Let $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^+$ (resp. $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^-$) be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $e_{k,r}^{(n)}$ (resp. $f_{k,r}^{(n)}$) for $k \in I$, $r \in \mathbb{Z}$, $n \in Z_{>0}$. We have $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^{\pm} \subset \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$. Let $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^0$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by q^h, the coefficients of $p_k^{\pm}(z)$ and

$$\begin{bmatrix} q^{h_k}; n \\ r \end{bmatrix} \overset{\text{def.}}{=} \prod_{s=1}^{r} \frac{q^{(\alpha_k,\alpha_k)h_k/2} q_k^{n-s+1} - q^{-(\alpha_k,\alpha_k)h_k/2} q_k^{-n+s-1}}{q_k^s - q_k^{-s}}$$

for all $h \in P$, $k \in I$, $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$. One can easily show that $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^0 \subset \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ (see, e.g., [36, 3.1.9]).

For $\varepsilon \in \mathbb{C}^*$, let $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ be the *specialized quantum loop algebra* defined by $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ via the algebra homomorphism $\mathbb{Z}[q,q^{-1}] \to \mathbb{C}$ that takes q to ε. We assume ε is *not* a root of unity in this paper. Let $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^{\pm}$ and $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ be the specializations of $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^{\pm}$ and $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^0$ respectively. We have a weak form of the triangular decomposition

$$\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- \cdot \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0 \cdot \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^+, \tag{1.2.13}$$

which follows from the definition (cf. [12, 6.1]).

We say a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M is an *l-highest weight module* (‘l’ stands for the loop) with *l-highest weight* $(\Lambda, (\Psi_k^{\pm}(z))_k)$ (where $\Lambda \in P$, $(\Psi_k^{\pm}(z))_k \in \mathbb{C}[[z^{\mp}]]^I$) if there exists a vector $m_0 \in M$ such that

(1.2.14)
$$e_{k,r} * m_0 = 0, \qquad \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * m_0 = M,$$

(1.2.15)
$$q^h * m_0 = \varepsilon^{\langle h,\Lambda\rangle} m_0 \quad \text{for } h \in P^*, \qquad \psi_k^{\pm}(z) * m_0 = \Psi_k^{\pm}(z) m_0 \quad \text{for } k \in I.$$

By using (1.2.13) and a standard argument, one can show that there is a simple l-highest weight module M of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ with l-highest weight vector m_0 satisfying the above for any $(\Lambda, (\Psi_k^{\pm}(z))_k)$ with $\Psi_k^+(\infty) = (\alpha_k,\alpha_k)\langle\Lambda, h_k\rangle/2$, $\Psi_k^-(0) =$

$-(\alpha_k, \alpha_k)\langle \Lambda, h_k\rangle/2$. Moreover, such M is unique up to isomorphism. For abuse of notation, we denote the pair $(\Lambda, (\Psi_k^\pm(z))_k)$ simply by the symbol $\Psi^\pm(z)$.

A $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M is said to be *l-integrable* if

(a) M has a weight space decomposition $M = \bigoplus_{\lambda\in P} M_\lambda$ as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module such that $\dim M_\lambda < \infty$,
(b) for any $m \in M$, there exists $n_0 \geq 1$ such that $e_{k,r_1}\cdots e_{k,r_n} * m = f_{k,r_1}\cdots f_{k,r_n} * m = 0$ for all $r_1, \dots, r_n \in \mathbb{Z}$, $k \in I$ and $n \geq n_0$.

For example, if $\mathfrak{g}$ is finite dimensional, and M is a finite dimensional module, then M satisfies the above conditions after twisting with a certain automorphism of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ ([10, 12.2.3]).

Proposition 1.2.16. *Assume that $\mathfrak{g}$ is symmetric. The simple l-highest weight $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M with l-highest weight $\Psi^\pm(z)$ is l-integrable if and only if Λ is dominant and there exist polynomials $P_k(u) \in \mathbb{C}[u]$ for $k \in I$ with $P_k(0) = 1$ such that*

$$\Psi_k^\pm(z) = \varepsilon_k^{\deg P_k}\left(\frac{P_k(\varepsilon_k^{-1}/z)}{P_k(\varepsilon_k/z)}\right)^\pm, \tag{1.2.17}$$

where $\varepsilon_k = \varepsilon^{(\alpha_k,\alpha_k)/2}$, and $(\ )^\pm \in \mathbb{C}[[z^\mp]]$ denotes the expansion at $z = \infty$ and 0 respectively.

This result was announced by Drinfel'd for the Yangian [15]. The proof of the 'only if' part when $\mathfrak{g}$ is finite dimensional was given by Chari-Pressley [10, 12.2.6]. Since the proof is based on a reduction to the case $\mathfrak{g} = \mathfrak{sl}_2$, it can be applied to a general Kac-Moody algebra $\mathfrak{g}$ (not necessarily symmetric). The 'if' part was proved by them later in [11] when $\mathfrak{g}$ is finite dimensional, again not necessarily symmetric. As an application of the main result of this paper, we will prove the converse for a symmetric Kac-Moody algebra $\mathfrak{g}$ in §13. Our proof is independent of Chari-Pressley's proof.

Remark 1.2.18. The polynomials P_k are called *Drinfel'd polynomials.*

When the Drinfel'd polynomials are given by

$$P_k(u) = \begin{cases} 1 - su & \text{if } k \neq k_0, \\ 1 & \text{otherwise}, \end{cases}$$

for some $k_0 \in I$, $s \in \mathbb{C}^*$, the corresponding simple l-highest weight module is called an *l-fundamental representation.* When $\mathfrak{g}$ is finite dimensional, $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is a Hopf algebra since Drinfel'd [15] announced and Beck [5] proved that $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ can be identified with (a quotient of) the specialized quantized enveloping algebra associated with Cartan data of affine type. Thus a tensor product of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules is again a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module. We have the following:

Proposition 1.2.19 ([10, 12.2.6,12.2.8])**.** *Suppose $\mathfrak{g}$ is finite dimensional.*

(1) *If M and N are simple l-highest weight $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules with Drinfel'd polynomials $P_{k,M}$, $P_{k,N}$ such that $M \otimes N$ is simple, then its Drinfel'd polynomial $P_{k,M\otimes N}$ is given by*

$$P_{k,M\otimes N} = P_{k,M}P_{k,N}.$$

(2) *Every simple l-highest weight $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module is a subquotient of a tensor product of l-fundamental representations.*

Unfortunately the coproduct is not defined for general $\mathfrak{g}$ as far as the author knows. Thus the above results do not make sense for general $\mathfrak{g}$.

1.3. **An $\boldsymbol{l}$-weight space decomposition.** Let M be an l-integrable $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module with the weight space decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$. Since the commutative subalgebra $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ preserves each M_λ, we can further decompose M into a sum of generalized simultaneous eigenspaces for $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$:

$$M = \bigoplus M_{\Psi^\pm}, \tag{1.3.1}$$

where $\Psi^\pm(z)$ is a pair $(\Lambda, (\Psi_k^\pm(z))_k)$ as before and

$$M_{\Psi^\pm} \overset{\text{def.}}{=} \left\{ m \in M \;\middle|\; \begin{array}{l} q^h * m = \varepsilon^{\langle h, \Lambda \rangle} m \quad \text{for } h \in P^*, \\ (\psi_k^\pm(z) - \Psi_k^\pm(z)\,\mathrm{Id})^N * m = 0 \\ \qquad\qquad\qquad\qquad \text{for } k \in I \text{ and sufficiently large } N \end{array} \right\}.$$

If $M_{\Psi^\pm} \neq 0$, we call $M_{\Psi^\pm}$ an *l-weight space*, and the corresponding $\Psi^\pm(z)$ an *l-weight*. This is a refinement of the weight space decomposition. A further study of the l-weight space decomposition will be given in §13.5.

Motivated by Proposition 1.2.16, we introduce the following notion:

Definition 1.3.2. An l-weight $\Psi^\pm(z) = (\Lambda, (\Psi_k^\pm(z))_k)$ is said to be *l-dominant* if Λ is dominant and there exists a polynomial $P(u) = (P_k(u))_k \in \mathbb{C}[u]^I$ for with $P_k(0) = 1$ such that (1.2.17) holds.

Thus Proposition 1.2.16 means that an l-highest weight module is l-integrable if and only if the l-highest weight is l-dominant.

2. Quiver variety

2.1. **Notation.** Suppose that a finite graph is given and assume that there are no edge loops, i.e., no edges joining a vertex with itself. Let I be the set of vertices and E the set of edges. Let $\mathbf{A}$ be the adjacency matrix of the graph, namely

$$\mathbf{A} = (\mathbf{A}_{kl})_{k,l \in I}, \qquad \text{where } \mathbf{A}_{kl} \text{ is the number of edges joining } k \text{ and } l.$$

We associate with the graph (I, E) a symmetric generalized Cartan matrix $\mathbf{C} = 2\mathbf{I} - \mathbf{A}$, where $\mathbf{I}$ is the identity matrix. This gives a bijection between the finite graphs without edge loops and symmetric Cartan matrices. We have the corresponding symmetric Kac-Moody algebra $\mathfrak{g}$, the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g})$, the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ and the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. Let H be the set of pairs consisting of an edge together with its orientation. For $h \in H$, we denote by $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) the incoming (resp. outgoing) vertex of h. For $h \in H$ we denote by $\overline{h}$ the same edge as h with the reverse orientation. Choose and fix an orientation Ω of the graph, i.e., a subset $\Omega \subset H$ such that $\overline{\Omega} \cup \Omega = H$, $\Omega \cap \overline{\Omega} = \emptyset$. The pair (I, Ω) is called a *quiver*. Let us define matrices $\mathbf{A}_\Omega$ and $\mathbf{A}_{\overline{\Omega}}$ by

$$\begin{aligned} (\mathbf{A}_\Omega)_{kl} &\overset{\text{def.}}{=} \#\{h \in \Omega \mid \mathrm{in}(h) = k, \mathrm{out}(h) = l\}, \\ (\mathbf{A}_{\overline{\Omega}})_{kl} &\overset{\text{def.}}{=} \#\{h \in \overline{\Omega} \mid \mathrm{in}(h) = k, \mathrm{out}(h) = l\}. \end{aligned} \tag{2.1.1}$$

So we have $\mathbf{A} = \mathbf{A}_\Omega + \mathbf{A}_{\overline{\Omega}}$, ${}^t\mathbf{A}_\Omega = \mathbf{A}_{\overline{\Omega}}$.

Let $V = (V_k)_{k \in I}$ be a collection of finite-dimensional vector spaces over $\mathbb{C}$ for each vertex $k \in I$. The dimension of V is a vector

$$\dim V = (\dim V_k)_{k \in I} \in \mathbb{Z}^I_{\geq 0}.$$

If V^1 and V^2 are such collections, we define vector spaces by

$$\begin{aligned} \mathrm{L}(V^1, V^2) &\overset{\text{def.}}{=} \bigoplus_{k \in I} \mathrm{Hom}(V^1_k, V^2_k), \\ \mathrm{E}(V^1, V^2) &\overset{\text{def.}}{=} \bigoplus_{h \in H} \mathrm{Hom}(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)}). \end{aligned} \tag{2.1.2}$$

For $B = (B_h) \in \mathrm{E}(V^1, V^2)$ and $C = (C_h) \in \mathrm{E}(V^2, V^3)$, let us define a multiplication of B and C by

$$CB \overset{\text{def.}}{=} (\sum_{\mathrm{in}(h)=k} C_h B_{\overline{h}})_k \in \mathrm{L}(V^1, V^3).$$

Multiplications ba, Ba of $a \in \mathrm{L}(V^1, V^2)$, $b \in \mathrm{L}(V^2, V^3)$, $B \in \mathrm{E}(V^2, V^3)$ are defined in an obvious manner. If $a \in \mathrm{L}(V^1, V^1)$, its trace $\mathrm{tr}(a)$ is understood as $\sum_k \mathrm{tr}(a_k)$.

For two collections V, W of vector spaces with $\mathbf{v} = \dim V$, $\mathbf{w} = \dim W$, we consider the vector space given by

$$\mathbf{M} \equiv \mathbf{M}(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \mathrm{E}(V, V) \oplus \mathrm{L}(W, V) \oplus \mathrm{L}(V, W), \tag{2.1.3}$$

where we use the notation $\mathbf{M}$ unless we want to specify dimensions of V, W. The above three components for an element of $\mathbf{M}$ will be denoted by B, i, j respectively. An element of $\mathbf{M}$ will be called an *ADHM datum*.

Usually a point in $\bigoplus_{h \in \Omega} \mathrm{Hom}(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)})$ is called a *representation of the quiver* (I, Ω) in the literature. Thus $\mathrm{E}(V, V)$ is the product of the space of representations of (I, Ω) and that of $(I, \overline{\Omega})$. On the other hand, the factor $\mathrm{L}(W, V)$ or $\mathrm{L}(V, W)$ has never appeared in the literature.

Convention 2.1.4. When we relate the quiver varieties to the quantum affine algebra, the dimension vectors will be mapped into the weight lattice in the following way:

$$\mathbf{v} \mapsto \sum_k v_k \alpha_k, \quad \mathbf{w} \mapsto \sum_k w_k \Lambda_k,$$

where v_k (resp. w_k) is the kth component of $\mathbf{v}$ (resp. $\mathbf{w}$). Since $\{\alpha_k\}$ and $\{\Lambda_k\}$ are both linearly independent, these maps are injective. We consider $\mathbf{v}$ and $\mathbf{w}$ as elements of the weight lattice P in this way hereafter.

For a collection $S = (S_k)_{k \in I}$ of subspaces of V_k and $B \in \mathrm{E}(V, V)$, we say S is *B-invariant* if $B_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$.

Fix a function $\varepsilon \colon H \to \mathbb{C}^*$ such that $\varepsilon(h) + \varepsilon(\overline{h}) = 0$ for all $h \in H$. In [44, 45], it was assumed that ε takes its value ± 1, but this assumption is not necessary as remarked by Lusztig [38]. For $B \in \mathrm{E}(V^1, V^2)$, let us denote by $\varepsilon B \in \mathrm{E}(V^1, V^2)$ data given by $(\varepsilon B)_h = \varepsilon(h) B_h$ for $h \in H$.

Let us define a symplectic form ω on $\mathbf{M}$ by

$$\omega((B, i, j), (B', i', j')) \overset{\text{def.}}{=} \mathrm{tr}(\varepsilon B\, B') + \mathrm{tr}(ij' - i'j). \tag{2.1.5}$$

Let G be the algebraic group defined by

$$G \equiv G_{\mathbf{v}} \overset{\text{def.}}{=} \prod_k \mathrm{GL}(V_k),$$

where we use the notation $G_{\mathbf{v}}$ when we want to emphasize the dimension. It acts on $\mathbf{M}$ by

$$(B, i, j) \mapsto g \cdot (B, i, j) \overset{\text{def.}}{=} (gBg^{-1}, gi, jg^{-1}), \tag{2.1.6}$$

preserving the symplectic form ω. The moment map $\mu\colon \mathbf{M} \to \mathrm{L}(V, V)$ vanishing at the origin is given by

$$\mu(B, i, j) = \varepsilon BB + ij, \tag{2.1.7}$$

where the dual of the Lie algebra of G is identified with the Lie algebra via the trace. Let $\mu^{-1}(0)$ be an affine algebraic variety (not necessarily irreducible) defined as the zero set of μ.

For $(B, i, j) \in \mu^{-1}(0)$, we consider the complex

$$\mathrm{L}(V, V) \xrightarrow{\iota} \mathrm{E}(V, V) \oplus \mathrm{L}(W, V) \oplus \mathrm{L}(V, W) \xrightarrow{d\mu} \mathrm{L}(V, V), \tag{2.1.8}$$

where $d\mu$ is the differential of μ at (B, i, j), and ι is given by

$$\iota(\xi) = (B\xi - \xi B) \oplus (-\xi i) \oplus j\xi.$$

If we identify $\mathrm{E}(V, V) \oplus \mathrm{L}(W, V) \oplus \mathrm{L}(V, W)$ with its dual via the symplectic form ω, ι is the transpose of $d\mu$.

2.2. **Two quotients $\mathfrak{M}_0$ and $\mathfrak{M}$.** We consider two types of quotients of $\mu^{-1}(0)$ by the group G. The first one is the affine algebro-geometric quotient given as follows. Let $A(\mu^{-1}(0))$ be the coordinate ring of the affine algebraic variety $\mu^{-1}(0)$. Then $\mathfrak{M}_0$ is defined as a variety whose coordinate ring is the invariant part of $A(\mu^{-1}(0))$:

$$\mathfrak{M}_0 \equiv \mathfrak{M}_0(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \mu^{-1}(0)/\!\!/G = \operatorname{Spec} A(\mu^{-1}(0))^G. \tag{2.2.1}$$

As before, we use the notation $\mathfrak{M}_0$ unless we need to specify the dimension vectors $\mathbf{v}$, $\mathbf{w}$. By the geometric invariant theory [43], this is an affine algebraic variety. It is also known that the geometric points of $\mathfrak{M}_0$ are closed G-orbits.

For the second quotient we follow A. King's approach [27]. Let us define a character $\chi\colon G \to \mathbb{C}^*$ by $\chi(g) = \prod_k \det g_k^{-1}$ for $g = (g_k)$. Set

$$A(\mu^{-1}(0))^{G,\chi^n} \overset{\text{def.}}{=} \{\, f \in A(\mu^{-1}(0)) \mid f(g(B, i, j)) = \chi(g)^n f(B, i, j) \,\}.$$

The direct sum with respect to $n \in \mathbb{Z}_{\geq 0}$ is a graded algebra, hence we can define

$$\mathfrak{M} \equiv \mathfrak{M}(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \operatorname{Proj} \bigoplus_{n \geq 0} A(\mu^{-1}(0))^{G,\chi^n}. \tag{2.2.2}$$

These are what we call *quiver varieties*.

2.3. **Stability condition.** In this subsection, we shall give a description of the quiver variety $\mathfrak{M}$ which is easier to deal with. We again follow King's work [27].

Definition 2.3.1. A point $(B, i, j) \in \mu^{-1}(0)$ is said to be *stable* if the following condition holds:

> if a collection $S = (S_k)_{k \in I}$ of subspaces of $V = (V_k)_{k \in I}$ is B-invariant and contained in $\operatorname{Ker} j$, then $S = 0$.

Let us denote by $\mu^{-1}(0)^{\mathrm{s}}$ the set of stable points.

Clearly, the stability condition is invariant under the action of G. Hence we may say an orbit is stable or not.

Let us lift the G-action on $\mu^{-1}(0)$ to the trivial line bundle $\mu^{-1}(0) \times \mathbb{C}$ by $g \cdot (B, i, j, z) = (g \cdot (B, i, j), \chi^{-1}(g)z)$.

We have the following:

Proposition 2.3.2. (1) *A point (B, i, j) is stable if and only if the closure of $G \cdot (B, i, j, z)$ does not intersect with the zero section of $\mu^{-1}(0) \times \mathbb{C}$ for $z \neq 0$.*

(2) *If (B, i, j) is stable, then the differential $d\mu \colon \mathbf{M} \to \mathrm{L}(V, V)$ is surjective. In particular, $\mu^{-1}(0)^{\mathrm{s}}$ is a nonsingular variety.*

(3) *If (B, i, j) is stable, then ι in* (2.1.8) *is injective.*

(4) *The quotient $\mu^{-1}(0)^{\mathrm{s}}/G$ has a structure of nonsingular quasi-projective variety of dimension $(\mathbf{v}, 2\mathbf{w} - \mathbf{v})$, and $\mu^{-1}(0)^{\mathrm{s}}$ is a principal G-bundle over $\mu^{-1}(0)^{\mathrm{s}}/G$.*

(5) *The tangent space of $\mu^{-1}(0)^{\mathrm{s}}/G$ at the orbit $G \cdot (B, i, j)$ is isomorphic to the middle cohomology group of* (2.1.8).

(6) *The variety $\mathfrak{M}$ is isomorphic to $\mu^{-1}(0)^{\mathrm{s}}/G$.*

(7) *$\mu^{-1}(0)^{\mathrm{s}}/G$ has a holomorphic symplectic structure as a symplectic quotient.*

Proof. See [45, 3.ii] and [44, 2.8]. □

Notation 2.3.3. For a stable point $(B, i, j) \in \mu^{-1}(0)$, its G-orbit considered as a geometric point in the quiver variety $\mathfrak{M}$ is denoted by $[B, i, j]$. If $(B, i, j) \in \mu^{-1}(0)$ has a closed G-orbit, then the corresponding geometric point in $\mathfrak{M}_0$ will be denoted also by $[B, i, j]$.

From the definition, we have a natural projective morphism (see [45, 3.18])

$$\pi \colon \mathfrak{M} \to \mathfrak{M}_0. \tag{2.3.4}$$

If $\pi([B, i, j]) = [B^0, i^0, j^0]$, then $G \cdot (B^0, i^0, j^0)$ is the unique closed orbit contained in the closure of $G \cdot (B, i, j)$. For $x \in \mathfrak{M}_0$, let

$$\mathfrak{M}_x \overset{\text{def.}}{=} \pi^{-1}(x). \tag{2.3.5}$$

If we want to specify the dimension, we denote the above by $\mathfrak{M}(\mathbf{v}, \mathbf{w})_x$. Unfortunately, this notation conflicts with the previous notation $\mathfrak{M}_0$ when $x = 0$. And the central fiber $\pi^{-1}(0)$ plays an important role later. We shall always write $\mathfrak{L} \equiv \mathfrak{L}(\mathbf{v}, \mathbf{w})$ for $\pi^{-1}(0)$ and not use the notation (2.3.5) with $x = 0$.

In order to explain a more precise relation between $[B, i, j]$ and $[B^0, i^0, j^0]$, we need the following notion.

Definition 2.3.6. Suppose that $(B, i, j) \in \mathbf{M}$ and a B-invariant increasing filtration

$$0 = V^{(-1)} \subset V^{(0)} \subset \cdots \subset V^{(N)} = V$$

with $\operatorname{Im} i \subset V^{(0)}$ are given. Then set $\operatorname{gr}_m V = V^{(m)}/V^{(m-1)}$ and $\operatorname{gr} V = \bigoplus \operatorname{gr}_m V$. Let $\operatorname{gr}_m B$ denote the endomorphism which B induces on $\operatorname{gr}_m V$. For $m = 0$, let $\operatorname{gr}_0 i \in \mathrm{L}(W, V^{(0)})$ be such that its composition with the inclusion $V^{(0)} \subset V$ is i, and let $\operatorname{gr}_0 j$ be the restriction of j to $V^{(0)}$. For $m \neq 0$, set $\operatorname{gr}_m i = 0$ and $\operatorname{gr}_m j = 0$. Let $\operatorname{gr}(B, i, j)$ be the direct sum of $(\operatorname{gr}_m B, \operatorname{gr}_m i, \operatorname{gr}_m j)$ considered as data on $\operatorname{gr} V$.

Proposition 2.3.7. *Suppose $\pi(x) = y$. Then there exist a representative (B, i, j) of x and a B-invariant increasing filtration $V^{(*)}$ as in Definition 2.3.6 such that $\operatorname{gr}(B, i, j)$ is a representative of y on $\operatorname{gr} V$.*

Proof. See [45, 3.20] ☐

Proposition 2.3.8. *$\mathfrak{L}$ is a Lagrangian subvariety which is homotopic to $\mathfrak{M}$.*

Proof. See [44, 5.5, 5.8]. ☐

2.4. **Hyper-Kähler structure.** We briefly recall hyper-Kähler structures on $\mathfrak{M}$, $\mathfrak{M}_0$. This viewpoint was used for the study of $\mathfrak{M}$, $\mathfrak{M}_0$ in [44]. (**Caution**: The following notation is different from the original one. $K_{\mathbf{v}}$ and $G_{\mathbf{v}}$ were denoted by $G_{\mathbf{v}}$ and $G_{\mathbf{v}}^{\mathbb{C}}$ respectively in [44]. μ in (2.1.7) was denoted by $\mu_{\mathbb{C}}$ and the pair $(\mu_{\mathbb{R}}, \mu)$ was denoted by μ in [44].)

Put and fix hermitian inner products on V and W. They together with an orientation Ω induce a hermitian inner product and a quaternion structure on $\mathbf{M}$ ([44, p.370]). Let $K_{\mathbf{v}}$ be a compact Lie group defined by $K_{\mathbf{v}} = \prod_k \mathrm{U}(V_k)$. This is a maximal compact subgroup of $G_{\mathbf{v}}$, and acts on $\mathbf{M}$ preserving the hermitian and quaternion structures. The corresponding hyper-Kähler moment map vanishing at the origin decomposes into the complex part $\mu\colon \mathbf{M} \to \bigoplus_k \mathfrak{gl}(V_k) = \mathrm{L}(V,V)$ (defined in (2.1.7)) and the real part $\mu_{\mathbb{R}}\colon \mathbf{M} \to \bigoplus_k \mathfrak{u}(V_k)$, where

$$\mu_{\mathbb{R}}(B,i,j) = \frac{i}{2}\left(\sum_{h\in H: k=\mathrm{in}(h)} B_h B_h^\dagger - B_{\overline{h}}^\dagger B_{\overline{h}} + i_k i_k^\dagger - j_k^\dagger j_k\right)_k .$$

Proposition 2.4.1. (1) *A $G_{\mathbf{v}}$-orbit $[B,i,j]$ in $\mu^{-1}(0)$ intersects with $\mu_{\mathbb{R}}^{-1}(0)$ if and only if it is closed. The map*

$$\left(\mu_{\mathbb{R}}^{-1}(0)\cap\mu^{-1}(0)\right)/K_{\mathbf{v}} \to \mu^{-1}(0)/\!\!/G_{\mathbf{v}} = \mathfrak{M}_0(\mathbf{v},\mathbf{w})$$

is a homeomorphism.

(2) *Choose a parameter $\zeta_{\mathbb{R}} = (\zeta_{\mathbb{R}}^{(k)})_k \in \mathbb{R}^I$ so that $\zeta_{\mathbb{R}}^{(k)} \in \sqrt{-1}\mathbb{R}_{>0}$. Then a $G_{\mathbf{v}}$-orbit $[B,i,j]$ in $\mu^{-1}(0)$ intersects with $\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})$ if and only if it is stable. The map*

$$\left(\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})\cap\mu^{-1}(0)\right)/K_{\mathbf{v}} \to \mu^{-1}(0)^{\mathrm{s}}/G = \mathfrak{M}(\mathbf{v},\mathbf{w})$$

is a homeomorphism.

Proof. See [44, 3.1,3.2,3.5]. ☐

2.5. Suppose $V = (V_k)_{k\in I}$ is a collection of subspaces of $V' = (V'_k)_{k\in I}$ and $(B,i,j) \in \mu^{-1}(0) \subset \mathbf{M}(\mathbf{v},\mathbf{w})$ is given. We can extend (B,i,j) to $\mathbf{M}(\mathbf{v}',\mathbf{w})$ by letting it equal 0 on a complementary subspace of V in V'. This operation induces a natural morphism

$$\mu^{-1}(0) \text{ in } \mathbf{M}(\mathbf{v},\mathbf{w}) \to \mu^{-1}(0) \text{ in } \mathbf{M}(\mathbf{v}',\mathbf{w}), \tag{2.5.1}$$

where $\mathbf{v}' = \dim V'$. This induces a morphism

$$\mathfrak{M}_0(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}',\mathbf{w}). \tag{2.5.2}$$

Moreover, we also have a map

$$\mu^{-1}(0)\cap\mu_{\mathbb{R}}^{-1}(0) \text{ in } \mathbf{M}(\mathbf{v},\mathbf{w}) \to \mu^{-1}(0)\cap\mu_{\mathbb{R}}^{-1}(0) \text{ in } \mathbf{M}(\mathbf{v}',\mathbf{w}).$$

Thus closed $G_{\mathbf{v}}$-orbits in $\mu^{-1}(0) \subset \mathbf{M}(\mathbf{v},\mathbf{w})$ are mapped to closed $G_{\mathbf{v}'}$-orbits in $\mu^{-1}(0) \subset \mathbf{M}(\mathbf{v}',\mathbf{w})$ by Proposition 2.4.1(1).

The following lemma was stated in [45, p.529] without proof.

Lemma 2.5.3. *The morphism* (2.5.2) *is injective.*

Proof. Suppose x^1, $x^2 \in \mathfrak{M}_0(\mathbf{v},\mathbf{w})$ have the same image under (2.5.2). We choose representatives (B^1, i^1, j^1), (B^2, i^2, j^2) which have closed $G_{\mathbf{v}}$-orbits.

Let us define $S^a = (S^a_k)_{k\in I}$ $(a = 1, 2)$ by

$$S^a_k \overset{\text{def.}}{=} \operatorname{Im}\left(\sum_{\operatorname{in}(h)=k} \varepsilon(h) B^a_h + i^a_k\right).$$

Choose complementary subspaces T^a_k of S^a_k in V_k. We choose a 1-parameter subgroup $\lambda^a \colon \mathbb{C}^* \to G_{\mathbf{v}}$ as follows: $\lambda^a(t) = 1$ on S^a_k and $\lambda^a(t) = t^{-1}$ on T^a_k. Then the limit $\lambda^a(t)\cdot(B^a, i^a, j^a)$ exists and its restriction to T^a_k is 0. Since (B^a, i^a, j^a) has a closed orbit, we may assume that the restriction of (B^a, i^a, j^a) to T^a_k is 0. Note that S^a is a subspace of V by the construction.

Suppose that there exists $g' \in G_{\mathbf{v}'}$ such that $g'\cdot(B^1, i^1, j^1) = (B^2, i^2, j^2)$. We want to construct $g \in G_{\mathbf{v}}$ such that $g\cdot(B^1, i^1, j^1) = (B^2, i^2, j^2)$. Since we have $g'(S^1) = S^2$, the restriction of g' to S^1 is invertible. Let g be an extension of the restriction $g'|_{S^1}$ to V so that T^1 is mapped to T^2. Then $g \in G_{\mathbf{v}}$ maps (B^1, i^1, j^1) to (B^2, i^2, j^2). □

Hereafter, we consider $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ as a subset of $\mathfrak{M}_0(\mathbf{v}',\mathbf{w})$. It is clearly a closed subvariety. Let

$$\mathfrak{M}_0(\infty,\mathbf{w}) \overset{\text{def.}}{=} \bigcup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v},\mathbf{w}). \tag{2.5.4}$$

If the graph is of finite type, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ stabilizes at some $\mathbf{v}$ (see Proposition 2.6.3 and Lemma 2.9.4(2) below). This is *not* true in general. However, it presents no harm in this paper. We use $\mathfrak{M}_0(\infty,\mathbf{w})$ to simplify the notation, and do not need any structures on it. We can always work on individual $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$, not on $\mathfrak{M}_0(\infty,\mathbf{w})$.

Later, we shall also study $\mathfrak{M}(\mathbf{v},\mathbf{w})$ for various $\mathbf{v}$ simultaneously. We introduce the following notation:

$$\mathfrak{M}(\mathbf{w}) \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v},\mathbf{w}), \qquad \mathfrak{L}(\mathbf{w}) \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v},\mathbf{w}).$$

Note that there are no obvious morphisms between $\mathfrak{M}(\mathbf{v},\mathbf{w})$ and $\mathfrak{M}(\mathbf{v}',\mathbf{w})$ since the stability condition is *not* preserved under (2.5.1).

2.6. **Definition of $\mathfrak{M}_0^{\mathrm{reg}}$.** Let us introduce an open subset of $\mathfrak{M}_0$ (possibly empty):

$$\begin{aligned} \mathfrak{M}_0^{\mathrm{reg}} &\equiv \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}) \\ &\overset{\text{def.}}{=} \{\, [B,i,j] \in \mathfrak{M}_0 \mid (B,i,j) \text{ has the trivial stabilizer in } G \,\}. \end{aligned} \tag{2.6.1}$$

Proposition 2.6.2. *If $[B,i,j] \in \mathfrak{M}_0^{\mathrm{reg}}$, then it is stable. Moreover, π induces an isomorphism $\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}) \simeq \mathfrak{M}_0^{\mathrm{reg}}$.*

Proof. See [45, 3.24] or [44, 4.1(2)]. □

As in §2.5, we consider $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ as a subset of $\mathfrak{M}_0(\mathbf{v}',\mathbf{w})$ when $\mathbf{v}'-\mathbf{v} \in \sum_k \mathbb{Z}_{\ge 0}\alpha_k$. Then we have

Proposition 2.6.3. *If the graph is of type ADE, then*

$$\mathfrak{M}_0(\mathbf{v}',\mathbf{w}) = \bigcup_{\mathbf{v}} \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}),$$

where the summation runs over the set of $\mathbf{v}$ such that $\mathbf{v}'-\mathbf{v} \in \sum_k \mathbb{Z}_{\ge 0}\alpha_k$.

Proof. See [44, 6.7], [45, 3.28]. $\square$

Definition 2.6.4. We say a point $x \in \mathfrak{M}_0(\infty, \mathbf{w})$ is *regular* if it is contained in $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}, \mathbf{w})$ for some $\mathbf{v}$. The above proposition says that all points are regular if the graph is of type ADE. But this is *not* true in general (see [45, 10.10]).

2.7. **$G_{\mathbf{w}} \times \mathbb{C}^*$-action.** Let us define a $G_{\mathbf{w}} \times \mathbb{C}^*$-action on $\mathfrak{M}$ and $\mathfrak{M}_0$, where $G_{\mathbf{w}} = \prod_{k \in I} \mathrm{GL}(W_k)$. (**Caution**: We use the same notation $G_{\mathbf{v}}$ and $G_{\mathbf{w}}$, but their roles are totally different.)

The $G_{\mathbf{w}}$-action is simply defined by its natural action on $\mathbf{M} = \mathrm{E}(V, V) \oplus \mathrm{L}(W, V) \oplus \mathrm{L}(V, W)$. It preserves the equation $\varepsilon BB + ij = 0$ and commutes with the G-action given by (2.1.6). Hence it induces an action on $\mathfrak{M}$ and $\mathfrak{M}_0$.

The $\mathbb{C}^*$-action is slightly different from the one given in [45, 3.iv], and we need extra data. For each pair $k, l \in I$ such that $b' = -(\alpha_k, \alpha_l) \geq 1$, we introduce and fix a numbering $1, 2, \dots, b'$ on edges joining k and l. It induces a numbering $h_1, \dots, h_{b'} \in \Omega$, $\overline{h_1}, \dots, \overline{h_{b'}} \in \overline{\Omega}$ on oriented edges between k and l. Let us define $m\colon H \to \mathbb{Z}$ by

$$m(h_p) = b' + 1 - 2p, \quad m(\overline{h}_p) = -b' - 1 + 2p. \tag{2.7.1}$$

Then we define a $\mathbb{C}^*$-action on $\mathbf{M}$ by

$$B_h \mapsto t^{m(h)+1} B_h, \quad i \mapsto ti, \quad j \mapsto tj \qquad \text{for } t \in \mathbb{C}^*. \tag{2.7.2}$$

The equation $\varepsilon BB + ij = 0$ is preserved since the left hand side is multiplied by t^2. It commutes with the G-action and preserves the stability condition. Hence it induces a $\mathbb{C}^*$-action on $\mathfrak{M}$ and $\mathfrak{M}_0$. This $G_{\mathbf{w}} \times \mathbb{C}^*$-action makes the projective morphism $\pi\colon \mathfrak{M} \to \mathfrak{M}_0$ equivariant.

In order to distinguish this $G_{\mathbf{w}} \times \mathbb{C}^*$-action from the $G_{\mathbf{v}}$-action (2.1.6), we denote it as

$$(B, i, j) \mapsto h \star (B, i, j) \qquad (h \in G_{\mathbf{w}} \times \mathbb{C}^*).$$

2.8. **Notation for $\mathbb{C}^*$-action.** For an integer m, we define a $\mathbb{C}^*$-module structure on $\mathbb{C}$ by

$$t \cdot v \overset{\text{def.}}{=} t^m v, \qquad t \in \mathbb{C}^*, v \in \mathbb{C}, \tag{2.8.1}$$

and denote it by $L(m)$. For a $\mathbb{C}^*$-module V, we use the following notational convention:

$$q^m V \overset{\text{def.}}{=} L(m) \otimes V. \tag{2.8.2}$$

We use the same notation later when V is an element of $\mathbb{C}^*$-equivariant K-theory.

2.9. **Tautological bundles.** By the construction of $\mathfrak{M}$, we have a natural vector bundle

$$\mu^{-1}(0)^{\mathrm{s}} \times_G V_k \to \mathfrak{M}$$

associated with the principal G-bundle $\mu^{-1}(0)^{\mathrm{s}} \to \mathfrak{M}$. For abuse of notation, we denote it also by V_k. It naturally has the structure of a $\mathbb{C}^*$-equivariant vector bundle. Letting $G_{\mathbf{w}}$ act trivially, we make it a $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant vector bundle.

The vector space W_k is also considered as a $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant vector bundle, where $G_{\mathbf{w}}$ acts naturally and $\mathbb{C}^*$ acts trivially.

We call V_k and W_k *tautological bundles.*

We consider $\mathrm{E}(V,V)$, $\mathrm{L}(W,V)$, $\mathrm{L}(V,W)$ as vector bundles defined by the same formula as in (2.1.2). By the definition of tautological bundles, B, i, j can be considered as sections of those bundles. Those bundles naturally have structures of $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant vector bundles. But we modify the $\mathbb{C}^*$-action on $\mathrm{E}(V,V)$ by letting $t \in \mathbb{C}^*$ act by $t^{m(h)+1}$ on the component $\mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)})$. This makes B an equivariant section of $\mathrm{E}(V,V)$.

We consider the following $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant complex $C_k^\bullet \equiv C_k^\bullet(\mathbf{v},\mathbf{w})$ over $\mathfrak{M} \equiv \mathfrak{M}(\mathbf{v},\mathbf{w})$ (cf. [45, 4.2]):

(2.9.1)

$$C_k^\bullet \equiv C_k^\bullet(\mathbf{v},\mathbf{w}) : q^{-2} V_k \xrightarrow{\sigma_k} q^{-1}\left(\bigoplus_{l : k \neq l} [-\langle h_k, \alpha_l\rangle]_q V_l \oplus W_k\right) \xrightarrow{\tau_k} V_k,$$

where

$$\sigma_k = \bigoplus_{\mathrm{in}(h)=k} B_{\overline{h}} \oplus j_k, \qquad \tau_k = \sum_{\mathrm{in}(h)=k} \varepsilon(h) B_h + i_k.$$

Let us explain the factor $[-\langle h_k, \alpha_l\rangle]_q V_l$. Set $b' = -\langle h_k, \alpha_l\rangle$. Since the $\mathbb{C}^*$-action in (2.7.2) is defined so that

$$\bigoplus_{h : \substack{\mathrm{in}(h)=k \\ \mathrm{out}(h)=l}} \mathrm{Hom}(V_k, V_l) = \mathrm{Hom}(V_k, V_l)^{\oplus b'}$$

has weights $b', b'-2, \dots, 2-b'$, the above can be written as

$$\left(q^{b'} + q^{b'-2} + \cdots + q^{2-b'}\right) \mathrm{Hom}(V_k, V_l) = q[b']_q \mathrm{Hom}(V_k, V_l)$$

in the notation (2.8.2). By the same reason $C_k^\bullet$ is an equivariant complex.

We assign degree 0 to the middle term. (This complex is the complex in [45, 4.2] with a modification of the $G_{\mathbf{w}} \times \mathbb{C}^*$-action.)

Lemma 2.9.2. *Fix a point $[B,i,j]$ and consider $C_k^\bullet$ as a complex of vector spaces. Then σ_k is injective.*

Proof. See [45, p.530]. (Lemma 54 therein is a misprint of Lemma 5.2.) □

Note that τ_k is *not* surjective in general. In fact, the following notion will play a crucial role later. Let X be an irreducible component of $\pi^{-1}(x)$ for $x \in \mathfrak{M}_0$. Considering τ_k at a generic element $[B,i,j]$ of X, we set

(2.9.3) $$\varepsilon_k(X) \overset{\text{def.}}{=} \mathrm{codim}_{V_k} \mathrm{Im}\, \tau_k.$$

Lemma 2.9.4. (1) *Take and fix a point $[B,i,j] \in \mathfrak{M}(\mathbf{v},\mathbf{w})$. Let τ_k be as in (2.9.1). If $[B,i,j] \in \pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}))$, then we have*

(2.9.5) $$\mathrm{Im}\, \tau_k = V_k \qquad \text{for any } k \in I.$$

Moreover, the converse holds if we assume $\pi([B,i,j])$ is regular in the sense of Definition 2.6.4. Namely under this assumption, $[B,i,j] \in \pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}))$ if and only if (2.9.5) *holds.*

(2) *If $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}) \neq \emptyset$, then $\mathbf{w} - \mathbf{v}$ is dominant.*

Proof. (1) See [45, 4.7] for the first assertion. During the proof of [45, 7.2], we have shown the second assertion, using [45, 3.10] = Proposition 2.3.7.

(2) Consider the alternating sum of dimensions of the complex $C_k^\bullet$. It is equal to the alternating sum of dimensions of cohomology groups. It is nonnegative, if $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}) \neq \emptyset$ by Lemma 2.9.2 and (1). On the other hand, it is equal to

$$\sum_{h:\mathrm{out}(h)=k} \dim V_{\mathrm{in}(h)} + W_k - 2\dim V_k = \langle \mathbf{w}-\mathbf{v}, h_k\rangle.$$

Thus we have the assertion. □

3. Stratification of $\mathfrak{M}_0$

As was shown in [44, §6] and [45, 3.v], there exists a natural stratification of $\mathfrak{M}_0$ by conjugacy classes of stabilizers. A local topological structure of a neighborhood of a point in a stratum (e.g., the homology group of the fiber of π) was studied in [44, 6.10]. We give a refinement in this section. We define a slice to a stratum, and study a local structure as a complex analytic space. Our technique is based on work of Sjamaar-Lerman [50] in the symplectic geometry and hence our transversal slice may not be algebraic. It is desirable to have a purely algebraic construction of a transversal slice, as Maffei did in a special case [42].

We fix dimension vectors $\mathbf{v}$, $\mathbf{w}$ and denote $\mathbf{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}(\mathbf{v},\mathbf{w})$ by $\mathbf{M}$, $\mathfrak{M}$ in this section.

3.1. **Stratification.**

Definition 3.1.1 (cf. Sjamaar-Lerman [50])**.** For a subgroup $\widehat{G}$ of G denote by $\mathbf{M}_{(\widehat{G})}$ the set of all points in $\mathbf{M}$ whose stabilizer is conjugate to $\widehat{G}$. A point $[(B,i,j)] \in \mathfrak{M}_0$ is said to be of *G-orbit type* $(\widehat{G})$ if its representative (B,i,j) is in $\mathbf{M}_{(\widehat{G})}$. The set of all points of orbit type $(\widehat{G})$ is denoted by $(\mathfrak{M}_0)_{(\widehat{G})}$.

The stratum $(\mathfrak{M}_0)_{(1)}$ corresponding to the trivial subgroup 1 is $\mathfrak{M}_0^{\mathrm{reg}}$ by definition. We have the following decomposition of $\mathfrak{M}_0$:

$$\mathfrak{M}_0 = \bigcup_{(\widehat{G})} (\mathfrak{M}_0)_{(\widehat{G})},$$

where the summation runs over the set of all conjugacy classes of subgroups of G.

For a more detailed description of $(\mathfrak{M}_0)_{(\widehat{G})}$, see [44, 6.5], [45, 3.27].

3.2. **Local normal form of the moment map.**

Let us recall the local normal form of the moment map following Sjamaar-Lerman [50].

Take $x \in \mathfrak{M}_0$ and fix its representative $m = (B,i,j) \in \mu^{-1}(0)$. We suppose m has a closed G-orbit and satisfies $\mu_{\mathbb{R}}(m) = 0$ by Proposition 2.4.1(1). Let $\widehat{G}$ be the stabilizer of m. It is the complexification of the stabilizer in $K = \prod \mathrm{U}(V_k)$ (see, e.g., [51, 1.6]). Since $\mu(m) = 0$, the G-orbit $Gm = G/\widehat{G}$ through m is an isotropic submanifold of $\mathbf{M}$. Let $\widehat{\mathbf{M}}$ be the quotient vector space $(T_m Gm)^\omega / T_m Gm$, where $T_m Gm$ is the tangent space of the orbit Gm, and $(T_m Gm)^\omega$ is the symplectic perpendicular of $T_m Gm$ in $T_m\mathbf{M}$, i.e., $\{v \in T_m\mathbf{M} \mid \omega(v,w) = 0 \text{ for all } w \in T_m Gm\}$. This is naturally a symplectic vector space. A vector bundle $T(Gm)^\omega/T(Gm)$ over Gm is called the *symplectic normal bundle*. (In general, the symplectic normal bundle of an isotropic submanifold S is defined by TS^ω/TS.) It is isomorphic to $G \times_{\widehat{G}} \widehat{\mathbf{M}}$. (In [44, p.388], $\widehat{\mathbf{M}}$ was defined as the orthogonal complement of the quaternion vector subspace spanned by $T_m Km$ with respect to the Riemannian metric.) The action of $\widehat{G}$ on $\widehat{\mathbf{M}}$ preserves the induced symplectic structure on $\widehat{\mathbf{M}}$. Let $\widehat{\mu}\colon \widehat{\mathbf{M}} \to \widehat{\mathfrak{g}}^*$ be the corresponding moment map vanishing at the origin.

We choose an $\operatorname{Ad}(\widehat{G})$-invariant splitting $\mathfrak{g} = \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}^{\perp}$ and its dual splitting $\mathfrak{g}^* = \widehat{\mathfrak{g}}^* \oplus \widehat{\mathfrak{g}}^{\perp *}$. Let us consider the natural action of $\widehat{G}$ on the product $T^*G \times \widehat{\mathbf{M}} = G \times \mathfrak{g}^* \times \widehat{\mathbf{M}}$. With the natural symplectic structure on $T^*G = G \times \mathfrak{g}^*$, we have the moment map

$$\begin{array}{rcl} \widetilde{\mu}\colon G \times \mathfrak{g}^* \times \widehat{\mathbf{M}} & \to & \widehat{\mathfrak{g}}^* \\ (g, \xi, \widehat{m}) & \mapsto & -\operatorname{pr}\xi + \widehat{\mu}(\widehat{m}), \end{array}$$

where $\operatorname{pr}\xi$ is the projection of $\xi \in \mathfrak{g}^*$ to $\widehat{\mathfrak{g}}^*$. Zero is a regular value of $\widetilde{\mu}$, hence the symplectic quotient $\widetilde{\mu}^{-1}(0)/\widehat{G}$ is a symplectic manifold. It can be identified with $G \times_{\widehat{G}} \left(\widehat{\mathfrak{g}}^{\perp *} \times \widehat{\mathbf{M}}\right)$ via the map

$$G \times_{\widehat{G}} \left(\widehat{\mathfrak{g}}^{\perp *} \times \widehat{\mathbf{M}}\right) \ni \widehat{G} \cdot (g, \xi, \widehat{m}) \longmapsto \widehat{G} \cdot (g, \xi + \widehat{\mu}(\widehat{m}), \widehat{m}) \in \widetilde{\mu}^{-1}(0)/\widehat{G}.$$

The embedding $G/\widehat{G}$ into $G \times_{\widehat{G}} \left(\widehat{\mathfrak{g}}^{\perp *} \times \widehat{\mathbf{M}}\right)$ is isotropic and its symplectic normal bundle is $G \times_{\widehat{G}} \widehat{\mathbf{M}}$. Thus two embeddings of $Gm \cong G/\widehat{G}$, one into $\mathbf{M}$ and the other into $G \times_{\widehat{G}} \left(\widehat{\mathfrak{g}}^{\perp *} \times \widehat{\mathbf{M}}\right)$, have the isomorphic symplectic normal bundles.

The G-equivariant version of Darboux-Moser-Weinstein's isotropic embedding theorem (a special case of [50, 2.2]) says the following:

Lemma 3.2.1. *A neighborhood of Gm (in $\mathbf{M}$) is G-equivalently symplectomorphic to a neighborhood of $G/\widehat{G}$ embedded as the zero section of $G \times_{\widehat{G}} \left(\widehat{\mathfrak{g}}^{\perp *} \times \widehat{\mathbf{M}}\right)$ with the G-moment map given by the formula*

$$\mu\left(\widehat{G} \cdot (g, \xi, \widehat{m})\right) = \operatorname{Ad}^*(g)\left(\xi + \widehat{\mu}(\widehat{m})\right).$$

(Here 'symplectomorphic' means that there exists a biholomorphism intertwining symplectic structures.)

Note that Sjamaar-Lerman worked on a *real* symplectic manifold with a *compact* Lie group action. Thus we need to take care when applying their result to our situation. Darboux-Moser-Weinstein's theorem is based on the inverse function theorem, which we have both in the category of C^∞-manifolds and in that of complex manifolds. A problem is that the domain of the symplectomorphism may not be chosen so that it covers the whole Gm as it is noncompact. We can overcome this problem by taking a symplectomorphism defined in a neighborhood of the compact orbit Km first, and then extending it to a neighborhood of Gm, as explained in the next three paragraphs. This approach is based on a result in [51].

A subset A of a G-space X is called *orbitally convex* with respect to the G-action if it is invariant under K (= maximal compact subgroup of G) and for all $x \in A$ and all $\xi \in \mathfrak{k}$ we have that both x and $\exp(\sqrt{-1}\xi)x$ in A implies that $\exp(\sqrt{-1}t\xi)x \in A$ for all $t \in [0,1]$. By [51, 1.4], if X and Y are complex manifolds with G-actions, and if A is an orbitally convex open subset of X and $f\colon A \to Y$ is a K-equivariant holomorphic map, then f can be uniquely extended to a G-equivariant holomorphic map.

Suppose that X is a Kähler manifold with a (real) moment map $\mu_{\mathbb{R}}\colon X \to \mathfrak{k}^*$ and that $x \in X$ is a point such that $\mu_{\mathbb{R}}(x)$ is fixed under the coadjoint action of K.

Then [51, Claim 1.13] says that the compact orbit Kx possesses a basis of orbitally convex open neighborhoods.

In our situation, we have a Kähler metric (§2.4) and we have assumed $\mu_{\mathbb{R}}(m) = 0$. Thus Km possesses a basis of orbitally convex open neighborhoods, and we have Lemma 3.2.1.

Now we want to study local structures of $\mathfrak{M}_0$, $\mathfrak{M}$ using Lemma 3.2.1. First the equation $\mu = 0$ implies $\xi = 0$, $\widehat{\mu}(\widehat{m}) = 0$. Thus $\mathfrak{M}_0$ and $\mathfrak{M}$ are locally isomorphic to 'quotients' of $G \times_{\widehat{G}} (\{0\} \times \widehat{\mu}^{-1}(0))$ by G, i.e., 'quotients' of $\widehat{\mu}^{-1}(0)$ by $\widehat{G}$. Here the 'quotients' are taken in the sense of the geometric invariant theory. Following Proposition 2.3.2(1), we say a point $\widehat{m} \in \widehat{\mu}^{-1}(0)$ is *stable* if the closure of $\widehat{G} \cdot (m, z)$ does not intersect with the zero section of $\widehat{\mu}^{-1}(0) \times \mathbb{C}$ for $z \neq 0$. Here we lift the $\widehat{G}$-action to the trivial line bundle $\widehat{\mu}^{-1}(0) \times \mathbb{C}$ by $\widehat{g} \cdot (\widehat{m}, z) = (\widehat{g} \cdot \widehat{m}, \chi(g)^{-1} z)$, where χ is the restriction of the one-parameter subgroup used in §2.2. Let $\widehat{\mu}^{-1}(0)^{\mathrm{s}}$ be the set of stable points. As in §2.3, we have a morphism $\widehat{\mu}^{-1}(0)^{\mathrm{s}}/\widehat{G} \to \widehat{\mu}^{-1}(0)/\!\!/\widehat{G}$, which we denote by $\widehat{\pi}$. By [51, Proposition 2.7], we may assume that the neighborhood of Gm in Lemma 3.2.1 is *saturated*, i.e., the closure of the G-orbit of a point in the neighborhood is contained in the neighborhood. Thus under the symplectomorphism in Lemma 3.2.1, (i) closed G-orbits are mapped to closed $\widehat{G}$-orbits, and (ii) the stability conditions are interchanged.

Proposition 3.2.2. *There exist a neighborhood U (resp. $U^\perp$) of $x \in \mathfrak{M}_0$ (resp. $0 \in \widehat{\mu}^{-1}(0)/\!\!/\widehat{G}$) and biholomorphic maps $\Phi\colon U \to U^\perp$, $\widetilde{\Phi}\colon \pi^{-1}(U) \to \widehat{\pi}^{-1}(U^\perp)$ such that the following diagram commutes*:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\cong]{\widetilde{\Phi}} & \widehat{\pi}^{-1}(U^\perp) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\widehat{\pi}} \\ U & \xrightarrow[\cong]{\Phi} & U^\perp \end{array}$$

In particular, $\pi^{-1}(x) = \mathfrak{M}_x$ is biholomorphic to $\widehat{\pi}^{-1}(0)$.

Furthermore, under Φ, a stratum $(\mathfrak{M}_0)_{(H)}$ of $\mathfrak{M}_0$ is mapped to a stratum $\left(\widehat{\mu}^{-1}(0)/\!\!/\widehat{G}\right)_{(H)}$, which is defined as in Definition 3.1.1. (If $(\mathfrak{M}_0)_{(H)}$ intersects with U, then H is conjugate to a subgroup of $\widehat{G}$.)

The above discussion shows Proposition 3.2.2 except for the last assertion. The last assertion follows from the argument in [50, p.386].

3.3. **Slice.** By [44, p.391], we have a $\widehat{G}$-invariant splitting $\widehat{\mathbf{M}} = T \times T^\perp$, where T is the tangent space $T_x(\mathfrak{M}_0)_{(\widehat{G})}$ of the stratum containing x, and $\widehat{G}$ acts trivially on T. Thus we have

$$\widehat{\mu}^{-1}(0)/\!\!/\widehat{G} \cong T \times \left(T^\perp \cap \widehat{\mu}^{-1}(0)\right)/\!\!/\widehat{G},$$
$$\widehat{\mu}^{-1}(0)^{\mathrm{s}}/\widehat{G} \cong T \times \left(T^\perp \cap \widehat{\mu}^{-1}(0)^{\mathrm{s}}\right)/\widehat{G}.$$

Furthermore, it was proved that $\left(T^\perp \cap \widehat{\mu}^{-1}(0)\right)/\!\!/\widehat{G}$ and $\left(T^\perp \cap \widehat{\mu}^{-1}(0)^{\mathrm{s}}\right)/\widehat{G}$ are quiver varieties associated with a certain graph possibly different from the original one, and possibly with edge loops. Replacing $U^\perp$ if necessary, we may assume that

$U^\perp$ is a product of a neighborhood U_T of 0 in T and $U_{\mathfrak{S}}$ of 0 in $\left(T^\perp \cap \widehat{\mu}^{-1}(0)\right) /\!\!/ \widehat{G}$. We define a *transversal slice* to $(\mathfrak{M}_0)_{(\widehat{G})}$ at x as

$$\mathfrak{S} \overset{\text{def.}}{=} \Phi^{-1}\left(U^\perp \cap \left(\{0\} \times \left(T^\perp \cap \widehat{\mu}^{-1}(0)\right) /\!\!/ \widehat{G}\right)\right) = \Phi^{-1}\left(\{0\} \times U_{\mathfrak{S}}\right).$$

Since Φ is a local biholomorphism, this slice $\mathfrak{S}$ satisfies the properties in [13, 3.2.19], i.e., there exists a biholomorphism

$$\left(U \cap (\mathfrak{M}_0)_{(\widehat{G})}\right) \times \mathfrak{S} \xrightarrow{\cong} U$$

which induces biholomorphisms between factors

$$\{x\} \times \mathfrak{S} \xrightarrow{\cong} \mathfrak{S}, \qquad \left(U \cap (\mathfrak{M}_0)_{(\widehat{G})}\right) \times \{x\} \xrightarrow{\cong} \left(U \cap (\mathfrak{M}_0)_{(\widehat{G})}\right).$$

Remark 3.3.1. Our construction gives a slice to a stratum in

$$\mathfrak{M}_{(0,\zeta_{\mathbb{C}})} = \mu^{-1}(-\zeta_{\mathbb{C}}) /\!\!/ G_{\mathbf{v}}$$

for general $\zeta_{\mathbb{C}}$. (See [44, p.371 and Theorem 3.1] for the definition of $\mathfrak{M}_{(0,\zeta_{\mathbb{C}})}$.) In particular, the fiber $\pi^{-1}(x)$ of $\pi\colon \mathfrak{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} \to \mathfrak{M}_{(0,\zeta_{\mathbb{C}})}$ is isomorphic to the fiber of $\left(T^\perp \cap \widehat{\mu}^{-1}(0)^{\mathrm{s}}\right)/\widehat{G} \to \left(T^\perp \cap \widehat{\mu}^{-1}(0)\right) /\!\!/ \widehat{G}$ at 0. This is a refinement of [44, 6.10], where an isomorphism between homology groups was obtained. We also remark that this gives a proof of smallness of

$$\pi\colon \bigsqcup_{\zeta_{\mathbb{C}}} \mathfrak{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} \to \bigsqcup_{\zeta_{\mathbb{C}}} \mathfrak{M}_{(0,\zeta_{\mathbb{C}})}$$

which was observed by Lusztig when $\mathfrak{g}$ is of type ADE [40]. An essential point is, as remarked in [44, 6.11], that $\left(T^\perp \cap \widehat{\mu}^{-1}(0)^{\mathrm{s}}\right)/\widehat{G}$ is diffeomorphic to an affine algebraic variety, and its homology group vanishes for degree greater than its complex dimension.

For our application, we only need the case when x is *regular*, i.e., $x \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ for some $\mathbf{v}^0$. Then, by [44, p.392], $\left(T^\perp \cap \widehat{\mu}^{-1}(0)\right) /\!\!/ \widehat{G}$ and $\left(T^\perp \cap \widehat{\mu}^{-1}(0)\right)^{\mathrm{s}}/\widehat{G}$ are isomorphic to the quiver varieties $\mathfrak{M}_0(\mathbf{v}_s, \mathbf{w}_s)$ and $\mathfrak{M}(\mathbf{v}_s, \mathbf{w}_s)$, associated with the original graph with dimension vector

$$\mathbf{v}_s = \mathbf{v} - \mathbf{v}^0, \qquad \mathbf{w}_s = \mathbf{w} - \mathbf{C}\mathbf{v}^0,$$

where

$$\mathbf{C}\mathbf{v}^0 = \sum_{k\in I} \left(2v_k^0 - a_{kl}\, v_l^0\right)\Lambda_k \qquad \text{if } \mathbf{v}^0 = \sum_{k\in I} v_k^0 \alpha_k$$

in Convention 2.1.4.

Theorem 3.3.2. *Suppose that $x \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ as above. Then there exist neighborhoods U, U_T, $U_{\mathfrak{S}}$ of $x \in \mathfrak{M}_0 = \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, $0 \in T$, $0 \in \mathfrak{M}_0(\mathbf{v}_s, \mathbf{w}_s)$ respectively, and biholomorphic maps $U \to U_T \times U_{\mathfrak{S}}$, $\pi^{-1}(U) \to U_T \times \pi^{-1}(U_{\mathfrak{S}})$ such that the following diagram commutes*:

$$\begin{array}{ccccc}
\mathfrak{M} \supset \pi^{-1}(U) & \xrightarrow[\cong]{} & U_T \times \pi^{-1}(U_{\mathfrak{S}}) & \subset & T \times \mathfrak{M}(\mathbf{v}_s, \mathbf{w}_s) \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\times\pi} & & \\
\mathfrak{M}_0 \supset U & \xrightarrow[\cong]{} & U_T \times U_{\mathfrak{S}} & \subset & T \times \mathfrak{M}_0(\mathbf{v}_s, \mathbf{w}_s)
\end{array}$$

In particular, $\pi^{-1}(x) = \mathfrak{M}_x$ is biholomorphic to $\mathfrak{L}(\mathbf{v}_s, \mathbf{w}_s)$.

Furthermore, a stratum of $\mathfrak{M}_0$ *is mapped to a product of* U_T *and a stratum of* $\mathfrak{M}_0(\mathbf{v}_s, \mathbf{w}_s)$.

Remark 3.3.3. Suppose that A is a subgroup of $G_{\mathbf{w}} \times \mathbb{C}^*$ fixing x. Since the A-action commutes with the G-action, $\widehat{\mathbf{M}}$ has an A-action. The above construction can be made A-equivariant. In particular, the diagram in Theorem 3.3.2 can be restricted to a diagram for A-fixed point sets.

4. Fixed point subvariety

Let A be an abelian reductive subgroup of $G_{\mathbf{w}} \times \mathbb{C}^*$. In this section, we study the A-fixed point subvarieties $\mathfrak{M}(\mathbf{v},\mathbf{w})^A$ and $\mathfrak{M}_0(\mathbf{v},\mathbf{w})^A$ of $\mathfrak{M}(\mathbf{v},\mathbf{w})$ and $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$.

4.1. **A homomorphism attached to a component of $\mathfrak{M}(\mathbf{v},\mathbf{w})^A$.** Suppose that $x \in \mathfrak{M}(\mathbf{v},\mathbf{w})^A$ is fixed by A. Take a representative $(B,i,j) \in \mu^{-1}(0)^{\mathrm{s}}$ of x. For every $a \in A$, there exists $\rho(a) \in G_{\mathbf{v}}$ such that

$$a \star (B,i,j) = \rho(a)^{-1} \cdot (B,i,j), \tag{4.1.1}$$

where the left hand side is the action defined in (2.7.2) and the right hand side is the action defined in (2.1.6). By the freeness of the $G_{\mathbf{v}}$-action on $\mu^{-1}(0)^{\mathrm{s}}$ (see Proposition 2.3.2), $\rho(a)$ is uniquely determined by a. In particular, the map $a \mapsto \rho(a)$ is a homomorphism.

Let $\mathfrak{M}(\rho) \subset \mathfrak{M}(\mathbf{v},\mathbf{w})^A$ be the set of fixed points x such that (4.1.1) holds for some representative (B,i,j) of x. Note that $\mathfrak{M}(\rho)$ depends only on the $G_{\mathbf{v}}$-conjugacy class of ρ. Since the $G_{\mathbf{v}}$-conjugacy class of ρ is locally constant on $\mathfrak{M}(\mathbf{v},\mathbf{w})^A$, $\mathfrak{M}(\rho)$ is a union of connected components of $\mathfrak{M}(\mathbf{v},\mathbf{w})^A$. Later we show that $\mathfrak{M}(\rho)$ is connected under some assumptions (see Theorem 5.5.6). As in Proposition 2.3.8, we have

Proposition 4.1.2. $\mathfrak{M}(\rho)$ *is homotopic to* $\mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{v},\mathbf{w})$.

We regard V as an A-module via ρ and consider the weight space which corresponds to $\lambda \in \operatorname{Hom}(A, \mathbb{C}^*)$:

$$V(\lambda) \overset{\text{def.}}{=} \{v \in V \mid \rho(a)\cdot v = \lambda(a)v\}.$$

We denote by $V_k(\lambda)$ the component of $V(\lambda)$ at the vertex k. We have $V = \bigoplus_\lambda V(\lambda)$. We regard W as an A-module via the composition

$$A \hookrightarrow G_{\mathbf{w}} \times \mathbb{C}^* \xrightarrow{\text{projection}} G_{\mathbf{w}}.$$

We also have the weight space decomposition $W = \bigoplus_\lambda W(\lambda)$, $W_k = \bigoplus_\lambda W_k(\lambda)$. We denote by q the composition

$$A \hookrightarrow G_{\mathbf{w}} \times \mathbb{C}^* \xrightarrow{\text{projection}} \mathbb{C}^*.$$

Then (4.1.1) is equivalent to

$$\begin{gathered} B_h(V_{\mathrm{out}(h)}(\lambda)) \subset V_{\mathrm{in}(h)}(q^{-m(h)-1}\lambda), \\ i_k(W_k(\lambda)) \subset V_k(q^{-1}\lambda), \quad j_k(V_k(\lambda)) \subset W_k(q^{-1}\lambda), \end{gathered} \tag{4.1.3}$$

where $m(h)$ is as in (2.7.1).

Lemma 4.1.4. *If* $V_k(\lambda) \neq 0$, *then* $W_l(q^n\lambda) \neq 0$ *for some* n *and* $l \in I$.

Proof. Consider λ satisfying $W_l(q^n\lambda) = 0$ for any $l \in I$, $n \in \mathbb{Z}$. If we set

$$S_k \overset{\text{def.}}{=} \bigoplus_{\lambda \text{ as above}} V_k(\lambda),$$

then $S = (S_k)_{k\in I}$ is B-invariant and contained in $\operatorname{Ker} j$ by (4.1.3). Thus we have $S_k = 0$ by the stability condition. □

The restriction of tautological bundles V_k, W_k to $\mathfrak{M}(\rho)$ are bundles of A-modules. We have the weight decomposition $V_k = \bigoplus V_k(\lambda)$, $W_k = \bigoplus W_k(\lambda)$. We consider $V_k(\lambda)$, $W_k(\lambda)$ as vector bundles over $\mathfrak{M}(\rho)$.

Similarly, the restriction of the complex $C_k^\bullet$ in (2.9.1) decomposes as $C_k^\bullet = \bigoplus_\lambda C_{k,\lambda}^\bullet$, where

(4.1.5)

$$C_{k,\lambda}^\bullet \equiv C_{k,\lambda}^\bullet(\rho) : V_k(q^2\lambda) \xrightarrow{\sigma_{k,\lambda}} \bigoplus_{h:\mathrm{in}(h)=k} V_{\mathrm{out}(h)}(q^{m(h)+1}\lambda) \oplus W_k(q\lambda) \xrightarrow{\tau_{k,\lambda}} V_k(\lambda).$$

Here $\sigma_{k,\lambda}$, $\tau_{k,\lambda}$ are restrictions of σ_k, τ_k. When we want to emphasize that this is a complex over $\mathfrak{M}(\rho)$, we denote this by $C_{k,\lambda}^\bullet(\rho)$.

The tangent space of $\mathfrak{M}(\rho)$ at $[B, i, j]$ is the A-fixed part of the tangent space of $\mathfrak{M}$. Since the latter is the middle cohomology group of (2.1.8), the former is the middle cohomology group of the complex

$$\bigoplus_{\lambda,k} \operatorname{End}(V_k(\lambda)) \longrightarrow \begin{matrix} \bigoplus_{\lambda,h} \operatorname{Hom}\left(V_{\mathrm{out}(h)}(\lambda), V_{\mathrm{in}(h)}(q^{-m(h)-1}\lambda)\right) \\ \oplus \\ \bigoplus_{\lambda,k} \operatorname{Hom}\left(W_k(\lambda), V_k(q^{-1}\lambda)\right) \\ \oplus \\ \bigoplus_{\lambda,k} \operatorname{Hom}\left(V_k(\lambda), W_k(q^{-1}\lambda)\right) \end{matrix} \longrightarrow \bigoplus_{\lambda,k} \operatorname{Hom}\left(V_k(\lambda), V_k(q^{-2}\lambda)\right),$$

where the differentials are the restrictions of ι, $d\mu$ in (2.1.8). Those restrictions are injective and surjective respectively by Proposition 2.3.2. Hence we have the following dimension formula:

$$\begin{aligned} &\dim \mathfrak{M}(\rho) \\ &= \sum_\lambda \Big[\sum_h \dim V_{\mathrm{out}(h)}(\lambda) \dim V_{\mathrm{in}(h)}(q^{-m(h)-1}\lambda) \\ &\qquad + \sum_k \dim W_k(\lambda) \left(\dim V_k(q^{-1}\lambda) + \dim V_k(q\lambda)\right) \\ &\qquad\qquad - \dim V_k(\lambda)^2 - \dim V_k(\lambda) \dim V_k(q^{-2}\lambda)\Big]. \end{aligned} \tag{4.1.6}$$

Recall that we have an isomorphism $\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}) \cong \mathfrak{M}_0^{\mathrm{reg}}$ (Proposition 2.6.2). Let

$$\mathfrak{M}_0^{\mathrm{reg}}(\rho) \overset{\text{def.}}{=} \pi\left(\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}) \cap \mathfrak{M}(\rho)\right) = \pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}) \cap \pi\left(\mathfrak{M}(\rho)\right). \tag{4.1.7}$$

By definition, $\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}(\rho)) = \pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}) \cap \mathfrak{M}(\rho)$ is an open subvariety of $\mathfrak{M}(\rho)$ which is isomorphic to $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ under π.

4.2. **A sufficient condition for $\mathfrak{M}_0^A = \{0\}$.** Let $a = (s, \varepsilon)$ be a semisimple element in $G_{\mathbf{w}} \times \mathbb{C}^*$ and let A be the Zariski closure of $\{a^n \mid n \in \mathbb{Z}\}$.

Definition 4.2.1. We say a is *generic* if $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})^A = \{0\}$ for any $\mathbf{v}$. (This condition depends on $\mathbf{w}$.)

Proposition 4.2.2. *Assume that there is at most one edge joining two vertices of I, and that*

$$\lambda/\lambda' \notin \{\varepsilon^n \mid n \in \mathbb{Z} \setminus \{0\}\}$$

for any pair of eigenvalues of $s \in G_{\mathbf{w}}$. (The condition for the special case $\lambda = \lambda'$ implies that ε is not a root of unity.) Then $a = (s, \varepsilon)$ is generic.

Proof. We prove $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})^A = \{0\}$ by induction on $\mathbf{v}$. The assertion is trivial when $\mathbf{v} = 0$.

Take a point in $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})^A$ and its representative (B, i, j). As in (4.1.1), there exists $g \in G_{\mathbf{v}}$ such that

$$a \star (B, i, j) = g \cdot (B, i, j).$$

We decompose V into eigenspaces of g:

$$V = \bigoplus V(\lambda), \qquad \text{where } V(\lambda) \overset{\text{def.}}{=} \{v \in V \mid g \cdot v = \lambda(a) v\}.$$

We also decompose W into eigenspaces of s as $\bigoplus W(\lambda)$. Then (4.1.3) holds where q is replaced by ε.

Choose and fix an eigenvalue μ of s. First suppose $V(\varepsilon^n \mu) \neq 0$ for some n. Let

$$n_0 \overset{\text{def.}}{=} \max\{n \mid V(\varepsilon^n \mu) \neq 0\}.$$

Since ε is not a root of unity, we have $\varepsilon^n \mu \neq \varepsilon^m \mu$ for $m \neq n$. Hence the above n_0 is well defined. By (4.1.3) (and $m(h) = 0$ from the assumption), we have $\operatorname{Im} B_h \cap V(\varepsilon^{n_0} \mu) = 0$. By the assumption, we have $W(\varepsilon^{n_0+1}\mu) = 0$, and hence $\operatorname{Im} i_k \cap V(\varepsilon^{n_0}\mu) = 0$ again by (4.1.3). Then we may assume the restriction of (B, i, j) to $V(\varepsilon^{n_0}\mu)$ is 0 as in the proof of Lemma 2.5.3.

Thus the data (B, i, j) is defined on the smaller subspace $V \ominus V(\varepsilon^{n_0}\mu)$. Thus $(B, i, j) = 0$ by the induction hypothesis.

If $V(\varepsilon^n \mu) = 0$ for any n, we replace μ. If we can find a μ' so that $V(\varepsilon^n \mu') \neq 0$ for some n, we are done. Otherwise, we have $V(\varepsilon^n \mu) = 0$ for any n, μ, and we have $i = j = 0$ by (4.1.3). Then we choose μ, which may not be an eigenvalue of s, so that $V(\mu) \neq 0$ and repeat the above argument. (This is possible since we may assume $V \neq 0$.) We have $\operatorname{Im} B_h \cap V(\varepsilon^{n_0}\mu) = 0$ and the data B is defined on the smaller subspace $V \ominus V(\varepsilon^{n_0}\mu)$ as above. □

5. Hecke correspondence and induction of quiver varieties

5.1. **Hecke correspondence.** Take dimension vectors $\mathbf{w}$, $\mathbf{v}^1$, $\mathbf{v}^2$ such that $\mathbf{v}^2 = \mathbf{v}^1 + \alpha_k$. Choose collections of vector spaces W, V^1, V^2, with $\dim W = \mathbf{w}$, $\dim V^a = \mathbf{v}^a$.

Let us consider the product $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$. We denote by V_k^1 (resp. V_k^2) the vector bundle $V_k \boxtimes \mathcal{O}_{\mathfrak{M}(\mathbf{v}^2, \mathbf{w})}$ (resp. $\mathcal{O}_{\mathfrak{M}(\mathbf{v}^1, \mathbf{w})} \boxtimes V_k$). A point in $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$ is denoted by $([B^1, i^1, j^1], [B^2, i^2, j^2])$. We regard B^a, i^a, j^a $(a = 1, 2)$ as homomorphisms between tautological bundles.

We define a three-term sequence of vector bundles over $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$ by

(5.1.1)
$$\mathrm{L}(V^1, V^2) \xrightarrow{\sigma} q\,\mathrm{E}(V^1, V^2) \oplus q\,\mathrm{L}(W, V^2) \oplus q\,\mathrm{L}(V^1, W) \xrightarrow{\tau} q^2\,\mathrm{L}(V^1, V^2) \oplus q^2 \mathcal{O},$$

where

$$\sigma(\xi) = (B^2\xi - \xi B^1) \oplus (-\xi i^1) \oplus j^2\xi,$$
$$\tau(C \oplus a \oplus b) = (\varepsilon B^2 C + \varepsilon C B^1 + i^2 b + a j^1) \oplus \left(\operatorname{tr}(i^1 b) + \operatorname{tr}(a j^2)\right).$$

This is a complex, that is, $\tau\sigma = 0$, thanks to the equations $\varepsilon BB + ij = 0$ and $\operatorname{tr}(i^1 j^2 \xi) = \operatorname{tr}(\xi i^1 j^2)$. Moreover, it is an equivariant complex with respect to the $G_{\mathbf{w}} \times \mathbb{C}^*$-action.

By [45, 5.2], σ is injective and τ is surjective. Hence the quotient $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ is a $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant vector bundle. Let us define an equivariant section s of $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ by

$$s = \left(0 \oplus (-i^2) \oplus j^1\right) \bmod \operatorname{Im}\sigma, \tag{5.1.2}$$

where $\tau(s) = 0$ follows from $\varepsilon BB + ij = 0$ and $\operatorname{tr}(\varepsilon B^1 B^1) = \operatorname{tr}(\varepsilon B^2 B^2) = 0$. The point $([B^1, i^1, j^1], [B^2, i^2, j^2])$ is contained in the zero locus $Z(s)$ of s if and only if there exists $\xi \in \mathrm{L}(V^1, V^2)$ such that

$$\xi B^1 = B^2 \xi, \quad \xi i^1 = i^2, \quad j^1 = j^2 \xi. \tag{5.1.3}$$

Moreover, $\operatorname{Ker}\xi$ is zero by the stability condition for B^2. Hence $\operatorname{Im}\xi$ is a subspace of V^2 with dimension $\mathbf{v}^1$ which is B^2-invariant and contains $\operatorname{Im} i^2$. Moreover, such ξ is unique if we fix representatives (B^1, i^1, j^1) and (B^2, i^2, j^2). Hence we have an isomorphism between $Z(s)$ and the variety of all pairs (B, i, j) and S (modulo $G_{\mathbf{v}^2}$-action) such that

(a) $(B, i, j) \in \mu^{-1}(0)$ is stable, and
(b) S is a B-invariant subspace containing the image of i with $\dim S = \mathbf{v}^1 = \mathbf{v}^2 - \alpha_k$.

Definition 5.1.4. We call $Z(s)$ the *Hecke correspondence*, and denote it by $\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$. It is a $G_{\mathbf{w}} \times \mathbb{C}^*$-invariant closed subvariety.

Introducing a connection ∇ on $\operatorname{Ker}\tau/\operatorname{Im}\sigma$, we consider the differential

$$\nabla s \colon T\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \oplus T\mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \to \operatorname{Ker}\tau/\operatorname{Im}\sigma$$

of the section s. Its restriction to $Z(s) = \mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$ is independent of the connection. By [45, 5.7], the differential ∇s is surjective over $\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$. Hence, $\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$ is nonsingular.

By the definition, the quotient V_k^2/V_k^1 defines a line bundle over $\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$.

5.2. **Hecke correspondence and fixed point subvariety.** Let A be as in §4 and let $\mathfrak{M}(\rho)$ be as in §4.1 for $\rho \in \operatorname{Hom}(A, G_{\mathbf{v}})$.

Let us consider the intersection $(\mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A) \cap \mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$. It decomposes as

$$(\mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A) \cap \mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) = \bigsqcup_{\rho^1, \rho^2} (\mathfrak{M}(\rho^1) \times \mathfrak{M}(\rho^2)) \cap \mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}).$$

Take a point $([B^1, i^1, j^1], [B^2, i^2, j^2]) \in (\mathfrak{M}(\rho^1) \times \mathfrak{M}(\rho^2)) \cap \mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$. Then we have

$$a \star (B^p, i^p, j^p) = \rho^p(a)^{-1} \cdot (B^p, i^p, j^p), \qquad a \in A \quad (p = 1, 2),$$

and there exists $\xi \in \mathrm{L}(V^1, V^2)$ such that

$$\xi B^1 = B^2 \xi, \quad \xi i^1 = i^2, \quad j^1 = j^2 \xi.$$

By the uniqueness of ξ, we must have $\rho^2(a)\xi = \xi\rho^1(a)$, that is, $\xi\colon V^1 \to V^2$ is A-equivariant. Since ξ is injective, V^1 can be considered as an A-submodule of V^2.

If $V^1 = \bigoplus V^1(\lambda)$ and $V^2 = \bigoplus V^2(\lambda)$ are the weight decompositions, then there exists λ_0 such that

(a) $V^1_l(\lambda) \xrightarrow{\xi} V^2_l(\lambda)$ is an isomorphism if $\lambda \neq \lambda_0$ or $l \neq k$,
(b) $V^1_k(\lambda_0) \xrightarrow{\xi} V^2_k(\lambda_0)$ is a codimension 1 embedding.

5.3. We introduce a generalization of the Hecke correspondence. Let us define $\mathfrak{P}^{(n)}_k(\mathbf{v},\mathbf{w})$ as

$$\mathfrak{P}^{(n)}_k(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \{(B,i,j,S) \mid (B,i,j) \in \mathbf{M}(\mathbf{v},\mathbf{w}),\ S \subset V \text{ as below}\}/G_{\mathbf{v}}, \tag{5.3.1}$$

(a) $(B,i,j) \in \mu^{-1}(0)^{\mathrm{s}}$,
(b) S is a B-invariant subspace containing the image of i with $\dim S = \mathbf{v} - n\alpha_k$.

For $n = 1$, it is just $\mathfrak{P}_k(\mathbf{v},\mathbf{w})$. We consider $\mathfrak{P}^{(n)}_k(\mathbf{v},\mathbf{w})$ as a closed subvariety of $\mathfrak{M}(\mathbf{v} - n\alpha_k,\mathbf{w}) \times \mathfrak{M}(\mathbf{v},\mathbf{w})$ by setting

$$(B^1,i^1,j^1) \overset{\text{def.}}{=} \text{the restriction of } (B,i,j) \text{ to } S,$$
$$(B^2,i^2,j^2) \overset{\text{def.}}{=} (B,i,j).$$

We have a vector bundle of rank n defined by V^2_k/V^1_k.

We shall show that $\mathfrak{P}^{(n)}_k(\mathbf{v},\mathbf{w})$ is nonsingular later (see the proof of Lemma 11.2.3).

5.4. **Induction.** We recall some results in [45, §4]. Let $Q_k(\mathbf{v},\mathbf{w})$ be the middle cohomology of the complex (2.9.1), i.e.,

$$Q_k(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \operatorname{Ker}\tau_k/\operatorname{Im}\sigma_k.$$

We introduce the following subsets of $\mathfrak{M}(\mathbf{v},\mathbf{w})$ (cf. [34, 12.2]):

$$\begin{gathered}\mathfrak{M}_{k;n}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \left\{ [B,i,j] \in \mathfrak{M}(\mathbf{v},\mathbf{w}) \;\middle|\; \operatorname{codim}_{V_k} \operatorname{Im}\tau_k = n \right\},\\ \mathfrak{M}_{k;\le n}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \bigcup_{m\le n} \mathfrak{M}_{k;m}(\mathbf{v},\mathbf{w}), \qquad \mathfrak{M}_{k;\ge n}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \bigcup_{m\ge n} \mathfrak{M}_{k;m}(\mathbf{v},\mathbf{w}).\end{gathered} \tag{5.4.1}$$

Since $\mathfrak{M}_{k;\le n}(\mathbf{v},\mathbf{w})$ is an open subset of $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_{k;n}(\mathbf{v},\mathbf{w})$ is a locally closed subvariety. The restriction of $Q_k(\mathbf{v},\mathbf{w})$ to $\mathfrak{M}_{k;n}(\mathbf{v},\mathbf{w})$ is a $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant vector bundle of rank $\langle h_k, \mathbf{w} - \mathbf{v}\rangle + n$, where we used Convention 2.1.4.

Replacing V_k by $\operatorname{Im}\tau_k$, we have a natural map

$$p\colon \mathfrak{M}_{k;n}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k,\mathbf{w}). \tag{5.4.2}$$

Note that the projection π (2.3.4) factors through p. In particular, the fiber of π is preserved under p.

Proposition 5.4.3. *Let $G(n, Q_k(\mathbf{v} - n\alpha_k,\mathbf{w})|_{\mathfrak{M}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w})})$ be the Grassmann bundle of n-planes in the vector bundle obtained by restricting $Q_k(\mathbf{v} - n\alpha_k,\mathbf{w})$*

to $\mathfrak{M}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w})$*. Then we have the following diagram:*

$$\begin{array}{ccc}
G(n, Q_k(\mathbf{v}-n\alpha_k,\mathbf{w})|_{\mathfrak{M}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w})}) & \xrightarrow{\pi} & \mathfrak{M}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w}) \\
\downarrow\cong & & \| \\
\mathfrak{M}_{k;n}(\mathbf{v},\mathbf{w}) & \xrightarrow{p} & \mathfrak{M}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w}) \\
{\scriptstyle p_2}\uparrow\cong & & \| \\
\mathfrak{P}_k^{(n)}(\mathbf{v},\mathbf{w}) \cap (\mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w}) \times \mathfrak{M}_{k;\le n}(\mathbf{v},\mathbf{w})) & \xrightarrow{p_1} & \mathfrak{M}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w}),
\end{array}$$

where π *is the natural projection, and* p_1 *and* p_2 *are restrictions of the projections to the first and second factors. The kernel of the natural surjective homomorphism* $p^*Q_k(\mathbf{v}-n\alpha_k,\mathbf{w}) \to Q_k(\mathbf{v},\mathbf{w})$ *is isomorphic to the tautological vector bundle of the Grassmann bundle of the first row, and also to the the restriction of the vector bundle* V_k^2/V_k^1 *over* $\mathfrak{P}_k^{(n)}(\mathbf{v},\mathbf{w})$ *in the third row.*

Proof. The proof is essentially contained in [45, 4.5]. See also Proposition 5.5.2 for a similar result. □

5.5. **Induction for fixed point subvarieties.** We consider an analogue of the results in the previous subsection for fixed point subvariety $\mathfrak{M}(\rho)$. Let us use notation as in §4.1, and suppose that A is the Zariski closure of a semisimple element $a = (s,\varepsilon) \in G_{\mathbf{w}} \times \mathbb{C}^*$.

Let $Q_{k,\lambda}(\rho)$ be the middle cohomology of the complex $C^\bullet_{k,\lambda}(\rho)$ in (4.1.5), i.e.,

$$Q_{k,\lambda}(\rho) \overset{\text{def.}}{=} \operatorname{Ker}\tau_{k,\lambda}/\operatorname{Im}\sigma_{k,\lambda}.$$

Let

$$\mathfrak{M}_{k;(n_\lambda)}(\rho) \overset{\text{def.}}{=} \left\{ [B,i,j] \in \mathfrak{M}(\rho) \;\middle|\; \operatorname{codim}_{V_k(\lambda)} \operatorname{Im}\tau_{k,\lambda} = n_\lambda \text{ for each } \lambda \right\}. \tag{5.5.1}$$

Replacing $V_k(\lambda)$ by $\operatorname{Im}\tau_{k,\lambda}$, we have a natural map

$$p^A \colon \mathfrak{M}_{k;(n_\lambda)}(\rho) \to \mathfrak{M}_{k;(0)}(\rho'),$$

where $\rho' \colon A \to G_{\mathbf{v}'}$ ($\mathbf{v}' = \mathbf{v} - \sum n_\lambda \alpha_k$) is the homomorphism obtained from $\rho\colon A \to G_{\mathbf{v}}$ by replacing $V_k(\lambda)$ by its codimension n_λ subspace. Its conjugacy class is independent of the choice of the subspace. This map is just the restriction of p in the previous subsection.

For each λ, let $G(n_\lambda, Q_{k,q^{-2}\lambda}(\rho')|_{\mathfrak{M}_{k;(0)}(\rho')})$ denote the Grassmann bundle of n_λ-planes in the vector bundle obtained by restricting $Q_{k,q^{-2}\lambda}(\rho')$ to $\mathfrak{M}_{k;(0)}(\rho')$. Let

$$\prod_\lambda G(n_\lambda, Q_{k,q^{-2}\lambda}(\rho')|_{\mathfrak{M}_{k;(0)}(\rho')})$$

be their fiber product over $\mathfrak{M}_{k;(0)}(\rho')$.

We have the following analogue of Proposition 5.4.3:

Proposition 5.5.2. *Suppose that $\varepsilon^2 \neq 1$. We have the following diagram:*

$$\begin{array}{ccc} \prod_\lambda G(n_\lambda, Q_{k,q^{-2}\lambda}(\rho')|_{\mathfrak{M}_{k;(0)}(\rho')}) & \xrightarrow{\ \pi\ } & \mathfrak{M}_{k;(0)}(\rho') \\ \downarrow\cong & & \| \\ \mathfrak{M}_{k;(n_\lambda)}(\rho) & \xrightarrow{\ p^A\ } & \mathfrak{M}_{k;(0)}(\rho'), \end{array}$$

where π is the natural projection. For each λ, the kernel of the natural surjective homomorphism $(p^A)^* Q_{k,q^{-2}\lambda}(\rho') \to Q_{k,q^{-2}\lambda}(\rho)$ is isomorphic to the tautological vector bundle of the Grassmann bundle. Moreover, we have

$$n_\lambda \geq \max\left(0, -\operatorname{rank} C^\bullet_{k,\lambda}(\rho)\right), \tag{5.5.3}$$

$$\dim \mathfrak{M}_{k;(n_\lambda)}(\rho) = \dim \mathfrak{M}(\rho) - \sum_\lambda n_\lambda \left(\operatorname{rank} C^\bullet_{k,\lambda}(\rho) + n_\lambda\right). \tag{5.5.4}$$

(*Here* rank *of a complex means the alternating sum of dimensions of cohomology groups.*)

Proof. We have a surjective homomorphism $(p^A)^* Q_{k,q^{-2}\lambda}(\rho') \to Q_{k,q^{-2}\lambda}(\rho)$ of codimension n_λ over $M_{k;(n_\lambda)}(\rho)$. This gives a morphism from $M_{k;(n_\lambda)}(\rho)$ to the fiber product of Grassmann bundles. By a straightforward modification of the arguments in [45, 4.5], one can show that it is an isomorphism. The details are left to the reader. The assumption $\varepsilon^2 \neq 1$ is used to distinguish $Q_{k,\lambda}$ and $Q_{k,q^{-2}\lambda}$.

Let us prove the remaining part (5.5.3), (5.5.4). First note that

$$\begin{aligned} \operatorname{rank} Q_{k,q^{-2}\lambda}(\rho')|_{\mathfrak{M}_{k;(0)}(\rho')} &= \operatorname{rank} C^\bullet_{k,q^{-2}\lambda}(\rho') \\ &= \operatorname{rank} C^\bullet_{k,q^{-2}\lambda}(\rho) + n_\lambda + n_{q^{-2}\lambda}. \end{aligned}$$

Since we have an n_λ-dimensional subspace in $Q_{k,q^{-2}\lambda}(\rho')|_{\mathfrak{M}_{k;(0)}(\rho')}$, we must have

$$n_{q^{-2}\lambda} + \operatorname{rank} C^\bullet_{k,q^{-2}\lambda} \geq 0.$$

Replacing $q^{-2}\lambda$ by λ, we get (5.5.3).

Moreover, we have

$$\dim \mathfrak{M}_{k;(n_\lambda)}(\rho) = \dim \mathfrak{M}_{k;(0)}(\rho') + \sum_\lambda n_\lambda \left(\operatorname{rank} C^\bullet_{k,q^{-2}\lambda}(\rho) + n_{q^{-2}\lambda}\right).$$

On the other hand, the dimension formula (4.1.6) implies

$$\dim \mathfrak{M}(\rho) - \dim \mathfrak{M}(\rho') = \sum_\lambda n_\lambda \left(\operatorname{rank} C^\bullet_{k,\lambda}(\rho) + \operatorname{rank} C^\bullet_{k,q^{-2}\lambda}(\rho) + n_\lambda + n_{q^{-2}\lambda}\right).$$

Since $\mathfrak{M}_{k;(0)}(\rho')$ is an open subset of $\mathfrak{M}(\rho')$, we get (5.5.4). ∎

Note that the inequality (5.5.3) implies that

$$n_\lambda \left(\operatorname{rank} C^\bullet_{k,\lambda}(\rho) + n_\lambda\right) \geq 0$$

and the equality holds if and only if

$$n_\lambda = \max\left(0, -\operatorname{rank} C^\bullet_{k,\lambda}(\rho)\right).$$

In particular, we have the following analog of [45, 4.6].

Corollary 5.5.5. *Suppose $\varepsilon^2 \neq 1$. On a nonempty open subset $\mathfrak{M}(\rho)$, we have*

$$\operatorname{codim}_{V_k(\lambda)} \operatorname{Im} \tau_{k,\lambda} = \max\left(0, -\operatorname{rank} C^\bullet_{k,\lambda}(\rho)\right)$$

for each λ. Also, the complement is a lower dimensional subvariety of $\mathfrak{M}(\rho)$.

As an application of this induction, we prove the following:

Theorem 5.5.6. *Assume that ε is not a root of unity and there is at most one edge joining two vertices of I. Then $\mathfrak{M}(\rho)$ is connected if it is a nonempty set.*

Proof. We prove the assertion by induction on $\dim V$, $\dim W$. (The result is trivial when $V = W = 0$.)

We first make a reduction to the case when

$$\operatorname{rank} C^{\bullet}_{k,\lambda}(\rho) < 0 \quad \text{for some } k,\ \lambda. \tag{5.5.7}$$

Fix a $\mu \in \operatorname{Hom}(A, \mathbb{C}^*)$ and consider

$$n_0 \overset{\text{def.}}{=} \max\left\{n \mid V_k(q^n\mu) \neq 0 \text{ or } W_k(q^n\mu) \neq 0 \text{ for some } k \in I\right\}.$$

Since ε is not a root of unity, we have $q^n\mu \neq q^m\mu$ for $m \neq n$. Hence the above n_0 is well defined.

Suppose $W_k(q^{n_0}\mu) \neq 0$. By (4.1.3) and the choice of n_0, we have

$$\operatorname{Im} j_k \cap W_k(q^{n_0}\mu) = \{0\}.$$

Let us replace $W_k(q^{n_0}\mu)$ by 0. Namely we change (restriction of i_k): $W_k(q^{n_0}\mu) \to V_k(q^{n_0-1}\mu)$ to 0 and all other data are unchanged. The equation $\mu(B, i, j) = 0$ and the stability condition are preserved by the replacement. Thus we have a morphism

$$\mathfrak{M}(\rho) \to \mathfrak{M}'(\rho),$$

where $\mathfrak{M}'(\rho)$ is a fixed point subvariety of $\mathfrak{M}(\mathbf{v}, \mathbf{w}')$ obtained by the replacement. (This notation will not be used elsewhere. The data $\mathbf{w}$ is fixed elsewhere.)

Conversely, we can put any homomorphism $W_k(q^{n_0}\mu) \to V_k(q^{n_0-1}\mu)$ to get a point in $\mathfrak{M}(\rho)$ starting from a point in $\mathfrak{M}'(\rho)$. This shows that $\mathfrak{M}(\rho)$ is the total space of the vector bundle $\operatorname{Hom}(W_k(q^{n_0}\mu), V_k(q^{n_0-1}\mu))$ over $\mathfrak{M}'(\rho)$, where $W_k(q^{n_0}\mu)$ is considered as a trivial bundle. In particular, $\mathfrak{M}(\rho)$ is (nonempty and) connected if and only if $\mathfrak{M}(\rho')$ is also. By the induction hypothesis, $\mathfrak{M}(\rho')$ is connected and we are done.

Thus we may assume $V_k(q^{n_0}\mu) \neq 0$. Then $C_{k,q^{n_0}\mu}(\rho)$ consists of the last term by the choice of n_0. (Note $m(h) = 0$ under the assumption that there is at most one edge joining two vertices of I.) Hence we have (5.5.7) with $\lambda = q^{n_0}\mu$.

Now let us prove the connectedness of $\mathfrak{M}(\rho)$ under (5.5.7). By Corollary 5.5.5, we have

$$\dim \mathfrak{M}_{k;(n_\lambda)}(\rho) < \dim \mathfrak{M}(\rho)$$

unless $n_\lambda = \max\left(0, -\operatorname{rank} C^{\bullet}_{k,\lambda}(\rho)\right)$ for each λ. Hence it is enough to prove the connectedness of $\mathfrak{M}_{k;(n^0_\lambda)}(\rho)$ for $n^0_\lambda = \max\left(0, -\operatorname{rank} C^{\bullet}_{k,\lambda}(\rho)\right)$.

Let us consider the map $p^A \colon \mathfrak{M}_{k;(n^0_\lambda)}(\rho) \to \mathfrak{M}_{k;(0)}(\rho')$. By (5.5.7), $\dim V$ becomes smaller for $\mathfrak{M}_{k;(0)}(\rho')$. Hence $\mathfrak{M}(\rho')$ is connected by the induction hypothesis. Again by Corollary 5.5.5, $\mathfrak{M}_{k;(0)}(\rho')$ is also connected. Since p^A is a fiber product of Grassmann bundles, $\mathfrak{M}_{k;(n^0_\lambda)}(\rho)$ is connected. □

6. Equivariant K-theory

In this section, we review the equivariant K-theory of a quasi-projective variety with a group action. See [13, Chapter 5] for further details.

6.1. **Definitions.** Let X be a quasi-projective variety over $\mathbb{C}$. Suppose that a linear algebraic group G acts algebraically on X. Let $K^G(X)$ be the Grothendieck group of the abelian category of G-equivariant coherent sheaves on X. It is a module over $R(G)$, the representation ring of G.

A class in $K^G(X)$ represented by a G-equivariant sheaf E will be denoted by $[E]$, or simply by E if there is no fear of confusion.

The trivial line bundle of rank 1, i.e., the structure sheaf, is denoted by $\mathcal{O}_X$. If the underlying space is clear, we simply write $\mathcal{O}$.

Let $K^0_G(X)$ be the Grothendieck group of the abelian category of G-equivariant algebraic vector bundles on X. This is also an $R(G)$-module. The tensor product $\otimes$ defines a structure of an $R(G)$-algebra on $K^0_G(X)$. Also, $K^G(X)$ has a structure of a $K^0_G(X)$-module by the tensor product:

$$K^0_G(X) \times K^G(X) \ni ([E],[F]) \mapsto [E \otimes F] \in K^G(X). \tag{6.1.1}$$

Suppose that Y is a G-invariant closed subvariety of X and let $U = X \setminus Y$ be the complement. Two inclusions

$$Y \xrightarrow{i} X \xleftarrow{j} U$$

induce an exact sequence

$$K^G(Y) \xrightarrow{\ i_*\ } K^G(X) \xrightarrow{\ j^*\ } K^G(U) \longrightarrow 0, \tag{6.1.2}$$

where i_* is given by $[E] \mapsto [i_*E]$ and j^* is given by $[E] \mapsto [E|_U]$. (See [53].)

Suppose that Y is a G-invariant closed subvariety of X and that X is nonsingular. Let $K^G(X;Y)$ be the Grothendieck group of the derived category of G-equivariant complexes $E^\bullet$ of algebraic vector bundles over X, which are exact outside Y (see [4, §1]). We have a natural homomorphism $K^G(X;Y) \to K^G(Y)$ by setting

$$[E^\bullet] \mapsto \sum_i (-1)^i [\operatorname{gr} H^i(E^\bullet)].$$

Here $H^i(E^\bullet)$ is the ith cohomology sheaf of $E^\bullet$, which is a G-equivariant coherent sheaf on X which is supported on Y. If $\mathfrak{I}_Y$ is the defining ideal of Y, we have $\mathfrak{I}_Y^N \cdot H^i(E^\bullet) = 0$ for sufficiently large N. Then

$$\operatorname{gr} H^i(E^\bullet) \overset{\text{def.}}{=} \bigoplus_j \mathfrak{I}_Y^j \cdot H^i(E^\bullet)/\mathfrak{I}_Y^{j+1} \cdot H^i(E^\bullet)$$

is a sheaf on Y, and defines an element in $K^G(Y)$. Conversely if a G-equivariant coherent sheaf F on Y is given, we can take a resolution by a finite G-equivariant complex of algebraic vector bundles:

$$0 \to E^{-n} \to E^{1-n} \to \cdots \to E^0 \to i_*F \to 0,$$

where $i\colon Y \to X$ denotes the inclusion. (See [13, 5.1.28].) This shows that the homomorphism $K^G(X;Y) \to K^G(Y)$ is an isomorphism. This relative K-group $K^G(X;Y)$ was not used in [13] explicitly, but many operations were defined by using it implicitly. When $Y = X$, $K^G(X;X)$ is isomorphic to $K^0_G(X)$. In particular, we have an isomorphism $K^0_G(X) \cong K^G(X)$ if X is nonsingular.

We shall also use equivariant topological K-homology $K^G_{\text{top}}(X)$. There are several approaches for the definition, but we take the one in [54, 5.3]. There is a comparison map

$$K^G(X) \to K^G_{\text{top}}(X)$$

which satisfies obvious functorial properties.

Occasionally, we also consider the *higher* equivariant topological K-homology group $K^G_{1,\mathrm{top}}(X)$. (See [54, 5.3] again.) In this circumstance, $K^G_{\mathrm{top}}(X)$ may be written as $K^G_{0,\mathrm{top}}(X)$. But we do not use higher equivariant algebraic K-homology $K^G_i(X)$.

Suppose that Y is a G-invariant closed subvariety of X and let $U = X \setminus Y$ be the complement. Two inclusions

$$Y \xrightarrow{i} X \xleftarrow{j} U$$

induce a natural exact hexagon

$$\tag{6.1.3} \begin{array}{ccccc} K^G_{0,\mathrm{top}}(Y) & \xrightarrow{\ i_*\ } & K^G_{0,\mathrm{top}}(X) & \xrightarrow{\ j^*\ } & K^G_{0,\mathrm{top}}(U) \\ \uparrow & & & & \downarrow \\ K^G_{1,\mathrm{top}}(U) & \xleftarrow{\ j^*\ } & K^G_{1,\mathrm{top}}(X) & \xleftarrow{\ i_*\ } & K^G_{1,\mathrm{top}}(Y), \end{array}$$

for suitably defined i_*, j^*.

6.2. **Operations on K-theory of vector bundles.** If E is a G-equivariant vector bundle, its rank and dual vector bundle will be denoted by $\operatorname{rank} E$ and E^* respectively.

We extend rank and $*$ to operations on $K^0_G(X)$:

$$\operatorname{rank}\colon K^0_G(X) \to \mathbb{Z}^{\pi_0(X)}, \qquad {}^*\colon K^0_G(X) \to K^0_G(X),$$

where $\pi_0(X)$ is the set of the connected components of X. Note that the rank of a vector bundle may not be a constant, when X has several connected components. But we assume X is connected in this subsection for simplicity. In general, operators below can be defined component-wisely.

If L is a G-equivariant line bundle, we define $L^{\otimes r} = (L^*)^{\otimes(-r)}$ for $r < 0$. Thus we have $L^{\otimes r} \otimes L^{\otimes s} = L^{\otimes(r+s)}$ for $r, s \in \mathbb{Z}$.

If E is a vector bundle, we define

$$\det E \overset{\text{def.}}{=} \bigwedge\nolimits^{\operatorname{rank} E} E, \qquad \bigwedge\nolimits_u E \overset{\text{def.}}{=} \sum_{i=0}^{\operatorname{rank} E} u^i \bigwedge\nolimits^i E.$$

These operations can be extended to $K^0_G(X)$ of G-equivariant algebraic vector bundles:

$$\det\colon K^0_G(X) \to K^0_G(X), \quad \bigwedge\nolimits_u \colon K^0_G(X) \to [\mathcal{O}_X] + K^0_G(X) \otimes u\mathbb{Z}[[u]].$$

This is well defined since we have $\det F = \det E \otimes \det G$, $\bigwedge_u F = \bigwedge_u E \otimes \bigwedge_u G$ for an exact sequence $0 \to E \to F \to G \to 0$.

Note the formula

$$\bigwedge\nolimits_u E = u^{\operatorname{rank} E} \det E \bigwedge\nolimits_{1/u} E^*$$

for a vector bundle E. Using this formula, we expand $\bigwedge_u E$ into the Laurent expansion also at $u = \infty$:

$$u^{-\operatorname{rank} E} (\det E)^* \bigwedge\nolimits_u E \in [\mathcal{O}_X] + K^0_G(X) \otimes u^{-1}\mathbb{Z}[[u^{-1}]].$$

6.3. **Tor-product.** (Cf. [4, 1.3], [13, 5.2.11].) Let X be a nonsingular quasi-projective variety with a G-action. Let Y_1, $Y_2 \subset X$ be G-invariant closed subvarieties of X. Suppose that $E_1^\bullet$ (resp. $E_2^\bullet$) is a G-equivariant complex of vector bundles over X which is exact outside Y_1 (resp. Y_2). Then we can construct a complex

$$\cdots \longrightarrow \bigoplus_{p+q=k} E_1^p \otimes E_2^q \longrightarrow \bigoplus_{p+q=k+1} E_1^p \otimes E_2^q \longrightarrow \cdots$$

with suitably defined differentials from the double complex $E_1^\bullet \otimes E_2^\bullet$. It is exact outside $Y_1 \cap Y_2$. This construction defines an $R(G)$-bilinear pairing

$$K^G(X;Y_1) \times K^G(X;Y_2) \to K^G(X;Y_1 \cap Y_2).$$

Since we assume X is nonsingular, we have $K^G(X;Y_1) \cong K^G(Y_1)$, $K^G(X;Y_2) \cong K^G(Y_2)$, $K^G(X;Y_1 \cap Y_2) \cong K^G(Y_1 \cap Y_2)$. Thus we also have an $R(G)$-bilinear pairing

$$K^G(Y_1) \times K^G(Y_2) \to K^G(Y_1 \cap Y_2).$$

We denote these operations by $\cdot \otimes_X^L \cdot$. (It is denoted by $\otimes$ in [13].)

Lemma 6.3.1 ([13, 5.4.10], [58, Lemma 1])**.** *Let Y_1, $Y_2 \subset X$ be nonsingular G-subvarieties with conormal bundles $T^*_{Y_1}X$, $T^*_{Y_2}X$. Suppose that $Y \overset{\text{def.}}{=} Y_1 \cap Y_2$ is nonsingular and $TY_1|_Y \cap TZ_2|_Y = TY$, where $|_Y$ means the restriction to Y. Then for any $E_1 \in K^0_G(Y_1) \cong K_0^G(Y_1)$, $E_2 \in K^0_G(Y_2) \cong K_0^G(Y_2)$, we have*

$$E_1 \otimes_X^L E_2 = \sum_i (-1)^i {\textstyle\bigwedge}^i N \otimes E_1|_Y \otimes E_2|_Y \in K^0_G(Y) \cong K_0^G(Y),$$

where $N \overset{\text{def.}}{=} T^*_{Y_1}X|_Y \cap T^*_{Y_2}X|_Y$.

6.4. **Pull-back with support.** (Cf. [4, 1.2], [13, 5.2.5].) Let $f\colon Y \to X$ be a G-equivariant morphism between *nonsingular* G-varieties. Suppose that X' and Y' are G-invariant closed subvarieties of X and Y respectively satisfying $f^{-1}(X') \subset Y'$. Then the pull-back

$$E^\bullet \mapsto f^*E^\bullet$$

induces a homomorphism $K^G(X;X') \to K^G(Y;Y')$. Via isomorphisms $K^G(X') \cong K^G(X;X')$, $K^G(Y') \cong K^G(Y;Y')$, we get a homomorphism $K^G(X') \to K^G(Y')$. Note that this depends on the ambient spaces X, Y.

Let $f\colon Y \to X$ as above. Suppose that $X_1', X_2' \subset X$, $Y_1', Y_2' \subset Y$ are G-invariant closed subvarieties such that $f^{-1}(X_a') \subset Y_a'$ for $a = 1, 2$. Then we have

$$f^*(E_1 \otimes_X^L E_2) = f^*(E_1) \otimes_Y^L f^*(E_2) \tag{6.4.1}$$

for $E_a \in K^G(X_a')$ $(a = 1, 2)$.

6.5. **Push-forward.** Let $f\colon X \to Y$ be a proper G-equivariant morphism between G-varieties (not necessarily nonsingular). Then we have a push-forward homomorphism $f_*\colon K^G(X) \to K^G(Y)$ defined by

$$f_*[E] \overset{\text{def.}}{=} \sum (-1)^i [R^i f_* E].$$

Suppose further that X and Y are nonsingular. If $X' \subset X$, $Y' \subset Y$ are G-invariant closed subvarieties, we have the following projection formula ([13, 5.3.13]):

$$f_*(E \otimes^L_X f^*F) = f_*E \otimes^L_Y F \in K^G(f(X') \cap Y') \tag{6.5.1}$$

for $E \in K^G(X')$, $F \in K^G(Y')$.

6.6. **Chow group and homology group.** Let $H_*(X, \mathbb{Z}) = \bigoplus_k H_k(X, \mathbb{Z})$ be the integral Borel-Moore homology of X. Let $A_*(X) = \bigoplus_k A_k(X)$ be the Chow group of X. We have a cycle map

$$A_*(X) \to H_*(X, \mathbb{Z}),$$

which has certain functorial properties (see [19, Chapter 19]).

If Y is a closed subvariety of X and $U = X \setminus Y$ is its complement, then we have exact sequences which are analogues of (6.1.2), (6.1.3):

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0, \tag{6.6.1}$$

$$\cdots \to H_k(Y, \mathbb{Z}) \xrightarrow{i_*} H_k(X, \mathbb{Z}) \xrightarrow{j^*} H_k(U, \mathbb{Z}) \xrightarrow{\partial_*} H_{k-1}(Y, \mathbb{Z}) \to \cdots . \tag{6.6.2}$$

We have operations on $A_*(X)$ and $H_*(X, \mathbb{Z})$ which are analogues of those in §6.3, §6.4, §6.5. (See [19].)

In the next section, we prove results for K-homology and the Chow group in parallel arguments. It is the reason why we avoid higher algebraic K-homology. There is no analogue for the Chow group.

7. Freeness

7.1. **Properties (S), (T), (T').** Following [14, 39], we say that an algebraic variety X *has property* (S) if

(a) $H_{\text{odd}}(X, \mathbb{Z}) = 0$ and $H_{\text{even}}(X, \mathbb{Z})$ is a free abelian group.
(b) The cycle map $A_*(X) \to H_{\text{even}}(X, \mathbb{Z})$ is an isomorphism.

Similarly, we say X *has property* (T) if

(a) $K_{1,\text{top}}(X) = 0$ and $K_{\text{top}}(X) = K_{0,\text{top}}(X)$ is a free abelian group.
(b) The comparison map $K(X) \to K_{\text{top}}(X)$ is an isomorphism.

Suppose that X is a closed subvariety of a nonsingular variety M. We have a diagram (see [4])

$$\begin{array}{ccc} K(X) \otimes \mathbb{Q} & \xrightarrow{\text{ch}} & A_*(X) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ K_{\text{top}}(X) \otimes \mathbb{Q} & \xrightarrow{\text{ch}} & H_{\text{even}}(X, \mathbb{Q}), \end{array}$$

where the horizontal arrows are local Chern character homomorphisms in algebraic and topological K-homologies respectively, the left vertical arrow is a comparison map, and the right vertical arrow is the cycle map. It is known that the upper horizontal arrow is an isomorphism ([19, 15.2.16]). Thus the composite $K(X) \otimes \mathbb{Q} \to H_{\text{even}}(X, \mathbb{Q})$ is an isomorphism if X has property (S).

Assume that X is nonsingular and projective. We define the bilinear pairing $K(X) \otimes K(X) \to \mathbb{Z}$ by

$$F \otimes F' \mapsto p_*(F \otimes^L_X F'), \tag{7.1.1}$$

where p is the canonical map from X to the point.

We say that X has property (T') if X has property (T) and the pairing (7.1.1) is perfect. (In [39], this property is called (S').)

Let G be a linear algebraic group. Let X be an algebraic variety with a G-action. We say that X *has property* (T_G) if

(a) $K^G_{1,\mathrm{top}}(X) = 0$ and $K^G_{\mathrm{top}}(X) = K^G_{0,\mathrm{top}}(X)$ is a free R_G-module.
(b) The natural map $K^G(X) \to K^G_{\mathrm{top}}(X)$ is an isomorphism.
(c) For a closed algebraic subgroup $H \subset G$, H-equivariant K-theories satisfy the above properties (a), (b), and the natural homomorphism $K^G(X) \otimes_{R(G)} R(H) \to K^H(X)$ is an isomorphism.

Suppose further that X is smooth and projective. By the same formula as (7.1.1), we have a bilinear pairing $K^G(X) \otimes K^G(X) \to R(G)$. We say that X *has property* (T'_G) if X has property (T_G) and this pairing is perfect.

A finite partition of a variety X into locally closed subvarieties is said to be an *α-partition* if the subvarieties in the partition can be indexed $X_1, \dots, X_n$ in such a way that $X_1 \cup X_2 \cup \cdots \cup X_i$ is closed in X for $i = 1, \dots, n$. The following is proved in [14, Lemma 1.8].

Lemma 7.1.2. *If X has an α-partition into pieces which have property (S), then X has property (S).*

The proof is based on exact sequences (6.6.1),(6.6.2) in homology groups and Chow groups. Since we have corresponding exact sequences (6.1.2),(6.1.3) in K-theory, we have the following.

Lemma 7.1.3. *Suppose that an algebraic variety X has an action of a linear algebraic group G. If X has an α-partition into G-invariant locally closed subvarieties which have property (T_G), then X has property (T_G).*

Lemma 7.1.4. *Let $\pi\colon E \to X$ be a G-equivariant fiber bundle with affine spaces as fibers. Suppose that π is locally a trivial G-equivariant vector bundle, i.e., a product of base and a vector space with a linear G-action. If X has property (T_G) (resp. (S)), then E also has property (T_G) (resp. (S)).*

Proof. We first show that $\pi^*\colon K^G(X) \to K^G(E)$ is surjective. Choose a closed subvariety Y of X so that E is a trivial G-bundle over $U = X \setminus Y$. There is a diagram

$$\begin{array}{ccccccc}
K^G(Y) & \longrightarrow & K^G(X) & \longrightarrow & K^G(U) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \\
K^G(\pi^{-1}(Y)) & \longrightarrow & K^G(E) & \longrightarrow & K^G(\pi^{-1}(U)) & \longrightarrow & 0,
\end{array}$$

with exact rows by (6.1.2). By a diagram chase it suffices to prove the surjectivity for the restrictions of E to U and to Y. By repeating the process on Y, it suffices to prove it for the case when E is a trivial G-equivariant bundle. By Thom isomorphism [53, 4.1] π^* is an isomorphism if E is a G-equivariant bundle. Thus we prove the assertion.

Let us repeat the same argument for $\pi^*\colon K^G_{0,\mathrm{top}}(X) \to K^G_{0,\mathrm{top}}(E)$ and $\pi^* : K^G_{1,\mathrm{top}}(X) \to K^G_{1,\mathrm{top}}(E)$ by replacing (6.1.2) by (6.1.3). By the five lemma both π^* are isomorphisms. In particular, we have $K^G_{1,\mathrm{top}}(E) \cong K^G_{1,\mathrm{top}}(X) = 0$ by assumption.

Consider the diagram

$$\begin{array}{ccc} K^G(X) & \xrightarrow{\pi^*} & K^G(E) \\ \big\downarrow & & \big\downarrow \\ K^G_{\mathrm{top}}(X) & \xrightarrow{\pi^*} & K^G_{\mathrm{top}}(E), \end{array}$$

where the vertical arrows are comparison maps. The left vertical arrow is an isomorphism by assumption. Thus the right vertical arrow is also an isomorphism by the commutativity of the diagram and what we just proved above. Condition (c) for (T_G) can be checked in the same way, and E has property (T_G).

Property (S) can be checked in the same way. $\square$

Lemma 7.1.5. *Let X be a nonsingular quasi-projective variety with $G\times\mathbb{C}^*$-action with a Kähler metric g such that*

(a) *g is complete,*
(b) *g is invariant under the maximal compact subgroup of $G\times\mathbb{C}^*$,*
(c) *there exists a moment map f associated with the Kähler metric g and the S^1-action (the maximal compact subgroup of the second factor), and it is proper.*

Let

$$L \stackrel{\text{def.}}{=} \{x\in X \mid \lim_{t\to\infty} t\cdot x \ \textit{exists}\}.$$

If the fixed point set $X^{\mathbb{C}^*}$ has property $(T_{G\times\mathbb{C}^*})$ (resp. (S)), then both X and L have property $(T_{G\times\mathbb{C}^*})$ (resp. (S)).

Furthermore, the bilinear pairing

$$K^{G\times\mathbb{C}^*}(X)\times K^{G\times\mathbb{C}^*}(L)\ni(F,F')\longmapsto p_*(F\otimes^L_X F')\in R(G\times\mathbb{C}^*) \tag{7.1.6}$$

is nondegenerate if $X^{\mathbb{C}^*}$ has property $(T'_{G\times\mathbb{C}^*})$. A similar intersection pairing between $A_*(X)$ and $A_*(L)$ is nondegenerate if $X^{\mathbb{C}^*}$ has property (S). Here p is the canonical map from X to the point.

Proof. By [2, 2.2] the moment map f is a Bott-Morse function, and critical manifolds are the fixed point $X^{\mathbb{C}^*}$. Let F_1, F_2, … be the components of $X^{\mathbb{C}^*}$. By [2, §3], stable and unstable manifolds for the gradient flow of $-f$ coincide with $(\pm)$-attracting sets of Bialynicki-Birula decomposition [7]:

$$S_k=\{x\in X\mid \lim_{t\to 0} t\cdot x\in F_k\},\qquad U_k=\{x\in X\mid \lim_{t\to\infty} t\cdot x\in F_k\}.$$

These are invariant under the G-action since the G-action commutes with the $\mathbb{C}^*$-action.

Note that results in [2] are stated for compact manifolds, but the argument can be modified to our setting. A difference is that $\bigcup_k U_k = L$ is not X unless X is compact. On the other hand, $\bigcup_k S_k$ is X since f is proper.

As in [3], we can introduce an ordering on the index set $\{k\}$ of components of $X^{\mathbb{C}^*}$ such that $X=\bigcup S_k$ is an α-partition and $L=\bigcup U_k$ is an α-partition with respect to the reversed order.

By [7] (see also [8] for analytic arguments), the maps

$$S_k\ni x\mapsto \lim_{t\to 0} t\cdot x\in F_k,\qquad U_k\ni x\mapsto \lim_{t\to\infty} t\cdot x\in F_k$$

are fiber bundles with affine spaces as fibers. Furthermore, S_k (resp. U_k) is locally isomorphic to a $G \times \mathbb{C}^*$-equivariant vector bundle by the proof. Thus S_k and U_k have properties (S) and $(T_{G\times\mathbb{C}^*})$ by Lemma 7.1.4. Hence X and L have properties (S) and $(T_{G\times\mathbb{C}^*})$ by Lemmas 7.1.2 and 7.1.3.

By the argument in [39, 1.7, 2.5], the pairing (7.1.6) can be identified with a pairing

$$\bigoplus_k K^{G\times\mathbb{C}^*}(F_k) \times \bigoplus_k K^{G\times\mathbb{C}^*}(F_k) \to R(G\times\mathbb{C}^*)$$

of the form

$$\left(\sum_k \xi_k, \sum_k \xi'_k\right) = \sum_{k\geq k'} (\xi_k, \xi'_{k'})_{k,k'}$$

for some pairing $(\ ,\ )_{k,k'}\colon K^{G\times\mathbb{C}^*}(F_k)\times K^{G\times\mathbb{C}^*}(F'_k) \to R(G\times\mathbb{C}^*)$ such that $(\ ,\ )_{k,k}$ is the pairing (7.1.1) for $X = F_k$. Since $(\ ,\ )_{k,k}$ is nondegenerate for all k by the assumption, (7.1.6) is also nondegenerate.

The proof of the statement for $A_*(X)$, $A_*(L)$ is similar. One uses the fact that the intersection pairing $A_*(F_k) \times A_*(F_k) \to \mathbb{Z}$ is nondegenerate under property (S). □

7.2. **Decomposition of the diagonal.**

Proposition 7.2.1 (cf. [16], [13, 5.6.1])**.** *Let X be a nonsingular projective variety.*

(1) *Let $\mathcal{O}_{\Delta X}$ be the structure sheaf of the diagonal and $[\mathcal{O}_{\Delta X}]$ the corresponding element in $K(X\times X)$. Assume that*

$$[\mathcal{O}_{\Delta X}] = \sum_i \alpha_i \boxtimes \beta_i \tag{7.2.2}$$

holds for some α_i, $\beta_i \in K(X)$. Then X has property (T').

(2) *Let G be a linear algebraic group. Suppose that X has G-action and that (7.2.2) holds in $K^G(X\times X)$ for some α_i, $\beta_i \in K^G(X)$. Then X has property (T'_G).*

(3) *Let $[\Delta X]$ be the class of the diagonal in $A(X\times X)$. Assume that*

$$[\Delta X] = \sum_i p_1^* a_i \cup p_2^* b_i \tag{7.2.3}$$

holds for some a_i, $b_i \in A(X)$. Then X has property (S).

Proof. Let $p_a\colon X\times X \to X$ denote the projection to the ath factor ($a = 1, 2$). Let Δ be the diagonal embedding $X \to X\times X$. Then we have $[\mathcal{O}_{\Delta X}] = \Delta_*[\mathcal{O}_X]$. Hence

$$\begin{aligned} p_{1*}\left(p_2^*F \otimes^L_{X\times X} [\mathcal{O}_{\Delta X}]\right) &= p_{1*}\left(p_2^*F \otimes^L_{X\times X} \Delta_*[\mathcal{O}_X]\right) \\ &= p_{1*}\Delta_*\left(\Delta^* p_2^* F \otimes^L_X [\mathcal{O}_X]\right) \quad \text{(by the projection formula)} \\ &= F \otimes^L_X [\mathcal{O}_X] \quad (p_1\circ\Delta = p_2\circ\Delta = \mathrm{id}_X) \\ &= F. \end{aligned}$$

If we substitute (7.2.2) into the above, we get

$$F = \sum_i p_{1*}\left(p_2^*F \otimes^L_{X\times X} p_1^*\alpha_i \otimes^L_{X\times X} p_2^*\beta_i\right) = \sum_i (F, \beta_i)\alpha_i. \tag{7.2.4}$$

In particular, $K(X)$ is spanned by α_i's.

If $mF = 0$ for some $m \in \mathbb{Z} \setminus \{0\}$, then $0 = (mF, \beta_i) = m(F, \beta_i)$. Hence we have $(F, \beta_i) = 0$. The above equality (7.2.4) implies $F = 0$. This means that $K(X)$ is torsion-free. Thus we could assume the α_i's are linearly independent in (7.2.2). Under this assumption, $\{\alpha_i\}$ is a basis of $K(X)$, and (7.2.4) implies that $\{\beta_i\}$ is the dual basis.

If we perform the same computation in $K_{0,\mathrm{top}}(X) \oplus K_{1,\mathrm{top}}(X)$, we get the same result. In particular, $\{\alpha_i\}$ is a basis of $K_{0,\mathrm{top}}(X) \oplus K_{1,\mathrm{top}}(X)$. However, α_i, β_i are in $K_{0,\mathrm{top}}(X)$, thus we have $K_{1,\mathrm{top}}(X) = 0$. We also have $K(X) \to K_{0,\mathrm{top}}(X)$ is an isomorphism. Thus X has property (T').

If X has G-action and (7.2.2) holds in the equivariant K-group, we do the same calculation in the equivariant K-groups. Then the same argument shows that X has property (T'_G).

The assertion for Chow groups and homology groups can be proved in the same way. $\square$

7.3. **Diagonal of the quiver variety.** Let us recall the decomposition of the diagonal of the quiver variety defined in [45, Sect. 6]. In this section, we fix dimension vectors $\mathbf{v}$, $\mathbf{w}$ and use the notation $\mathfrak{M}$ instead of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$.

Let us consider the product $\mathfrak{M} \times \mathfrak{M}$. We denote by V_k^1 (resp. V_k^2) the vector bundle $V_k \boxtimes \mathcal{O}_{\mathfrak{M}}$ (resp. $\mathcal{O}_{\mathfrak{M}} \boxtimes V_k$). A point in $\mathfrak{M} \times \mathfrak{M}$ is denoted by $([B^1, i^1, j^1], [B^2, i^2, j^2])$. We regard B^a, i^a, j^a $(a = 1, 2)$ as homomorphisms between tautological bundles.

We consider the following $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant complex of vector bundles over $\mathfrak{M} \times \mathfrak{M}$:

$$\mathrm{L}(V^1, V^2) \xrightarrow{\sigma} q\,\mathrm{E}(V^1, V^2) \oplus q\,\mathrm{L}(W, V^2) \oplus q\,\mathrm{L}(V^1, W) \xrightarrow{\tau} q^2\,\mathrm{L}(V^1, V^2), \tag{7.3.1}$$

where

$$\sigma(\xi) \overset{\text{def.}}{=} (B^2\xi - \xi B^1) \oplus (-\xi i^1) \oplus j^2\xi,$$
$$\tau(C \oplus a \oplus b) \overset{\text{def.}}{=} (\varepsilon B^2 C + \varepsilon C B^1 + i^2 b + a j^1).$$

It was shown that σ is injective and τ is surjective (cf. [45, 5.2]). Thus $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ is an equivariant vector bundle. We define an equivariant section s of $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ by

$$s \overset{\text{def.}}{=} \bigl(0 \oplus (-i^2) \oplus j^1\bigr) \bmod \operatorname{Im}\sigma.$$

Then $([B^1, i^1, j^1], [B^2, i^2, j^2])$ is contained in the zero locus $Z(s)$ of s if and only if there exists $\xi \in \mathrm{L}(V^1, V^2)$ such that

$$\xi B^1 = B^2\xi, \quad \xi i^1 = i^2, \quad j^1 = j^2\xi.$$

Moreover ξ is an isomorphism by the stability condition. Hence $Z(s)$ is equal to the diagonal $\Delta\mathfrak{M}$. If ∇ is a connection on $\operatorname{Ker}\tau/\operatorname{Im}\sigma$, the differential $\nabla s \colon T(\mathfrak{M}\times\mathfrak{M}) \to \operatorname{Ker}\tau/\operatorname{Im}\sigma$ is surjective on $Z(s) = \Delta\mathfrak{M}$ (cf. [45, 5.7]). In particular, we have an exact sequence

$$0 \to \textstyle\bigwedge^{\max}(\operatorname{Ker}\tau/\operatorname{Im}\sigma)^* \to \cdots \to (\operatorname{Ker}\tau/\operatorname{Im}\sigma)^* \to \mathcal{O}_{\mathfrak{M}\times\mathfrak{M}} \to \mathcal{O}_{\Delta\mathfrak{M}} \to 0.$$

In $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}\times\mathfrak{M})$, $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ is equal to the alternating sum of terms of (7.3.1) which has a form $\sum \alpha_i \boxtimes \beta_i$ for some $\alpha_i, \beta_i \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M})$. Hence $\mathfrak{M}$ satisfies the conditions of Proposition 7.2.1 except the projectivity. Unfortunately, the projectivity is essential in the proof of Proposition 7.2.1. (We could not define

$(p_1)_*$ otherwise.) Thus Proposition 7.2.1 is not directly applicable to $\mathfrak{M}$. In order to get rid of this difficulty, we consider the fixed point set $\mathfrak{M}^{\mathbb{C}^*}$ with respect to the $\mathbb{C}^*$-action.

For technical reasons, we need to use a $\mathbb{C}^*$-action, which is different from (2.7.2). Let $\mathbb{C}^*$ act on $\mathbf{M}$ by

$$B_h \mapsto tB_h, \quad i \mapsto ti, \quad j \mapsto tj \qquad \text{for } t \in \mathbb{C}^*. \tag{7.3.2}$$

This induces a $\mathbb{C}^*$-action on $\mathfrak{M}$ and $\mathfrak{M}_0$ which commutes with the previous $G_{\mathbf{w}} \times \mathbb{C}^*$-action. (If the adjacency matrix satisfies $\mathbf{A}_{kl} \le 1$ for any $k, l \in I$, then the new $\mathbb{C}^*$-action coincides with the old one.) The tautological bundles V_k, W_k become $\mathbb{C}^*$-equivariant vector bundles as before.

We consider the fixed point set $\mathfrak{M}^{\mathbb{C}^*}$. $[B, i, j] \in \mathfrak{M}$ is a fixed point if and only if there exists a homomorphism $\rho\colon \mathbb{C}^* \to G_{\mathbf{v}}$ such that

$$t \diamond (B, i, j) = \rho(t)^{-1} \cdot (B, i, j)$$

as in §4.1. Here $\diamond$ denotes the new $\mathbb{C}^*$-action. We decompose the fixed point set $\mathfrak{M}^{\mathbb{C}^*}$ according to the conjugacy class of ρ:

$$\mathfrak{M}^{\mathbb{C}^*} = \bigsqcup \mathfrak{M}[\rho].$$

Lemma 7.3.3. $\mathfrak{M}[\rho]$ *is a nonsingular projective variety.*

Proof. Since $\mathfrak{M}[\rho]$ is a union of connected components (possibly single component) of the fixed point set of the $\mathbb{C}^*$-action on a nonsingular variety $\mathfrak{M}$, $\mathfrak{M}[\rho]$ is nonsingular.

Suppose that $[B, i, j] \in \mathfrak{M}_0$ is a fixed point of the $\mathbb{C}^*$-action. It means that (tB, ti, tj) lies in the closed orbit $G \cdot (B, i, j)$. But (tB, ti, tj) converges to 0 as $t \to 0$. Hence the closed orbit must be $\{0\}$. Since $\pi\colon \mathfrak{M} \to \mathfrak{M}_0$ is equivariant, $\mathfrak{M}^{\mathbb{C}^*}$ is contained in $\pi^{-1}(0)$. In particular, $\mathfrak{M}[\rho]$ is projective. □

This lemma is not true for the original $\mathbb{C}^*$-action.

We restrict the complex (7.3.1) to $\mathfrak{M}[\rho] \times \mathfrak{M}[\rho]$. Then fibers of V^1 and V^2 become $\mathbb{C}^*$-modules and hence we can take the $\mathbb{C}^*$-fixed part of (7.3.1):

$$\mathrm{L}(V^1, V^2)^{\mathbb{C}^*} \xrightarrow{\sigma^{\mathbb{C}^*}} \left(q\, \mathrm{E}(V^1, V^2) \oplus q\, \mathrm{L}(W, V^2) \oplus q\, \mathrm{L}(V^1, W)\right)^{\mathbb{C}^*} \xrightarrow{\tau^{\mathbb{C}^*}} (q^2\, \mathrm{L}(V^1, V^2))^{\mathbb{C}^*},$$

where $\sigma^{\mathbb{C}^*}$ (resp. $\tau^{\mathbb{C}^*}$) is the restriction of σ (resp. τ) to the $\mathbb{C}^*$-fixed part. Then $\sigma^{\mathbb{C}^*}$ is injective and $\tau^{\mathbb{C}^*}$ is surjective, and $\operatorname{Ker} \tau^{\mathbb{C}^*} / \operatorname{Im} \sigma^{\mathbb{C}^*}$ is a vector bundle which is the $\mathbb{C}^*$-fixed part of $\operatorname{Ker} \tau / \operatorname{Im} \sigma$.

The section s takes values in $\operatorname{Ker} \tau^{\mathbb{C}^*} / \operatorname{Im} \sigma^{\mathbb{C}^*} = (\operatorname{Ker} \tau / \operatorname{Im} \sigma)^{\mathbb{C}^*}$. Considering it as a section of $\operatorname{Ker} \sigma^{\mathbb{C}^*} / \operatorname{Im} \tau^{\mathbb{C}^*}$, we denote it by $s^{\mathbb{C}^*}$. The zero locus $Z(s^{\mathbb{C}^*})$ is $Z(s) \cap (\mathfrak{M}[\rho] \times \mathfrak{M}[\rho])$ which is the diagonal $\Delta\mathfrak{M}[\rho]$ of $\mathfrak{M}[\rho] \times \mathfrak{M}[\rho]$. Furthermore, the differential $\nabla s^{\mathbb{C}^*}\colon T(\mathfrak{M}[\rho] \times \mathfrak{M}[\rho]) \to (\operatorname{Ker} \tau / \operatorname{Im} \sigma)^{\mathbb{C}^*}$ is surjective on $Z(s^{\mathbb{C}^*}) = \Delta\mathfrak{M}[\rho]$.

Our original $G_{\mathbf{w}} \times \mathbb{C}^*$-action (defined in §2.7) commutes with the new $\mathbb{C}^*$-action. Thus $\mathfrak{M}[\rho]$ has an induced $G_{\mathbf{w}} \times \mathbb{C}^*$-action. By the construction, $\operatorname{Ker} \tau^{\mathbb{C}^*} / \operatorname{Im} \sigma^{\mathbb{C}^*}$ is a $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant vector bundle, and $s^{\mathbb{C}^*}$ is an equivariant section.

Proposition 7.3.4. $\mathfrak{M}[\rho]$ *has properties* (S) *and* $(T'_{G_{\mathbf{w}} \times \mathbb{C}^*})$. *Moreover,* $\mathfrak{M}[\rho]$ *is connected.*

Proof. Let $\mathcal{O}_{\Delta\mathfrak{M}[\rho]}$ be the structure sheaf of the diagonal considered as a sheaf on $\mathfrak{M}[\rho]\times\mathfrak{M}[\rho]$. By the above argument, the Koszul complex of $s^{\mathbb{C}^*}$ gives a resolution of $\mathcal{O}_{\Delta\mathfrak{M}[\rho]}$:

$$0 \to \bigwedge^{\max}(\operatorname{Ker}\tau^{\mathbb{C}^*}/\operatorname{Im}\sigma^{\mathbb{C}^*})^* \to \cdots \to (\operatorname{Ker}\tau^{\mathbb{C}^*}/\operatorname{Im}\sigma^{\mathbb{C}^*})^* \\ \to \mathcal{O}_{\mathfrak{M}[\rho]\times\mathfrak{M}[\rho]} \to \mathcal{O}_{\Delta\mathfrak{M}[\rho]} \to 0,$$

where $\max = \operatorname{rank}\operatorname{Ker}\sigma^{\mathbb{C}^*}/\operatorname{Im}\tau^{\mathbb{C}^*}$. Thus we have the following equality in the Grothendieck group $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}[\rho]\times\mathfrak{M}[\rho])$:

$$[\mathcal{O}_{\Delta\mathfrak{F}}] = \bigwedge\nolimits_{-1}[(\operatorname{Ker}\tau^{\mathbb{C}^*}/\operatorname{Im}\sigma^{\mathbb{C}^*})^*].$$

Since $\sigma^{\mathbb{C}^*}$ is injective and $\tau^{\mathbb{C}^*}$ is surjective, we have

$$\begin{aligned}&[\operatorname{Ker}\tau^{\mathbb{C}^*}/\operatorname{Im}\sigma^{\mathbb{C}^*}]\\ &\quad= -[\mathrm{L}(V^1,V^2)^{\mathbb{C}^*}] + \Big[(q\,\mathrm{E}(V^1,V^2)\oplus q\,\mathrm{L}(W,V^2)\oplus q\,\mathrm{L}(V^1,W))^{\mathbb{C}^*}\Big]\\ &\qquad - [(q^2\,\mathrm{L}(V^1,V^2))^{\mathbb{C}^*}].\end{aligned}$$

Each factor of the right hand side can be written in the form $\sum_i \alpha_i\boxtimes\beta_i$ for some $\alpha_i,\beta_i\in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}[\rho])$. For example, the first factor is equal to

$$\mathrm{L}(V^1,V^2)^{\mathbb{C}^*} = \bigoplus_m \mathrm{L}(V^1(m),V^2(m)),$$

where $V^a(m)$ is the weight space of V^a, i.e.,

$$V^a(m) = \{v\in V^a \mid t\diamond v = t^m v\}.$$

The remaining factors have a similar description. Thus by Proposition 7.2.1, $\mathfrak{M}[\rho]$ has property $(T'_{G_{\mathbf{w}}\times\mathbb{C}^*})$.

Moreover, the above shows that $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}[\rho])$ is generated by exterior powers of $V_k(m)$, $W_k(m)$ and its duals (as an $R(G_{\mathbf{w}}\times\mathbb{C}^*)$-algebra). Note that these bundles have constant rank on $\mathfrak{M}[\rho]$. If $\mathfrak{M}[\rho]$ have components M_1, M_2, …, the structure sheaf of M_1 (extended to $\mathfrak{M}[\rho]$ by setting 0 outside) cannot be represented by $V_k(m)$, $W_k(m)$. This contradiction shows that $\mathfrak{M}[\rho]$ is connected.

The assertion for Chow groups can be proved in exactly the same way. By the above argument, the fundamental class $[\Delta\mathfrak{M}[\rho]]$ is the top Chern class of $\operatorname{Ker}\tau^{\mathbb{C}^*}/\operatorname{Im}\sigma^{\mathbb{C}^*}$, which can be represented as $\sum_i p_1^*a_i\cup p_2^*b_i$ for some a_i, $b_i\in A(X)$. □

Theorem 7.3.5. *$\mathfrak{M}$ and $\mathfrak{L}$ have properties (S) and $(T_{G_{\mathbf{w}}\times\mathbb{C}^*})$. Moreover, the bilinear pairing*

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M})\times K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L})\ni(F,F')\longmapsto p_*(F\otimes^L_{\mathfrak{M}}F')\in R(G_{\mathbf{w}}\times\mathbb{C}^*)$$

is nondegenerate. A similar pairing between $A_*(\mathfrak{M})$ and $A_*(\mathfrak{L})$ is also nondegenerate. Here p is the canonical map from $\mathfrak{M}$ to the point.

Proof. We apply Lemma 7.1.5. By [44, 2.8], the metric on $\mathfrak{M}$ defined in §2.4 is complete. By the construction, it is invariant under $K_{\mathbf{w}}\times S^1$, where $K_{\mathbf{w}}=\prod\mathrm{U}(W_k)$ is the maximal compact subgroup of $G_{\mathbf{w}}$. (Note that the hyper-Kähler structure is *not* invariant under the S^1-action, but the metric is invariant.) The moment map for the S^1-actions is given by

$$\frac{1}{2}\left(\sum_h \|B_h\|^2 + \sum_k \left(\|i_k\|^2 + \|j_k\|^2\right)\right).$$

This is a proper function on $\mathfrak{M}$. Thus Lemma 7.1.5 is applicable. Note that we have $\mathfrak{L} = \{x \in \mathfrak{M} \mid \lim_{t\to\infty} t \diamond x \text{ exists}\}$ as in [44, 5.8]. (Though our $\mathbb{C}^*$-action is different from the one in [44], the same proof works.) $\square$

7.4. **Fixed point subvariety.** Let A be an abelian reductive subgroup of $G_{\mathbf{w}} \times \mathbb{C}^*$ as in §4. Let $\mathfrak{M}^A$ and $\mathfrak{L}^A$ be the fixed point set in $\mathfrak{M}$ and $\mathfrak{L}$ respectively. Exactly as in the previous subsection, we have the following generalization of Theorem 7.3.5.

Theorem 7.4.1. *$\mathfrak{M}^A$ and $\mathfrak{L}^A$ have properties (T) and (S). Moreover, the bilinear pairing*

$$K(\mathfrak{M}^A) \times K(\mathfrak{L}^A) \ni (F, F') \longmapsto p_*(F \otimes^L_{\mathfrak{M}^A} F') \in \mathbb{Z}$$

is nondegenerate. A similar pairing between $A_*(\mathfrak{M}^A)$ and $A_*(\mathfrak{L}^A)$ is also nondegenerate. Here p is the canonical map from $\mathfrak{M}^A$ to the point.

7.5. **Connectedness of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$.** Let us consider a natural homomorphism

$$R(G_{\mathbf{w}} \times \mathbb{C}^* \times G_{\mathbf{v}}) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}), \tag{7.5.1}$$

which sends representations to bundles associated with tautological bundles. If we can apply Proposition 7.2.1 to $\mathfrak{M}$, then this homomorphism is surjective. Unfortunately we cannot apply Proposition 7.2.1 since $\mathfrak{M}$ is not projective. However, it seems reasonable to conjecture that the homomorphism (7.5.1) is surjective. In particular, it implies that $\mathfrak{M}$ is connected as in the proof of Proposition 7.3.4. This was stated in [45, 6.2]. But the proof contains a gap since the function $\|s_1\|$ may not be proper in general.

8. Convolution

Let X_1, X_2, X_3 be a nonsingular quasi-projective variety, and write $p_{ab}\colon X_1 \times X_2 \times X_3 \to X_a \times X_b$ for the projection $((a,b) = (1,2),(2,3),(1,3))$.

Suppose Z_{12} (resp. Z_{23}) is a closed subvariety of $X_1 \times X_2$ (resp. $X_2 \times X_3$) such that the restriction of the projection $p_{13}\colon p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to X_1 \times X_3$ is proper. Let $Z_{12} \circ Z_{23} = p_{13}\left(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})\right)$. We can define the *convolution product* $*\colon K(Z_{12}) \otimes K(Z_{23}) \to K(Z_{12} \circ Z_{23})$ by

$$K_{12} * K_{23} \overset{\text{def.}}{=} p_{13*}\left(p_{12}^* K_{12} \otimes^L_{X_1\times X_2\times X_3} p_{23}^* K_{23}\right)$$

for $K_{12} \in K(Z_{12}), K_{23} \in K(Z_{23})$.

Note that the convolution product depends on the ambient spaces X_1, X_2 and X_3. When we want to specify them, we say the convolution product relative to X_1, X_2, X_3.

In this section, we study what happens when X_1, X_2, X_3 are replaced by

(a) submanifolds S_1, S_2, S_3 of X_1, X_2, X_3,
(b) principal G-bundles P_1, P_2, P_3 over X_1, X_2, X_3.

Although we work on nonequivariant K-theory, the results extend to the case of equivariant K-theory, the Borel-Moore homology group, or any other reasonable theory in a straightforward way.

8.1. Before studying the above problem, we recall the following lemma which will be used several times.

Lemma 8.1.1. *In the above setting, we further assume that $X_1 = X_2$ and $Z_{12} = \operatorname{Image}\Delta_{X_1}$, where Δ_{X_1} is the diagonal embedding $X_1 \to X_1 \times X_2$. Then we have*

$$(\Delta_{X_1})_*[E] * K_{23} = p_2^*[E] \otimes K_{23}$$

for a vector bundle E over X_1, where $p_2 \colon X_2 \times X_3 \to X_2 = X_1$ is the projection, and $\otimes$ in the right hand side is the tensor product (6.1.1) between $K^0(Z_{23})$ and $K(Z_{23})$.

The proof is obvious from the definition, and is omitted.

8.2. **Restriction of the convolution to submanifolds.** Suppose we have non-singular closed submanifolds S_1, S_2, S_3 of X_1, X_2, X_3 such that

$$(S_1 \times X_2) \cap Z_{12} \subset S_1 \times S_2, \qquad (S_2 \times X_3) \cap Z_{23} \subset S_2 \times S_3. \tag{8.2.1}$$

By this assumption, we have

$$(S_1 \times X_3) \cap (Z_{12} \circ Z_{23}) \subset S_1 \times S_3. \tag{8.2.2}$$

Let Z'_{12} (resp. Z'_{23}) be the intersection $(S_1 \times S_2) \cap Z_{12}$ (resp. $(S_2 \times S_3) \cap Z_{23}$). By (8.2.2), we have $Z'_{12} \circ Z'_{23} = (S_1 \times X_3) \cap (Z_{12} \circ Z_{23})$. We have the convolution product $*' \colon K(Z'_{12}) \otimes K(Z'_{23}) \to K(Z'_{12} \circ Z'_{23})$ relative to S_1, S_2, S_3:

$$K'_{12} *' K'_{23} \overset{\text{def.}}{=} p'_{13*}\left(p'^*_{12} K'_{12} \otimes^L_{S_1 \times S_2 \times S_3} p'^*_{23} K'_{23}\right),$$

where p'_{ab} is the projection $S_1 \times S_2 \times S_3 \to S_a \times S_b$.

We want to relate two convolution products $*$ and $*'$ via pull-back homomorphisms. For this purpose, we consider the inclusion $i_a \times \mathrm{id}_{X_b} \colon S_a \times X_b \to X_a \times X_b$, where i_a is the inclusion $S_a \hookrightarrow X_a$ ($(a,b) = (1,2), (2,3), (1,3)$). By (8.2.1), we have a pull-back homomorphism

$$K(Z_{12}) \cong K(X_1 \times X_2; Z_{12}) \xrightarrow{(i_1 \times \mathrm{id}_{X_2})^*} K(S_1 \times X_2; Z_{12} \cap S_1 \times X_2) \cong K(Z'_{12}).$$

Similarly, we have

$$K(Z_{23}) \xrightarrow{(i_2 \times \mathrm{id}_{X_3})^*} K(Z'_{23}), \qquad K(Z_{12} \circ Z_{23}) \xrightarrow{(i_1 \times \mathrm{id}_{X_3})^*} K(Z'_{12} \circ Z'_{23}).$$

Proposition 8.2.3. *For $K_{12} \in K(Z_{12}), K_{23} \in K(Z_{23})$, we have*

$$(i_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) = ((i_1 \times \mathrm{id}_{X_2})^* K_{12}) *' ((i_2 \times \mathrm{id}_{X_3})^* K_{23}). \tag{8.2.4}$$

Namely, the following diagram commutes:

$$\begin{tikzcd}
K(Z_{12}) \otimes K(Z_{23}) \arrow[r, "*"] \arrow[d, "(i_1 \times \mathrm{id}_{X_2})^* \otimes (i_2 \times \mathrm{id}_{X_3})^*"'] & K(Z_{12} \circ Z_{23}) \arrow[d, "(i_1 \times \mathrm{id}_{X_3})^*"] \\
K(Z'_{12}) \otimes K(Z'_{23}) \arrow[r, "*'"] & K(Z'_{12} \circ Z'_{23}).
\end{tikzcd}$$

Example 8.2.5. Suppose $X_1 = X_2$, $S_1 = S_2$ and $Z_{12} = \operatorname{Image}\Delta_{X_1}$, where Δ_{X_1} is the diagonal embedding $X_1 \to X_1 \times X_2$. Then the above assumption $S_1 \times X_2 \cap Z_{12} \subset S_1 \times S_2$ is satisfied, and we have $Z'_{12} = \operatorname{Image}\Delta_{S_1}$, where Δ_{S_1} is the diagonal embedding $S_1 \to S_1 \times S_2$. If E is a vector bundle over X_1, we have

$(i_1 \times \mathrm{id}_{X_2})^*(\Delta_{X_1})_*[E] = (\Delta_{S_1})_*[i_1^* E]$ by the base change [13, 5.3.15]. By Lemma 8.1.1, we have

$$(\Delta_{X_1})_*[E] * K_{23} = p_2^*[E] \otimes K_{23},$$
$$(\Delta_{S_1})_*[i^* E] *' (i_2 \times \mathrm{id}_{X_3})^* K_{23} = p_2'^* i_1^*[E] \otimes (i_2 \times \mathrm{id}_{X_3})^* K_{23},$$

where $p_2 \colon X_2 \times X_3 \to X_2$, $p_2' \colon S_2 \times X_3 \to S_2$ are the projections. Note

$$\begin{aligned} p_2'^* i^*[E] \otimes (i_2 \times \mathrm{id}_{X_3})^* K_{23} &= (i_2 \times \mathrm{id}_{X_3})^* p_2^*[E] \otimes (i_2 \times \mathrm{id}_{X_3})^* K_{23} \\ &= (i_2 \times \mathrm{id}_{X_3})^* \left(p_2^*[E] \otimes K_{23}\right) \end{aligned}$$

by (6.4.1). Hence we have (8.2.4) in this case.

Proof of Proposition 8.2.3. In order to relate $*$ relative to X_1, X_2, X_3 and $*'$ relative to S_1, S_2, S_3, we replace X_a by S_a factor by factor.

Step 1. First we want to replace X_1 by S_1. We consider the following fiber square:

$$\begin{tikzcd} S_1 \times X_2 \times X_3 \arrow[r, "i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3}"] \arrow[d, "p''_{13}"'] & X_1 \times X_2 \times X_3 \arrow[d, "p_{13}"] \\ S_1 \times X_3 \arrow[r, "i_1 \times \mathrm{id}_{X_3}"'] & X_1 \times X_3, \end{tikzcd}$$

where p''_{13} is the projection. We have

(8.2.6)
$$\begin{aligned} (i_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) &= (i_1 \times \mathrm{id}_{X_3})^* p_{13*}\left(p_{12}^* K_{12} \otimes^L_{X_1 \times X_2 \times X_3} p_{23}^* K_{23}\right) \\ &= p''_{13*}(i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* \left(p_{12}^* K_{12} \otimes^L_{X_1 \times X_2 \times X_3} p_{23}^* K_{23}\right) \\ &= p''_{13*}\left((i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* p_{12}^* K_{12} \otimes^L_{S_1 \times X_2 \times X_3} (i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* p_{23}^* K_{23}\right), \end{aligned}$$

where we have used the base change ([13, 5.3.15]) in the second equality and (6.4.1) in the third equality. If $p''_{12} \colon S_1 \times X_2 \times X_3 \to S_1 \times X_2$ denotes the projection, we have $p_{12} \circ (i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3}) = (i_1 \times \mathrm{id}_{X_2}) \circ p''_{12}$. Hence we get

$$(i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* p_{12}^* K_{12} = p''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12}.$$

Similarly, we have

$$(i_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* p_{23}^* K_{23} = p''^*_{23} K_{23},$$

where $p''_{23} \colon S_1 \times X_2 \times X_3 \to X_2 \times X_3$ is the projection. Substituting this into (8.2.6), we obtain

(8.2.7) $$(i_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) = p''_{13*}\left(p''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times X_2 \times X_3} p''^*_{23} K_{23}\right).$$

Step 2. Next we replace X_2 by S_2. By (8.2.1), we have a homomorphism

$$(\mathrm{id}_{S_1} \times i_2)_* \colon K(Z'_{12}) \cong K(S_1 \times S_2; Z'_{12}) \to K(S_1 \times X_2; Z'_{12}) \cong K(Z'_{12}),$$

which is just the identity operator. We will consider $(i_1 \times \mathrm{id}_{X_2})^* K_{12} \in K(Z'_{12})$ as an element of $K(S_1 \times S_2; Z'_{12})$ or $K(S_1 \times X_2; Z'_{12})$ interchangeably. We consider the fiber square

$$\begin{CD} S_1 \times S_2 \times X_3 @>{p'''_{12}}>> S_1 \times S_2 \\ @V{\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3}}VV @VV{\mathrm{id}_{S_1} \times i_2}V \\ S_1 \times X_2 \times X_3 @>>{p''_{12}}> S_1 \times X_2, \end{CD}$$

where p'''_{12} is the projection. By base change [13, 5.3.15], we get

$$\begin{aligned} p''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} &= p''^*_{12}(\mathrm{id}_{S_1} \times i_2)_*(i_1 \times \mathrm{id}_{X_2})^* K_{12} \\ &= (\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3})_* p'''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12}. \end{aligned}$$

By the projection formula (6.5.1), we get

$$\begin{aligned} &p''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times X_2 \times X_3} p''^*_{23} K_{23} \\ &= (\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3})_* \left(p'''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times S_2 \times X_3} (\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3})^* p''^*_{23} K_{23}\right). \end{aligned}$$

Substituting this into (8.2.7), we have

$$\begin{aligned} &(i_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) \\ &\quad = p''_{13*}(\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3})_* \\ &\qquad \times \left(p'''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times S_2 \times X_3} (\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3})^* p''^*_{23} K_{23}\right) \\ &\quad = p'''_{13*}\left(p'''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times S_2 \times X_3} p'''^*_{23}(i_2 \times \mathrm{id}_{X_3})^* K_{23}\right), \end{aligned} \tag{8.2.8}$$

where $p'''_{13}\colon S_1 \times S_2 \times X_3 \to S_1 \times X_3$, $p'''_{23}\colon S_1 \times S_2 \times X_3 \to S_2 \times X_3$ are the projections, and we have used $p'''_{13} = p''_{13} \circ (\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3})$ and $p''_{23} \circ (\mathrm{id}_{S_1} \times i_2 \times \mathrm{id}_{X_3}) = (i_2 \times \mathrm{id}_{X_3}) \circ p'''_{23}$.

Step 3. We finally replace X_3 by S_3. By (8.2.1), we have a homomorphism

$$(\mathrm{id}_{S_2} \times i_3)_*\colon K(Z'_{23}) \cong K(S_2 \times S_3; Z'_{23}) \to K(S_2 \times X_3; Z'_{23}) \cong K(Z'_{23}),$$

which is just the identity operator. We consider the fiber square

$$\begin{CD} S_1 \times S_2 \times S_3 @>{p'_{23}}>> S_2 \times S_3 \\ @V{\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3}VV @VV{\mathrm{id}_{S_2} \times i_3}V \\ S_1 \times S_2 \times X_3 @>>{p'''_{23}}> S_2 \times X_3. \end{CD}$$

By base change [13, 5.3.15], we get

$$\begin{aligned} p'''^*_{23}(i_2 \times \mathrm{id}_{X_3})^* K_{23} &= p'''^*_{23}(\mathrm{id}_{S_2} \times i_3)_*(i_2 \times \mathrm{id}_{X_3})^* K_{23} \\ &= (\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3)_* p'^*_{23}(i_2 \times \mathrm{id}_{X_3})^* K_{23}. \end{aligned}$$

Substituting this into (8.2.8), we obtain

$$\begin{aligned} &(i_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) \\ &\quad = p'''_{13*}\left(p'''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times S_2 \times X_3} (\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3)_* p'^*_{23}(i_2 \times \mathrm{id}_{X_3})^* K_{23}\right) \\ &\quad = p'''_{13*}(\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3)_* \\ &\qquad \times \left((\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3)^* p'''^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times S_2 \times S_3} p'^*_{23}(i_2 \times \mathrm{id}_{X_3})^* K_{23}\right) \\ &\quad = (\mathrm{id}_{S_1} \times i_3)_* p'_{13*}\left(p'^*_{12}(i_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{S_1 \times S_2 \times S_3} p'^*_{23}(i_2 \times \mathrm{id}_{X_3})^* K_{23}\right), \end{aligned}$$

where we have used (6.5.1) in the second equality and $p'''_{13} \circ (\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3) = (\mathrm{id}_{S_1} \times i_3) \circ p'_{13}$, $p'''_{12} \circ (\mathrm{id}_{S_1} \times \mathrm{id}_{S_2} \times i_3) = p'_{12}$ in the third equality. Finally, by (8.2.2), the homomorphism $(\mathrm{id}_{S_1} \times i_3)_*$ is just the identity operator $K(Z'_{12} \circ Z'_{23}) \to K(Z'_{12} \circ Z'_{23})$. Thus we have the assertion. □

8.3. **Convolution and principal bundles.** Let G be a linear algebraic group and suppose that we have principal G-bundles $\pi_a \colon P_a \to X_a$ over X_a for $a = 1, 2, 3$. Consider the restriction of the principal G-bundle $\pi_a \times \mathrm{id}_{X_b} \colon P_a \times X_b \to X_a \times X_b$ to Z_{ab} for $(a, b) = (1, 2), (2, 3)$. Then the pull-back homomorphism gives a canonical isomorphism

$$(\pi_a \times \mathrm{id}_{X_b})^* \colon K(Z_{ab}) \xrightarrow{\cong} K^G((\pi_a \times \mathrm{id}_{X_b})^{-1} Z_{ab}). \tag{8.3.1}$$

Similarly we have an isomorphism

$$(\pi_1 \times \mathrm{id}_{X_3})^* \colon K(Z_{12} \circ Z_{23}) \xrightarrow{\cong} K^G((\pi_1 \times \mathrm{id}_{X_3})^{-1}(Z_{12} \circ Z_{23})). \tag{8.3.2}$$

Let G act on $P_a \times P_b$ diagonally. We assume that there exists a closed G-invariant subvariety Z'_{ab} of $P_a \times P_b$ such that

$$\begin{aligned} &\text{the restriction of } \mathrm{id}_{P_a} \times \pi_b \colon P_a \times P_b \to P_a \times X_b \text{ to } Z'_{ab} \text{ is proper, and} \\ &(\mathrm{id}_{P_a} \times \pi_b)(Z'_{ab}) = (\pi_a \times \mathrm{id}_{X_b})^{-1} Z_{ab} \end{aligned} \tag{8.3.3}$$

for $(a, b) = (1, 2), (2, 3)$.

Let p'_{ab} be the projection $P_1 \times P_2 \times P_3 \to P_a \times P_b$. Since the restriction of the projection $p'_{13} \colon p'^{-1}_{12}(Z'_{12}) \cap p'^{-1}_{23}(Z'_{23}) \to P_1 \times P_3$ is proper, we have the convolution product $*' \colon K^G(Z'_{12}) \otimes K^G(Z'_{23}) \to K^G(Z'_{12} \circ Z'_{23})$ given by

$$K'_{12} *' K'_{23} \stackrel{\text{def.}}{=} p'_{13*}\left(p'^*_{12} K'_{12} \otimes^L_{P_1 \times P_2 \times P_3} p'^*_{23} K'_{23}\right)$$

for $K'_{12} \in K^G(Z'_{12})$, $K'_{23} \in K^G(Z'_{23})$. We want to compare this convolution product with that on $K(Z_{12}) \otimes K(Z_{23})$.

By (8.3.3), we have

$$(\mathrm{id}_{P_a} \times \pi_b)_* K'_{ab} \in K^G((\mathrm{id}_{P_a} \times \pi_b)(Z'_{ab})) = K^G((\pi_a \times \mathrm{id}_{X_b})^{-1} Z_{ab}).$$

Via (8.3.1), we define

$$K_{ab} \stackrel{\text{def.}}{=} ((\pi_a \times \mathrm{id}_{X_b})^*)^{-1} (\mathrm{id}_{P_a} \times \pi_b)_* K'_{ab} \in K(Z_{ab}). \tag{8.3.4}$$

Thus we can consider the convolution product $K_{12} * K_{23}$.

By the construction, we have

$$(\mathrm{id}_{P_1} \times \pi_3)(Z'_{12} \circ Z'_{23}) = (\pi_1 \times \mathrm{id}_{X_3})^{-1}(Z_{12} \circ Z_{23}).$$

Noticing that the restriction of $\mathrm{id}_{P_1} \times \pi_3$ to $Z'_{12} \circ Z'_{23}$ is proper, we have

$$(\mathrm{id}_{P_1} \times \pi_3)_*(K'_{12} *' K'_{23}) \in K^G((\pi_1 \times \mathrm{id}_{X_3})^{-1}(Z_{12} \circ Z_{23}).$$

Combining this with (8.3.2), we have

$$((\pi_1 \times \mathrm{id}_{X_3})^*)^{-1} (\mathrm{id}_{P_1} \times \pi_3)_*(K'_{12} *' K'_{23}) \in K(Z_{12} \circ Z_{23}).$$

Proposition 8.3.5. *In the above setup, we have*

$$(\pi_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) = (\mathrm{id}_{P_1} \times \pi_3)_*(K'_{12} *' K'_{23}). \tag{8.3.6}$$

Namely the following diagram commutes:

$$\begin{CD} K^G(Z'_{12}) \otimes K^G(Z'_{23}) @>{*'}>> K^G(Z'_{12} \circ Z'_{23}) \\ @VVV @VVV \\ K(Z_{12}) \otimes K(Z_{23}) @>{*}>> K(Z_{12} \circ Z_{23}), \end{CD}$$

where the left vertical arrow is

$$((\pi_1 \times \mathrm{id}_{X_2})^*)^{-1} (\mathrm{id}_{P_1} \times \pi_2)_* \otimes ((\pi_2 \times \mathrm{id}_{X_3})^*)^{-1} (\mathrm{id}_{P_2} \times \pi_3)_*,$$

and the right vertical arrow is

$$((\pi_1 \times \mathrm{id}_{X_3})^*)^{-1} (\mathrm{id}_{P_1} \times \pi_3)_*.$$

Example 8.3.7. Suppose $X_1 = X_2$, $P_1 = P_2$ and $Z_{12} = \mathrm{Image}\,\Delta_{X_1}$, where Δ_{X_1} is the diagonal embedding $X_1 \to X_1 \times X_2$. If we take $Z'_{12} = \mathrm{Image}\,\Delta_{P_1}$, where Δ_{P_1} is the diagonal embedding $P_1 \to P_1 \times P_2$, the assumption (8.3.3) is satisfied. In fact, the restriction of $\mathrm{id}_{P_1} \times \pi_2$ to Z'_{12} is an isomorphism. Take a vector bundle E and consider $K_{12} = \Delta_{X_1 *}[E]$. By the isomorphism $K(X_1) \xrightarrow[\cong]{\pi_1^*} K^G(P_1)$, we can define $K'_{12} = \Delta_{P_1 *}\pi_1^*[E]$. Then both $(\mathrm{id}_{P_1} \times \pi_2)_* K'_{12}$ and $(\pi_1 \times \mathrm{id}_{X_2})^* K_{12}$ is $\Delta'_*[E]$ where $\Delta' \colon P_1 \to (\mathrm{id}_{P_1} \times \pi_2)\Delta_{P_1} = (\pi_1 \times \mathrm{id}_{X_2})^{-1}\Delta_{X_1}$ is the natural isomorphism. Hence (8.3.4) holds for K_{12} and K'_{12}. By Lemma 8.1.1, we have

$$\begin{aligned} &(\Delta_{X_1})_*[E] * K_{23} = p_2^*[E] \otimes K_{23}, \\ &(\Delta_{P_1})_*[\pi_1^* E] *' K'_{23} = p_2'^* \pi_2^*[E] \otimes K'_{23}, \end{aligned}$$

where $p_2 \colon X_2 \times X_3 \to X_2$, $p'_2 \colon P_2 \times X_3 \to P_2$ are the projections. We can directly check (8.3.6) in this case.

Proof of Proposition 8.3.5. As in the proof of Proposition 8.2.3, we replace X_a by P_a factor by factor.

Step 1. First we replace X_1 by P_1. Consider the following fiber square:

$$\begin{CD} P_1 \times X_2 \times X_3 @>{\pi_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3}}>> X_1 \times X_2 \times X_3 \\ @V{p''_{13}}VV @VV{p_{13}}V \\ P_1 \times X_3 @>>{\pi_1 \times \mathrm{id}_{X_3}}> X_1 \times X_3, \end{CD}$$

where p''_{13} is the projection. By base change [13, 5.3.15] and (6.4.1), we have

(8.3.8)

$$\begin{aligned} (\pi_1 \times \mathrm{id}_{X_3})^*(K_{12} * K_{23}) &= (\pi_1 \times \mathrm{id}_{X_3})^* p_{13*} \left(p_{12}^* K_{12} \otimes^L_{X_1 \times X_2 \times X_3} p_{23}^* K_{23}\right) \\ &= p''_{13*}(\pi_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* \left(p_{12}^* K_{12} \otimes^L_{X_1 \times X_2 \times X_3} p_{23}^* K_{23}\right) \\ &= p''_{13*} \left((\pi_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* p_{12}^* K_{12} \otimes^L_{P_1 \times X_2 \times X_3} (\pi_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_{X_3})^* p_{23}^* K_{23}\right) \\ &= p''_{13*} \left(p''^*_{12}(\pi_1 \times \mathrm{id}_{X_2})^* K_{12} \otimes^L_{P_1 \times X_2 \times X_3} p''^*_{23} K_{23}\right), \end{aligned}$$

where $p''_{12} \colon P_1 \times X_2 \times X_3 \to P_1 \times X_2$ and $p''_{23} \colon P_1 \times X_2 \times X_3 \to X_2 \times X_3$ are projections.

Step 2. Consider the fiber square

$$\begin{array}{ccc} P_1 \times P_2 \times X_3 & \xrightarrow{p_{12}'''} & P_1 \times P_2 \\ {\scriptstyle \mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}_{P_1} \times \pi_2} \\ P_1 \times X_2 \times X_3 & \xrightarrow[p_{12}'']{} & P_1 \times X_2, \end{array}$$

where p_{12}''' is the projection. By base change [13, 5.3.15], we have

$$(\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3})_* p_{12}'''^* K_{12}' = p_{12}''^* (\mathrm{id}_{P_1} \times \pi_2)_* K_{12}' = p_{12}''^* (\pi_1 \times \mathrm{id}_{X_2})^* K_{12},$$

where we have used (8.3.4) for $(a,b) = (1,2)$. Substituting this into (8.3.8), we get
(8.3.9)

$$\begin{aligned} &(\pi_1 \times \mathrm{id}_{X_3})^* (K_{12} * K_{23}) \\ &\quad = p_{13*}'' \left((\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3})_* p_{12}'''^* K_{12}' \otimes^L_{P_1 \times X_2 \times X_3} p_{23}''^* K_{23} \right) \\ &\quad = p_{13*}'' (\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3})_* \left(p_{12}'''^* K_{12}' \otimes^L_{P_1 \times P_2 \times X_3} (\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3})^* p_{23}''^* K_{23} \right), \end{aligned}$$

where we have used (6.5.1) in the second equality. Let $p_{23}''' \colon P_1 \times P_2 \times X_3 \to P_2 \times X_3$ be the projection. By $p_{23}'' \circ (\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3}) = (\pi_2 \times \mathrm{id}_{X_3}) \circ p_{23}'''$, we have

$$(\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3})^* p_{23}''^* = p_{23}'''^* (\pi_2 \times \mathrm{id}_{X_3})^*.$$

We also have

$$p_{13*}'' (\mathrm{id}_{P_1} \times \pi_2 \times \mathrm{id}_{X_3})_* = p_{13*}''',$$

where $p_{13} \colon P_1 \times P_2 \times X_3 \to P_1 \times X_3$ is the projection. Substituting these two equalities into (8.3.9), we get
(8.3.10)

$$(\pi_1 \times \mathrm{id}_{X_3})^* (K_{12} * K_{23}) = p_{13*}''' \left(p_{12}'''^* K_{12}' \otimes^L_{P_1 \times P_2 \times X_3} p_{23}'''^* (\pi_2 \times \mathrm{id}_{X_3})^* K_{23} \right).$$

Step 3. Consider the fiber square

$$\begin{array}{ccc} P_1 \times P_2 \times P_3 & \xrightarrow{p_{23}'} & P_2 \times P_3 \\ {\scriptstyle \mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}_{P_2} \times \pi_3} \\ P_1 \times P_2 \times X_3 & \xrightarrow{p_{23}'''} & P_2 \times X_3. \end{array}$$

By base change [13, 5.3.15], we have

$$(\mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3)_* p_{23}'^* K_{23}' = p_{23}'''^* (\mathrm{id}_{P_2} \times \pi_3)_* K_{23}' = p_{23}'''^* (\pi_2 \times \mathrm{id}_{X_3})^* K_{23},$$

where we have used (8.3.4) for $(a,b) = (2,3)$ in the second equality. Substituting this into (8.3.10), we have

$$\begin{aligned} &(\pi_1 \times \mathrm{id}_{X_3})^* (K_{12} * K_{23}) \\ &\quad = p_{13*}''' \left(p_{12}'''^* K_{12}' \otimes^L_{P_1 \times P_2 \times X_3} (\mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3)_* p_{23}'^* K_{23}' \right) \\ &\quad = p_{13*}''' (\mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3)_* \left((\mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3)^* p_{12}'''^* K_{12}' \otimes^L_{P_1 \times P_2 \times P_3} p_{23}'^* K_{23}' \right). \end{aligned}$$

By $p_{13}''' \circ (\mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3) = (\mathrm{id}_{P_1} \times \pi_3) \circ p_{13}'$ and $p_{12}''' \circ (\mathrm{id}_{P_1} \times \mathrm{id}_{P_2} \times \pi_3) = p_{12}'$, we get

$$(\pi_1 \times \mathrm{id}_{X_3})^* (K_{12} * K_{23}) = (\mathrm{id}_{P_1} \times \pi_3)_* \circ p_{13*}' \left(p_{12}'^* K_{12}' \otimes^L_{P_1 \times P_2 \times P_3} p_{23}'^* K_{23}' \right).$$

This proves our assertion. ∎

9. A homomorphism $\mathbf{U}_q(\mathbf{L}\mathfrak{g}) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$

9.1. Let us define an analogue of the Steinberg variety $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$ by

$$\{(x^1, x^2) \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \mid \pi(x^1) = \pi(x^2)\}. \tag{9.1.1}$$

Here $\pi(x^1) = \pi(x^2)$ means that $\pi(x^1)$ is equal to $\pi(x^2)$ if we regard both as elements of $\mathfrak{M}_0(\infty, \mathbf{w})$ by (2.5.4). This is a closed subvariety of $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$.

The map $p_{13}\colon p_{12}^{-1}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \cap p_{23}^{-1}(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})) \to \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$ is proper and its image is contained in $Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w})$. Hence we can define the convolution product on the equivariant K-theory:

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \otimes K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w})).$$

Let

$$\prod_{\mathbf{v}^1,\mathbf{v}^2}{}' K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$$

be the subspace of $\prod_{\mathbf{v}^1,\mathbf{v}^2} K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ consisting of elements $(F_{\mathbf{v}^1,\mathbf{v}^2})$ such that

(1) for fixed $\mathbf{v}^1$, $F_{\mathbf{v}^1,\mathbf{v}^2} = 0$ for all but finitely many choices of $\mathbf{v}^2$,
(2) for fixed $\mathbf{v}^2$, $F_{\mathbf{v}^1,\mathbf{v}^2} = 0$ for all but finitely many choices of $\mathbf{v}^1$.

The convolution product $*$ is well defined on $\prod'_{\mathbf{v}^1,\mathbf{v}^2} K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$. When the underlying graph is of type ADE, $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is empty for all but finitely many choices of $\mathbf{v}$, so $\prod'$ is just the direct product $\prod$.

Let $Z(\mathbf{w})$ denote the disjoint union $\bigsqcup_{\mathbf{v}^1,\mathbf{v}^2} Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$. When we write

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})),$$

we mean $\prod'_{\mathbf{v}^1,\mathbf{v}^2} K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ as convention.

The second projection $G_{\mathbf{w}} \times \mathbb{C}^* \to \mathbb{C}^*$ induces a homomorphism $R(\mathbb{C}^*) \to R(G_{\mathbf{w}} \times \mathbb{C}^*)$. Thus $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ is an $R(\mathbb{C}^*)$-algebra. Moreover, $R(\mathbb{C}^*)$ is isomorphic to $\mathbb{Z}[q, q^{-1}]$ where q^m corresponds to $L(m)$ in (2.8.1). Thus $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ is a $\mathbb{Z}[q, q^{-1}]$-algebra.

The aim of this section and the next two sections is to define the homomorphism from $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ into $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. We first define the map on generators of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, and then check the defining relation.

9.2. First we want to define the image of q^h, $h_{k,m}$.

Let $C_k^\bullet(\mathbf{v}, \mathbf{w})$ be the $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant complex over $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ defined in (2.9.1). We consider $C_k^\bullet(\mathbf{v}, \mathbf{w})$ as an element of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$ by identifying it with the alternating sum

$$-[q^{-2}V_k] + \left[q^{-1}\left(\bigoplus_{l:k\neq l}[-\langle h_k, \alpha_l\rangle]_q V_l \oplus W_k\right)\right] - [V_k].$$

The rank of the complex (2.9.1), as an element of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$ (see §6.2), is given by

$$\operatorname{rank} C_k^\bullet(\mathbf{v}, \mathbf{w}) = -\sum_{l:k\neq l}(\alpha_k, \alpha_l)\dim V_l + \dim W_k - 2\dim V_k = \langle h_k, \mathbf{w} - \mathbf{v}\rangle.$$

Let Δ denote the diagonal embedding $\mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}(\mathbf{v},\mathbf{w}) \times \mathfrak{M}(\mathbf{v},\mathbf{w})$;

$$
\begin{aligned}
q^h &\longmapsto \sum_{\mathbf{v}} q^{\langle h,\mathbf{w}-\mathbf{v}\rangle} \Delta_* \mathcal{O}_{\mathfrak{M}(\mathbf{v},\mathbf{w})}, \\
p_k^+(z) &\longmapsto \sum_{\mathbf{v}} \Delta_* \left(\bigwedge\nolimits_{-1/z}(C_k^\bullet(\mathbf{v},\mathbf{w})) \right)^+, \\
p_k^-(z) &\longmapsto \sum_{\mathbf{v}} \Delta_* \left((-z)^{\operatorname{rank} C_k^\bullet(\mathbf{v},\mathbf{w})} \det C_k^\bullet(\mathbf{v},\mathbf{w})^* \bigwedge\nolimits_{-1/z}(C_k^\bullet(\mathbf{v},\mathbf{w})) \right)^-,
\end{aligned}
\tag{9.2.1}
$$

where $(\ )^\pm$ denotes the expansion at $z=\infty$, 0 respectively. Note that

$$
\psi_k^\pm(z) = q^{\pm h_k} \frac{p_k^\pm(qz)}{p_k^\pm(q^{-1}z)} \longmapsto \sum_{\mathbf{v}} q^{\operatorname{rank} C_k^\bullet(\mathbf{v},\mathbf{w})} \Delta_* \left(\frac{\bigwedge_{-1/qz}(C_k^\bullet(\mathbf{v},\mathbf{w}))}{\bigwedge_{-q/z}(C_k^\bullet(\mathbf{v},\mathbf{w}))} \right)^\pm .
$$

9.3. Next we define the images of $e_{k,r}$ and $f_{k,r}$. They are given by line bundles over Hecke correspondences.

Let $\mathbf{v}^1$, $\mathbf{v}^2$ and $\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w})$ be as in §5.1. By the definition, the quotient V_k^2/V_k^1 defines a line bundle over $\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w})$. The generator $e_{k,r}$ is very roughly defined as the rth power of V_k^2/V_k^1, but we need a certain modification in order to have the correct commutation relation.

For the modification, we need to consider the following variants of $C_k^\bullet(\mathbf{v},\mathbf{w})$:

$$
C_k'^\bullet(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \operatorname{Coker} \sigma_k, \qquad C_k''^\bullet(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} V_k[-1], \tag{9.3.1}
$$

where $V_k[-1]$ means that we consider the complex consisting of V_k in degrees 1 and 0 for other degrees. Since α is injective, we have $C_k^\bullet(\mathbf{v},\mathbf{w}) = C_k'^\bullet(\mathbf{v},\mathbf{w}) + C_k''^\bullet(\mathbf{v},\mathbf{w})$ in $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))$. We have the corresponding decomposition of the Cartan matrix $\mathbf{C}$:

$$
\mathbf{C} = \mathbf{C}' + \mathbf{C}'', \qquad \text{where } \mathbf{C}' \overset{\text{def.}}{=} \mathbf{C} - \mathbf{I},\ \mathbf{C}'' \overset{\text{def.}}{=} \mathbf{I}.
$$

We identify $\mathbf{C}'$ (resp. $\mathbf{C}''$) with a map given by

$$
\mathbf{v} = \sum v_k \alpha_k \in \bigoplus \mathbb{Z}\alpha_k \mapsto \sum v_k(\alpha_k - \Lambda_k) \quad \left(\text{resp. } \sum v_k \Lambda_k\right) \in P.
$$

We also need matrices $\mathbf{C}_\Omega$, $\mathbf{C}_{\overline{\Omega}}$ given by

$$
\mathbf{C}_\Omega \overset{\text{def.}}{=} \mathbf{I} - \mathbf{A}_\Omega, \qquad \mathbf{C}_{\overline{\Omega}} \overset{\text{def.}}{=} \mathbf{I} - \mathbf{A}_{\overline{\Omega}},
$$

where $\mathbf{A}_\Omega$, $\mathbf{A}_{\overline{\Omega}}$ are as in (2.1.1). We also identify them with maps $\bigoplus \mathbb{Z}\alpha_k \to P$ exactly as above.

Let $\omega\colon \mathfrak{M}(\mathbf{v}^1,\mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2,\mathbf{w}) \to \mathfrak{M}(\mathbf{v}^2,\mathbf{w}) \times \mathfrak{M}(\mathbf{v}^1,\mathbf{w})$ be the exchange of factors. Let us denote $\omega\left(\mathfrak{P}_k(\mathbf{v}^2+\alpha_k,\mathbf{w})\right) \subset \mathfrak{M}(\mathbf{v}^2+\alpha_k,\mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2,\mathbf{w})$ by $\mathfrak{P}_k^-(\mathbf{v}^2,\mathbf{w})$. As on $\mathfrak{P}_k(\mathbf{v},\mathbf{w})$, we have a natural line bundle over $\mathfrak{P}_k^-(\mathbf{v}^2,\mathbf{w})$. Let us denote it by V_k^1/V_k^2.

Now we define the images of $e_{k,r}$, $f_{k,r}$ by

(9.3.2)

$$
\begin{aligned}
e_{k,r} &\longmapsto \sum_{\mathbf{v}^2} (-1)^{-\langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^2\rangle} \left[i_* \left(q^{-1} V_k^2/V_k^1\right)^{\otimes r - \langle h_k, \mathbf{C}''\mathbf{v}^2\rangle} \otimes \det C_k'^\bullet(\mathbf{v}^2,\mathbf{w})^* \right], \\
f_{k,r} &\longmapsto \sum_{\mathbf{v}^2} (-1)^{\langle h_k, \mathbf{w} - \mathbf{C}_\Omega\mathbf{v}^2\rangle} \left[i_*^- \left(q^{-1} V_k^1/V_k^2\right)^{\otimes r + \langle h_k, \mathbf{w} - \mathbf{C}'\mathbf{v}^2\rangle} \otimes \det C_k''^\bullet(\mathbf{v}^2,\mathbf{w})^* \right],
\end{aligned}
$$

where $i\colon \mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}) \to Z(\mathbf{v}^2-\alpha_k,\mathbf{v}^2;\mathbf{w})$ and $i^-\colon \mathfrak{P}_k^-(\mathbf{v}^2,\mathbf{w}) \to Z(\mathbf{v}^2+\alpha_k,\mathbf{v}^2;\mathbf{w})$ are the inclusions. Hereafter, we may omit i_* or i_*^-, hoping that it causes no confusion.

9.4.

Theorem 9.4.1. *The assignments* (9.2.1), (9.3.2) *define a homomorphism*

$$\mathbf{U}_q(\mathbf{L}\mathfrak{g}) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$$

of $\mathbb{Q}(q)$*-algebras.*

We need to check the defining relations (1.2.1)–(1.2.11). We do not need to consider the relations (1.2.1), (1.2.5) because we are considering $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ instead of $\mathbf{U}_q(\widehat{\mathfrak{g}})$. The relations (1.2.2), (1.2.3) and (1.2.4) follow from Lemma 8.1.1 and the fact that $E\otimes F \cong F\otimes E$. The relation (1.2.6) also follows from Lemma 8.1.1. The remaining relations will be checked in the next two sections.

10. Relations (I)

10.1. **Relation (1.2.7).** Fix a vertex $k\in I$ and take $\mathbf{v}^1$, $\mathbf{v}^2=\mathbf{v}^1+\alpha_k$. Let i be the inclusion $\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}) \to Z(\mathbf{v}^1,\mathbf{v}^2;\mathbf{w})$ and let p_1 and p_2 be the projections $\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}) \to \mathfrak{M}(\mathbf{v}^1,\mathbf{w})$ and $\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}) \to \mathfrak{M}(\mathbf{v}^2,\mathbf{w})$ respectively.

By Lemma 8.1.1, we have

(10.1.1)

$$\begin{aligned}
&\Delta_*{\textstyle\bigwedge}_{-1/z}(C_l^\bullet(\mathbf{v}^1,\mathbf{w})) * i_*\left[\sum_{r=-\infty}^{\infty}\left(q^{-1}V_k^2/V_k^1\right)^{\otimes r} w^{-r}\right] * \Delta_*\left({\textstyle\bigwedge}_{-1/z}C_l^\bullet(\mathbf{v}^2,\mathbf{w})\right)^{-1}\\
&= i_*\left[{\textstyle\bigwedge}_{-1/z}p_1^*(C_l^\bullet(\mathbf{v}^1,\mathbf{w}))\otimes\left({\textstyle\bigwedge}_{-1/z}p_2^*(C_l^\bullet(\mathbf{v}^2,\mathbf{w}))\right)^{-1}\otimes\sum_{r=-\infty}^{\infty}\left(q^{-1}V_k^2/V_k^1\right)^{\otimes r} w^{-r}\right].
\end{aligned}$$

We have the following equality in $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}))$:

$$V_k^1 = V_k^2 - V_k^2/V_k^1.$$

Hence we have

$${\textstyle\bigwedge}_{-1/z}p_1^*C_l^\bullet(\mathbf{v}^1,\mathbf{w})\otimes\left({\textstyle\bigwedge}_{-1/z}p_2^*C_l^\bullet(\mathbf{v}^2,\mathbf{w})\right)^{-1} = {\textstyle\bigwedge}_{-1/z}[\langle h_k,\alpha_l\rangle]_q\left(q^{-1}V_k^2/V_k^1\right).$$

Substituting this into (10.1.1), we get

$$\begin{aligned}
&\Delta_*{\textstyle\bigwedge}_{-1/z}(C_l^\bullet(\mathbf{v}^1,\mathbf{w})) * i_*\left[\sum_{r=-\infty}^{\infty}\left(q^{-1}V_k^2/V_k^1\right)^{\otimes r} w^{-r}\right] * \Delta_*\left({\textstyle\bigwedge}_{-1/z}C_l^\bullet(\mathbf{v}^2,\mathbf{w})\right)^{-1}\\
&= \begin{cases}
\left(1-\dfrac{wq}{z}\right)\left(1-\dfrac{w}{zq}\right) i_*\left[\displaystyle\sum_{r=-\infty}^{\infty}\left(q^{-1}V_k^2/V_k^1\right)^{\otimes r} w^{-r}\right] & \text{if } k=l,\\
\displaystyle\prod_{p=0}^{-\langle h_k,\alpha_l\rangle-1}\left(1-\dfrac{q^{\langle h_k,\alpha_l\rangle+1+2p}w}{z}\right)^{-1} i_*\left[\displaystyle\sum_{r=-\infty}^{\infty}\left(q^{-1}V_k^2/V_k^1\right)^{\otimes r} w^{r}\right] & \text{otherwise.}
\end{cases}
\end{aligned}$$

This is equivalent to (1.2.7) for $x_l^+(w)$. The relation (1.2.7) for $x_l^-(w)$ can be proved in the same way.

10.2. **Relation (1.2.8) for $k \neq l$.** Fix two vertices $k \neq l$. Let $\mathbf{v}^1$, $\mathbf{v}^2$, $\mathbf{v}^3$, $\mathbf{v}^4$ be dimension vectors such that

$$\mathbf{v}^2 = \mathbf{v}^1 + \alpha_k = \mathbf{v}^3 + \alpha_l, \quad \mathbf{v}^4 = \mathbf{v}^1 - \alpha_l = \mathbf{v}^3 - \alpha_k.$$

We want to compute $e_{k,r} * f_{l,s}$ and $f_{l,s} * e_{k,r}$ in the component $K^{G_{\mathbf{w}}\times\mathbb{C}}(Z(\mathbf{v}^1, \mathbf{v}^3, \mathbf{w}))$.

Let us consider the intersection

$$p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l^-(\mathbf{v}^3, \mathbf{w}) \qquad (\text{resp. } p_{12}^{-1}\mathfrak{P}_l^-(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3, \mathbf{w}))$$

in $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$ (resp. $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^4, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$). On the intersection, we have the inclusion of restrictions of tautological bundles

$$V^1 \subset V^2 \supset V^3 \qquad (\text{resp. } V^1 \supset V^4 \subset V^3).$$

Lemma 10.2.1. *The above two intersections are transversal, and there is a $G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant isomorphism between them such that*

(a) *it is the identity operators on the factor $\mathfrak{M}(\mathbf{v}^1, \mathbf{w})$ and $\mathfrak{M}(\mathbf{v}^3, \mathbf{w})$,*
(b) *it induces isomorphisms $V_k^2/V_k^1 \cong V_k^3/V_k^4$ and $V_l^2/V_l^3 \cong V_l^1/V_l^4$.*

Proof. See [45, Lemmas 9.8, 9.9, 9.10 and their proofs]. □

Since the intersection is transversal, we have

$$e_{k,r} * f_{l,s} = (-1)^{-\langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^2\rangle + \langle h_l, \mathbf{w} - \mathbf{C}_\Omega \mathbf{v}^3\rangle} p_{13*}[\mathcal{L}], \tag{10.2.2}$$

where $\mathcal{L}$ is the following line bundle over $p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l^-(\mathbf{v}^3, \mathbf{w})$:

$$\left(q^{-1}V_k^2/V_k^1\right)^{\otimes r - \langle h_k, \mathbf{C}''\mathbf{v}^2\rangle} \otimes \left(q^{-1}V_l^2/V_l^3\right)^{\otimes s + \langle h_l, \mathbf{w} - \mathbf{C}'\mathbf{v}^3\rangle} \\ \otimes \det C_k'^{\bullet}(\mathbf{v}^2, \mathbf{w})^* \otimes \det C_l''^{\bullet}(\mathbf{v}^3, \mathbf{w})^*.$$

Similarly, we have

$$f_{l,s} * e_{k,r} = (-1)^{\langle h_l, \mathbf{w} - \mathbf{C}_\Omega \mathbf{v}^4\rangle - \langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^3\rangle} p_{13*}[\mathcal{L}'], \tag{10.2.3}$$

where $\mathcal{L}'$ is the following line bundle over $p_{12}^{-1}\mathfrak{P}_l^-(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3, \mathbf{w})$:

$$\left(q^{-1}V_l^1/V_l^4\right)^{\otimes s + \langle h_l, \mathbf{w} - \mathbf{C}'\mathbf{v}^4\rangle} \otimes \left(q^{-1}V_k^3/V_k^4\right)^{\otimes r - \langle h_k, \mathbf{C}''\mathbf{v}^3\rangle} \\ \otimes \det C_l''^{\bullet}(\mathbf{v}^4, \mathbf{w})^* \otimes \det C_k'^{\bullet}(\mathbf{v}^3, \mathbf{w})^*.$$

Let us compare (10.2.2) and (10.2.3). On $p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l^-(\mathbf{v}^3, \mathbf{w})$, we have

$$\begin{aligned} \det C_k'^{\bullet}(\mathbf{v}^2, \mathbf{w}) &= \det C_k'^{\bullet}(\mathbf{v}^3, \mathbf{w}) \otimes \det\left([-(\alpha_k, \alpha_l)]_q (q^{-1}V_l^2/V_l^3)\right) \\ &= \det C_k'^{\bullet}(\mathbf{v}^3, \mathbf{w}) \otimes \left(q^{-1}V_l^2/V_l^3\right)^{\otimes -(\alpha_k, \alpha_l)}. \end{aligned}$$

On the other hand, we have

$$\det C_l''^{\bullet}(\mathbf{v}^3, \mathbf{w}) = \det C_l''^{\bullet}(\mathbf{v}^4, \mathbf{w})$$

on $p_{12}^{-1}\mathfrak{P}_l^-(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3, \mathbf{w})$. Hence under the isomorphism in Lemma 10.2.1, we obtain

$$\mathcal{L} \cong \mathcal{L}',$$

where we have used

$$\langle h_l, \mathbf{C}'\mathbf{v}^3\rangle = \langle h_l, \mathbf{C}'\mathbf{v}^4\rangle - \mathbf{A}_{lk}, \quad \langle h_k, \mathbf{C}''\mathbf{v}^2\rangle = \langle h_k, \mathbf{C}''\mathbf{v}^3\rangle. \tag{10.2.4}$$

By

$$\langle h_l, \mathbf{C}_\Omega \mathbf{v}^3\rangle = \langle h_l, \mathbf{C}_\Omega \mathbf{v}^4\rangle - (\mathbf{A}_\Omega)_{lk},$$
$$\langle h_k, \mathbf{C}_{\overline{\Omega}} \mathbf{v}^2\rangle = \langle h_k, \mathbf{C}_{\overline{\Omega}} \mathbf{v}^3\rangle - (\mathbf{A}_{\overline{\Omega}})_{kl}, \quad (\mathbf{A}_\Omega)_{lk} = (\mathbf{A}_{\overline{\Omega}})_{kl},$$

we have

$$(-1)^{-\langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^2\rangle + \langle h_l, \mathbf{w} - \mathbf{C}_\Omega \mathbf{v}^3\rangle} = (-1)^{\langle h_l, \mathbf{w}-\mathbf{C}_\Omega\mathbf{v}^4\rangle - \langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^3\rangle}.$$

Thus we have $[e_{k,r}, f_{l,s}] = 0$.

10.3. **Relation (1.2.10).** We give the proof of (1.2.10) for $\pm = +$ in this subsection. The relation (1.2.10) for $\pm = -$ can be proved in a similar way, and hence is omitted.

Fix two vertices $k \neq l$. Let $\mathbf{v}^1$, $\mathbf{v}^2$, $\mathbf{v}^3$, $\mathbf{v}^4$ be dimension vectors such that

$$\mathbf{v}^2 = \mathbf{v}^1 + \alpha_k, \quad \mathbf{v}^4 = \mathbf{v}^1 + \alpha_l, \quad \mathbf{v}^3 = \mathbf{v}^2 + \alpha_l = \mathbf{v}^4 + \alpha_k = \mathbf{v}^1 + \alpha_k + \alpha_l.$$

We want to compute $e_{k,r} * e_{l,s}$ and $e_{l,s} * e_{k,r}$ in the component $K^{G_\mathbf{w}\times\mathbb{C}}(Z(\mathbf{v}^1, \mathbf{v}^3, \mathbf{w}))$.

Let us consider the intersection

$$p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l(\mathbf{v}^3, \mathbf{w}) \qquad (\text{resp. } p_{12}^{-1}\mathfrak{P}_l(\mathbf{v}^4, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3, \mathbf{w}))$$

in $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$ (resp. $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^4, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$).

Lemma 10.3.1. *The above two intersections are transversal respectively.*

Proof. The proof below is modeled on [45, 9.8, 9.9]. We give the proof for $p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l(\mathbf{v}^3, \mathbf{w})$. Then the same result for $p_{12}^{-1}\mathfrak{P}_l(\mathbf{v}^4, \mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3, \mathbf{w})$ follows by $k \leftrightarrow l$, $\mathbf{v}^2 \leftrightarrow \mathbf{v}^4$.

We consider the complex (5.1.1) for $\mathfrak{P}_k(\mathbf{v}^2, \mathbf{w})$ and $\mathfrak{P}_l(\mathbf{v}^3, \mathbf{w})$:

$$\mathrm{L}(V^1, V^2) \xrightarrow{\sigma^{12}} \mathrm{E}(V^1, V^2) \oplus \mathrm{L}(W, V^2) \oplus \mathrm{L}(V^1, W) \xrightarrow{\tau^{12}} \mathrm{L}(V^1, V^2) \oplus \mathcal{O},$$
$$\mathrm{L}(V^2, V^3) \xrightarrow{\sigma^{23}} \mathrm{E}(V^2, V^3) \oplus \mathrm{L}(W, V^3) \oplus \mathrm{L}(V^2, W) \xrightarrow{\tau^{23}} \mathrm{L}(V^2, V^3) \oplus \mathcal{O},$$

where we use suffixes 12, 23 to distinguish endomorphisms. We have sections s^{12} and s^{23} of $\operatorname{Ker}\tau^{12}/\operatorname{Im}\sigma^{12}$ and $\operatorname{Ker}\tau^{23}/\operatorname{Im}\sigma^{23}$ respectively.

Identifying these vector bundles and sections with those of pull-backs to $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$, we consider their zero loci $Z(s^{12}) = \mathfrak{P}_k(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^3, \mathbf{w})$ and $Z(s^{23}) = \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{P}_l(\mathbf{v}^3, \mathbf{w})$.

As in the proof of [45, 5.7], we consider the transpose of ∇s^{12}, ∇s^{23} via the symplectic form. Their sum gives a vector bundle endomorphism

$$\begin{aligned}{}^t(\nabla s^{12}) + {}^t(\nabla s^{22})\colon\ & \operatorname{Ker}{}^t\sigma^{12}/\operatorname{Im}{}^t\tau^{12} \oplus \operatorname{Ker}{}^t\sigma^{23}/\operatorname{Im}{}^t\tau^{23} \\ & \to T\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \oplus T\mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \oplus T\mathfrak{M}(\mathbf{v}^3, \mathbf{w}).\end{aligned}$$

It is enough to show that the kernel of ${}^t(\nabla s^{12}) + {}^t(\nabla s^{23})$ is zero at (x^1, x^2, x^3).

Take representatives (B^a, i^a, j^a) of x^a $(a = 1, 2, 3)$. Then we have ξ^{12}, ξ^{23} which satisfy (5.1.3) for (B^a, i^a, j^a). Suppose that

$$(C'^{12}, a'^{12}, b'^{12}) \pmod{\operatorname{Im}{}^t\tau^{12}} \oplus (C'^{23}, a'^{23}, b'^{23}) \pmod{\operatorname{Im}{}^t\tau^{23}}$$

lies in the kernel. Then there exist $\gamma^a \in \mathrm{L}(V^a, V^a)$ $(a = 1, 2, 3)$ such that

$$
(10.3.2)\qquad
\begin{cases}
\varepsilon C'^{12}\,\xi^{12} = \gamma^1\,B^1 - B^1\,\gamma^1,\\
b'^{12} = \gamma^1\,i^1,\\
-a'^{12}\,\xi^{12} = -j^1\,\gamma^1,
\end{cases}
\qquad
\begin{cases}
\varepsilon\,\xi^{23} C'^{23} = \gamma^3\,B^3 - B^3\,\gamma^3,\\
\xi^{23}\,b'^{23} = \gamma^3\,i^3,\\
-a'^{23} = -j^3\,\gamma^3,
\end{cases}
$$

$$
\begin{cases}
\varepsilon(\xi^{12}\,C'^{12} + C'^{23}\,\xi^{23}) = \gamma^2\,B^2 - B^2\,\gamma^2,\\
\xi^{12}\,b'^{12} + b'^{23} = \gamma^2\,i^2,\\
-a'^{12} - a'^{23}\xi^{23} = -j^2\gamma^2.
\end{cases}
$$

Then we have

$$
B^3\left(\xi^{23}(\gamma^2\xi^{12} - \xi^{12}\gamma^1) - \gamma^3\xi^{23}\xi^{12}\right) = \left(\xi^{23}(\gamma^2\xi^{12} - \xi^{12}\gamma^1) - \gamma^3\xi^{23}\xi^{12}\right)B^1,
$$
$$
j^3\left(\xi^{23}(\gamma^2\xi^{12} - \xi^{12}\gamma^1) - \gamma^3\xi^{23}\xi^{12}\right) = 0.
$$

Hence we have

$$
(10.3.3)\qquad \xi^{23}(\gamma^2\xi^{12} - \xi^{12}\gamma^1) - \gamma^3\xi^{23}\xi^{12} = 0
$$

by the stability condition.

Consider the equation (10.3.3) at the vertex l. Since $k \neq l$, ξ^{12}_l is an isomorphism. Hence (10.3.3) implies that $\operatorname{Im}\xi^{23}_l$ is invariant under γ^3_l. Since $\operatorname{Im}\xi^{23}_l$ is a codimension 1 subspace, the induced map $\gamma^3_l\colon V^3_l/\operatorname{Im}\xi^{23}_l \to V^3_l/\operatorname{Im}\xi^{23}_l$ is a scalar which we denote by λ^{23}. Moreover, there exists a homomorphism $\zeta^{23}_l\colon V^3_l \to V^2_l$ such that

$$
\gamma^3_l - \lambda^{23}\,\mathrm{id}_{V^3_l} = \xi^{23}_l\zeta^{23}_l.
$$

For another vertex $l' \neq l$, $\xi^{23}_{l'}$ is an isomorphism, hence we can define $\zeta^{23}_{l'}$ so that the same equation holds also for the vertex l'. Thus we have

$$
(10.3.4)\qquad \gamma^3 - \lambda^{23}\,\mathrm{id}_{V^3} = \xi^{23}\zeta^{23}.
$$

Substituting (10.3.4) into (10.3.2) and using the injectivity of ξ^{23}, we get

$$
\begin{cases}
C'^{23} = \varepsilon(\zeta^{23}B^3 - B^2\,\zeta^{23}),\\
a'^{23} = j^3(\xi^{23}\zeta^{23} + \lambda^{23}\,\mathrm{id}_{V^3}),\\
b'^{23} = (\zeta^{23}\xi^{23} + \lambda^{23}\,\mathrm{id}_{V^2})i^2.
\end{cases}
$$

This means that $(C'^{23}, a'^{23}, b'^{23}) = {}^t\tau^{23}(\zeta^{23} \oplus \lambda^{23})$.

Substituting (10.3.4) into (10.3.3) and noticing ξ^{23} is injective, we obtain

$$
(10.3.5)\qquad \left(\gamma^2 - (\zeta^{23}\xi^{23} + \lambda^{23}\,\mathrm{id}_{V^2})\right)\xi^{12} = \xi^{12}\gamma^1.
$$

Thus $\operatorname{Im}\xi^{12}$ is invariant under $\gamma^2 - (\zeta^{23}\xi^{23} + \lambda^{23}\,\mathrm{id}_{V^2})$. Arguing as above, we can find a constant λ^{12} and a homomorphism $\zeta^{12}\colon V^2 \to V^1$ such that

$$
(10.3.6)\qquad \gamma^2 - (\zeta^{23}\xi^{23} + \lambda^{23}\,\mathrm{id}_{V^2}) - \lambda^{12}\,\mathrm{id}_{V^2} = \xi^{12}\zeta^{12}.
$$

Substituting this equation into (10.3.2), we get

$$
\begin{cases}
C'^{12} = \varepsilon(\zeta^{12}B^2 - B^1\,\zeta^{12}),\\
a'^{12} = j^2(\xi^{12}\zeta^{12} + \lambda^{12}\,\mathrm{id}_{V^2}),\\
b'^{12} = (\zeta^{12}\xi^{12} + \lambda^{12}\,\mathrm{id}_{V^1})i^1.
\end{cases}
$$

This means that $(C'^{12}, a'^{12}, b'^{12}) = {}^t\tau^{12}(\zeta^{12} \oplus \lambda^{12})$. Hence ${}^t(\nabla s^{12}) + {}^t(\nabla s^{23})$ is injective. $\square$

Let us consider the variety $\widetilde{Z}_{kl}$ (resp. $\widetilde{Z}_{lk}$) of all pairs $(B,i,j) \in \mu^{-1}(0)^{\mathrm{s}}$ and $S \subset V^3$ satisfying the following:

(a) S is a subspace with $\dim S = \mathbf{v}^1 = \mathbf{v}^3 - \alpha_k - \alpha_l$,
(b) S is B-stable,
(c) $\operatorname{Im} i \subset S$,
(d) the induced homomorphism $B_h \colon V_k^3/S_k \to V_l^3/S_l$ (resp. $B_{\overline{h}} \colon V_l^3/S_l \to V_k^3/S_k$) is zero for h with $\operatorname{out}(h) = k$, $\operatorname{in}(h) = l$.

Then $\widetilde{Z}_{kl}/G_{\mathbf{v}^3}$ (resp. $\widetilde{Z}_{lk}/G_{\mathbf{v}^3}$) is isomorphic to $p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l(\mathbf{v}^3,\mathbf{w})$ (resp. $p_{12}^{-1}\mathfrak{P}_l(\mathbf{v}^2,\mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3,\mathbf{w})$). The isomorphism is given by defining

$$\begin{aligned}
(B^1, i^1, j^1) &\overset{\text{def.}}{=} \text{the restriction of } (B,i,j) \text{ to } S,\\
(B^2, i^2, j^2) &\overset{\text{def.}}{=} \text{the restriction of } (B,i,j) \text{ to } S',\\
\Big(\text{resp. } (B^4, i^4, j^4) &\overset{\text{def.}}{=} \text{the restriction of } (B,i,j) \text{ to } S''\Big),\\
(B^3, i^3, j^3) &\overset{\text{def.}}{=} (B,i,j),
\end{aligned}$$

where S' (resp. S'') is given by

$$S'_m \overset{\text{def.}}{=} \begin{cases} V_m & \text{if } m \neq l \\ S_l & \text{if } m = l \end{cases} \qquad \left(\text{resp. } S''_m \overset{\text{def.}}{=} \begin{cases} V_m & \text{if } m \neq k \\ S_k & \text{if } m = k \end{cases}\right).$$

It is also clear that the restriction of p_{13} to $\widetilde{Z}_{kl}/G_{\mathbf{v}^3}$ (resp. $\widetilde{Z}_{lk}/G_{\mathbf{v}^3}$) is an isomorphism onto its image. Hereafter, we identify $\widetilde{Z}_{kl}/G_{\mathbf{v}^3}$ (resp. $\widetilde{Z}_{lk}/G_{\mathbf{v}^3}$) with the image. Then $\widetilde{Z}_{kl}/G_{\mathbf{v}^3}$ and $\widetilde{Z}_{lk}/G_{\mathbf{v}^3}$ are closed subvarieties of $Z(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})$. Let $i_{kl} \colon \widetilde{Z}_{kl}/G_{\mathbf{v}^3} \to Z(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})$ (resp. $i_{lk} \colon \widetilde{Z}_{lk}/G_{\mathbf{v}^3} \to Z(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})$) denote the inclusion.

The quotient V_k^3/S_k (resp. V_l^3/S_l) forms a line bundle over

$$\widetilde{Z}_{kl}/G_{\mathbf{v}^3} = p_{12}^{-1}\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_l(\mathbf{v}^3,\mathbf{w})$$

(resp. $\widetilde{Z}_{lk}/G_{\mathbf{v}^3} = p_{12}^{-1}\mathfrak{P}_l(\mathbf{v}^2,\mathbf{w}) \cap p_{23}^{-1}\mathfrak{P}_k(\mathbf{v}^3,\mathbf{w})$). By the above consideration, $e_{k,r} * e_{l,s}$ (resp. $e_{l,s} * e_{k,r}$) is represented by

(10.3.7)
$$\begin{aligned}
(-1)^{\langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^2\rangle + \langle h_l, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^3\rangle} i_{kl*}\big[&(q^{-1}V_k^3/S_k)^{\otimes r - \langle h_k, \mathbf{C}''\mathbf{v}^2\rangle} \otimes (q^{-1}V_l^3/S_l)^{\otimes s - \langle h_l, \mathbf{C}''\mathbf{v}^3\rangle}\\
&\otimes \det C_k'^{\bullet}(\mathbf{v}^2,\mathbf{w})^* \otimes \det C_l'^{\bullet}(\mathbf{v}^3,\mathbf{w})^*\big]
\end{aligned}$$

(resp.

$$\begin{aligned}
(-1)^{\langle h_l, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^4\rangle + \langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^3\rangle} i_{lk*}\big[&(q^{-1}V_l^3/S_l)^{\otimes s - \langle h_l, \mathbf{C}''\mathbf{v}^4\rangle} \otimes (q^{-1}V_k^3/S_k)^{\otimes r - \langle h_k, \mathbf{C}''\mathbf{v}^3\rangle}\\
&\otimes \det C_l'^{\bullet}(\mathbf{v}^4,\mathbf{w})^* \otimes \det C_k'^{\bullet}(\mathbf{v}^3,\mathbf{w})^*\big]).
\end{aligned}$$

Note that we have

(10.3.8)
$$\begin{gathered}
\langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^2\rangle + \langle h_l, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^3\rangle = \langle h_l, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^4\rangle + \langle h_k, \mathbf{C}_{\overline{\Omega}}\mathbf{v}^3\rangle \pm (\alpha_k, \alpha_l),\\
\langle h_k, \mathbf{C}''\mathbf{v}^2\rangle = \langle h_k, \mathbf{C}''\mathbf{v}^3\rangle, \qquad \langle h_l, \mathbf{C}''\mathbf{v}^3\rangle = \langle h_l, \mathbf{C}''\mathbf{v}^4\rangle,\\
\det C_k'^{\bullet}(\mathbf{v}^2,\mathbf{w}) = \det C_k'^{\bullet}(\mathbf{v}^3,\mathbf{w}) \otimes (q^{-1}V_l^3/S_l)^{\otimes(\alpha_k,\alpha_l)},\\
\det C_l'^{\bullet}(\mathbf{v}^4,\mathbf{w}) = \det C_l'^{\bullet}(\mathbf{v}^3,\mathbf{w}) \otimes (q^{-1}V_k^3/S_k)^{\otimes(\alpha_k,\alpha_l)}.
\end{gathered}$$

Set $b' = -(\alpha_k, \alpha_l)$. We consider

$$\bigoplus_{\mathrm{out}(h)=k,\mathrm{in}(h)=l} B_{\overline{h}}$$

as a section of the vector bundle $q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k)$ over $\widetilde{Z}_{kl}/G_{\mathbf{v}^3}$. Let us denote it by s_{kl}. Similarly $\bigoplus_{\mathrm{out}(h)=k,\mathrm{in}(h)=l} B_h$ is a section (denoted by s_{lk}) of the vector bundle $q[b']_q \operatorname{Hom}(V_k^3/S_k, V_l^3/S_l)$ over $\widetilde{Z}_{lk}/G_{\mathbf{v}^3}$.

Lemma 10.3.9. *The section s_{kl} (resp. s_{lk}) is transversal to the zero section (if it vanishes somewhere).*

Proof. Fix a subspace $S \subset V^3$ with $\dim S = \mathbf{v}^1$. Let P be the parabolic subgroup of $G_{\mathbf{v}^3}$ consisting of elements which preserve S. We also fix a complementary subspace T. Thus we have $V^3 = S \oplus T$. We will check the assertion for s_{kl}. The assertion for s_{lk} follows if we exchange k and l.

We consider

$$\widetilde{\mathbf{M}} \overset{\text{def.}}{=} \left\{ (B,i,j) \in \mathbf{M}(\mathbf{v}^3,\mathbf{w}) \;\middle|\; \begin{array}{l} B(S) \subset S,\ \operatorname{Im} i \subset S, \\ B_h \colon V_k^3/S_k \to V_l^3/S_l \text{ is } 0 \text{ for } h \\ \hfill \text{with } \mathrm{out}(h)=k,\ \mathrm{in}(h)=l \end{array} \right\}.$$

It is a linear subspace of $\mathbf{M}(\mathbf{v}^3,\mathbf{w})$. Let $\widetilde{\mu}\colon \widetilde{\mathbf{M}} \to \mathrm{L}(V^3,S)$ be the composition of the restriction of the moment map $\mu\colon \mathbf{M}(\mathbf{v}^3,\mathbf{w}) \to \mathrm{L}(V^3,V^3)$ to $\widetilde{\mathbf{M}}$ and the projection $\mathrm{L}(V^3,V^3) \to \mathrm{L}(V^3,S)$. Let $\widetilde{\mu}^{-1}(0)^{\mathrm{s}}$ denote the set of $(B,i,j) \in \widetilde{\mu}^{-1}(0)$ which is stable. It is preserved under the action of P and we have a $G_{\mathbf{v}^3}$-equivariant isomorphism

$$G_{\mathbf{v}^3} \times_P \widetilde{\mu}^{-1}(0)^{\mathrm{s}} \cong \widetilde{Z}_{kl}.$$

Note that the $\mathrm{L}(V^3,T)$-part of the moment map μ vanishes on $\widetilde{\mathbf{M}}$ thanks to the definition of $\widetilde{\mathbf{M}}$.

The assertion follows if we check that

$$d\widetilde{\mu} \oplus \Pi \colon \widetilde{\mathbf{M}} \to \mathrm{L}(V^3,S) \oplus q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k)$$

is surjective at $(B,i,j) \in \widetilde{\mu}^{-1}(0)^{\mathrm{s}}$. Here $\Pi\colon \widetilde{\mathbf{M}} \to q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k)$ is the natural projection. Thus it is enough to show that $d\widetilde{\mu}\colon \widetilde{\mathbf{M}} \cap \operatorname{Ker}\Pi \to \mathrm{L}(V^3,S)$ is surjective.

Suppose that $\zeta \in \mathrm{L}(S,V^3)$ is orthogonal to $d\widetilde{\mu}(\widetilde{\mathbf{M}} \cap \operatorname{Ker}\Pi)$, namely

$$\operatorname{tr}\left(\varepsilon(\delta B\, B + B\, \delta B)\zeta + \delta i\, j\, \zeta + i\, \delta j\, \zeta\right) = 0$$

for any $(\delta B, \delta i, \delta j) \in \widetilde{\mathbf{M}} \cap \operatorname{Ker}\Pi$. Hence we have

$$0 = j\zeta \in \mathrm{L}(S,W), \qquad 0 = B\zeta - \zeta B|_S \in \mathrm{L}(S,V^3),$$

where $B|_S$ is the restriction of B to S. Therefore the image of ζ is invariant under B and contained in $\operatorname{Ker} j$. By the stability condition, we have $\zeta = 0$. Thus we have proved the assertion. $\square$

Let $Z(s_{kl})$ (resp. $Z(s_{lk})$) be the zero locus of s_{kl} (resp. s_{lk}). By Lemma 10.3.9, we have the following exact sequence (Koszul complex) on $\widetilde{Z}_{kl}/G_{\mathbf{v}^3}$ (resp. $\widetilde{Z}_{lk}/G_{\mathbf{v}^3}$):

$$0 \to \bigwedge^{\max}(q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k))^* \to \cdots \to (q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k))^* \\ \to \mathcal{O}_{\widetilde{Z}_{kl}/G_{\mathbf{v}^3}} \to \mathcal{O}_{Z(s_{kl})} \to 0$$

(resp.

$$0 \to \bigwedge^{\max}(q[b']_q \operatorname{Hom}(V_k^3/S_k, V_l^3/S_l))^* \to \cdots \to (q[b']_q \operatorname{Hom}(V_k^3/S_k, V_l^3/S_l))^* \\ \to \mathcal{O}_{\widetilde{Z}_{lk}/G_{\mathbf{v}^3}} \to \mathcal{O}_{Z(s_{lk})} \to 0).$$

Hence we have the following equality in $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{Z}_{kl}/G_{\mathbf{v}^3})$ (resp. $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{Z}_{lk}/G_{\mathbf{v}^3})$):

$$\mathcal{O}_{Z(s_{kl})} = \bigwedge\nolimits_{-1}(q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k))^* \\ \Big(\text{resp. } \mathcal{O}_{Z(s_{lk})} = \bigwedge\nolimits_{-1}(q[b']_q \operatorname{Hom}(V_k^3/S_k, V_l^3/S_l))^*\Big).$$

Both $Z(s_{kl})$ and $Z(s_{lk})$ consist of all pairs $(B,i,j) \in \mu^{-1}(0)^{\mathrm{s}}$ and $S \subset V^3$ satisfying the following:

(a) S is a subspace with $\dim S = \mathbf{v}^1 = \mathbf{v}^3 - \alpha_k - \alpha_l$,
(b) S is B-stable,
(c) $\operatorname{Im} i \subset S$,
(d) the induced homomorphism $B\colon V/S \to V/S$ is zero,

modulo the action of $G_{\mathbf{v}^3}$. In particular, we have $Z(s_{kl}) = Z(s_{lk})$. Hence we have

$$i_{kl*}\left[\bigwedge\nolimits_{-1}(q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k))^*\right] = i_{lk*}\left[\bigwedge\nolimits_{-1}(q[b']_q \operatorname{Hom}(V_k^3/S_k, V_l^3/S_l))^*\right].$$

This implies

(10.3.10)
$$i_{kl*}\left[(q^{-1}V_k^3/S_k)^{\otimes r} \otimes (q^{-1}V_l^3/S_l)^{\otimes s} \otimes \bigwedge\nolimits_{-1}(q[b']_q \operatorname{Hom}(V_l^3/S_l, V_k^3/S_k))^*\right] \\ = i_{lk*}\left[(q^{-1}V_k^3/S_k)^{\otimes r} \otimes (q^{-1}V_l^3/S_l)^{\otimes s} \otimes \bigwedge\nolimits_{-1}(q[b']_q \operatorname{Hom}(V_k^3/S_k, V_l^3/S_l))^*\right]$$

by the projection formula (6.5.1). Multiplying this equality by $z^{-r}w^{-s}$ and taking the sum with respect to r and s, we get

$$\prod_{p=1}^{b'}(1 - q^{b'-2p}\frac{w}{z}) \sum_{r,s=-\infty}^{\infty} i_{kl*}\left[(q^{-1}V_k^3/S_k)^{\otimes r} \otimes (q^{-1}V_l^3/S_l)^{\otimes s}\right] z^{-r}w^{-s} \\ = \prod_{p=1}^{b'}(1 - q^{b'-2p}\frac{z}{w}) \sum_{r,s=-\infty}^{\infty} i_{lk*}\left[(q^{-1}V_k^3/S_k)^{\otimes r} \otimes (q^{-1}V_l^3/S_l)^{\otimes s}\right] z^{-r}w^{-s}.$$

Comparing this with (10.3.7) and using (10.3.8), we get (1.2.10).

10.4. **Relation (1.2.11).** We give the proof of (1.2.11) for $\pm = +$ in this subsection, assuming other relations. (The relations (1.2.8) with $k = l$ and (1.2.9) will be checked in the next section, but its proof is independent of results in this subsection.) The relation (1.2.11) for $\pm = -$ can be proved in a similar way, and hence is omitted.

By the proof of [45, 9.3], operators $e_{k,0}$, $f_{k,0}$ acting on $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ are locally nilpotent. (See also Lemma 13.2.4 below.) It is known that the constant term of

(1.2.11), i.e.,

$$\sum_{p=0}^{b}(-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_{q_k} e_{k,0}^{p} e_{l,0} e_{k,0}^{b-p} = 0, \tag{10.4.1}$$

can be deduced from the other relations and the local nilpotency of $e_{k,0}$, $f_{k,0}$ (see, e.g., [13, 4.3.2] for the proof for $q = 1$). Thus our task is to reduce

$$\sum_{\sigma\in S_b}\sum_{p=0}^{b}(-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_{q_k} e_{k,r_{\sigma(1)}}\cdots e_{k,r_{\sigma(p)}} e_{l,s} e_{k,r_{\sigma(p+1)}}\cdots e_{k,r_{\sigma(b)}} = 0 \tag{10.4.2}$$

to (10.4.1). This reduction was done by Grojnowski [23], but we reproduce it here for the sake of completeness.

For $p \in \{0, 1, \dots, b\}$, let $\mathbf{v}^0, \dots, \mathbf{v}^{b+1}$ be dimension vectors with

$$\mathbf{v}^i = \begin{cases} \mathbf{v}^{i-1} + \alpha_k & \text{if } i \neq p+1, \\ \mathbf{v}^{i-1} + \alpha_l & \text{if } i = p+1. \end{cases}$$

Let

$$\pi_{ij}\colon \mathfrak{M}(\mathbf{v}^0, \mathbf{w}) \times \cdots \times \mathfrak{M}(\mathbf{v}^{b+1}, \mathbf{w}) \to \mathfrak{M}(\mathbf{v}^i, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^j, \mathbf{w})$$

be the projection. Let

(10.4.3)

$$\begin{aligned}\widehat{\mathfrak{P}}_p \overset{\text{def.}}{=} &\pi_{12}^{-1}(\mathfrak{P}_k(\mathbf{v}^1, \mathbf{w})) \cap \cdots \cap \pi_{p-1,p}^{-1}(\mathfrak{P}_k(\mathbf{v}^p, \mathbf{w})) \\ &\cap \pi_{p,p+1}^{-1}(\mathfrak{P}_l(\mathbf{v}^{p+1}, \mathbf{w})) \cap \pi_{p+1,p+2}^{-1}(\mathfrak{P}_k(\mathbf{v}^{p+2}, \mathbf{w})) \cap \cdots \cap \pi_{b,b+1}^{-1}(\mathfrak{P}_k(\mathbf{v}^{b+1}, \mathbf{w})).\end{aligned}$$

This is equal to

$$\{(B, i, j, V^0 \subset \cdots \subset V^{b+1}) \mid \text{as below}\}/G_{\mathbf{v}^{b+1}},$$

(a) $(B, i, j) \in \mu^{-1}(0)^{\mathrm{s}}$ (in $\mathbf{M}(\mathbf{v}^{b+1}, \mathbf{w})$),
(b) V^i is a B-invariant subspace containing the image of i with $\dim V^i = \mathbf{v}^i$.

In particular, we have line bundles V^i/V^{i-1} on $\widehat{\mathfrak{P}}_p$ ($i = 1, \dots, b+1$). By the definition, there exists a line bundle $\mathfrak{L}_p(r_1, \dots, r_b; s) \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\widehat{\mathfrak{P}}_p)$ such that

$$e_{k,r_1}\cdots e_{k,r_p} e_{l,s} e_{k,r_{p+1}}\cdots e_{k,r_b} = \pi_{0,b+1*}\mathfrak{L}_p(r_1, \dots, r_b; s).$$

Moreover, we have

(10.4.4)

$$\begin{aligned}\mathfrak{L}_p(r_1, \dots, r_b; s) = &\left(q^{-1}V_k^1/V_k^0\right)^{\otimes r_1} \otimes \cdots \otimes \left(q^{-1}V_k^p/V_k^{p-1}\right)^{\otimes r_p} \otimes \left(q^{-1}V_l^{p+1}/V_l^p\right)^{\otimes s} \\ &\otimes \left(q^{-1}V_k^{p+2}/V_k^{p+1}\right)^{\otimes r_{p+1}} \otimes \cdots \otimes \left(q^{-1}V_k^{b+1}/V_k^b\right)^{\otimes r_b} \otimes \mathfrak{L}_p(0, \dots, 0; 0).\end{aligned}$$

Now consider the symmetrization. By (10.4.4), we have

$$\sum_{\sigma\in S_b} \mathfrak{L}_p(r_{\sigma(1)}, \dots, r_{\sigma(b)}; s) = T(V_k^{b+1}/V_k^0) \otimes \left(q^{-1}V_l^{p+1}/V_l^p\right)^{\otimes s} \otimes \mathfrak{L}_p(0, \dots, 0; 0)$$

for some tensor product $T(V_k^{b+1}/V_k^0)$ of exterior products of the bundle V_k^{b+1}/V_k^0 and its dual. (In the notation in §11.4 below, $T(V_k^{b+1}/V_k^0)$ corresponds to the symmetric function $\sum_{\sigma\in S_b} x_1^{r_{\sigma(1)}}\cdots x_b^{r_{\sigma(b)}}$.) Note that we have $q^{-1}V_l^{p+1}/V_l^p =$

$q^{-1}V_l^{b+1}/V_l^0$. Thus $T(V_k^{b+1}/V_k^0) \otimes \left(q^{-1}V_l^{b+1}/V_l^0\right)^{\otimes s}$ can be considered as a vector bundle over $\pi_{0,b+1}(\widehat{\mathfrak{P}}_p)$. Then the projection formula implies that

$$\sum_{\sigma\in S_b} \pi_{0,b+1*}\mathfrak{L}_p(r_{\sigma(1)},\dots,r_{\sigma(b)};s) = T(V_k^{b+1}/V_k^0) \otimes \left(q^{-1}V_l^{b+1}/V_l^0\right)^{\otimes s} \otimes \pi_{0,b+1*}\mathfrak{L}_p(0,\dots,0;0).$$

Noticing that $T(V_k^{b+1}/V_k^0) \otimes \left(q^{-1}V_l^{b+1}/V_l^0\right)^{\otimes s}$ is independent of p, we can derive (10.4.2) from (10.4.1).

11. Relations (II)

The purpose of this section is to check the relations (1.2.8) with $k = l$ and (1.2.9). Our strategy is the following. We first reduce the computation of the convolution product to the case of the graph of type A_1 using results in §8 and introducing modifications of quiver varieties and Hecke correspondences. Then we perform the computation using the explicit description of the equivariant K-theory for quiver varieties for the graph of type A_1.

In this section we fix a vertex k.

11.1. **Modifications of quiver varieties.** We take a collection of vector spaces $V = (V_l)_{l\in I}$ with $\dim V = \mathbf{v}$. Let μ_k be the $\operatorname{Hom}(V_k,V_k)$-component of $\mu\colon \mathbf{M}(\mathbf{v},\mathbf{w}) \to \mathrm{L}(V,V)$, i.e.,

$$\mu_k(B,i,j) \overset{\text{def.}}{=} \sum_{\operatorname{in}(h)=k} \varepsilon(h)B_hB_{\overline{h}} + i_kj_k.$$

Let

$$\widetilde{\mathfrak{M}}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \left\{(B,i,j)\in\mu_k^{-1}(0) \,\middle|\, \bigoplus_{\operatorname{out}(h)=k} B_h \oplus j_k\colon V_k \to \bigoplus_{\operatorname{out}(h)=k} V_{\operatorname{in}(h)} \oplus W_k \text{ is injective}\right\}\Big/ \mathrm{GL}(V_k).$$

This is a product of the quiver variety for the graph of type A_1 and the affine space:

$$\widetilde{\mathfrak{M}}(\mathbf{v},\mathbf{w}) = \mathfrak{M}(v_k,N)\times \mathbf{M}'(\mathbf{v},\mathbf{w}),$$

where

$$v_k = \dim V_k, \qquad N = -\sum_{l:l\neq k}(\alpha_k,\alpha_l)\dim V_l + \dim W_k,$$

$$\mathbf{M}'(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \bigoplus_{h\colon \substack{\operatorname{in}(h)\neq k\\ \operatorname{out}(h)\neq k}} \operatorname{Hom}(V_{\operatorname{out}(h)},V_{\operatorname{in}(h)}) \oplus \bigoplus_{l\colon l\neq k}\operatorname{Hom}(W_l,V_l)\oplus\operatorname{Hom}(V_l,W_l).$$

Moreover, the variety $\mathfrak{M}(v_k,N)$ is isomorphic to the cotangent bundle of the Grassmann manifold $G(v_k,N)$ of v_k-dimensional subspaces in the N-dimensional space. (See [44, Chap. 7] for details.) The isomorphism is given as follows: $\mathfrak{M}(v_k,N)$ is the set of $\mathrm{GL}(v_k,\mathbb{C})$-orbits of $i\colon\mathbb{C}^N\to\mathbb{C}^{v_k}$, $j\colon\mathbb{C}^{v_k}\to\mathbb{C}^N$ such that

(a) $ij = 0$,
(b) j is injective.

The action is given by $(i,j) \mapsto (gi, jg^{-1})$. Then

$$\mathfrak{M}(v_k, N) \ni G\cdot(i,j) \mapsto (\operatorname{Image} j \subset \mathbb{C}^N)$$

defines a map $\mathfrak{M}(v_k, N) \to G(v_k, N)$, and the linear map

$$ji\colon \mathbb{C}^N / \operatorname{Image} j \longrightarrow \operatorname{Image} j$$

defines a cotangent vector at $\operatorname{Image} j$.

Let $\mu^{-1}(0)^{\mathrm{s}}$ be as in Definition 2.3.1 and let $\mu_k^{-1}(0)^{\mathrm{s}}$ be the set of stable points in $\mu_k^{-1}(0)$. Although the stability condition (2.3.1) was defined only for $(B,i,j) \in \mu^{-1}(0)$, it can be defined for any $(B,i,j) \in \mathbf{M}(\mathbf{v},\mathbf{w})$. Let

$$\widetilde{\mathfrak{M}}^{\circ}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \mu_k^{-1}(0)^{\mathrm{s}} / \operatorname{GL}(V_k), \qquad \widehat{\mathfrak{M}}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \mu^{-1}(0)^{\mathrm{s}} / \operatorname{GL}(V_k).$$

We have a natural action of

$$G'_{\mathbf{v}} \overset{\text{def.}}{=} \prod_{l: l\neq k} \operatorname{GL}(V_l)$$

on $\widetilde{\mathfrak{M}}(\mathbf{v},\mathbf{w})$, $\widetilde{\mathfrak{M}}^{\circ}(\mathbf{v},\mathbf{w})$ and $\widehat{\mathfrak{M}}(\mathbf{v},\mathbf{w})$. We have the following relations between these varieties:

(a) $\widetilde{\mathfrak{M}}^{\circ}(\mathbf{v},\mathbf{w})$ is an open subvariety of $\widetilde{\mathfrak{M}}(\mathbf{v},\mathbf{w})$,

(b) $\widehat{\mathfrak{M}}(\mathbf{v},\mathbf{w})$ is a nonsingular closed subvariety of $\widetilde{\mathfrak{M}}^{\circ}(\mathbf{v},\mathbf{w})$ (defined by the equation $\mu_l = 0$ for $l \neq k$),

(c) $\widehat{\mathfrak{M}}(\mathbf{v},\mathbf{w})$ is a principal $G'_{\mathbf{v}}$-bundle over $\mathfrak{M}(\mathbf{v},\mathbf{w})$.

The vector space V_k defines vector bundles $\widetilde{\mathfrak{M}}(\mathbf{v},\mathbf{w})$, $\widetilde{\mathfrak{M}}^{\circ}(\mathbf{v},\mathbf{w})$, $\widehat{\mathfrak{M}}(\mathbf{v},\mathbf{w})$ and $\mathfrak{M}(\mathbf{v},\mathbf{w})$. We denote all of them by V_k for brevity, hoping that it causes no confusion.

11.2. **Modifications of Hecke correspondences.** Fix $n \in \mathbb{Z}_{>0}$. Take collections of vector spaces $V^1 = (V_l^1)_{l\in I}$, $V^2 = (V_l^2)_{l\in I}$ whose dimension vectors $\mathbf{v}^1$, $\mathbf{v}^2$ satisfy $\mathbf{v}^2 = \mathbf{v}^1 + n\alpha_k$. (For the proof of Theorem 9.4.1, it is enough to consider the case $n = 1$. But we study general n for a later purpose.) These data will be fixed throughout this subsection, and we use the following notation:

$$\widetilde{\mathfrak{M}}_1 = \widetilde{\mathfrak{M}}(\mathbf{v}^1,\mathbf{w}),\ \widetilde{\mathfrak{M}}_2 = \widetilde{\mathfrak{M}}(\mathbf{v}^2,\mathbf{w}),\ \widetilde{\mathfrak{M}}_1^{\circ} = \widetilde{\mathfrak{M}}^{\circ}(\mathbf{v}^1,\mathbf{w}),\ \widetilde{\mathfrak{M}}_2^{\circ} = \widetilde{\mathfrak{M}}^{\circ}(\mathbf{v}^2,\mathbf{w}),$$
$$\widehat{\mathfrak{M}}_1 = \widehat{\mathfrak{M}}(\mathbf{v}^1,\mathbf{w}),\ \widehat{\mathfrak{M}}_2 = \widehat{\mathfrak{M}}(\mathbf{v}^2,\mathbf{w}),\ \mathfrak{M}_1 = \mathfrak{M}(\mathbf{v}^1,\mathbf{w}),\ \mathfrak{M}_2 = \mathfrak{M}(\mathbf{v}^2,\mathbf{w}).$$

Consider $\widetilde{\mathfrak{M}}_1$ and $\widetilde{\mathfrak{M}}_2$. These varieties are products of quiver varieties for the graph of type A_1 and the affine space. We fix an isomorphism $V_l^1 \cong V_l^2$ for $l \neq k$. Then we have identifications $G'_{\mathbf{v}^1} \cong G'_{\mathbf{v}^2}$ and $\mathbf{M}'(\mathbf{v}^1,\mathbf{w}) \cong \mathbf{M}'(\mathbf{v}^2,\mathbf{w})$. We write them as G' and $\mathbf{M}'$ respectively for brevity. We write V_l for V_l^1 and V_l^2 for $l \neq k$. Let us define a subvariety $\widetilde{\mathfrak{P}}_k^{(n)} \subset \widetilde{\mathfrak{M}}_1 \times \widetilde{\mathfrak{M}}_2$ as the product of the Hecke correspondence for the graph of type A_1 and the diagonal for the affine space. Namely

$$\begin{array}{ccc} \widetilde{\mathfrak{P}}_k^{(n)} & \overset{\text{def.}}{=} & \mathfrak{P}_k^{(n)}(v_k^2, N) \times \Delta\mathbf{M}' \\ \cap & & \cap \\ \widetilde{\mathfrak{M}}_1 \times \widetilde{\mathfrak{M}}_2 & = & \mathfrak{M}(v_k^2 - n, N) \times \mathfrak{M}(v_k^2, N) \times \mathbf{M}' \times \mathbf{M}', \end{array}$$

where $v_k^2 = \dim V_k^2$ and

$$N = -\sum_{l: l\neq k} (\alpha_k, \alpha_l) \dim V_l^1 + \dim W_k = -\sum_{l: l\neq k} (\alpha_k, \alpha_l) \dim V_l^2 + \dim W_k.$$

Here $\mathfrak{P}_k^{(n)}(v_k^2, N) \subset \mathfrak{M}(v_k^2 - n, N) \times \mathfrak{M}(v_k^2, N)$ is the generalization of the Hecke correspondence introduced in (5.3.1). Since the graph is of type A_1, it is isomorphic to the conormal bundle of

$$O^{(n)}(v_k^2, N) \overset{\text{def.}}{=} \{(V_k^1, V_k^2) \in G(v_k^2 - n, N) \times G(v_k^2, N) \mid V_k^1 \subset V_k^2\}.$$

The quotient V_k^2/V_k^1 defines a vector bundle over $\mathfrak{P}_k^{(n)}(v_k^2, N)$ of rank n.

We have

$$\widetilde{\mathfrak{P}}_k^{(n)} \cap \left(\widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2\right) \subset \widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2^\circ, \qquad \widetilde{\mathfrak{P}}_k^{(n)} \cap \left(\widetilde{\mathfrak{M}}_1 \times \widetilde{\mathfrak{M}}_2^\circ\right) \subset \widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2^\circ. \tag{11.2.1}$$

The latter inclusion is obvious from the definition of stability, and the former one follows from the argument in [45, Proof of 4.5]. Let

$$\widetilde{\mathfrak{P}}_k^{(n)\circ} \overset{\text{def.}}{=} \widetilde{\mathfrak{P}}_k^{(n)} \cap \left(\widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2^\circ\right).$$

For $([B^1, i^1, j^1], [B^2, i^2, j^2]) \in \widetilde{\mathfrak{P}}_k^{(n)}$, the first factor $[B^1, i^1, j^1]$ satisfies $\mu(B^1, i^1, j^1) = 0$ if and only if the other factor $[B^2, i^2, j^2]$ also satisfies $\mu(B^2, i^2, j^2) = 0$. This implies that

(11.2.2)
$$\widetilde{\mathfrak{P}}_k^{(n)\circ} \cap \left(\widehat{\mathfrak{M}}_1 \times \widetilde{\mathfrak{M}}_2^\circ\right) \subset \widehat{\mathfrak{M}}_1 \times \widehat{\mathfrak{M}}_2, \qquad \widetilde{\mathfrak{P}}_k^{(n)\circ} \cap \left(\widetilde{\mathfrak{M}}_1^\circ \times \widehat{\mathfrak{M}}_2\right) \subset \widehat{\mathfrak{M}}_1 \times \widehat{\mathfrak{M}}_2.$$

Let

$$\widehat{\mathfrak{P}}_k^{(n)} \overset{\text{def.}}{=} \widetilde{\mathfrak{P}}_k^{(n)\circ} \cap \left(\widehat{\mathfrak{M}}_1 \times \widehat{\mathfrak{M}}_2\right).$$

The quotient V_k^2/V_k^1 defines vector bundles over $\widetilde{\mathfrak{P}}_k^{(n)}$, $\widetilde{\mathfrak{P}}_k^{(n)\circ}$ and $\widehat{\mathfrak{P}}_k^{(n)}$. For brevity, all are simply denoted by V_k^2/V_k^1.

Let us denote by i_a the inclusion $\widehat{\mathfrak{M}}_a \subset \widetilde{\mathfrak{M}}_a^\circ$ for $a = 1, 2$. By (11.2.2), the inclusion map $i_1 \times \mathrm{id}_{\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^2, \mathbf{w})} \colon \widehat{\mathfrak{M}}(\mathbf{v}^1, \mathbf{w}) \times \widetilde{\mathfrak{M}}^\circ(\mathbf{v}^2, \mathbf{w}) \to \widetilde{\mathfrak{M}}^\circ(\mathbf{v}^1, \mathbf{w}) \times \widetilde{\mathfrak{M}}^\circ(\mathbf{v}^2, \mathbf{w})$ induces the pull-back homomorphism with support

$$(i_1 \times \mathrm{id}_{\widetilde{\mathfrak{M}}_2^\circ})^* \colon K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\widetilde{\mathfrak{P}}_k^{(n)\circ}) \to K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\widehat{\mathfrak{P}}_k^{(n)}).$$

Similarly, we have

$$(\mathrm{id}_{\widetilde{\mathfrak{M}}_1^\circ} \times i_2)^* \colon K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\widetilde{\mathfrak{P}}_k^{(n)\circ}) \to K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\widehat{\mathfrak{P}}_k^{(n)}).$$

Lemma 11.2.3. *We have*

$$(i_1 \times \mathrm{id}_{\widetilde{\mathfrak{M}}_2^\circ})^*[\mathcal{O}_{\widetilde{\mathfrak{P}}_k^{(n)\circ}}] = [\mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}], \qquad (\mathrm{id}_{\widetilde{\mathfrak{M}}_2^\circ} \times i_2)^*[\mathcal{O}_{\widetilde{\mathfrak{P}}_k^{(n)\circ}}] = [\mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}].$$

More generally, if $T(V_k^2/V_k^1)$ denotes a tensor product of exterior products of the bundle V_k^2/V_k^1 and its dual, we have

$$(i_1 \times \mathrm{id}_{\widetilde{\mathfrak{M}}_2^\circ})^*[T\left(V_k^2/V_k^1\right)] = [T\left(V_k^2/V_k^1\right) \otimes \mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}],$$
$$(\mathrm{id}_{\widetilde{\mathfrak{M}}_1^\circ} \times i_2)^*[T\left(V_k^2/V_k^1\right)] = [T\left(V_k^2/V_k^1\right) \otimes \mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}].$$

Proof. The latter statement follows from the first statement and the formula (6.4.1). Thus it is enough to check the first statement, and the first statement follows from the transversality of intersections (11.2.2) in $\widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2^\circ$.

Let $\mu' \colon \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \bigoplus_{l \neq k} \mathfrak{gl}(V_l)$ be the $\bigoplus_{l \neq k} \mathfrak{gl}(V_l)$-part of the moment map μ. It induces a map $\widetilde{\mathfrak{M}}_a^\circ \to \bigoplus_{l \neq k} \mathfrak{gl}(V_l)$ for $a = 1, 2$. Let us denote it by μ'_a. Thus we have $\widehat{\mathfrak{M}}_a = {\mu'_a}^{-1}(0)$. Composing μ'_a with the projection $p_a \colon \widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2^\circ \to \widetilde{\mathfrak{M}}_a^\circ$, we have a map $\mu'_a \circ p_a \colon \widetilde{\mathfrak{M}}_1^\circ \times \widetilde{\mathfrak{M}}_2^\circ \to \bigoplus_{l\neq k} \mathfrak{gl}(V_l)$ for $a = 1, 2$. We denote it also by μ'_a for brevity. It is enough to show that the restriction of the differential $d\mu'_a$ to $T\widetilde{\mathfrak{P}}_k^{(n)\circ}$ is surjective on $\widehat{\mathfrak{P}}_k^{(n)} = \widetilde{\mathfrak{P}}_k^{(n)\circ} \cap (\widetilde{\mathfrak{M}}_1^\circ \times \widehat{\mathfrak{M}}_2) = \widetilde{\mathfrak{P}}_k^{(n)\circ} \cap (\widehat{\mathfrak{M}}_1 \times \widetilde{\mathfrak{M}}_2^\circ)$.

We consider the homomorphisms σ_k, τ_k defined in (2.9.1) where (B, i, j) is replaced by (B^a, i^a, j^a) $(a = 1, 2)$. We denote them by σ_k^a and τ_k^a respectively.

Take a point $([B^1, i^1, j^1], [B^2, i^2, j^2]) \in \widehat{\mathfrak{P}}_k^{(n)}$. Then

$$B_h^1 = B_h^2 \quad (\mathrm{in}(h) \neq k, \mathrm{out}(h) \neq k), \quad i_l^1 = i_l^2, \quad j_l^1 = j_l^2 \quad (l \neq k),$$

and there exists $\xi_k \colon V_k^1 \to V_k^2$ such that

$$\xi_k B_h^1 = B_h^2, \quad B_{\overline{h}}^1 \xi_k = B_{\overline{h}}^2 \quad (\mathrm{in}(h) = k), \quad \xi_k i_k^1 = i_k^2, \quad j_k^1 = j_k^2 \xi_k.$$

The tangent space $T\widetilde{\mathfrak{P}}_k^{(n)\circ}$ is isomorphic to the space of $(\delta B^1, \delta i^1, \delta j^1, \delta B^2, \delta i^2, \delta j^2) \in \mathbf{M}(\mathbf{v}^1, \mathbf{w}) \times \mathbf{M}(\mathbf{v}^2, \mathbf{w})$ such that

$$\begin{gathered} \delta B_h^1 = \delta B_h^2 \quad (\text{if } \mathrm{in}(h) \neq k,\ \mathrm{out}(h) \neq k), \\ \delta i_l^1 = \delta i_l^2, \quad \delta j_l^1 = \delta j_l^2 \quad (\text{for } l \neq k), \\ \tau_k^a\, \delta\sigma_k^a + \delta\tau_k^a\, \sigma_k^a = 0 \qquad (a = 1, 2), \\ \xi_k \circ \delta\tau_k^1 = \delta\tau_k^2, \qquad \delta\sigma_k^1 = \delta\sigma_k^2 \circ \xi_k \end{gathered} \tag{11.2.4}$$

modulo the image of

$$\begin{aligned} &\{(\delta\xi_k^1, \delta\xi_k^2) \in \mathfrak{gl}(V_k^1) \times \mathfrak{gl}(V_k^2) \mid \delta\xi_k^2\, \xi_k = \xi_k \delta\xi_k^1\} \\ &\longmapsto \begin{cases} \delta\xi_k^1 B_h^1,\ -B_{\overline{h}}^1 \delta\xi_k^1,\ \delta\xi_k^2 B_h^2,\ -B_{\overline{h}}^2 \delta\xi_k^2 \quad (\mathrm{in}(h) = k), \\ \delta\xi_k^1 i_k^1,\ -j_k^1 \delta\xi_k^1,\ \delta\xi_k^2 i_k^2,\ -j_k^2 \delta\xi_k^2 \\ \text{other components are } 0, \end{cases} \end{aligned} \tag{11.2.5}$$

where

$$\delta\sigma_k^a = \bigoplus_{\mathrm{in}(h)=k} \delta B_{\overline{h}}^a \oplus \delta j_k^a, \quad \delta\tau_k^a = \bigoplus_{\mathrm{in}(h)=k} \varepsilon(h) \delta B_h^a + \delta i_k^a \quad (a = 1, 2),$$

and we have used the identification $V_l^1 \cong V_l^2$ for $l \neq k$.

Now suppose that $(\zeta_l)_{l\neq k} \in \bigoplus_{l\neq k} \mathfrak{gl}(V_l)$ is orthogonal to the image of $d\mu'_a|_{T\widetilde{\mathfrak{P}}_k^{(n)\circ}}$. Putting $\zeta_k = 0$, we consider $\zeta = (\zeta_l)$ as an element of $\mathrm{L}(V^a, V^a)$. Then

$$\operatorname{tr}(\varepsilon \delta B^a (B^a \zeta - \zeta B^a) + \delta i^a\, j^a \zeta + \zeta i^a\, \delta j^a) = 0$$

for any $(\delta B^1, \delta i^1, \delta j^1, \delta B^2, \delta i^2, \delta j^2) \in T\widetilde{\mathfrak{P}}_k^{(n)\circ}$. Since the image of (11.2.5) lies in the kernel of $d\mu'_a$, the above equality holds for any $(\delta B^1, \delta i^1, \delta j^1, \delta B^2, \delta i^2, \delta j^2)$ satisfying (11.2.4).

Taking $(\delta B^1, \delta i^1, \delta j^1, \delta B^2, \delta i^2, \delta j^2)$ from $\Delta\mathbf{M}'(\mathbf{v}, \mathbf{w})$, we find

$$\begin{gathered} B_h^a \zeta_{\mathrm{out}(h)} = \zeta_{\mathrm{in}(h)} B_h^a \qquad \text{if } \mathrm{in}(h) \neq k,\ \mathrm{out}(h) \neq k, \\ \zeta_l i_l^2 = 0, \quad j_l^2 \zeta_l = 0 \qquad \text{if } l \neq k. \end{gathered} \tag{11.2.6}$$

Next taking $(\delta B^1, \delta i^1, \delta j^1, \delta B^2, \delta i^2, \delta j^2)$ from the other component (data related to the vertex k), we get

$$\sigma_k^a \circ \left(\sum_{\mathrm{in}(h)=k} \varepsilon(h) B_h^a \zeta_{\mathrm{out}(h)}, 0 \right) = \left(\bigoplus_{\mathrm{in}(h)=k} \zeta_{\mathrm{out}(h)} B_{\overline{h}}^a \oplus 0 \right) \circ \tau_k^a \in \operatorname{End} \left(\bigoplus_{\mathrm{in}(h)=k} V_{\mathrm{out}(h)} \oplus W_k \right).$$

Comparing $\operatorname{Hom}(V_{\mathrm{out}(h)}, W_k)$-components, we find

$$\varepsilon(h) j_k B_h^a \zeta_{\mathrm{out}(h)} = 0. \tag{11.2.7}$$

Comparing $\operatorname{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{out}(h')})$-components, we have

$$B_{\overline{h'}}^a \varepsilon(h) B_h^a \zeta_{\mathrm{out}(h)} = \zeta_{\mathrm{out}(h')} B_{\overline{h'}}^a \varepsilon(h) B_h^a. \tag{11.2.8}$$

If we define

$$S_l \overset{\text{def.}}{=} \begin{cases} \operatorname{Im} \zeta_l & \text{if } l \neq k, \\ \sum_{\mathrm{in}(h)=k} \operatorname{Im}\left(B_h^a \zeta_{\mathrm{out}(h)}\right) & \text{if } l = k, \end{cases}$$

(11.2.6), (11.2.7) and (11.2.8) imply that $S = (S_l)$ is B^a-invariant and contained in $\operatorname{Ker} j$. Thus $S = 0$ by the stability condition. In particular, we have $\zeta = 0$. This means that $d\mu'_a|_{T\widetilde{\mathfrak{P}}_k^{(n)\circ}}$ is surjective. □

Let $\pi_a \colon \widehat{\mathfrak{M}}_a \to \mathfrak{M}_a$ be the projection ($a = 1, 2$). Then we have

$$\begin{gathered} \text{the restriction of } \mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2 \colon \widehat{\mathfrak{M}}_1 \times \widehat{\mathfrak{M}}_2 \to \widehat{\mathfrak{M}}_1 \times \mathfrak{M}_2 \text{ to } \widehat{\mathfrak{P}}_k^{(n)} \text{ is proper,} \\ (\mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2)(\widehat{\mathfrak{P}}_k^{(n)}) = (\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{-1}(\mathfrak{P}_k^{(n)}), \\ \text{the restriction of } \pi_1 \times \mathrm{id}_{\widehat{\mathfrak{M}}_2} \colon \widehat{\mathfrak{M}}_1 \times \widehat{\mathfrak{M}}_2 \to \mathfrak{M}_1 \times \widehat{\mathfrak{M}}_2 \text{ to } \widehat{\mathfrak{P}}_k^{(n)} \text{ is proper,} \\ (\pi_1 \times \mathrm{id}_{\widehat{\mathfrak{M}}_2})(\widehat{\mathfrak{P}}_k^{(n)}) = (\mathrm{id}_{\mathfrak{M}_1} \times \pi_2)^{-1}(\mathfrak{P}_k^{(n)}). \end{gathered} \tag{11.2.9}$$

By these properties, we have homomorphisms

$$\begin{aligned} (\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{*-1} \left(\mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2 \right)_* &\colon K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\widehat{\mathfrak{P}}_k^{(n)}) \to K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{P}_k^{(n)}), \\ (\mathrm{id}_{\mathfrak{M}_1} \times \pi_1)^{*-1} \left(\pi_1 \times \mathrm{id}_{\widehat{\mathfrak{M}}_2} \right)_* &\colon K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\widehat{\mathfrak{P}}_k^{(n)}) \to K^{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{P}_k^{(n)}). \end{aligned}$$

Lemma 11.2.10. *We have*

$$\begin{aligned} (\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{*-1} \left(\mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2 \right)_* [\mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}] &= [\mathcal{O}_{\mathfrak{P}_k^{(n)}}], \\ (\mathrm{id}_{\mathfrak{M}_1} \times \pi_2)^{*-1} \left(\pi_1 \times \mathrm{id}_{\widehat{\mathfrak{M}}_2} \right)_* [\mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}] &= [\mathcal{O}_{\mathfrak{P}_k^{(n)}}]. \end{aligned}$$

More generally, if $T(V_k^2/V_k^1)$ denotes a tensor product of exterior products of the bundle V_k^2/V_k^1 and its dual, we have

$$\begin{aligned} (\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{*-1} \left(\mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2 \right)_* [(\pi_1 \times \pi_2)^* T\left(V_k^2/V_k^1\right)] &= [T\left(V_k^2/V_k^1\right)], \\ (\mathrm{id}_{\mathfrak{M}_1} \times \pi_2)^{*-1} \left(\pi_1 \times \mathrm{id}_{\widehat{\mathfrak{M}}_2} \right)_* [(\pi_1 \times \pi_2)^* T\left(V_k^2/V_k^1\right)] &= [T\left(V_k^2/V_k^1\right)]. \end{aligned}$$

Proof. The latter statement follows from the former one together with the projection formula (6.5.1). Thus it is enough to prove the former statement.

By definition, $(\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{-1}(\mathfrak{P}_k^{(n)})$ consists of

$$(\mathrm{GL}(V_k^1) \cdot (B^1, i^1, j^1), G_{\mathbf{v}^2} \cdot (B^2, i^2, j^2)) \in \widehat{\mathfrak{M}}_1 \times \mathfrak{M}_2$$

such that there exists $\xi \in \mathrm{L}(V^1, V^2)$ satisfying (5.1.3). We fix representatives (B^1, i^1, j^1), (B^2, i^2, j^2). Then the above ξ is uniquely determined. Recall that we have chosen the identification $V_l^1 \cong V_l^2$ for $l \neq k$ over $\widehat{\mathfrak{M}}_1 \times \widehat{\mathfrak{M}}_2$. Let us define $\xi' \in \mathrm{L}(V^2, V^2)$ by

$$\xi'_l \overset{\text{def.}}{=} \begin{cases} \mathrm{id} & \text{if } l = k, \\ \xi_l & \text{otherwise.} \end{cases}$$

We define a new datum

$$(B^3, i^3, j^3) \overset{\text{def.}}{=} (\xi'^{-1} B^2 \xi', \xi'^{-1} i^2, j^2 \xi').$$

By definition, we have

$$\begin{gathered} \xi_k B_h^1 = B_h^3, \quad B_{\overline{h}}^1 = B_{\overline{h}}^3 \xi_k \quad (\mathrm{in}(h) = k), \quad \xi_k i_k^1 = i_k^3, \quad j_k^1 = j_k^3 \xi_k, \\ B_h^1 = B_h^3 \quad (\mathrm{in}(h) \neq k, \mathrm{out}(h) \neq k), \quad i_l^1 = i_l^3, \quad j_l^1 = j_l^3 \quad (l \neq k). \end{gathered}$$

Hence $(\mathrm{GL}(V_k^1) \cdot (B^1, i^1, j^1), \mathrm{GL}(V_k^2) \cdot (B^3, i^3, j^3))$ is contained in $\widehat{\mathfrak{P}}_k$. Moreover, $\mathrm{GL}(V_k^2) \cdot (B^3, i^3, j^3)$ is independent of the choice of the representative (B^2, i^2, j^2). Thus we have defined a map $(\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{-1}(\mathfrak{P}_k^{(n)}) \to \widehat{\mathfrak{P}}_k^{(n)}$ by

$$\begin{aligned} &(\mathrm{GL}(V_k^1) \cdot (B^1, i^1, j^1), G_{\mathbf{v}^2} \cdot (B^2, i^2, j^2)) \\ &\qquad \mapsto (\mathrm{GL}(V_k^1) \cdot (B^1, i^1, j^1), \mathrm{GL}(V_k^2) \cdot (B^3, i^3, j^3)), \end{aligned}$$

which is the inverse of the restriction of $\mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2$. In particular, this implies

$$\left(\mathrm{id}_{\widehat{\mathfrak{M}}_1} \times \pi_2\right)_* [\mathcal{O}_{\widehat{\mathfrak{P}}_k^{(n)}}] = [\mathcal{O}_{(\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{-1}(\mathfrak{P}_k^{(n)})}].$$

Since $\pi_1 \times \mathrm{id}_{\mathfrak{M}_2} \colon (\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{-1}(\mathfrak{P}_k^{(n)}) \to \mathfrak{P}_k^{(n)}$ is a principal G'-bundle, we have

$$(\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^* [\mathcal{O}_{\mathfrak{P}_k^{(n)}}] = [\mathcal{O}_{(\pi_1 \times \mathrm{id}_{\mathfrak{M}_2})^{-1}(\mathfrak{P}_k^{(n)})}].$$

Thus we have proved the first equation. The second equation can be proved in a similar way. □

11.3. **Reduction to the rank 1 case.** First consider the relation (1.2.8) for $k = l$. Let $\mathbf{v}^1$, $\mathbf{v}^2$, $\mathbf{v}^3$, $\mathbf{v}^4$ be dimension vectors such that

$$\mathbf{v}^1 = \mathbf{v}^3, \quad \mathbf{v}^2 = \mathbf{v}^1 + \alpha_k, \quad \mathbf{v}^4 = \mathbf{v}^1 - \alpha_k.$$

We want to compute $x_k^+(z) * x_k^-(w)$ and $x_k^-(w) * x_k^+(z)$ in the component

$$K^{G_{\mathbf{w}} \times \mathbb{C}}(Z(\mathbf{v}^1, \mathbf{v}^3, \mathbf{w})),$$

and then compare it with the right hand side of (1.2.8) with $k = l$ in the same component.

Let $\widehat{\mathfrak{M}}(\mathbf{v}^a, \mathbf{w})$, $\widetilde{\mathfrak{M}}(\mathbf{v}^a, \mathbf{w})$, $\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^a, \mathbf{w})$, $G'_{\mathbf{v}^a}$ and $\mathbf{M}'(\mathbf{v}^a, \mathbf{w})$ be as in §11.1. Let $\mathfrak{P}_k(\mathbf{v}^a, \mathbf{w})$, $\widehat{\mathfrak{P}}_k(\mathbf{v}^a, \mathbf{w})$, $\widetilde{\mathfrak{P}}_k(\mathbf{v}^a, \mathbf{w})$, $\widetilde{\mathfrak{P}}_k^\circ(\mathbf{v}^a, \mathbf{w})$ be the Hecke correspondence and its modifications introduced in §11.2. (We drop the superscript (n) and write the dimension vector $\mathbf{v}^a$, $\mathbf{w}$.) Let ω be the exchange of factors as before. Let $\widehat{Z}(\mathbf{v}^a, \mathbf{v}^b; \mathbf{w})$,

$\widetilde{Z}(\mathbf{v}^a,\mathbf{v}^b;\mathbf{w})$, $\widetilde{Z}^\circ(\mathbf{v}^a,\mathbf{v}^b;\mathbf{w})$ be subvarieties in $\widehat{\mathfrak{M}}(\mathbf{v}^a,\mathbf{w})\times\widehat{\mathfrak{M}}(\mathbf{v}^b,\mathbf{w})$, $\widetilde{\mathfrak{M}}(\mathbf{v}^a,\mathbf{w})\times\widetilde{\mathfrak{M}}(\mathbf{v}^b,\mathbf{w})$, $\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^a,\mathbf{w})\times\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^b,\mathbf{w})$ defined in the same way as $Z(\mathbf{v}^a,\mathbf{v}^b;\mathbf{w})$.

We have the following commutative diagram:

$$\begin{array}{ccc}
K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}))\times K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\omega\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w})) & \longrightarrow & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})) \\
\uparrow & & \uparrow \\
K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\widehat{\mathfrak{P}}_k(\mathbf{v}^2,\mathbf{w}))\times K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\omega\widehat{\mathfrak{P}}_k(\mathbf{v}^2,\mathbf{w})) & \longrightarrow & K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\widehat{Z}(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})) \\
\uparrow & & \uparrow \\
K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{\mathfrak{P}}^\circ_k(\mathbf{v}^2,\mathbf{w}))\times K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\omega\widetilde{\mathfrak{P}}^\circ_k(\mathbf{v}^2,\mathbf{w})) & \longrightarrow & K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{Z}^\circ(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})) \\
\uparrow & & \uparrow \\
K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{\mathfrak{P}}_k(\mathbf{v}^2,\mathbf{w}))\times K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\omega\widetilde{\mathfrak{P}}_k(\mathbf{v}^2,\mathbf{w})) & \longrightarrow & K^{G'\times G_{\mathbf{w}}\times\mathbb{C}^*}(\widetilde{Z}(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})).
\end{array}$$

The horizontal arrows are convolution products relative to

(1) $\mathfrak{M}(\mathbf{v}^1,\mathbf{w})$, $\mathfrak{M}(\mathbf{v}^2,\mathbf{w})$, $\mathfrak{M}(\mathbf{v}^3,\mathbf{w})$,
(2) $\widehat{\mathfrak{M}}(\mathbf{v}^1,\mathbf{w})$, $\widehat{\mathfrak{M}}(\mathbf{v}^2,\mathbf{w})$, $\widehat{\mathfrak{M}}(\mathbf{v}^3,\mathbf{w})$,
(3) $\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^1,\mathbf{w})$, $\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^2,\mathbf{w})$, $\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^3,\mathbf{w})$,
(4) $\widetilde{\mathfrak{M}}(\mathbf{v}^1,\mathbf{w})$, $\widetilde{\mathfrak{M}}(\mathbf{v}^2,\mathbf{w})$, $\widetilde{\mathfrak{M}}(\mathbf{v}^3,\mathbf{w})$.

The vertical arrows between the first and the second rows are homomorphisms given in Proposition 8.3.5. The arrows between the second and the third are homomorphisms given in Proposition 8.2.3. By the properties (11.2.2), (11.2.9) and

(a) $\widetilde{Z}^\circ(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})\cap\left(\widehat{\mathfrak{M}}(\mathbf{v}^1,\mathbf{w})\times\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^2,\mathbf{w})\right)\subset\widehat{Z}(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})$,
(b) the restriction of $\pi_1\times\mathrm{id}_{\widehat{\mathfrak{M}}(\mathbf{v}^3,\mathbf{w})}\colon\widehat{\mathfrak{M}}(\mathbf{v}^1,\mathbf{w})\times\widehat{\mathfrak{M}}(\mathbf{v}^3,\mathbf{w})\to\mathfrak{M}(\mathbf{v}^1,\mathbf{w})\times\widehat{\mathfrak{M}}(\mathbf{v}^3,\mathbf{w})$ to $\widehat{Z}(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w})$ is proper,
(c) $(\pi_1\times\mathrm{id}_{\widehat{\mathfrak{M}}(\mathbf{v}^3,\mathbf{w})})(\widehat{Z}(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w}))\subset(\mathrm{id}_{\mathfrak{M}(\mathbf{v}^1,\mathbf{w})}\times\pi_3)^{-1}(Z(\mathbf{v}^1,\mathbf{v}^3;\mathbf{w}))$,

those homomorphisms can be defined. Finally the arrows between the third and the fourth are restrictions to open subvarieties.

The commutativity for the first and the second squares follows from Propositions 8.3.5 and 8.2.3 respectively. The last square is also commutative since $\widetilde{\mathfrak{M}}^\circ(\mathbf{v}^a,\mathbf{w})$ is an open subvariety of $\widetilde{\mathfrak{M}}(\mathbf{v}^a,\mathbf{w})$ and since we have (11.2.1).

Recall that the modified Hecke correspondence in the last row is the product of the Hecke correspondence for type A_1 and the diagonal $\Delta\mathbf{M}'(\mathbf{v}^1,\mathbf{w})$. Under the composite of vertical homomorphisms, $e_{k,r}$, $f_{k,s}$ at the upper left are the images of the exterior products of the corresponding elements for type A_1 and $\mathcal{O}_{\Delta\mathbf{M}'(\mathbf{v}^1,\mathbf{w})}$ at the lower left, except for the following two differences:

(a) the groups acting on varieties are different,
(b) the sign factors in (9.3.2), which involve the orientation Ω, are different.

For the quiver varieties of type A_1, the group is

$$\mathrm{GL}\left(\bigoplus_{h:\mathrm{out}(h)=k}V_{\mathrm{in}(h)}\oplus W_k\right)\times\mathbb{C}^*\cong\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*.$$

But, if we define a homomorphism $G' \times G_{\mathbf{w}} \times \mathbb{C}^* \to \mathrm{GL}_N(\mathbb{C}) \times \mathbb{C}^*$ by

$$((g_l)_{l \in I: l \neq k}, (h_{l'})_{l' \in I}, q) \longmapsto \left(\bigoplus_{h: \mathrm{out}(h) = k} q^{m(h)} g_{\mathrm{in}(h)} \oplus h_k, q \right),$$

we have an induced homomorphism in equivariant K-groups: $K^{\mathrm{GL}_N(\mathbb{C}) \times \mathbb{C}^*}(\ ) \to K^{G' \times G_{\mathbf{w}} \times \mathbb{C}^*}(\ )$. (Here $m(h)$ is as in (2.7.1).) It is compatible with the convolution product, hence it is enough to check the relation in $K^{\mathrm{GL}_N(\mathbb{C}) \times \mathbb{C}^*}(\ )$.

Furthermore, the sign factor cancels out in $e_{k,r} * f_{k,s}$. Thus the above differences make no effect when we check the relation (1.2.8).

By the commutativity of the diagram, $e_{k,r} * f_{k,s}$ is the image of the corresponding element in the lower right.

We have a similar commutative diagram to compute $f_{k,s} * e_{k,r}$. Hence the commutator $[e_{k,r}, f_{k,s}]$ is the image of the corresponding commutator in the lower right. In the next section, we will check the relation (1.2.8) for type A_1. In particular, the commutator in the lower right is represented by tautological bundles, considered as an element of the K-theory of the diagonal $\Delta\widetilde{\mathfrak{M}}(\mathbf{v}^1, \mathbf{w})$. Note that $\mathcal{O}_{\Delta\widetilde{\mathfrak{M}}(\mathbf{v}^1, \mathbf{w})}$ is mapped to $\mathcal{O}_{\Delta\mathfrak{M}(\mathbf{v}^1, \mathbf{w})}$ by examples in §8, and that the tautological bundles on $\widetilde{\mathfrak{M}}(\mathbf{v}^1, \mathbf{w})$ are restricted to tautological bundles on $\mathfrak{M}(\mathbf{v}^1, \mathbf{w})$. Hence we have exactly the same relation (1.2.8) for the general case.

Similarly, we can reduce the check of the relation (1.2.9) to the case of type A_1.

11.4. **Rank 1 case.** In this subsection, we check the relation when the graph is of type A_1. This calculation is essentially the same as the one by Vasserot [58], but we reproduce it here for the convenience of the reader. (Remark that our $\mathbb{C}^*$-action is different from the one in [58]. The definition of $e_{k,r}$, etc. is also different.) We drop the subscript $k \in I$ as usual.

We prepare several notations. For $a, b \in \mathbb{Z}$, let

$$[a, b] \overset{\text{def.}}{=} \begin{cases} \{a, a+1, \dots, b\} & \text{if } b \geq a, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let

$$\mathbf{R} \overset{\text{def.}}{=} \mathbb{Z}[q, q^{-1}][x_1^{\pm}, \dots, x_N^{\pm}].$$

For a partition $I = (I_1, I_2)$ of the set $\{1, \dots, N\}$ into 2 subsets, let $S_I = S_{I_1} \times S_{I_2}$ be the subgroups of S_N consisting of permutations which preserve each subset. For a subgroup $G \subset S_N$, let $\mathbf{R}^G$ be the subring of $\mathbf{R}$ consisting of elements which are fixed by the action of G. If J is another partition of $\{1, \dots, N\}$, we define the symmetrizer $\mathfrak{S}_I^J : \mathcal{R}^{S_I \cap S_J} \to \mathcal{R}^{S_J}$ by

$$f \mapsto \sum_{\sigma \in S_J / S_I \cap S_J} \sigma(f),$$

where $\mathcal{R}$ is the quotient field of $\mathbf{R}$. For each $v \in [0, N]$, let $[v]$ be the partition $([1, v], [v+1, N])$. If $I = (I_1, I_2)$ is a partition of $\{1, \dots, N\}$ into 2 subsets and $k \in I_1$ (resp. $k \in I_2$), we define a new partition $\tau_k^+(I)$ (resp. $\tau_k^-(I)$) by

$$\tau_k^+(I) \overset{\text{def.}}{=} (I_1 \setminus \{k\}, I_2 \cup \{k\}) \quad \left(\text{resp. } \tau_k^-(I) \overset{\text{def.}}{=} (I_1 \cup \{k\}, I_2 \setminus \{k\})\right).$$

If $I = (I_1, I_2)$ is a partition and $f \in \mathbf{R}^{S_{[v]}}$, we define

$$f(x_I) \overset{\text{def.}}{=} f(x_{i_1}, \dots, x_{i_v}, x_{j_1}, \dots, x_{j_{N-v}}),$$

where $I_1 = \{i_1, \dots, i_v\}$, $I_2 = \{j_1, \dots, j_{N-v}\}$.

Let $\mathfrak{M}(v, N)$ be the quiver variety for the graph of type A_1 with dimension vectors v, N. It is isomorphic to the cotangent bundle of the Grassmann variety of v-dimensional subspaces of an N-dimensional space. Let $G(v, N)$ denote the Grassmann variety contained in $\mathfrak{M}(v, N)$ as the 0-section. Let $Z(v^1, v^2; N)$ be the analogue of Steinberg's variety as before. The following lemma is crucial.

Lemma 11.4.1 ([58, Lemma 13], [13, Claim 7.6.7])**.** *The representation of*

$$\bigoplus K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(Z(v^1, v^2; N))$$

on $\bigoplus K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v, N))$ *by convolution is faithful.*

Thus it is enough to check the relation in $\bigoplus K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v, N))$.

Let $\mathfrak{P}^{(n)}(v, N) \subset \mathfrak{M}(v - n, N) \times \mathfrak{M}(v, N)$ be as in (5.3.1). It is the conormal bundle of

$$O^{(n)}(v, N) \overset{\text{def.}}{=} \{(V^1, V^2) \in G(v - n, N) \times G(v, N) \mid V^1 \subset V^2\}.$$

We denote the projections for $\mathfrak{P}(v, N)$ by p_1, p_2, and the projections for $O(v, N)$ by P_1, P_2. Note that both P_1, P_2 are smooth and proper. Let π_1, π_2 denote the projections $T^*G(v - n, N) \to G(v - 1, N)$, $T^*G(v, N) \to G(v, N)$.

Lemma 11.4.2 ([58, Corollary 4])**.** *For* $E \in K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v, N))$ *(resp.* $E \in K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v - n, N))$*), we have*

$$[\mathcal{O}_{\mathfrak{P}^{(n)}(v,N)}] * \pi_2^* E = \sum_i (-q^{-2})^i \pi_1^* P_{1*}\left([{\textstyle\bigwedge}^i TP_1] \otimes P_2^* E\right)$$

$$\left(\text{resp. } [\mathcal{O}_{\mathfrak{P}^{(n)}(v,N)}] * \pi_1^* E = \sum_i (-q^{-2})^i \pi_2^* P_{2*}\left([{\textstyle\bigwedge}^i TP_2] \otimes P_1^* E\right)\right),$$

where TP_1 *(resp.* TP_2*) is the relative tangent bundle along the fibers of* P_1 *(resp.* P_2*).*

Proof. As explained in [58, Corollary 4], the result follows from Lemma 6.3.1. The factor q^{-2} is introduced to make the differential in the Koszul complex *equivariant*. □

By the Thom isomorphism [13, 5.4.17],

$$\pi^* \colon K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v, N)) \to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v, N))$$

is an isomorphism. Moreover, we have the following explicit description of the K-group of the Grassmann variety (cf. [13, 6.1.6]):

$$K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v, N)) \cong R(\mathbb{C}^* \times \mathrm{GL}_v(\mathbb{C}) \times \mathrm{GL}_{N-v}(\mathbb{C})) \cong \mathbf{R}^{S_{[v]}},$$

where $S_{[v]} = S_v \times S_{N-v}$ acts as permutations of $x_1, \dots, x_v$ and $x_{v+1}, \dots, x_N$. If E denotes the tautological rank v vector bundle over $G(v, N)$ and Q denotes the quotient bundle $\mathcal{O}^{\oplus N}/E$, the isomorphism is given by

$$e_i(x_1, \dots, x_v) \mapsto {\textstyle\bigwedge}^i E, \qquad e_i(x_1^{-1}, \dots, x_v^{-1}) \mapsto {\textstyle\bigwedge}^i E^*,$$
$$e_i(x_{v+1}, \dots, x_N) \mapsto {\textstyle\bigwedge}^i Q, \quad e_i(x_{v+1}^{-1}, \dots, x_N^{-1}) \mapsto {\textstyle\bigwedge}^i Q^*,$$

where e_i denotes the ith elementary symmetric polynomial.

The tautological vector bundle V is isomorphic to qE, and W is isomorphic to the trivial bundle $\mathcal{O}^{\oplus N}$. Let $C^\bullet(v,N)$ (resp. $C'^\bullet(v,N)$, $C''^\bullet(v,N)$) be the complex (2.9.1) (resp. (9.3.1)) over $\mathfrak{M}(v,N)$. In the description above, we have

$$\begin{gathered}\textstyle\bigwedge_{-1/z} C^\bullet(v,N) = \left(\prod_{u\in[1,v]}(1-z^{-1}qx_u)\right)^{-1}\prod_{t\in[v+1,N]}(1-z^{-1}q^{-1}x_t),\\ \det C'^\bullet(v,N) = \prod_{t\in[v+1,N]} q^{-1}x_t, \quad \det C''^\bullet(v,N) = \prod_{u\in[1,v]} q^{-1}x_u^{-1}.\end{gathered} \tag{11.4.3}$$

We also have

$$K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(O^{(n)}(v,N)) \cong \mathbf{R}^{S_{[v-n]}\cap S_{[v]}},$$

where

$$S_{[v-n]}\cap S_{[v]} \cong S_{v-n}\times S_n\times S_{N-v}$$

acts as permutations of $x_1,\dots,x_{v-n}$, $x_{v-n+1},\dots,x_v$, and $x_{v+1},\dots,x_N$. The natural vector bundle V^2/V^1 is $q(x_{v-n+1}+\dots+x_v)$. The relative tangent bundles TP_1, TP_2 are

$$[TP_1] = \sum_{t=v+1}^{N}\sum_{k=v-n+1}^{v}\frac{x_t}{x_k}, \qquad [TP_2] = \sum_{u=1}^{v-n}\sum_{k=v-n+1}^{v}\frac{x_k}{x_u}.$$

Lemma 11.4.4 ([58, Proposition 6])**.** (1) *The pull-back homomorphisms*

$$\begin{aligned}&P_1^*\colon K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v-n,N)) \to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(O(v,N)),\\ &P_2^*\colon K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v,N)) \to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(O(v,N))\end{aligned}$$

are identified with the natural homomorphisms

$$\mathbf{R}^{S_{[v-n]}} \to \mathbf{R}^{S_{[v-n]}\cap S_{[v]}}, \quad \mathbf{R}^{S_{[v]}} \to \mathbf{R}^{S_{[v-n]}\cap S_{[v]}}$$

respectively.

(2) *The push-forward homomorphisms*

$$\begin{aligned}&P_{1*}\colon K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(O(v,N)) \to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v-n,N)),\\ &P_{2*}\colon K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(O(v,N)) \to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v,N))\end{aligned}$$

are identified with the natural homomorphisms

$$\begin{aligned}&\mathbf{R}^{S_{[v-n]}\cap S_{[v]}} \ni f \mapsto \mathfrak{S}^{[v-n]}_{[v]}\left(f\prod_{t=v+1}^{N}\prod_{k=v-n+1}^{v}(1-\frac{x_k}{x_t})^{-1}\right),\\ &\mathbf{R}^{S_{[v-n]}\cap S_{[v]}} \ni f \mapsto \mathfrak{S}^{[v]}_{[v-n]}\left(f\prod_{u=1}^{v-1}\prod_{k=v-n+1}^{v}(1-\frac{x_u}{x_k})^{-1}\right)\end{aligned}$$

respectively. (*The right hand sides are a priori in $\mathcal{R}$, but they are in fact in $\mathbf{R}$.*)

Using the above lemmas, we can write the operators $x^+(z)$ explicitly as:

$$\begin{aligned} x^+(z)f &= (-1)^{N-v}\mathfrak{S}_{[v]}^{[v-1]}\Biggl(f\sum_{r=-\infty}^{\infty}\left(\frac{x_v}{z}\right)^r x_v^{-v} \\ &\qquad\qquad \times\prod_{t\in[v+1,N]} qx_t^{-1}\left(1-\frac{x_v}{x_t}\right)^{-1}\left(1-q^{-2}\frac{x_t}{x_v}\right)\Biggr) \\ &= \sum_{k\in[v,N]} f(x_{\tau_k^-[v-1]})\sum_{r=-\infty}^{\infty}\left(\frac{x_k}{z}\right)^r x_k^{-N}\prod_{t\in[v,N]\setminus\{k\}}\frac{qx_k-q^{-1}x_t}{x_k-x_t}, \end{aligned}$$

for $f\in\mathbf{R}^{S_{[v]}}$. Similarly,

$$\begin{aligned} x^-(w)g &= (-1)^{1-v}\mathfrak{S}_{[v-1]}^{[v]}\Biggl(g\sum_{s=-\infty}^{\infty}\left(\frac{x_v}{w}\right)^s x_v^{N-v+1} \\ &\qquad\qquad \times\prod_{u\in[1,v-1]} qx_u\left(1-\frac{x_u}{x_v}\right)^{-1}\left(1-q^{-2}\frac{x_v}{x_u}\right)\Biggr) \\ &= \sum_{l\in[1,v]} g(x_{\tau_l^+[v]})\sum_{s=-\infty}^{\infty}\left(\frac{x_l}{w}\right)^s x_l^{N}\prod_{u\in[1,v]\setminus\{l\}}\frac{q^{-1}x_l-qx_u}{x_l-x_u} \end{aligned}$$

for $g\in\mathbf{R}^{S_{[v-1]}}$.

Let us compare $x^+(z)x^+(w)$ with $x^+(w)x^+(z)$ in the component

$$K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v,N))\to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v-2,N))$$

as follows:

(11.4.5)

$$\begin{aligned} &x^+(z)x^+(w)f \\ &\quad= -\sum_{l\in[v-1,N]}\sum_{k\in[v-1,N]\setminus\{l\}} f(x_{\tau_k^-\tau_l^-[v-2]})\sum_{r=-\infty}^{\infty}\left(\frac{x_k}{w}\right)^r\sum_{s=-\infty}^{\infty}\left(\frac{x_l}{z}\right)^s x_k^{-N}x_l^{-N} \\ &\qquad\qquad \times\prod_{t\in[v-1,N]\setminus\{k,l\}}\frac{qx_k-q^{-1}x_t}{x_k-x_t}\prod_{u\in[v-1,N]\setminus\{l\}}\frac{qx_l-q^{-1}x_u}{x_l-x_u}, \\ &x^+(w)x^+(z)f \\ &\quad= -\sum_{k\in[v-1,N]}\sum_{l\in[v-1,N]\setminus\{k\}} f(x_{\tau_l^-\tau_k^-[v-2]})\sum_{r=-\infty}^{\infty}\left(\frac{x_k}{w}\right)^r\sum_{s=-\infty}^{\infty}\left(\frac{x_l}{z}\right)^s x_k^{-N}x_l^{-N} \\ &\qquad\qquad \times\prod_{t\in[v-1,N]\setminus\{k\}}\frac{qx_k-q^{-1}x_t}{x_k-x_t}\prod_{u\in[v-1,N]\setminus\{k,l\}}\frac{qx_l-q^{-1}x_u}{x_l-x_u}. \end{aligned}$$

Hence we have

$$x^+(w)x^+(z)=\frac{qw-q^{-1}z}{q^{-1}w-qz}x^+(z)x^+(w).$$

The relation (1.2.9) for $x^-(z)$, $x^-(w)$ can be proved in the same way.

Let us compare $x^-(w)x^+(z)$ and $x^+(z)x^-(w)$ in the component

$$K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v,N))\to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v,N))$$

as follows:

$$x^-(w)x^+(z)f = \sum_{l\in[1,v]}\sum_{k\in[v+1,N]\cup\{l\}} f(x_{\tau_k^-\tau_l^+[v]}) \sum_{r,s=-\infty}^{\infty} \left(\frac{x_k}{z}\right)^r \left(\frac{x_l}{w}\right)^s \left(\frac{x_l}{x_k}\right)^N$$
$$\times \prod_{t\in([v+1,N]\cup\{l\})\setminus\{k\}} \frac{qx_k - q^{-1}x_t}{x_k - x_t} \prod_{u\in[1,v]\setminus\{l\}} \frac{q^{-1}x_l - qx_u}{x_l - x_u},$$

$$x^+(z)x^-(w)f = \sum_{k\in[v+1,N]}\sum_{l\in[1,v]\cup\{k\}} f(x_{\tau_l^+\tau_k^-[v]}) \sum_{r,s=-\infty}^{\infty} \left(\frac{x_k}{z}\right)^r \left(\frac{x_l}{w}\right)^s \left(\frac{x_l}{x_k}\right)^N$$
$$\times \prod_{t\in[v+1,N]\setminus\{k\}} \frac{qx_k - q^{-1}x_t}{x_k - x_t} \prod_{u\in([1,v]\cup\{k\})\setminus\{l\}} \frac{q^{-1}x_l - qx_u}{x_l - x_u}.$$

Terms with $k \neq l$ cancel out for $x^+(z)x^-(w)f$ and $x^-(w)x^+(z)f$. Thus

$$\begin{aligned}
&\left[x^+(z), x^-(w)\right] \\
&= \sum_{k\in[v+1,N]} \sum_{r,s=-\infty}^{\infty} \left(\frac{x_k}{z}\right)^r \left(\frac{x_k}{w}\right)^s \prod_{t\in[v+1,N]\setminus\{k\}} \frac{qx_k - q^{-1}x_t}{x_k - x_t} \prod_{u\in[1,v]} \frac{q^{-1}x_k - qx_u}{x_k - x_u} \\
&\quad - \sum_{l\in[1,v]} \sum_{r,s=-\infty}^{\infty} \left(\frac{x_l}{z}\right)^r \left(\frac{x_l}{w}\right)^s \prod_{t\in[v+1,N]} \frac{qx_l - q^{-1}x_t}{x_l - x_t} \prod_{u\in[1,v]\setminus\{l\}} \frac{q^{-1}x_l - qx_u}{x_l - x_u}.
\end{aligned}$$

Let

$$A(x) \overset{\text{def.}}{=} \prod_{u\in[1,v]} (x - x_u) \prod_{t\in[v+1,N]} (x - x_t),$$
$$B(x) \overset{\text{def.}}{=} \prod_{u\in[1,v]} (q^{-1}x - qx_u) \prod_{t\in[v+1,N]} (qx - q^{-1}x_t).$$

Then we have

$$\left[x^+(z), x^-(w)\right] = \frac{1}{q-q^{-1}} \sum_{m\in[1,N]} \sum_{r,s=-\infty}^{\infty} \left(\frac{x_m}{z}\right)^r \left(\frac{x_m}{w}\right)^s \frac{x_m^{-1}B(x_m)}{A'(x_m)}.$$

Applying the residue theorem to $\frac{1}{q-q^{-1}} \sum_{r,s=-\infty}^{\infty} \left(\frac{x}{z}\right)^r \left(\frac{x}{w}\right)^s \frac{B(x)}{A(x)} \frac{dx}{x}$, we get

$$\left[x^+(z), x^-(w)\right] = \frac{1}{q-q^{-1}} \left(\sum_{r=-\infty}^{\infty} \left(\frac{z}{w}\right)^r \left(\frac{B(z)}{A(z)}\right)^+ - \sum_{r=-\infty}^{\infty} \left(\frac{z}{w}\right)^r \left(\frac{B(z)}{A(z)}\right)^- \right),$$

where $\left(\frac{B(z)}{A(z)}\right)^\pm \in \mathbb{C}[[z^\mp]]$ denotes the Laurent expansion of $\frac{B(z)}{A(z)}$ at $z = \infty$ and 0 respectively. Since

$$\frac{B(z)}{A(z)} = q^{N-2v} \frac{\bigwedge_{-1/(qz)} C^\bullet(v,N)}{\bigwedge_{-q/z} C^\bullet(v,N)}$$

by (11.4.3), we have completed the proof of Theorem 9.4.1.

12. Integral structure

In this section, we compare $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ with $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$. In the case of the affine Hecke algebra, the equivariant K-group of the Steinberg variety is isomorphic to the integral form of the affine Hecke algebra (see [13, 7.2.5]). We shall prove a weaker form of the corresponding result for quiver varieties in this section.

12.1. **Rank 1 case.** We first consider the case when the graph is of type A_1. We drop the subscript k. We use the notation in §11.4. We also consider $\omega(\mathfrak{P}^{(n)}(v,N))$ where $\mathfrak{P}^{(n)}(v,N)$ is as in (5.3.1) and $\omega\colon \mathfrak{M}(v-n,N)\times\mathfrak{M}(v,N)\to\mathfrak{M}(v,N)\times\mathfrak{M}(v-n,N)$ is the exchange of factors. We identify its equivariant K-group with $\mathbf{R}^{S_{[v-n]}\cap S_{[v]}}$ as in §11.4. In particular, the vector bundle V^1/V^2 is identified with $q(x_{v-n+1}+\cdots+x_v)$.

Lemma 12.1.1. (1) *Let $p_1<\cdots<p_s$ be an increasing sequence of integers and let $n_1,\dots,n_s$ be a sequence of positive integers such that $\sum n_i = n$. Let λ be the partition*

$$((p_2-p_1)^{n_2}\cdots(p_s-p_1)^{n_s}).$$

Then for $g\in K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(T^*G(v-n,N))$, we have

$$\begin{aligned} &f_{p_1}^{(n_1)}f_{p_2}^{(n_2)}\cdots f_{p_s}^{(n_s)}g\\ &\quad=\pm q^L\sum_{\{l_1,\dots,l_n\}} g(x_{\tau_{l_1}^+\cdots\tau_{l_n}^+[v]})(x_{l_1}\cdots x_{l_n})^{N+p_1}P_\lambda(x_{l_1},\dots,x_{l_n};q^2)\\ &\qquad\qquad\times\prod_{\substack{i=1,\dots,n\\ u\in[1,v]\setminus\{l_1,\dots,l_n\}}}\frac{q^{-1}x_{l_i}-qx_u}{x_{l_i}-x_u}\end{aligned}$$

for some $L\in\mathbb{Z}$. Here P_λ is the Hall-Littlewood polynomial (*see* [41, III(2.1)]), *and the summation runs over the set of unordered n-tuples $\{l_1,\dots,l_n\}\subset[1,v]$ such that $l_i\neq l_j$ for $i\neq j$.*

(2) *Let us consider a tensor product $T(V^1/V^2)$ of exterior products of the bundle V^1/V^2 and its dual over $\omega(\mathfrak{P}^{(n)}(v,N))$, and denote by $T(x_{v-n+1},\dots,x_v)\in\mathbb{Z}[x_{v-n+1}^\pm,\dots,x_v^\pm]^{S_n}\subset\mathbf{R}^{S_{[v-n]}\cap S_{[v]}}$ the corresponding element in the equivariant K-group. Then for $g\in\mathbf{R}^{S_{[v-n]}}$, we have the following formula:*

$$\begin{aligned} &\left[T(V^1/V^2)\otimes\det C''^{\bullet}(v-n,N)^{\otimes -n}\right]*g\\ &\quad=\pm\sum_{\{l_1,\dots,l_n\}} g(x_{\tau_{l_1}^+\cdots\tau_{l_n}^+[v]})\,(x_{l_1}\dots x_{l_n})^{v-n}\,T(x_{l_1},\dots,x_{l_n})\\ &\qquad\qquad\times\prod_{\substack{i=1,\dots,n\\ u\in[1,v]\setminus\{l_1,\dots,l_n\}}}\frac{q^{-1}x_{l_i}-qx_u}{x_{l_i}-x_u},\end{aligned}$$

where the summation runs over the set of unordered n-tuples $\{l_1,\dots,l_n\}\subset[1,v]$ such that $l_i\neq l_j$ for $i\neq j$.

Proof. (1) Generalizing (11.4.5), we have the following formula for $f_{r_1} f_{r_2} \dots f_{r_n}$: $K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v-n,N)) \to K^{\mathrm{GL}_N(\mathbb{C})\times\mathbb{C}^*}(G(v,N))$:

$$f_{r_1} f_{r_2} \dots f_{r_n} g = \pm \sum_{(l_1,\dots,l_n)} g(x_{\tau^+_{l_1}\cdots\tau^+_{l_n}[v]}) x_{l_1}^{r_1} \cdots x_{l_n}^{r_n} (x_{l_1}\cdots x_{l_n})^N$$
$$\times \prod_{\substack{i=1,\dots,n \\ t\in[1,v]\setminus\{l_1,\dots,l_n\}}} \frac{q^{-1}x_{l_i} - q x_u}{x_{l_i} - x_u} \prod_{i>j} \frac{q^{-1}x_{l_i} - q x_{l_j}}{x_{l_i} - x_{l_j}},$$

where the summation runs over the set of *ordered* n-tuples $(l_1,\dots,l_n)$ such that $l_i \in [1,v]$, $l_i \neq l_j$ for $i \neq j$.

Choose $r_1 \le r_2 \le \dots \le r_n$ so that

$$(r_1, r_2, \dots, r_n) = (\underbrace{p_1,\dots,p_1}_{n_1 \text{ times}}, \underbrace{p_2,\dots,p_2}_{n_2 \text{ times}}, \dots).$$

Consider the following term which appeared in the above formula:

$$\sum_{\sigma\in S_n} x_{l_{\sigma(1)}}^{r_1} \cdots x_{l_{\sigma(n)}}^{r_n} \prod_{i>j} \frac{q^{-1}x_{l_{\sigma(i)}} - q x_{l_{\sigma(j)}}}{x_{l_{\sigma(i)}} - x_{l_{\sigma(j)}}}$$
$$= (x_{l_1}\cdots x_{l_n})^{r_1} \sum_{\sigma\in S_n} x^0_{l_{\sigma(1)}} x^{r_2-r_1}_{l_{\sigma(2)}} \cdots x^{r_n-r_1}_{l_{\sigma(n)}} \prod_{i>j} \frac{q^{-1}x_{l_{\sigma(i)}} - q x_{l_{\sigma(j)}}}{x_{l_{\sigma(i)}} - x_{l_{\sigma(j)}}}.$$

By [41, III(2.1)] it is equal to

$$(x_{l_1}\cdots x_{l_n})^{r_1} q^{-n(n-1)/2} v_\lambda(q^2) P_\lambda(x_{l_1},\dots,x_{l_n}; q^2),$$

where P_λ is the Hall-Littlewood polynomial and

$$v_\lambda(q^2) = q^{n_1(n_1-1)/2} [n_1]_q! q^{n_2(n_2-1)/2} [n_2]_q! \cdots q^{n_s(n_s-1)/2} [n_s]_q!\ .$$

Thus we have the assertion.

(2) By Lemmas 11.4.2, 11.4.4 we have

$$\left[T(V^1/V^2) \otimes \det C''^{\bullet}(v-n,N)^{\otimes -n}\right] * g$$
$$= \mathfrak{S}^{[v]}_{[v-n]} \Bigg(g\, T(x_{v-n+1},\dots,x_v) \prod_{u\in[1,v-n]} (q x_u)^n$$
$$\times \prod_{l\in[v-n+1,v]} \left(1 - \frac{x_u}{x_l}\right)^{-1} \left(1 - q^{-2}\frac{x_l}{x_u}\right) \Bigg)$$
$$= \pm \sum_{\{l_1,\dots,l_n\}} g(x_{\tau^+_{l_1}\cdots\tau^+_{l_n}[v]})\, (x_{l_1}\dots x_{l_n})^{v-n}\, T(x_{l_1},\dots,x_{l_n})$$
$$\times \prod_{\substack{i=1,\dots,n \\ u\in[1,v]\setminus\{l_1,\dots,l_n\}}} \frac{q^{-1}x_{l_i} - q x_u}{x_{l_i} - x_u}.$$

∎

12.2. Let $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))/\text{torsion}$ be

$$\operatorname{Image}\Big(K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)\Big).$$

(It seems reasonable to conjecture that $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}^1,\mathbf{v}^2;\mathbf{w}))$ is free over $R(G_{\mathbf{w}}\times\mathbb{C}^*)$ since it is true for type A_n. But I do not know how to prove it in general.)

Theorem 12.2.1. *The homomorphism in Theorem 9.4.1 induces a homomorphism* $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))/\text{torsion}$.

Remark 12.2.2. The homomorphism is neither injective nor surjective. It is likely that there exists a surjective homomorphism from a modification of $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ to $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z^{\text{reg}}(\mathbf{w}))/\text{torsion}$ for a suitable subset $Z^{\text{reg}}(\mathbf{w})$ of $Z(\mathbf{w})$, as in [45, 9.5, 10.15].

Proof of Theorem 12.2.1. It is enough to check that $e_{k,r}^{(n)}$, $f_{k,r}^{(n)}$, q^h and the coefficients of $p_k^{\pm}(z)$ are mapped to $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$. For q^h and the coefficients of $p_k^{\pm}(z)$, the assertion is clear from the definition.

For $e_{k,r}^{(n)}$ and $f_{k,r}^{(n)}$, we can use a reduction to the rank 1 case as in §11. Namely, it is enough to show the assertion when the graph is of type A_1.

Now if the graph is of type A_1, Lemma 12.1.1 together with Lemma 11.4.1 show that $f_r^{(n)}$ is represented by a certain line bundle over $\omega(\mathfrak{P}^{(n)}(v,N))$ extended to $Z(v,v-n;N)$ by 0. We leave the proof for $e_r^{(n)}$ as an exercise. The only thing we need is to write down an analogue of Lemma 12.1.1 for $e_r^{(n)}$. It is straightforward. ∎

12.3. **The module $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$.** By Theorems 12.2.1 and 7.3.5, $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ is a $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$-module. We show that it is an l-highest weight module in this subsection.

Lemma 12.3.1. *Let $\mathfrak{P}_k^{(n)}(\mathbf{v},\mathbf{w})$ be as in* (5.3.1) *and let $\omega\colon \mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w})\times\mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}(\mathbf{v},\mathbf{w})\times\mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w})$ denote the exchange of factors. Let $T(V_k^1/V_k^2)$ be a tensor product of exterior products of the vector bundle V_k^1/V_k^2 and its dual over $\omega(\mathfrak{P}_k^{(n)}(\mathbf{v},\mathbf{w}))$. Let us consider it as an element of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v},\mathbf{v}-n\alpha_k;\mathbf{w}))$. Then $\left[T(V_k^1/V_k^2)\otimes\det C_k''^{\bullet}(\mathbf{v}-n\alpha_k,\mathbf{w})^{\otimes -n}\right]$ can be written as a linear combination* (*over* $\mathbb{Z}[q,q^{-1}]$) *of elements of the form*

$$f_{k,p_1}^{(n_1)} f_{k,p_2}^{(n_2)} \cdots f_{k,p_s}^{(n_s)} * [\mathcal{O}_{\Delta\mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w})}] \qquad (n_1+n_2+\cdots+n_s=n, \quad p_i \text{ distinct}),$$

where $\Delta\mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w})$ is the diagonal in $\mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w})\times\mathfrak{M}(\mathbf{v}-n\alpha_k,\mathbf{w})$.

Proof. As in §11, we may assume that the graph is of type A_1. Now Lemma 12.1.1 together with the fact that Hall-Littlewood polynomials form a basis of symmetric polynomials implies the assertion. ∎

Proposition 12.3.2. *Let $[0]\in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(0,\mathbf{w}))$ be the class represented by the structure sheaf of $\mathfrak{M}(0,\mathbf{w})\cong\mathfrak{L}(0,\mathbf{w}) = \text{point}$. Then*

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) = \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * (R(G_{\mathbf{w}}\times\mathbb{C}^*)[0]).$$

Proof. The following proof is an adaptation of the proof of [45, 10.2], which was inspired by [35, 3.6] in turn.

We need the following notation:

$$\mathfrak{L}_{k;n}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \mathfrak{L}(\mathbf{v},\mathbf{w}) \cap \mathfrak{M}_{k;n}(\mathbf{v},\mathbf{w}), \qquad \mathfrak{L}_{k;\le n}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \mathfrak{L}(\mathbf{v},\mathbf{w}) \cap \mathfrak{M}_{k;\le n}(\mathbf{v},\mathbf{w}),$$

$$\mathfrak{L}_{k;\ge n}(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \mathfrak{L}(\mathbf{v},\mathbf{w}) \cap \mathfrak{M}_{k;\ge n}(\mathbf{v},\mathbf{w}).$$

We prove $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v},\mathbf{w})) \subset \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * (R(G_{\mathbf{w}}\times\mathbb{C}^*)[0])$ by induction on the dimension vector $\mathbf{v}$. When $\mathbf{v}=0$, the result is trivial since $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\text{point}) = R(G_{\mathbf{w}}\times\mathbb{C}^*)$. Consider $\mathfrak{L}(\mathbf{v},\mathbf{w})$ and suppose that

$$\text{(12.3.3)} \qquad \begin{aligned} &\text{if } \mathbf{v}-\mathbf{v}' \in \bigoplus \mathbb{Z}_{\ge 0}\alpha_k \setminus \{0\}, \\ &\text{then } K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v}',\mathbf{w})) \subset \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * (R(G_{\mathbf{w}}\times\mathbb{C}^*)[0]). \end{aligned}$$

Take $[E] \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v},\mathbf{w}))$. We want to show

$$[E] \in \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * (R(G_{\mathbf{w}}\times\mathbb{C}^*)[0]) .$$

We may assume that the support of E is contained in an irreducible component of $\mathfrak{L}(\mathbf{v},\mathbf{w})$ without loss of generality. In fact, suppose that $\operatorname{Supp} E \subset X\cup Y$ such that X is an irreducible component. Since Y is a closed subvariety of $X\cup Y$ and since $X\cap Y$ is a closed subvariety of X, we have the diagram

$$\begin{array}{ccccccc} K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Y) & \xrightarrow{i_*} & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(X\cup Y) & \xrightarrow{j^*} & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(X\setminus Y) & \longrightarrow & 0 \\ \uparrow & & \uparrow{\scriptstyle i''_*} & & \| & & \\ K^{G_{\mathbf{w}}\times\mathbb{C}^*}(X\cap Y) & \xrightarrow{i'_*} & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(X) & \xrightarrow{j'^*} & K^{G_{\mathbf{w}}\times\mathbb{C}^*}(X\setminus Y) & \longrightarrow & 0, \end{array}$$

where the first and the second row are exact by (6.1.2). Thus there exists $[E'] \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(X)$ such that $j'^*[E'] = j^*[E]$. Then $j^*([E]-i''_*[E'])=0$, and therefore there exists $E'' \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Y)$ such that $[E] = i_*[E''] + i''_*[E']$. By induction on the number of irreducible components in the support, we may assume that the support of E is contained in an irreducible component, which is denoted by X_E.

Let us consider ε_k defined in (2.9.3). If $\varepsilon_k(X_E)=0$ for all $k\in I$, X_E must be $\mathfrak{L}(0,\mathbf{w})$ by Lemma 2.9.4. We have nothing to prove in this case. Thus there exists k such that $\varepsilon_k(X_E)>0$. Set $n=\varepsilon_k(X_E)$. By the descending induction on ε_k, we may assume that

$$\text{(12.3.4)} \quad \text{if } \operatorname{Supp}(E') \subset \mathfrak{L}_{k;\ge n+1}(\mathbf{v},\mathbf{w}), \text{ then } [E'] \in \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * (R(G_{\mathbf{w}}\times\mathbb{C}^*)[0]).$$

Since $\mathfrak{L}_{k;\le n}(\mathbf{v},\mathbf{w})$ is an open subvariety of $\mathfrak{L}(\mathbf{v},\mathbf{w})$, we have an exact sequence

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}_{k;\ge n+1}(\mathbf{v},\mathbf{w})) \xrightarrow{a_*} K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v},\mathbf{w})) \xrightarrow{b^*} K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}_{k;\le n}(\mathbf{v},\mathbf{w})) \to 0$$

by (6.1.2). Consider $b^*[E]$. By (12.3.4), it is enough to show that

$$\text{(12.3.5)} \qquad \text{there exists } [\widetilde{E}] \in \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * [0] \text{ such that } b^*[E] = b^*[\widetilde{E}].$$

Since $X_E \cap \mathfrak{L}_{k;\le n}(\mathbf{v},\mathbf{w}) \subset \mathfrak{L}_{k;n}(\mathbf{v},\mathbf{w})$, the support of $b^*(E)$ is contained in $\mathfrak{L}_{k;n}(\mathbf{v},\mathbf{w})$. We have a map

$$P\colon \mathfrak{L}_{k;n}(\mathbf{v},\mathbf{w}) \to \mathfrak{L}_{k;0}(\mathbf{v}-n\alpha_k,\mathbf{w})$$

which is the restriction of the map (5.4.2). Recall that this map is a Grassmann bundle (see Proposition 5.4.3). Let us denote its tautological bundle by S. Then $b^*[E]$ can be written as a linear combination of elements of the form

$$[T(S)] \otimes P^*[E_0],$$

where $T(S)$ is a tensor product of exterior powers of the tautological bundle S and $E_0 \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w}))$. Since the homomorphism

$$b'^*\colon K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v} - n\alpha_k, \mathbf{w})) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w}))$$

is surjective by (6.1.2), there exists $[E_1] \in K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{v} - n\alpha_k, \mathbf{w}))$ such that $b'^*[E_1] = [E_0]$.

Consider $\mathfrak{P}_k^{(n)\prime}(\mathbf{v}-n\alpha_k, \mathbf{w})\cap(\mathfrak{L}_{k;\le n}(\mathbf{v}, \mathbf{w})\times\mathfrak{L}(\mathbf{v}-n\alpha_k, \mathbf{w}))$. By Proposition 5.4.3, it is isomorphic to $\mathfrak{L}_{k;n}(\mathbf{v}, \mathbf{w})$ and the map P can be identified with the projection to the second factor. Moreover, the tautological bundle S is identified with the restriction of the natural vector bundle V_k^1/V_k^2. Hence we have

$$[T(S)] \otimes P^*[E_0] = b^*\left(T(V_k^1/V_k^2) * [E_1]\right),$$

where $T(V_k^1/V_k^2)$ is considered as an element of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{v}, \mathbf{v} - n\alpha_k; \mathbf{w}))$. By Lemma 12.3.1, $T(V_k^1/V_k^2)$ can be written as a linear combination of elements

$$f_{k,p_1}^{(n_1)} f_{k,p_2}^{(n_2)} \dots f_{k,p_s}^{(n_s)} * [p_2^* \det C_k''^{\bullet}(\mathbf{v} - n\alpha_k, \mathbf{w})^{\otimes n}],$$

where $p_2\colon \Delta\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \to \mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w})$ is the projection. By (12.3.3),

$$[\det C_k''^{\bullet}(\mathbf{v} - n\alpha_k, \mathbf{w})^{\otimes n} \otimes E_1] \in \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- * (R(G_{\mathbf{w}} \times \mathbb{C}^*)[0]).$$

Hence $T(V_k^1/V_k^2)*[E_1] \in \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^-*(R(G_{\mathbf{w}} \times \mathbb{C}^*)[0])$. Thus we have shown (12.3.5). $\square$

13. Standard modules

In this section, we start the study of the representation theory of $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ using $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ and the homomorphism in (12.2.1). We shall define certain modules called *standard modules*, and study their properties. Results in this section hold even if ε is a root of unity.

Note that $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ is contained in the center of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ by

$$R(G_{\mathbf{w}} \times \mathbb{C}^*) \ni \rho \mapsto \rho \otimes \sum_{\mathbf{v}} [\mathcal{O}_{\Delta\mathfrak{M}(\mathbf{v},\mathbf{w})}].$$

Hence a $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$/torsion-module M (over $\mathbb{C}$), which is l-integrable as a $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$-module, decomposes as $M = \bigoplus M_\chi$, where χ is a homomorphism from $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ to $\mathbb{C}$ and M_χ is the corresponding simultaneous generalized eigenspace, i.e., some powers of the kernel of χ act as 0 on M_χ. Such a homomorphism χ is given by the evaluation of the character at a semisimple element $a = (s, \varepsilon)$ in $G_{\mathbf{w}} \times \mathbb{C}^*$. (This gives us a bijection between homomorphisms and semisimple elements.)

What is the meaning of the choice of $a = (s, \varepsilon)$ when we consider M as a $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$-module? The role of ε is clear. It is a specialization $q \to \varepsilon$, and we get $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules. It will become clear later that s corresponds to the Drinfel'd polynomials by

$$P_k(u) = \text{(a normalization of) the characteristic polynomial of } s_k.$$

13.1. **Fixed data.** Let $a = (s, \varepsilon)$ be a semisimple element in $G_{\mathbf{w}} \times \mathbb{C}^*$ and let A be the Zariski closure of $\{a^n \mid n \in \mathbb{Z}\}$. Let $\chi_a \colon R(A) \to \mathbb{C}$ be the homomorphism given by the evaluation at a. Considering $\mathbb{C}$ as an $R(A)$-module by this evaluation homomorphism, we denote it by $\mathbb{C}_a$. Via the homomorphism $R(G_{\mathbf{w}} \times \mathbb{C}^*) \to R(A)$, we consider $\mathbb{C}_a$ also as an $R(G_{\mathbf{w}} \times \mathbb{C}^*)$-module. We consider $R(A)$ as a $\mathbb{Z}[q, q^{-1}]$-algebra, where $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ is a $\mathbb{Z}[q, q^{-1}]$-algebra as in §9.1.

Let $\mathfrak{M}(\mathbf{w})^A$, $\mathfrak{M}_0(\infty, \mathbf{w})^A$ be the fixed point subvarieties of $\mathfrak{M}(\mathbf{w})$, $\mathfrak{M}_0(\infty, \mathbf{w})$ respectively. Let us take a point $x \in \mathfrak{M}_0(\infty, \mathbf{w})^A$ which is regular, i.e., $x \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ for some $\mathbf{v}^0$.

The data x, a will be fixed throughout this section.

13.2. **Definition.** As in (2.3.5), let $\mathfrak{M}(\mathbf{v}, \mathbf{w})_x$ denote the inverse image of $x \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}) \subset \mathfrak{M}_0(\infty, \mathbf{w})$ under the map $\pi \colon \mathfrak{M}(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w}) \hookrightarrow \mathfrak{M}_0(\infty, \mathbf{w})$. It is invariant under the A-action. Let $\mathfrak{M}(\mathbf{w})_x$ be $\bigsqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})_x$. We set

$$K^A(\mathfrak{M}(\mathbf{w})_x) \overset{\text{def.}}{=} \bigoplus_{\mathbf{v}} K^A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x)$$

as convention.

Let $K^A(Z(\mathbf{w}))/\text{torsion}$ be

$$\operatorname{Image}\left(K^A(Z(\mathbf{w})) \to K^A(Z(\mathbf{w})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)\right).$$

Let

$$M_{x,a} \overset{\text{def.}}{=} K^A(\mathfrak{M}(\mathbf{w})_x) \otimes_{R(A)} \mathbb{C}_a. \tag{13.2.1}$$

By Theorem 7.3.5 together with Theorem 3.3.2, $K^A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x)$ is a free $R(A)$-module. Thus the $K^A(Z(\mathbf{w}))$-module structure on $K^A(\mathfrak{M}(\mathbf{w})_x)$ descends to a $K^A(Z(\mathbf{w}))/\text{torsion}$-module structure. Hence $M_{x,a}$ is a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module via the composition of

$$\begin{aligned} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) &\to K^{G_{\mathbf{w}} \times \mathbb{C}^*}(Z(\mathbf{w}))/\text{torsion} \otimes_{R(G_{\mathbf{w}} \times \mathbb{C}^*)} \mathbb{C}_a \\ &\to K^A(Z(\mathbf{w}))/\text{torsion} \otimes_{R(A)} \mathbb{C}_a. \end{aligned} \tag{13.2.2}$$

We call $M_{x,a}$ the *standard module*.

It has a decomposition $M_{x,a} = \bigoplus K^A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x) \otimes_{R(A)} \mathbb{C}_a$, and each summand is a weight space:

$$q^h * v = \varepsilon^{\langle h, \mathbf{w} - \mathbf{v} \rangle} v \qquad \text{for } v \in K^A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x) \otimes_{R(A)} \mathbb{C}_a. \tag{13.2.3}$$

Thus $M_{x,a}$ has the weight decomposition as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module.

In the remainder of this section, we study properties of $M_{x,a}$. The first one is the following.

Lemma 13.2.4. *As a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module, $M_{x,a}$ is l-integrable.*

Proof. The assertion is proved exactly as in [45, 9.3]. Note that the regularity assumption of x is *not* used here. □

13.3. **Highest weight vector.** Recall that $\pi \colon \mathfrak{M}(\mathbf{v}^0, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}^0, \mathbf{w})$ is an isomorphism on $\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}))$ (Proposition 2.6.2). Under this isomorphism, we can consider x as a point in $\mathfrak{M}(\mathbf{v}^0, \mathbf{w})$. Then $\mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x$ consists of the single point x, thus we have a canonical generator of $K^A(\mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x)$. We denote it by $[x]$.

Since x is fixed by A, the fibers $(V_k)_x$, $(W_k)_x$ of tautological bundles at x are A-modules. Then the restriction of the complex $C_k^\bullet(\mathbf{v}^0, \mathbf{w})$ to x can be considered as a complex of A-modules. In particular, it defines an element in $R(A)$. Let us denote it by $C^\bullet_{k,x}$.

Let us spell out $C^\bullet_{k,x}$ more explicitly. Since x is fixed by A, we have a homomorphism $\rho\colon A \to G_{\mathbf{v}^0}$ by §4.1. It is uniquely determined by x up to the conjugacy. Then a virtual $G_{\mathbf{v}^0} \times G_{\mathbf{w}} \times \mathbb{C}^*$-module

$$q^{-1}\left(\bigoplus_l [-\langle h_k, \alpha_l\rangle]_q V_l^0 \oplus W_k\right)$$

can be considered as a virtual A-module via $\rho \times (\text{inclusion})\colon A \to G_{\mathbf{v}^0} \times G_{\mathbf{w}} \times \mathbb{C}^*$. Its isomorphism class is independent of ρ and coincides with $C^\bullet_{k,x}$. Note that the first and third terms in (2.9.1) are absorbed in the term $l = k$.

Proposition 13.3.1. *The standard module $M_{x,a}$ is an l-highest weight module with l-highest weight $P_k(u) \overset{\text{def.}}{=} \chi_a\left(\bigwedge_{-u} C^\bullet_{k,x}\right)$. Namely, the following hold*:

(1) *$P_k(u)$ is a polynomial in u of degree $\langle h_k, \mathbf{w} - \mathbf{v}^0\rangle$.*

(2)

$$x_k^+(z) * [x] = 0, \quad q^h * [x] = \varepsilon^{\langle h, \mathbf{w}-\mathbf{v}^0\rangle}[x],$$

$$p_k^+(z) * [x] = P_k(1/z)[x], \quad p_k^-(z) * [x] = (-z)^{\operatorname{rank} C^\bullet_{k,x}} P_k(1/z)\, \chi_a\left((\det C^\bullet_{k,x})^*\right)[x].$$

(3) $M_{x,a} = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * [x]$.

Proof. (1) If we restrict the complex $C^\bullet_k(\mathbf{v}^0, \mathbf{w})$ to x, τ_k is surjective and σ_k is injective by Lemma 2.9.2. Thus $C^\bullet_{k,x}$ is represented by a genuine A-module, and $\chi_a\left(\bigwedge_{-u} C^\bullet_{k,x}\right)$ is a polynomial in u. The degree is equal to $\langle h_k, \mathbf{w} - \mathbf{v}^0\rangle$ by the definition of $C_{k,x}$.

(2) The first equation is the consequence of $\mathfrak{M}(\mathbf{v} - \alpha^k, \mathbf{w})_x = \emptyset$, which follows from Lemma 2.9.4. The remaining equations follow from the definition and Lemma 8.1.1.

(3) The assertion is proved exactly as in Proposition 12.3.2. Note that the assumption $x \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ is used here in order to apply Lemma 2.9.4. □

Remark 13.3.2. $P_k(u)$ is the Drinfel'd polynomial attached to the simple quotient of $M_{x,a}$, which we will study later.

We give a proof of Proposition 1.2.16 as promised:

Proof of Proposition 1.2.16. It is enough to show that there exists a simple l-integrable l-highest module with given Drinfel'd polynomials $P_k(u)$. We can construct it as the quotient of the standard module $M_{0,a}$ by the unique maximal proper submodule. (The uniqueness can be proved as in the case of Verma modules.) Here the parameter $a = (s, \varepsilon)$ is chosen so that $P_k(u) = \chi_a\left(\bigwedge_{-u} q^{-1} W_k\right)$, i.e., $P_k(u)$ is a normalization of the characteristic polynomial of s_k. □

13.4. **Localization.** Let $R(A)_a$ denote the localization of $R(A)$ with respect to $\operatorname{Ker}\chi_a$.

Let $Z(\mathbf{w})^A$ denote the fixed point set of A on $Z(\mathbf{w})$, and let $i\colon \mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A \to \mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w})$ be the inclusion. Note that it induces an inclusion $Z(\mathbf{w})^A \to Z(\mathbf{w})$ which we also denote by i. By the concentration theorem [53]

$$i_*\colon K^A(Z(\mathbf{w})^A) \otimes_{R(A)} R(A)_a \to K^A(Z(\mathbf{w})) \otimes_{R(A)} R(A)_a$$

is an isomorphism. Let

$$\begin{aligned} i^*\colon K^A(Z(\mathbf{w})) &\cong K^A(\mathfrak{M}(\mathbf{w})\times\mathfrak{M}(\mathbf{w}); Z(\mathbf{w})) \\ &\longrightarrow K^A(\mathfrak{M}(\mathbf{w})^A\times\mathfrak{M}(\mathbf{w})^A; Z(\mathbf{w})^A) \cong K^A(Z(\mathbf{w})^A) \end{aligned}$$

be the pull-back with support map. Then $i^* i_*$ is given by multiplication by $\bigwedge_{-1} N^* \boxtimes \bigwedge_{-1} N^*$, where N is the normal bundle of $\mathfrak{M}(\mathbf{w})^A$ in $\mathfrak{M}(\mathbf{w})$. By [13, 5.11.3], $\bigwedge_{-1} N^*$ becomes invertible in the localized K-group. Thus i^* is an isomorphism on the localized K-group. As in [13, 5.11.10], we introduce a correction factor to i^*:

$$r_a \overset{\text{def.}}{=} (1\boxtimes(\textstyle\bigwedge_{-1} N^*)^{-1})\circ i^*\colon K^A(Z(\mathbf{w}))\otimes_{R(A)} R(A)_a \to K^A(Z(\mathbf{w})^A)\otimes_{R(A)} R(A)_a.$$

Then r_a is an algebra isomorphism with respect to the convolution.

Since A acts trivially on $Z(\mathbf{w})^A$, we have

$$K^A(Z(\mathbf{w})^A) \cong K(Z(\mathbf{w})^A)\otimes R(A). \tag{13.4.1}$$

Thus we have the evaluation map

$$\mathrm{ev}_a\colon K^A(Z(\mathbf{w})^A)\otimes_{R(A)} R(A)_a \cong K(Z(\mathbf{w})^A)\otimes R(A)_a \to K(Z(\mathbf{w})^A)\otimes\mathbb{C},$$

by sending $F\otimes(f/g)$ to $F\otimes(\chi_a(f)/\chi_a(g))$.

By the bivariant Riemann-Roch theorem [13, 5.11.11],

$$\mathrm{RR} \overset{\text{def.}}{=} (1\boxtimes \mathrm{Td}_{\mathfrak{M}(\mathbf{w})^A})\cup \mathrm{ch}\colon K(Z(\mathbf{w})^A)\to H_*(Z(\mathbf{w})^A,\mathbb{Q})$$

is an algebra homomorphism with respect to the convolution. Here ch is the local Chern character homomorphism with respect to $Z(\mathbf{w})^A\subset \mathfrak{M}(\mathbf{w})^A\times\mathfrak{M}(\mathbf{w})^A$ and $\mathrm{Td}_{\mathfrak{M}(\mathbf{w})^A}$ is the Todd genus of $\mathfrak{M}(\mathbf{w})^A$.

Composing (13.2.2) with all these homomorphisms, we have a homomorphism

$$\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to H_*(Z(\mathbf{w})^A,\mathbb{C}). \tag{13.4.2}$$

Note that the torsion part in (13.2.2) disappears in the right hand side of (13.4.1) after tensoring with $R(A)_a$.

We have similar $\mathbb{C}$-linear maps for $\mathfrak{M}(\mathbf{w})_x$:

$$\begin{aligned} M_{x,a} = K^A(\mathfrak{M}(\mathbf{w})_x)\otimes_{R(A)}\mathbb{C}_a &\xrightarrow[\cong]{i^*} K^A(\mathfrak{M}(\mathbf{w})_x^A)\otimes_{R(A)}\mathbb{C}_a \\ &\xrightarrow[\cong]{\mathrm{ev}_a} K(\mathfrak{M}(\mathbf{w})_x^A)\otimes\mathbb{C} \xrightarrow[\cong]{\mathrm{ch}} H_*(\mathfrak{M}(\mathbf{w})_x^A,\mathbb{C}), \end{aligned} \tag{13.4.3}$$

where i^* is an isomorphism by the concentration theorem [53] and the invertibility of $\bigwedge_{-1} N^*$ in the localized K-homology group, ev_a is an isomorphism since A acts trivially on $\mathfrak{M}(\mathbf{w})_x^A$, and ch is an isomorphism by Theorem 7.4.1 and Theorem 3.3.2. The composition is compatible with the $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module structure, where $H_*(\mathfrak{M}(\mathbf{w})_x^A,\mathbb{C})$ is a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module via the convolution together with (13.4.2).

Recall that we have the decomposition

$$\mathfrak{M}(\mathbf{w})^A = \bigsqcup_\rho \mathfrak{M}(\rho),$$

where ρ runs over the set of homomorphisms $A\to G_{\mathbf{v}}$ (with various $\mathbf{v}$) (§4.1). Let

$$\mathfrak{M}(\rho)_x \overset{\text{def.}}{=} \mathfrak{M}(\rho)\cap\mathfrak{M}(\mathbf{w})_x^A.$$

Thus we have the canonical decomposition

$$M_{x,a} = H_*(\mathfrak{M}(\mathbf{w})_x^A, \mathbb{C}) = \bigoplus_\rho H_*(\mathfrak{M}(\rho)_x, \mathbb{C}). \tag{13.4.4}$$

Each summand $H_*(\mathfrak{M}(\rho)_x, \mathbb{C})$ in (13.4.4) is an l-weight space with respect to the $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-action in the sense that operators $\psi_k^\pm(z)$ act on $H_*(\mathfrak{M}(\rho)_x, \mathbb{C})$ as scalars plus nilpotent transformations. More precisely, we have

Proposition 13.4.5. (1) *Let V_k be the tautological vector bundle over $\mathfrak{M}(\mathbf{v},\mathbf{w})$. Viewing $\bigwedge_u V_k$ as an element of $K^A(\Delta\mathfrak{M}(\mathbf{v},\mathbf{w}))[u]$, we consider it as an operator on $M_{x,a}$. Then we have*

$$\begin{aligned} H_*(\mathfrak{M}(\rho)_x, \mathbb{C}) = \{m \in M_{x,a} \mid (\textstyle\bigwedge_u V_k - \chi_a(\bigwedge_u V_k)\,\mathrm{Id})^N * m = 0 \\ \text{for } k \in I \text{ and sufficiently large } N\}, \end{aligned}$$

where $\chi_a(\bigwedge_u V_k)$ is the evaluation at a of $\bigwedge_u V_k$, considered as an A-module via $\rho\colon A \to G_{\mathbf{v}}$.

(2) *Let us consider*

$$C_k^\bullet(\mathbf{v},\mathbf{w}) = q^{-1}\left(\bigoplus_l [-\langle h_k, \alpha_l\rangle]_q V_l \oplus W_k\right)$$

as a virtual A-module via $\rho \times (\text{inclusion})\colon A \to G_{\mathbf{v}} \times G_{\mathbf{w}} \times \mathbb{C}^*$. Then operators $\psi_k^\pm(z)$ act on $H_*(\mathfrak{M}(\rho)_x, \mathbb{C})$ by

$$\varepsilon^{\operatorname{rank} C_k^\bullet(\mathbf{v},\mathbf{w})}\chi_a\left(\frac{\bigwedge_{-1/qz} C_k^\bullet(\mathbf{v},\mathbf{w})}{\bigwedge_{-q/z} C_k^\bullet(\mathbf{v},\mathbf{w})}\right)^\pm \tag{13.4.6}$$

plus nilpotent transformations.

Proof. (2) follows from (1). We show (1).

Note that $\mathcal{O}_{\Delta\mathfrak{M}(\mathbf{v},\mathbf{w})}$ is mapped to $\mathcal{O}_{\Delta\mathfrak{M}(\mathbf{v},\mathbf{w})^A}$ under r_a, and $\mathcal{O}_{\Delta\mathfrak{M}(\mathbf{v},\mathbf{w})^A}$ is mapped to the fundamental class $[\Delta\mathfrak{M}(\mathbf{v},\mathbf{w})^A]$ under RR. Combining with the projection formula (6.5.1), we find that the operator $\bigwedge_u V_k$ is mapped to

$$(\mathrm{ch} \circ \mathrm{ev}_a \circ i^* \textstyle\bigwedge_u V_k) \cap [\Delta\mathfrak{M}(\mathbf{v},\mathbf{w})^A]$$

under the homomorphism (13.4.2). Thus as an operator on $H_*(\mathfrak{M}(\rho)_x, \mathbb{C})$, it is equal to

$$m \mapsto (j^* \circ \mathrm{ch} \circ \mathrm{ev}_a \circ i^* \textstyle\bigwedge_u V_k) \cap m, \tag{13.4.7}$$

where $j\colon \mathfrak{M}(\rho)_x \to \mathfrak{M}(\mathbf{v},\mathbf{w})^A$ is the inclusion.

Now, on a connected space X, any $\alpha \in H^*(X,\mathbb{C})$ acts on $H_*(X,\mathbb{C})$ as a scalar plus nilpotent operator, where the scalar is the $H^0(X,\mathbb{C})(\cong \mathbb{C})$-part of α. In our situation, the H^0-part of (13.4.7) is given by $\chi_a(\bigwedge_u V_k)$. (Although we do not prove $\mathfrak{M}(\rho)_x$ is connected, the H^0-part is the same on any component.)

Furthermore, $\chi_a(\bigwedge_u V_k)$ determines all eigenvalues of the operator a acting on V_k. Hence it determines the conjugacy class of the homomorphism $\rho\colon A \to G_{\mathbf{v}}$. Thus the generalized eigenspace of $\bigwedge_u V_k$ with the eigenvalue $\chi_a(\bigwedge_u V_k)$ coincides with $H_*(\mathfrak{M}(\rho)_x, \mathbb{C})$. $\square$

13.5. **Frenkel-Reshetikhin's q-character.** In this subsection, we study Frenkel-Reshetikhin's q-character for the standard module $M_{x,a}$. The result is a simple application of Proposition 13.4.5. Results in this subsection will not be used in the rest of the paper.

We assume $\mathfrak{g}$ is of type ADE and ε is not a root of unity in this subsection.

Let us recall the definition of q-character. It is a map from the Grothendieck group of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules M. As we shall see later in §14.3, standard modules $M_{0,a}$ ($x = 0$ is fixed, $\mathbf{w}$ and $a = (s,\varepsilon) \in G_{\mathbf{w}} \times \mathbb{C}^*$ are moving) give a basis of the Grothendieck group, thus it is enough to define the q-character for standard modules $M_{0,a}$. We decompose $M = M_{0,a}$ as

$$M = \bigoplus M_{\Psi^\pm},$$

as in (1.3.1). Moreover, by Proposition 13.4.5, $\Psi_k^\pm(z)$ have the form

$$\Psi_k^\pm(z) = \varepsilon^{\deg Q_k - \deg R_k} \frac{Q_k(1/\varepsilon z) R_k(\varepsilon/z)}{R_k(1/\varepsilon z) Q_k(\varepsilon/z)}, \tag{13.5.1}$$

where $Q_k(u)$, $R_k(u)$ are polynomials in u with constant term 1. (Compare with [18, Proposition 1]. Note $u = 1/z$.) Suppose

$$\frac{Q_k(u)}{R_k(u)} = \frac{\prod_r (1 - u a_{kr})}{\prod_s (1 - u b_{ks})}.$$

Then the q-character is defined by

$$\chi_q(M_{0,a}) \overset{\text{def.}}{=} \sum_{\Psi^\pm(z)} \dim V_{\Psi_k^\pm(z)} \prod_{k\in I} \prod_r Y_{k,a_{kr}} \prod_s Y_{k,b_{ks}}^{-1},$$

where $Y_{k,a_{kr}}$, $Y_{k,b_{ks}}$ are formal variables and χ_q takes its value in $\mathbb{Z}[Y_{k,c}^\pm]_{k\in I, c\in\mathbb{C}^*}$. ($\chi_q$ should not be confused with χ_a.)

Let

$$A_{k,a} \overset{\text{def.}}{=} Y_{k,a\varepsilon} Y_{k,a\varepsilon^{-1}} \prod_{h:\mathrm{in}(h)=k} Y_{\mathrm{out}(h), a\varepsilon^{m(h)}}^{-1}.$$

Proposition 13.5.2 (cf. Conjecture 1 in [18])**.** *Let $M_{0,a}$ be a standard module with $x = 0$. Suppose that $P_k(u)$ in Proposition 13.3.1 equals*

$$P_k(u) = \prod_{i=1}^{n_k} (1 - u a_i^{(k)})$$

for $k \in I$. Then the q-character of $M_{0,a}$ has the following form:

$$\prod_{k\in I} \prod_{i=1}^{n_k} Y_{k,a_i^{(k)}} \left(1 + \sum M_p'\right),$$

where each M_p' is a product of $A_{l,c}^{-1}$ with $c \in \bigcup a_i^{(k)} \varepsilon^{\mathbb{Z}}$.

Proof. By Proposition 13.4.5, $H_*(\mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})$ is a generalized eigenspace for $\psi_k^\pm(z)$ for a homomorphism $\rho\colon A \to G_{\mathbf{v}}$. Thus it is enough to study the eigenvalue. We consider V_k, W_k as A-modules via $\rho \times (\text{inclusion})\colon A \to G_{\mathbf{v}} \times G_{\mathbf{w}} \times \mathbb{C}^*$ as before. Let $V_k(\lambda)$, $W_k(\lambda)$ be weight spaces as in §4.1.

By the definition of $C_k^\bullet(\mathbf{v},\mathbf{w})$ and $P_k(u)$, we have

$$\chi_a\left(\frac{\bigwedge_{-1/qz} C_k^\bullet(\mathbf{v},\mathbf{w})}{\bigwedge_{-q/z} C_k^\bullet(\mathbf{v},\mathbf{w})}\right) = \chi_a\left(\frac{\bigwedge_{-1/qz} q^{-1}W_k}{\bigwedge_{-q/z} q^{-1}W_k}\prod_l \frac{\bigwedge_{-1/qz} q^{-1}[-\langle h_k,\alpha_l\rangle]_q V_l}{\bigwedge_{-q/z} q^{-1}[-\langle h_k,\alpha_l\rangle]_q V_l}\right),$$

$$P_k(u) = \chi_a\left(\bigwedge\nolimits_{-u} q^{-1}W_k\right).$$

By Proposition 13.4.5 we have

$$\frac{Q_k(u)}{R_k(u)} = P_k(u)\ \chi_a\left(\prod_l \bigwedge\nolimits_{-u} q^{-1}[-\langle h_k,\alpha_l\rangle]_q V_l\right),$$

where $Q_k(u)$, $R_k(u)$ are defined by (13.5.1).

Let $\{c_t^{(l)}\}$ be the set of eigenvalues of $a \in A$ on V_l counted with multiplicities. Then we have

$$\chi_a(\bigwedge\nolimits_{-u} q^{-1}[-\langle h_k,\alpha_l\rangle]_q V_l) = \begin{cases} \left(\prod_t (1-uc_t^{(k)})(1-u\varepsilon^2 c_t^{(k)})\right)^{-1} & \text{if } k = l,\\ \prod_{h:\substack{\mathrm{in}(h)=k\\ \mathrm{out}(h)=l}} \prod_t (1-u\varepsilon^{m(h)+1}c_t^{(l)}) & \text{otherwise.}\end{cases}$$

Thus we have

$$\chi_q(M_{0,a}) = \sum_\rho \dim H_*(\mathfrak{M}(\rho)\cap\mathfrak{L}(\mathbf{w}),\mathbb{C}) \prod_{k\in I}\prod_{i=1}^{n_k} Y_{k,a_i^{(k)}} \prod_t A^{-1}_{k,\varepsilon c_t^{(k)}}.$$

Note that the term for ρ with $\mathbf{v}=0$ has the contribution

$$\prod_{k\in I}\prod_{i=1}^{n_k} Y_{k,a_i^{(k)}},$$

and any other terms are monomials of $A^{-1}_{k,\varepsilon c_t^{(k)}}$ which are not constant.

Moreover, we have $c_t^{(k)} \in \bigcup a_i^{(k)}\varepsilon^{\mathbb{Z}}$ by Lemma 4.1.4. This completes the proof. □

14. Simple modules

The purpose of this section is to study simple modules of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. Our discussion relies on Ginzburg's classification of simple modules of the convolution algebra [13, Chapter 9]. (See also [37].) He applied his classification to the affine Hecke algebra. However, unlike the case of the affine Hecke algebra, his classification does not directly imply a classification of simple modules of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, and we need an extra argument. A difficulty lies in the fact that the homomorphism $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to H_*(Z(\mathbf{w})^A,\mathbb{C})$ in (13.4.2) is not necessarily an isomorphism. Our additional input is Proposition 13.3.1(3). In order to illustrate its usage, we first consider the special case when $a = (s,\varepsilon)$ is generic in the first subsection. In this case, Ginzburg's classification becomes trivial. Then we shall review Ginzburg's classification in §14.2, and finally we shall study the general case in the last subsection.

We preserve the setup in §13.

14.1. Let us identify $e_{k,r}$, $f_{k,r}$ with their image under (13.4.2). Let $1_\rho \in H_*(Z(\mathbf{w})^A)$ denote the fundamental class $[\Delta\mathfrak{M}(\rho)]$ of the diagonal of $\mathfrak{M}(\rho)\times\mathfrak{M}(\rho)$.

Lemma 14.1.1. *Let us consider*

$$(\mathfrak{M}(\rho^1)\times\mathfrak{M}(\rho^2))\cap\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w})\subset(\mathfrak{M}(\mathbf{v}^1,\mathbf{w})^A\times\mathfrak{M}(\mathbf{v}^2,\mathbf{w})^A)\cap\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}),$$

and let λ_0 be the weight of A determined by ρ^1 and ρ^2 as in §5.2. Then we have the following equality in $H_*((\mathfrak{M}(\rho^1)\times\mathfrak{M}(\rho^2))\cap\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}),\mathbb{C})$:

$$\begin{aligned}&p_2^*\operatorname{ch}\left(\textstyle\bigwedge_{-u}V_l^2(\lambda)\right)\cap(1_{\rho^1}x_k^+(z)1_{\rho^2})\\&\quad=\begin{cases}p_1^*\operatorname{ch}\left(\bigwedge_{-u}V_l^1(\lambda)\right)\cap(1_{\rho^1}x_k^+(z)1_{\rho^2}) & \text{if } l\neq k \text{ or } \lambda\neq\lambda_0,\\(1-\frac{uq}{z})p_1^*\operatorname{ch}\left(\bigwedge_{-u}V_l^1(\lambda)\right)\cap(1_{\rho^1}x_k^+(z)1_{\rho^2}) & \text{if } l=k,\ \lambda=\lambda_0.\end{cases}\end{aligned}$$

Proof. We have the following equality in $K^0((\mathfrak{M}(\rho^1)\times\mathfrak{M}(\rho^2))\cap\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w}))$:

$$p_2^*V_l^2(\lambda)=\begin{cases}p_1^*V_l^1(\lambda) & \text{if } l\neq k \text{ or } \lambda\neq\lambda_0,\\p_1^*V_k^1(\lambda_0)+(V^2/V^1) & \text{if } l=k,\ \lambda=\lambda_0,\end{cases}$$

where V^2/V^1 is (the restriction of) the natural line bundle over $\mathfrak{P}_k(\mathbf{v}^2,\mathbf{w})$. The assertion follows immediately. □

Theorem 14.1.2. *Suppose that $a=(s,\varepsilon)$ is generic in the sense of Definition 4.2.1. (Hence, $\mathfrak{L}(\mathbf{w})^A=\mathfrak{M}(\mathbf{w})^A$.) Then the standard module*

$$M_{0,a}=K^A(\mathfrak{M}(\mathbf{w})^A)\otimes_{R(A)}\otimes\mathbb{C}_a\cong H_*(\mathfrak{M}(\mathbf{w})^A,\mathbb{C})$$

is a simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module. Its Drinfel'd polynomial is given by

$$P_k(u)=\det(1-u\varepsilon^{-1}s_k),$$

where s_k is the $\mathrm{GL}(W_k)$-component of $s\in G_{\mathbf{w}}$. Moreover, $M_{0,a}$ is isomorphic to a tensor product of l-fundamental representations when $\mathfrak{g}$ is finite dimensional.

Proof. Recall that we have a distinguished vector (we denote it by $[0]$) in the standard module $M_{0,a}$ (§13.3). It has the properties listed in Proposition 13.3.1. In particular, it is the eigenvector for $p_k^\pm(z)$, and the eigenvalues are given in terms of $P_k(u)$ therein. In the present setting, $P_k(u)$ is equal to $\det(1-u\varepsilon^{-1}s_k)$.

Let

$$M_{0,a}^\circ\overset{\text{def.}}{=}\{m\in M_{0,a}\mid e_{k,r}*m=0 \text{ for any } k\in I,\ r\in\mathbb{Z}\}.$$

We have $[0]\in M_{0,a}^\circ$. We want to show that any nonzero submodule M' of $M_{0,a}$ is $M_{0,a}$ itself. The weight space decomposition (as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module) (13.2.3) of $M_{0,a}$ induces that of M'. Since the set of weights of M' is bounded from $\mathbf{w}$ with respect to the dominance order, there exists a maximal weight of M'. Then a vector in the corresponding weight space is killed by all $e_{k,r}$ by the maximality. Thus M' contains a nonzero vector $m\in M_{0,a}^\circ$. Hence it is enough to show that $M_{0,a}^\circ=\mathbb{C}[0]$ since we have already shown that $M_{0,a}=\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^-*[0]$ in Proposition 13.3.1(3).

Let us consider the operator

$$[\Delta_*\textstyle\bigwedge_u V_l]\in K^A(Z(\mathbf{v},\mathbf{v};\mathbf{w}))$$

where $\Delta\colon\mathfrak{M}(\mathbf{v},\mathbf{w})\to Z(\mathbf{v},\mathbf{v};\mathbf{w})$ is the diagonal embedding. If we consider such operators for various $\mathbf{v}$, $l\in I$, they form a commuting family. Moreover, $M_{0,a}^\circ$ is invariant under them since we have the relation

$$e_{k,r} * \left[\Delta_* \textstyle\bigwedge_u V_l^2\right] = \begin{cases} \left[\Delta_* \textstyle\bigwedge_u V_l^1\right] * e_{k,r} & \text{if } k \neq l, \\ \left[\Delta_* \textstyle\bigwedge_u V_k^1\right] * (e_{k,r} + uq e_{k,r+1}) & \text{if } k = l, \end{cases}$$

where V_k^1, V_k^2 are tautological bundles over $\mathfrak{M}(\mathbf{v}^1, \mathbf{w})$, $\mathfrak{M}(\mathbf{v}^2, \mathbf{w})$ respectively. (Here $\mathbf{v}^2 = \mathbf{v}^1 + \alpha_k$.) Thus $M_{0,a}^\circ$ is a direct sum of generalized eigenspaces for $\Delta_* \bigwedge_u V_l$. Let us take a direct summand $M_{0,a}^{\circ\circ}$. By Proposition 13.4.5(1), $M_{0,a}^{\circ\circ}$ is contained in $H_*(\mathfrak{M}(\rho), \mathbb{C})$ for some $\rho\colon A \to G_{\mathbf{v}}$. If we can show $M_{0,a}^{\circ\circ} = \mathbb{C}[0]$, then we get $M_{0,a}^\circ = \mathbb{C}[0]$ since $M_{0,a}^{\circ\circ}$ is an *arbitrary* direct summand.

Since a is generic, we have $\mathfrak{M}_0(\infty, \mathbf{w})^A = \{0\}$. Hence

$$Z(\mathbf{w})^A = \mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A,$$

and $\mathfrak{M}(\mathbf{w})^A$ is a nonsingular projective variety (having possibly infinitely many components). By the Poincaré duality, the intersection pairing

$$(\ ,\ )\colon H_*(\mathfrak{M}(\mathbf{w})^A, \mathbb{C}) \otimes H_*(\mathfrak{M}(\mathbf{w})^A, \mathbb{C}) \to \mathbb{C}$$

is nondegenerate.

Let ${}^t f_{k,r}$ denote the transpose of $f_{k,r}$ with respect to the pairing $(\ ,\ )$, namely

$$(f_{k,r} * m, m') = (m, {}^t f_{k,r} * m') \qquad \text{for } m,\, m' \in H_*(\mathfrak{M}(\mathbf{w})^A, \mathbb{C}).$$

By the definition of the convolution, ${}^t f_{k,r}$ is equal to $\omega_* f_{k,r}$ where $\omega\colon \mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A \to \mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A$ is the map exchanging the first and second factors and ω_* is the induced homomorphism on $H_*(\mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A, \mathbb{C})$.

Let us consider $1_\rho f_{k,r} 1_{\rho'}$, where ρ is as above and ρ' is any other homomorphism. It is just the projection of $f_{k,r}$ to the component $H_*(\mathfrak{M}(\rho) \times \mathfrak{M}(\rho'), \mathbb{C})$. We have

$$1_{\rho'}\ {}^t f_{k,r}\, 1_\rho = 1_{\rho'}\, \omega_* f_{k,r}\, 1_\rho = (p_1^* \alpha \cup p_2^* \beta) \cap 1_{\rho'}\, e_{k,r'}\, 1_\rho$$

for some $r' \in \mathbb{Z}$, $\alpha \in H_*(\mathfrak{M}(\rho), \mathbb{C})$, and $\beta \in H_*(\mathfrak{M}(\rho'), \mathbb{C})$. These α and β come from asymmetry in the definition of $f_{k,r}$, $e_{k,r}$ and in the homomorphism (13.4.2). We do not give their explicit forms, although it is possible. What we need is for β to be written by tensor powers of exterior products of $V_k^2(\lambda)$ for various k, λ. Thus we can write

$$(p_1^* \alpha \cup p_2^* \beta) \cap 1_{\rho'}\, e_{k,r'}\, 1_\rho = \sum_{r''} p_1^* \alpha_{r''} \cap 1_{\rho'}\, e_{k,r''}\, 1_\rho$$

for some $\alpha_{r''} \in H_*(\mathfrak{M}(\rho), \mathbb{C})$ by Lemma 14.1.1. Therefore, for $m \in M_{0,a}^{\circ\circ}$, we have

$$(f_{k,r} * m', m) = (1_\rho\, f_{k,r}\, 1_{\rho'} * m', m) = \left(m', \sum_{r''} p_1^* \alpha_{r''} \cap 1_{\rho'}\, e_{k,r''}\, 1_\rho * m \right) = 0$$

for any $k \in I$, $r \in \mathbb{Z}$, ρ', $m' \in H_*(\mathfrak{M}(\rho'), \mathbb{C})$. Here we have used $1_\rho * m = m$, $1_{\rho'} * m' = m'$, $e_{k,r''} * m = 0$. Since $H_*(\mathfrak{M}(\mathbf{w})^A, \mathbb{C}) = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * [0]$, we have one of the following:

(a) $(m', m) = 0$ for any $m' \in H_*(\mathfrak{M}(\mathbf{w})^A, \mathbb{C})$,
(b) $m \in \mathbb{C}[0]$.

The first case is excluded by the nondegeneracy of $(\ ,\ )$. Thus we have $m \in \mathbb{C}[0]$.

Let us prove the last assertion. First consider the case $\mathbf{w} = \Lambda_k$ for some k. If ε is not a root of unity, $a = (s, \varepsilon) \in \mathbb{C}^* \times \mathbb{C}^*$ is generic for the quiver variety $\mathfrak{M}(\Lambda_k)$. Hence the above shows that the standard module for $\mathfrak{M}(\Lambda_k)$ is simple, and hence gives an l-fundamental representation.

Let us return to the case for general $\mathbf{w} = \sum_k w_k \Lambda_k$. Let $a_k^1, \dots, a_k^{w_k}$ be eigenvalues of s_k counted with multiplicities. By Proposition 1.2.19, it is enough to show that

$$\dim M_{0,a} = \prod_k \prod_{i=1}^{w_k} \dim M_{0,a_k^i}(\Lambda_k), \tag{14.1.3}$$

where $M_{0,a_k}(\Lambda_k)$ is the standard module for $\mathfrak{M}(\Lambda_k)$ with $a_k = (s_k, \varepsilon)$. Since $\mathfrak{M}(\mathbf{w})^A$ has no odd homology groups (Theorem 7.4.1), we have

$$\dim M_{0,a} = \operatorname{Euler}(\mathfrak{M}(\mathbf{w})^A),$$

where Euler( ) denotes the topological Euler number. By a property of the Euler number, we have

$$\operatorname{Euler}(\mathfrak{M}(\mathbf{w})^A) = \operatorname{Euler}(\mathfrak{M}(\mathbf{w})).$$

If we take a maximal torus T of $G_{\mathbf{w}}$, the fixed point set $\mathfrak{M}(\mathbf{w})^T$ is isomorphic to $\prod_k \mathfrak{M}(\Lambda_k)^{w_k}$. Again by a property of the Euler number, we have

$$\operatorname{Euler}(\mathfrak{M}(\mathbf{w})) = \prod_k \operatorname{Euler}(\mathfrak{M}(\Lambda_k))^{w_k}.$$

Since we have

$$\dim M_{0,a_k^i}(\Lambda_k) = \operatorname{Euler}(\mathfrak{M}(\Lambda_k)),$$

we get (14.1.3). □

14.2. **Simple modules of the convolution algebra.** We briefly recall Ginzburg's classification of simple modules of the convolution algebra [13, §8.6]. (See also [37].)

Let X be a complex algebraic variety. We consider the derived category of complexes of sheaves with constructible cohomology sheaves, and denote it by $D^b(X)$. We use the notation in [13]. For example, we put

$$\operatorname{Ext}^k_{D^b(X)}(A, B) \overset{\text{def.}}{=} \operatorname{Hom}_{D^b(X)}(A, B[k]),$$

$$\operatorname{Ext}^*_{D^b(X)}(A, B) \overset{\text{def.}}{=} \bigoplus_k \operatorname{Ext}^k_{D^b(X)}(A, B).$$

$\operatorname{Ext}^*_{D^b(X)}(A, A)$ is an algebra by the Yoneda product. The Verdier duality operator is denoted by ${}^\vee$. Given graded vector spaces V, W, we write $V \doteq W$ if there exists a linear isomorphism which does *not* necessarily preserve the gradings. We will also use the same notation to denote that two objects are quasi-isomorphic up to a shift in the derived category.

Let $f\colon M \to X$ be a projective morphism between algebraic varieties M, X, and assume that M is nonsingular. Then we are in the setting for the convolution in §8 with $X_1 = X_2 = X_3$ and $Z_{12} = Z_{23} = Z$, where

$$Z \overset{\text{def.}}{=} M \times_X M = \{(m^1, m^2) \in M \times M \mid f(m^1) = f(m^2)\}.$$

Since $Z \circ Z = Z$, we have the convolution product

$$H_*(Z, \mathbb{C}) \otimes H_*(Z, \mathbb{C}) \to H_*(Z, \mathbb{C}).$$

Let $\mathcal{A}$ be the algebra $H_*(Z, \mathbb{C})$. Set $M_x = f^{-1}(x)$. Then the convolution defines an $\mathcal{A}$-module structure on $H_*(M_x, \mathbb{C})$. More generally, if Y is a locally closed subset of X, then $H_*(f^{-1}(Y), \mathbb{C})$ has an $\mathcal{A}$-module structure via convolution.

By [13, 8.6.7], we have an algebra isomorphism, which does not necessarily preserve gradings,

$$\mathcal{A} = H_*(Z, \mathbb{C}) \doteq \operatorname{Ext}^*_{D^b(X)}(f_*\mathbb{C}_M, f_*\mathbb{C}_M),$$

where $\mathbb{C}_M$ is the constant sheaf on M.

We apply the decomposition theorem [6] to $f_*\mathbb{C}_M$. There exists an isomorphism in $D^b(X)$:

$$f_*\mathbb{C}_M \cong \bigoplus_{\phi,k} L_{\phi,k} \otimes P_\phi[k], \tag{14.2.1}$$

where $\{P_\phi\}$ is the set of isomorphism classes of simple perverse sheaves on X such that some shift is a direct summand of $f_*\mathbb{C}_M$. We thus have an isomorphism

$$\mathcal{A} \doteq \bigoplus_{i,j,k,\phi,\psi} \operatorname{Hom}_{\mathbb{C}}(L_{\phi,i}, L_{\psi,j}) \otimes \operatorname{Ext}^k_{D^b(X)}(P_\phi, P_\psi).$$

Set $L_\phi \overset{\text{def.}}{=} \bigoplus_k L_{\phi,k}$ and

$$\mathcal{A}_k \overset{\text{def.}}{=} \bigoplus_{\phi,\psi} \operatorname{Hom}_{\mathbb{C}}(L_\phi, L_\psi) \otimes \operatorname{Ext}^k_{D^b(X)}(P_\phi, P_\psi)$$

so that $\mathcal{A} = \bigoplus \mathcal{A}_k$. By definition, $\mathcal{A}_k \cdot \mathcal{A}_l \subset \mathcal{A}_{k+l}$ under the multiplication of $\mathcal{A}$. By a property of perverse sheaves, we have $\mathcal{A}_k = 0$ for $k < 0$ and $\operatorname{Ext}^0_{D^b(X)}(P_\phi, P_\psi) = \mathbb{C}\delta_{\phi\psi}\,\mathrm{id}$. Hence,

$$\mathcal{A} = \mathcal{A}_0 \oplus \bigoplus_{k>0} \mathcal{A}_k; \qquad \mathcal{A}_0 \cong \bigoplus_\phi \operatorname{End}(L_\phi). \tag{14.2.2}$$

In particular, the projection $\mathcal{A} \to \mathcal{A}_0$ is an algebra homomorphism. Furthermore, $\mathcal{A}_0$ is a semisimple algebra, and the kernel of the projection, i.e., $\bigoplus_{k>0} \mathcal{A}_k$, consists of nilpotent elements, thus it is precisely the radical of $\mathcal{A}$. In particular,

$$\{L_\phi\}_\phi$$

is a complete set of mutually nonisomorphic simple $\mathcal{A}$-modules.

For $x \in X$, let $i_x \colon \{x\} \to X$ denote the inclusion. Then $H^*(i_x^! f_*\mathbb{C}_M)$ is an $\operatorname{Ext}^*_{D^b(X)}(f_*\mathbb{C}_M, f_*\mathbb{C}_M)$-module. More generally, if $i_Y \colon Y \hookrightarrow X$ is a locally closed embedding, then the hyper-cohomology groups $H^*(Y, i_Y^! f_*\mathbb{C}_M)$ and $H^*(Y, i_Y^*, f_*\mathbb{C}_M)$ are $\operatorname{Ext}^*_{D^b(X)}(f_*\mathbb{C}_M, f_*\mathbb{C}_M)$-modules. It is known (see [13, 8.6.16, 8.6.35]) that $H^*(Y, i_Y^! f_*\mathbb{C}_M)$ is isomorphic to $H_*(f^{-1}(Y), \mathbb{C})$ as an $\mathcal{A} \cong \operatorname{Ext}^*_{D^b(X)}(f_*\mathbb{C}_M, f_*\mathbb{C}_M)$-module.

For $C \in D^b(\{x\})$, we write $H^k(C)$ instead of $H^k(\{x\}, C)$, and $H^*(C)$ instead of $H^*(\{x\}, C)$. By applying $H^*(i_x^! \bullet)$ to (14.2.1), we get an isomorphism

$$H_*(M_x, \mathbb{C}) \doteq \bigoplus_{\phi,k} L_\phi \otimes H^k(i_x^! P_\phi).$$

Let

$$M_{\geq k}(i_x^! f_*\mathbb{C}_M) \overset{\text{def.}}{=} \bigoplus_{k' \geq k} L_\phi \otimes H^{k'}(i_x^! P_\phi).$$

By definition, we have $\mathcal{A}_k \cdot M_{\ge l}(i_x^! f_* \mathbb{C}_M) \subset M_{\ge k+l}(i_x^! f_* \mathbb{C}_M)$ under the $\mathcal{A}$-module on $H^*(i_x^! f_* \mathbb{C}_M)$. In particular, $M_{\ge k}(i_x^! f_* \mathbb{C}_M)$ is an $\mathcal{A}$-submodule for each k. Hence

$$\operatorname{gr} M(i_x^! f_* \mathbb{C}_M) \overset{\text{def.}}{=} \bigoplus_k M_{\ge k}(i_x^! f_* \mathbb{C}_M)/M_{\ge k+1}(i_x^! f_* \mathbb{C}_M)$$

is an $\mathcal{A}$-module, on which $\bigoplus_{k>0} \mathcal{A}_k$ acts as 0. By definition,

$$\operatorname{gr} M(i_x^! f_* \mathbb{C}_M) \cong \bigoplus_\phi L_\phi \otimes H^*(i_x^! P_\phi),$$

where the $\mathcal{A}$-module structure on the right hand side is given by $a\colon \xi \otimes \xi' \mapsto a\xi \otimes \xi'$. Thus we have

Theorem 14.2.3. *In the Grothendieck group of $\mathcal{A}$-modules of finite dimension over $\mathbb{C}$, we have*

$$H_*(M_x, \mathbb{C}) = \bigoplus_\phi L_\phi \otimes H^*(i_x^! P_\phi),$$

where the $\mathcal{A}$-module structure on the right hand side is given by $a\colon \xi \otimes \xi' \mapsto a\xi \otimes \xi'$.

Proof. Since $\operatorname{gr} M(i_x^! f_* \mathbb{C}_M)$ is equal to $M(i_x^! f_* \mathbb{C}_M)$ in the Grothendieck group, the assertion follows from the discussion above. □

14.3. In this subsection, we assume that the graph is of type ADE, and ε is not a root of unity. We apply the results in the previous subsection to our quiver varieties.

Recall

$$Z(\mathbf{w})^A = \{(x^1, x^2) \in \mathfrak{M}(\mathbf{w})^A \times \mathfrak{M}(\mathbf{w})^A \mid \pi^A(x^1) = \pi^A(x^2)\},$$

where $\pi^A\colon \mathfrak{M}(\mathbf{w})^A \to \mathfrak{M}_0(\infty, \mathbf{w})^A$ is the restriction of $\pi\colon \mathfrak{M}(\mathbf{w}) \to \mathfrak{M}_0(\infty, \mathbf{w})$. Thus the results in the previous subsection are applicable to this setting. We have an algebra isomorphism

$$H_*(Z(\mathbf{w})^A, \mathbb{C}) \doteq \operatorname{Ext}^*_{D^b(\mathfrak{M}_0(\infty,\mathbf{w})^A)}(\pi^A_* \mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}, \pi^A_* \mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}).$$

Let us denote this algebra by $\mathcal{A}$ as in the previous subsection.

Since the graph is of type ADE, we have $\mathfrak{M}_0(\infty, \mathbf{w}) = \bigsqcup_{\mathbf{v}} \mathfrak{M}_0^{\text{reg}}(\mathbf{v}, \mathbf{w})$ by Proposition 2.6.3. Thus we have the stratification $\mathfrak{M}_0(\infty, \mathbf{w})^A = \bigsqcup \mathfrak{M}_0^{\text{reg}}(\rho)$. (In fact, this holds under the same assumption as in Theorem 5.5.6: If we decompose $[B, i, j] \in \mathfrak{M}_0(\infty, \mathbf{w})^A$ as in [45, 3.27], then the restriction of B to V^i for $i > 0$ is zero by Proposition 4.2.2 with $\mathbf{w} = 0$.) Since the restriction

$$\pi^A|_{(\pi^A)^{-1}(\mathfrak{M}_0^{\text{reg}}(\rho))}\colon (\pi^A)^{-1}(\mathfrak{M}_0^{\text{reg}}(\rho)) \to \mathfrak{M}_0^{\text{reg}}(\rho)$$

is a locally trivial topological fibration by Theorem 3.3.2, all the complexes in the right hand side of (14.2.1) (applied to $f = \pi^A$, $M = \mathfrak{M}(\mathbf{w})^A$) have locally constant cohomology sheaves along each stratum $\mathfrak{M}_0^{\text{reg}}(\rho)$. Since $\mathfrak{M}_0^{\text{reg}}(\rho)$ is irreducible by Theorem 5.5.6, it implies that P_ϕ is the intersection cohomology complex $IC(\mathfrak{M}_0^{\text{reg}}(\rho), \phi)$ associated with an irreducible local system ϕ on $\mathfrak{M}_0^{\text{reg}}(\rho)$. Thus we have

$$\pi^A_* \mathbb{C}_{\mathfrak{M}(\mathbf{w})^A} \cong \bigoplus_{(\rho,\phi,k)} L_{(\rho,\phi,k)} \otimes IC(\mathfrak{M}_0^{\text{reg}}(\rho), \phi)[k] \tag{14.3.1}$$

for some finite dimensional vector space $L_{(\rho,\phi,k)}$. Let $L_{(\rho,\phi)} \overset{\text{def.}}{=} \bigoplus_k L_{(\rho,\phi,k)}$. By a discussion in the previous subsection, $\{L_{(\rho,\phi)}\}$ is a complete set of mutually non-isomorphic simple $\mathcal{A}$-modules. Via the homomorphism (13.4.2), $L_{(\rho,\phi)}$ is considered also as a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module.

Theorem 14.3.2. *Assume ε is not a root of unity.*

(1) *Simple perverse sheaves P_ϕ whose shift appears in a direct summand of $\pi^A_* \mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}$ are the intersection cohomology complexes $IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho))$ associated with the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(\rho)}$ on various $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$.*

(2) *Let us denote the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(\rho)}$ by $\mathbb{C}_\rho$ for simplicity. Then $L_{(\rho,\mathbb{C}_\rho)}$ is nonzero if and only if $\mathfrak{M}_0^{\mathrm{reg}}(\rho) \neq \emptyset$. Moreover, there is a bijection between the set $\{\rho \mid L_{(\rho,\mathbb{C}_\rho)} \neq 0\}$ and the set of l-weights of $M_{0,a}$ which are l-dominant.*

(3) *The simple $\mathcal{A} = H_*(Z(\mathbf{w})^A, \mathbb{C})$-module $L_{(\rho,\mathbb{C}_\rho)} = \bigoplus_k L_{(\rho,\mathbb{C}_\rho,k)}$ is also simple as a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module, and its Drinfel'd polynomial is $P_k(u) = \chi_a\left(\bigwedge_{-u} C^\bullet_{k,x}\right)$ in Proposition 13.3.1 for $x \in \mathfrak{M}_0^{\mathrm{reg}}(\rho)$.*

(4) *$L_{(\rho,\mathbb{C}_\rho)}$ is the simple quotient of $M_{x,a}$, where x is a point in a stratum $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$.*

(5) *Standard modules $M_{x,a}$ and $M_{y,a}$ are isomorphic as $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules if and only if x and y are contained in the same stratum.*

Proof. We use the transversal slice in §3.3. The idea to use transversal slices is taken from [13, §8.5].

Choose and fix a point $x \in \mathfrak{M}_0(\infty, \mathbf{w})^A$. Suppose that x is contained in a stratum $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$ for some ρ_x. We first show

Claim. If $\mathbb{C}_{\rho_x}$ denotes the constant local system on $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$, the corresponding vector space $L_{(\rho_x,\mathbb{C}_{\rho_x})}$ is nonzero.

If we restrict π^A to the component $\mathfrak{M}(\rho_x)$, then we have

$$\pi^A_* \mathbb{C}_{\mathfrak{M}(\rho_x)} \doteq \bigoplus_{(\rho,\phi)} L'_{(\rho,\phi)} \otimes IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho), \phi), \tag{14.3.3}$$

where $L'_{(\rho,\phi)}$ is a direct summand of $L_{(\rho,\phi)}$. The summation runs over the set of pairs (ρ,ϕ) such that $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ is contained in $\pi^A(\mathfrak{M}(\rho_x))$. (In fact, (14.3.1) was obtained by applying the decomposition theorem to each component $\mathfrak{M}(\rho_x)$ and taking the direct sum.) If we restrict (14.3.3) to the open stratum $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$ of $\pi^A(\mathfrak{M}(\rho_x))$, the right hand side of (14.3.3) becomes

$$\bigoplus_\phi L'_{(\rho_x,\phi)} \otimes \phi,$$

where the summation runs over the set of isomorphism classes of irreducible local systems ϕ on $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$. On the other hand, π^A induces an isomorphism between $(\pi^A)^{-1}(\mathfrak{M}_0^{\mathrm{reg}}(\rho_x))$ and $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$ by Proposition 2.6.2. This means that the restriction of the left hand side of (14.3.3) is the constant local system $\mathbb{C}_{\rho_x}$. Hence we have $L'_{(\rho_x,\mathbb{C}_{\rho_x})} \cong \mathbb{C}$, and $L_{(\rho_x,\mathbb{C}_{\rho_x})}$ is nonzero. This is the end of the proof of the claim.

The claim implies the first assertion of (2). Let us prove the latter assertion of (2). Suppose $\mathfrak{M}_0^{\mathrm{reg}}(\rho) \neq \emptyset$. Then we have $\mathfrak{M}(\rho) \neq \emptyset$ and $H_*(\mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C}) \neq 0$ by Proposition 4.1.2. By Proposition 13.4.5, the corresponding l-weight space is nonzero, where the l-weight $\Psi^\pm(z) = (\Psi^\pm_k(z))_k$ is given by (13.4.6). Furthermore, since $C_k^\bullet(\mathbf{v},\mathbf{w})$ can be represented by a genuine A-module over a point in $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ by Lemma 2.9.2, $\chi_a\left(\bigwedge_{-u} C_k^\bullet(\mathbf{v},\mathbf{w})\right)$ is a polynomial in u. Thus $\Psi^\pm(z)$ is l-dominant.

Conversely suppose that we have the l-weight space with the l-weight (13.4.6) nonzero. Since ε is *not* a root of unity, the ε-analogue of the Cartan matrix $[-\langle h_k,\alpha_l\rangle]_\varepsilon$ is invertible. Hence (13.4.6) determines $\chi_a(\bigwedge_u V_k)$, and a homomorphism $\rho\colon A\to G_{\mathbf{v}}$. Moreover, the l-weight space is precisely $H_*(\mathfrak{M}(\rho)\cap\mathfrak{L}(\mathbf{w}),\mathbb{C})$ by Proposition 13.4.5(1). In particular, we have $H_*(\mathfrak{M}(\rho)\cap\mathfrak{L}(\mathbf{w}),\mathbb{C})\neq 0$, and hence $\mathfrak{M}(\rho)\neq\emptyset$. Furthermore, if we decompose $C_k^\bullet(\mathbf{v},\mathbf{w})$ into $\bigoplus_\lambda C_{k,\lambda}^\bullet(\rho)$ as in §4.1, we have $\operatorname{rank} C_{k,\lambda}^\bullet(\rho)\geq 0$ since the l-weight (13.4.6) is l-dominant. By Corollary 5.5.5, $\tau_{k,\lambda}$ is surjective for any k, λ on a nonempty open subset of $\mathfrak{M}(\rho)$. By Lemma 2.9.4, $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ is nonempty. This shows the latter half of (2).

Let $\mathbf{v}_x$ denote the dimension vector corresponding to ρ_x, i.e., $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)\subset\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}_x,\mathbf{w})$. Take a transversal slice to $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}_x,\mathbf{w})$ at x as in §3.3. Let S be its intersection with $\mathfrak{M}_0(\infty,\mathbf{w})^A$. Since the transversal slice in §3.3 can be made A-equivariant (Remark 3.3.3), it is a transversal slice to $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$ (at x) in $\mathfrak{M}_0(\infty,\mathbf{w})^A$. Let $\widetilde{S}\overset{\text{def.}}{=}(\pi^A)^{-1}(S)$. Let $\varepsilon\colon S\to\mathfrak{M}_0(\infty,\mathbf{w})^A$, $\widetilde{\varepsilon}\colon\widetilde{S}\to\mathfrak{M}(\mathbf{w})^A$ be the inclusions.

The stratification $\mathfrak{M}_0(\mathbf{w})^A=\bigsqcup\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ induces by restriction a stratification $S=\bigsqcup S_\rho$ where $S_\rho=\mathfrak{M}_0^{\mathrm{reg}}(\rho)\cap S$. Any intersection complex $IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho),\phi)$ restricts (up to shift) to the intersection complex $IC(S_\rho,\phi|_{S_\rho})$ by transversality. Here $\phi|_{S_\rho}$ is the restriction of ϕ to S_ρ. Taking $\varepsilon^!$ of (14.3.1), we get

$$\varepsilon^!\left(\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}\right)\doteq\bigoplus_{(\rho,\phi)}L_{(\rho,\phi)}\otimes IC(S_\rho,\phi|_{S_\rho}). \tag{14.3.4}$$

Let $i_x^S\colon\{x\}\to S$ be the inclusion. It induces two pull-back homomorphisms $i_x^{S!}$, i_x^{S*}, and there is a natural morphism $i_x^{S!}E\to i_x^{S*}E$ for any $E\in D^b(S)$. We apply these functors to both sides of (14.3.4) and take cohomology groups. By a property of intersection cohomology sheaves (see [13, 8.5.3]), the homomorphism

$$H^*(i_x^{S!}IC(S_\rho,\phi|_{S_\rho}))\to H^*(i_x^{S*}IC(S_\rho,\phi|_{S_\rho})) \tag{14.3.5}$$

is zero unless $S_\rho=\{x\}$ (or equivalently $\rho=\rho_x$), in which case it is a quasi-isomorphism. Thus

$$\operatorname{Im}\left[H^*(i_x^{S!}\varepsilon^!\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A})\to H^*(i_x^{S*}\varepsilon^!\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A})\right]\doteq\bigoplus_\phi L_{(\rho_x,\phi)}\otimes\phi_x, \tag{14.3.6}$$

where the summation runs over isomorphism classes of irreducible local systems on $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$, and ϕ_x is the fiber of the local system ϕ at x. Moreover, (14.3.5) is a homomorphism of $\mathcal{A}$-modules, and (14.3.6) is an isomorphism of $\mathcal{A}$-modules, where the module structure on the right hand side is given by $a\colon\xi\otimes\xi'\mapsto a\xi\otimes\xi'$.

On the other hand, we have

$$H^*(i_x^{S!}\varepsilon^!\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A})=H^*(i_x^!\pi_*^A\mathbb{C}_{\widetilde{S}})\doteq H_*(\mathfrak{M}(\mathbf{w})_x^A,\mathbb{C}).$$

As shown in (13.4.3), the right hand side is isomorphic to the standard module $M_{x,a}$. Thus the left hand side of (14.3.6) is a quotient of $M_{x,a}$, and it is indecomposable by Proposition 13.3.1(3). Thus the right hand side of (14.3.6) consists of at most a single direct summand. Since we have already shown that $L_{(\rho_x,\mathbb{C}_{\rho_x})}\neq 0$ in the claim, we get $L_{(\rho_x,\phi)}=0$ if ϕ is a *nonconstant* irreducible local system. Since x was an arbitrary point, we have the statement (1).

Let us prove (3). For the proof, we need a further study of (14.3.6). By the above discussion, we have

$$\operatorname{Im}\left[H^*(i_x^{S!}\varepsilon^!\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}) \to H^*(i_x^{S*}\varepsilon^!\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A})\right] \doteq L_{(\rho_x,\mathbb{C}_{\rho_x})}. \tag{14.3.7}$$

By the base change theorem, we have $\varepsilon^!\left(\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}\right) = \pi_*^S\widetilde{\varepsilon}^!\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}$ where π^S is the restriction of π^A to $\widetilde{S}$. Further, we have $\widetilde{\varepsilon}^!\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A} \doteq \mathbb{C}_{\widetilde{S}}$ since $\widetilde{S}$ is a nonsingular submanifold of $\mathfrak{M}(\mathbf{w})^A$. Applying the Verdier duality, we have

$$\operatorname{Hom}\left(H^*(i_x^{S!}\varepsilon^!\pi_*^A\mathbb{C}_{\mathfrak{M}(\mathbf{w})^A}),\mathbb{C}\right) \doteq H^*((i_x^{S!}\pi_*^S\mathbb{C}_{\widetilde{S}})^\vee) \doteq H^*(i_x^{S*}\pi_*^S\mathbb{C}_{\widetilde{S}}).$$

Hence (14.3.7) becomes

$$\operatorname{Im}\left[M_{x,a} \to M_{x,a}^*\right] \doteq L_{(\rho_x,\mathbb{C}_{\rho_x})}, \tag{14.3.8}$$

where $M_{x,a}^*$ is the dual space of $M_{x,a}$ as a complex vector space. Let us introduce an $\mathcal{A}$-module on $M_{x,a}^*$ by

$$\langle a * h, \xi\rangle = \langle h, (\omega_* a) * \xi\rangle, \quad a \in \mathcal{A},\ h \in M_{x,a}^*,\ \xi \in M_{x,a},$$

where $\langle\ ,\ \rangle$ denotes the dual pairing, $\omega\colon Z(\mathbf{w})^A \to Z(\mathbf{w})^A$ is the exchange of two factors of $Z(\mathbf{w})^A = \mathfrak{M}(\mathbf{w})^A \times_{\mathfrak{M}_0(\infty,\mathbf{w})^A} \mathfrak{M}(\mathbf{w})^A$, and ω_* is the induced homomorphism on $\mathcal{A} = H_*(Z(\mathbf{w})^A,\mathbb{C})$. Then (14.3.8) is compatible with $\mathcal{A}$-module structures (cf. [13, paragraphs preceding 8.6.25]).

The decomposition (13.4.4) induces a similar one for $M_{x,a}^*$:

$$M_{x,a}^* = \bigoplus_\rho \operatorname{Hom}(H_*(\mathfrak{M}(\rho)_x,\mathbb{C}),\mathbb{C}).$$

The homomorphism $M_{x,a} \to M_{x,a}^*$ respects the decomposition, and induces a decomposition on (14.3.8).

Recall that we have the distinguished vector $[x]$ in $M_{x,a}$. The component $H_*(\mathfrak{M}(\rho_x),\mathbb{C})$ of $M_{x,a}$ is 1-dimensional space $\mathbb{C}[x]$. (See §13.3.) By the above discussion, $[x]$ is *not* annihilated by the above homomorphism $M_{x,a} \to M_{x,a}^*$. Thus we may consider $[x]$ also as an element of $M_{x,a}^*$.

We want to show that any nonzero $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-submodule L' of $L_{(\rho_x,\mathbb{C}_{\rho_x})}$ is $L_{(\rho_x,\mathbb{C}_{\rho_x})}$ itself. Our strategy is the same as in the proof of Theorem 14.1.2. Since we already show that $L_{(\rho_x,\mathbb{C}_{\rho_x})}$ is a quotient of $M_{x,a}$, Proposition 13.3.1(3) implies $L_{(\rho_x,\mathbb{C}_{\rho_x})} = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * [x]$. Thus it is enough to show that L' contains $[x]$. To show this, consider

$$M_{x,a}^{*\circ} \overset{\text{def.}}{=} \{m^* \in M_{x,a}^* \mid e_{k,r} * m^* = 0 \text{ for any } k \in I,\ r \in \mathbb{Z}\}.$$

By the argument as in the proof of Theorem 14.1.2, L' contains a nonzero vector in $M_{x,a}^{*\circ}$. Hence it is enough to show that $M_{x,a}^{*\circ} = \mathbb{C}[x]$.

As in the proof of Theorem 14.1.2 above, $M_{x,a}^{*\circ}$ is a direct sum of generalized eigenspaces for $\Delta_*\bigwedge_u V_l$. Let us choose and fix a direct summand $M_{x,a}^{*\circ\circ}$ contained in $H_*(\mathfrak{M}(\rho)_x,\mathbb{C})$. Then $m^* \in M_{x,a}^{*\circ\circ}$ satisfies

$$\langle f_{k,r} * m, m^*\rangle = 0$$

for any k, r. Since $M_{x,a} = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * [x]$ by Proposition 13.3.1(3), the above equation implies that $m^* \in \operatorname{Hom}(\mathbb{C}[x],\mathbb{C})$. Thus we get $M_{x,a}^{*\circ} = \mathbb{C}[x]$ as desired.

We have shown the statement (4) during the above discussion.

Let us prove (5). Since π^A is a locally trivial topological fibration on each stratum $\mathfrak{M}_0^{\text{reg}}(\rho)$, $M_{x,a}$ and $M_{y,a}$ are isomorphic if both x and y are contained in $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$. Conversely, if $M_{x,a}$ and $M_{y,a}$ are isomorphic as $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules, the corresponding l-highest weights $\chi_a\left(\bigwedge_{-u} C^\bullet_{k,x}\right)$ and $\chi_a\left(\bigwedge_{-u} C^\bullet_{k,y}\right)$ are equal. Since $\chi_a\left(\bigwedge_{-u} C^\bullet_{k,x}\right)$ determines the homomorphism ρ as in the proof of (2), x and y are in the same stratum. □

Remark 14.3.9. The assumption that ε is *not* a root of unity is used to apply Theorem 5.5.6 and to have the invertibility of the ε-analogue of the Cartan matrix. It seems likely that Theorem 5.5.6 holds even if ε is a root of unity. The latter condition was used to parametrize the index set of ρ (i.e., Theorem 14.3.2(2)). But one should have a similar parametrization if one replaces a notion of l-weights in a suitable way. Thus Theorem 14.3.2 should hold even if ε is a root of unity, if one replaces the statement (2).

Let $\mathcal{P} = \{P(u) = (P_k(u))_k\}$ be the set of l-weights of $M_{0,a}$, which are l-dominant. Since the index set $\{\rho\}$ of the stratum coincides with $\mathcal{P}$, we may write $L_{(\rho,\mathbb{C}_\rho)}$ as $L(P)$, when $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ corresponds to $P \in \mathcal{P}$. The standard module $M_{x,a}$ depends only on the stratum containing x, so we may also write $M_{x,a}$ as $M(P)$. We have an analogue of the Kazhdan-Lusztig multiplicity formula:

Theorem 14.3.10. *Assume ε is not a root of unity.*

For $x \in \mathfrak{M}_0(\infty, \mathbf{w})^A$, let $i_x\colon \{x\} \to \mathfrak{M}_0(\infty,\mathbf{w})^A$ denote the inclusion. Let $P \in \mathcal{P}$ be the l-weight corresponding to the stratum $\mathfrak{M}_0^{\mathrm{reg}}(\rho_x)$ containing x. In the Grothendieck group of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules, we have

$$M(P) = \bigoplus_{Q\in\mathcal{P}} L(Q) \otimes H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q))),$$

where $\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)$ is the stratum corresponding to $Q \in \mathcal{P}$, and $IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q))$ is the intersection cohomology complex attached to $\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)$ and the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)}$. Here the $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module structure on the right hand side is given by $a\colon \xi\otimes\xi' \mapsto a\xi\otimes\xi'$.

This follows from Theorem 14.3.2 and a result in the previous subsection.

Remark 14.3.11. By [13, 8.7.8] and Theorem 7.4.1 with Theorem 3.3.2, the cohomology group $H^{d_Q+n}(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)))$ vanishes for all odd n, where d_Q is the dimension of $\mathfrak{M}_0^{\mathrm{reg}}(\rho_Q)$.

15. The $\mathbf{U}_\varepsilon(\mathfrak{g})$-module structure

In this section, we assume the graph is of type ADE. The result of this section holds even if ε is a root of unity, if we replace the simple module $L(\Lambda)$ by the corresponding Weyl module (see [10, 11.2] for the definition).

15.1. For a given $\mathbf{w} \in \bigoplus \mathbb{Z}_{\ge 0}\Lambda_k$, let $\mathcal{V}_{\mathbf{v}^0}(\mathbf{w})$ be the finite set consisting of all $\mathbf{v} \in \bigoplus \mathbb{Z}_{\ge 0}\alpha_k$ such that $\mathbf{w}-\mathbf{v}$ is dominant and the weight space with weight $\mathbf{w}-\mathbf{v}^0$ is nonzero in the simple highest weight $\mathbf{U}_q(\mathfrak{g})$-module $L(\mathbf{w}-\mathbf{v})$.

Let $\mathcal{V}(\mathbf{w})$ be the union of all $\mathcal{V}_{\mathbf{v}^0}(\mathbf{w})$ for various $\mathbf{v}^0$. It is the set consisting of all $\mathbf{v}$ such that $\mathbf{w}-\mathbf{v}$ is dominant.

Since the graph is of type ADE, we have $\mathfrak{M}_0(\infty,\mathbf{w}) = \bigsqcup_{\mathbf{v}} \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$. Since $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ is isomorphic to an open subvariety of $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ is irreducible if $\mathfrak{M}(\mathbf{v},\mathbf{w})$ is connected. Although we do not know whether $\mathfrak{M}(\mathbf{v},\mathbf{w})$ is connected or not (see §7.5), we consider the intersection cohomology complex

$IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}))$ attached to $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ and the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})}$. It may not be a simple perverse sheaf if $\mathfrak{M}(\mathbf{v},\mathbf{w})$ is not connected.

We prove the following in this section:

Theorem 15.1.1. *As a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module, we have the following decomposition:*

$$\operatorname{Res} M_{x,a} = \bigoplus_{\mathbf{v}\in\mathcal{V}(\mathbf{w})} H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}))) \otimes L(\mathbf{w}-\mathbf{v}),$$

where $i_x\colon \{x\} \to \mathfrak{M}_0(\infty,\mathbf{w})$ is the inclusion, and $\mathbf{U}_\varepsilon(\mathfrak{g})$ acts trivially on the factor $H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w})))$.

Remark 15.1.2. By [13, 8.7.8] and Theorem 7.3.5 with Theorem 3.3.2, the cohomology group $H^{\mathrm{odd}}(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})))$ vanishes.

15.2. **Reduction to $\varepsilon = 1$.** Suppose that x is contained in a stratum $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$. Take a representative (B,i,j) of x and define $\rho(a)$ as in (4.1.1). Here a is fixed and we do not consider A. We choose $S \in \mathfrak{g}_{\mathbf{w}} = \operatorname{Lie} G_{\mathbf{w}}$, $R \in \mathfrak{g}_{\mathbf{v}} = \operatorname{Lie} G_{\mathbf{v}}$, $E \in \mathbb{C}$ so that $\exp S = s$, $\exp R = \rho(a)$, $\exp E = \varepsilon$, where $a = (s,\varepsilon)$. Let $a_t = (\exp tS, \exp tE)$ for $t \in \mathbb{C}$. Then we have

$$a_t * (B,i,j) = \exp(tR)^{-1}\cdot(B,i,j)$$

from (4.1.1). If A_x denotes the stabilizer of x in $G_{\mathbf{w}}\times\mathbb{C}^*$, the above equation means that $a_t \in A_x$.

Let us consider a $\mathbf{U}_{\exp tE}(\mathbf{L}\mathfrak{g})$-module

$$M_t \overset{\text{def.}}{=} K^{A_x}(\mathfrak{M}(\mathbf{w})_x) \otimes_{R(A_x)} \mathbb{C}_{a_t}$$

parametrized by $t \in \mathbb{C}$, where $\mathbb{C}_{a_t}$ is an $R(A_x)$-module given by the evaluation at a_t as in §13. When $t = 1$, we can replace A_x by A by Theorem 7.3.5 and Theorem 3.3.2, hence the module $M_{t=1}$ coincides with $M_{x,a}$. Moreover, it depends continuously on t, also by Theorem 7.3.5.

Let us consider M_t as a $\mathbf{U}_{\exp tE}(\mathfrak{g})$-module by the restriction. Since finite dimensional $\mathbf{U}_{\exp tE}(\mathfrak{g})$-modules are classified by discrete data (highest weights), it is independent of t. (Simple modules $L(\Lambda)$ of $\mathbf{U}_{\exp tE}(\mathfrak{g})$ depend continuously on t.) Thus it is enough to decompose M_t when $t = 0$, i.e., $s = 1$, $\varepsilon = 1$. By Theorem 7.3.5 and Theorem 3.3.2, $K^{A_x}(\mathfrak{M}(\mathbf{w})_x)$ is specialized to $H_*(\mathfrak{M}(\mathbf{w})_x,\mathbb{C})$ at $s = 1$, $\varepsilon = 1$. Thus our task now becomes the decomposition of $H_*(\mathfrak{M}(\mathbf{w})_x,\mathbb{C})$ into simple $\mathfrak{g}$-modules.

15.3. When Y is pure dimensional, we denote by $H_{\mathrm{top}}(Y,\mathbb{C})$ the top degree part of $H_*(Y,\mathbb{C})$, that is, the subspace spanned by the fundamental classes of irreducible components of Y. Suppose that Y has several connected components $Y_1, Y_2, \dots$ such that each Y_i is pure dimensional, but $\dim Y_i$ may change for different i. Then we define $H_{\mathrm{top}}(Y,\mathbb{C})$ as $\bigoplus H_{\mathrm{top}}(Y_i,\mathbb{C})$. Note that the degree top may differ for different i since the dimensions are changing.

By [45, 9.4], there is a homomorphism

$$\mathbf{U}_{\varepsilon=1}(\mathfrak{g}) \to H_{\mathrm{top}}(Z(\mathbf{w}),\mathbb{C}).$$

In fact, it is the restriction of the homomorphism in (13.4.2) for $A = \{1\}$, $\varepsilon = 1$, composed with the projection

$$H_*(Z(\mathbf{w}),\mathbb{C}) \to H_{\mathrm{top}}(Z(\mathbf{w}),\mathbb{C}).$$

For each $\mathbf{v}$, we take a point $x_{\mathbf{v}} \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$. (By Lemma 2.9.4(2), $\mathbf{w}-\mathbf{v}$ is dominant if $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ is nonempty.) By [45, 10.2], $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C})$ is the simple highest weight module $L(\mathbf{w}-\mathbf{v})$ via this homomorphism. (In fact, we have already proved a similar result, i.e., Proposition 13.3.1.)

Proposition 15.3.1. *Consider the map* $\pi\colon \mathfrak{M}(\mathbf{v}^0,\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}^0,\mathbf{w})$. *Then* π, *as a map into* $\pi(\mathfrak{M}(\mathbf{v},\mathbf{w}))$, *is semi-small and all strata are relevant, namely*

$$2\dim\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}} = \operatorname{codim}\mathfrak{M}_0(\mathbf{v},\mathbf{w}) \quad \textit{for } x_{\mathbf{v}} \in \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}),$$

where codim *is the codimension in* $\pi(\mathfrak{M}(\mathbf{v},\mathbf{w}))$.

Proof. See [44, 6.11] and [45, 10.11]. ∎

Proposition 15.3.2. *We have*

(15.3.3)
$$\pi_*\left(\mathbb{C}_{\mathfrak{M}(\mathbf{v}^0,\mathbf{w})}[\dim\mathfrak{M}(\mathbf{v}^0,\mathbf{w})]\right) = \bigoplus_{\mathbf{v}\in\mathcal{V}_{\mathbf{v}^0}(\mathbf{w})} H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C}) \otimes IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})),$$

where $x_{\mathbf{v}}$ *is taken from* $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$. (*By Theorem 3.3.2* $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C})$ *is independent of the choice of* $x_{\mathbf{v}}$.)

Proof. By the decomposition theorem for a semi-small map [13, 8.9.3], the left hand side of (15.3.3) decomposes as

$$\pi_*\left(\mathbb{C}_{\mathfrak{M}(\mathbf{v}^0,\mathbf{w})}[\dim\mathfrak{M}(\mathbf{v}^0,\mathbf{w})]\right) = \bigoplus_{\mathbf{v},\alpha,\phi} L_{(\mathbf{v},\alpha,\phi)} \otimes IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})^\alpha,\phi),$$

where $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})^\alpha$ is a component of $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ and $IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})^\alpha,\phi)$ is the intersection complex associated with an irreducible local system ϕ on $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})^\alpha$. Moreover, by [13, 8.9.9], we have

$$H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C}) = \bigoplus_{\phi} L_{(\mathbf{v},\alpha,\phi)}, \tag{15.3.4}$$

where ϕ runs over the set of irreducible local systems on the component of $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})^\alpha$ containing $x_{\mathbf{v}}$. But as argued in the proof of Theorem 14.3.2, the indecomposability of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C})$ implies that no intersection complex associated with a *nontrivial* local system appears in the summand. Moreover the left hand side of (15.3.4) is independent of the choice of the component by Theorem 3.3.2. Thus we can combine the summation over α together as

$$\bigoplus_{\alpha,\phi} L_{\mathbf{v},\alpha,\phi} \otimes IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})^\alpha,\phi) = H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C}) \otimes IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})).$$

Our remaining task is to identify the index set of $\mathbf{v}$. The fundamental class $[\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}}]$ is nonzero if $\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}}$ is nonempty. Thus $\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}}$ is nonempty if and only if

$$H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C}) \neq 0.$$

By [45, 10.2] and the construction, $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}^0,\mathbf{w})_{x_{\mathbf{v}}},\mathbb{C})$ is isomorphic to the weight space of weight $\mathbf{w}-\mathbf{v}^0$ in $L(\mathbf{w}-\mathbf{v})$. Thus it is nonzero if and only if $\mathbf{v}\in\mathcal{V}_{\mathbf{v}^0}(\mathbf{w})$. ∎

Take $x \in \mathfrak{M}(\infty, \mathbf{w})$ and consider the inclusion $i_x \colon \{x\} \to \mathfrak{M}_0(\infty, \mathbf{w})$. Applying $H^*(i_x^! \bullet)$ to (15.3.3) and then summing with respect to $\mathbf{v}^0$, we get

$$(15.3.5) \qquad H_*(\mathfrak{M}(\mathbf{w})_x, \mathbb{C}) = \bigoplus_{\mathbf{v} \in \mathcal{V}(\mathbf{w})} H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_{x_{\mathbf{v}}}, \mathbb{C}) \otimes H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}, \mathbf{w}))).$$

By the convolution product, $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_{x_{\mathbf{v}}}, \mathbb{C})$ is a module of $H_{\mathrm{top}}(Z(\mathbf{w}), \mathbb{C})$. By [13, §8.9], the decomposition (15.3.5) is compatible with the module structure, where $H_{\mathrm{top}}(Z(\mathbf{w}), \mathbb{C})$ acts on $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_{x_{\mathbf{v}}}, \mathbb{C}) \otimes H^*(i_x^! IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}, \mathbf{w})))$ by $z \colon \xi \otimes \xi' \mapsto z\xi \otimes \xi'$. This completes the proof of Theorem 15.1.1.

Added in proof

Crawley-Boevey recently proved that $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is connected (see §7.5) in "Geometry of the moment map for representations of quivers", to appear in Compositio Math.

References

[1] T. Akasaka and M. Kashiwara, *Finite-dimensional representations of quantum affine algebras*, Publ. RIMS **33** (1997), 839–867. MR **99d:**17017

[2] M.F. Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15. MR **83e:**53037

[3] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. London A **308** (1982), 524–615. MR **85k:**14006

[4] P. Baum, W. Fulton and R. MacPherson, *Riemann-Roch and topological K-theory for singular varieties*, Acta. Math. **143** (1979), 155–192. MR **82c:**14021

[5] J. Beck, *Braid group action and quantum affine algebras*, Comm. Math. Phys. **165** (1994), 555–568. MR **95i:**17011

[6] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982). MR **86g:**32015

[7] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. **98** (1973), 480–497. MR **51:**3186

[8] J. Carrell and A. Sommese, *$\mathbb{C}^*$-actions*, Math. Scand. **43** (1978/79), 49–59. MR **80h:**32053

[9] V. Chari and A. Pressley, *Fundamental representations of Yangians and singularities of R-matrices*, J. reine angew Math. **417** (1991), 87–128. MR **92h:**17010

[10] ———, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. MR **95j:**17010

[11] ———, *Quantum affine algebras and their representations* in "Representation of Groups", CMS Conf. Proc., **16**, AMS, 1995, 59–78. MR **96j:**17009

[12] ———, *Quantum affine algebras at roots of unity*, Representation Theory **1** (1997), 280–328. MR **98e:**17018

[13] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Progress in Math. Birkhäuser, 1997. MR **98i:**22021

[14] C. De Concini, G. Lusztig and C. Procesi, *Homology of the zero-set of a nilpotent vector field on a flag manifold*, J. Amer. Math. Soc. **1** (1988), 15–34. MR **89f:**14052

[15] V.G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **32** (1988), 212–216. MR **88j:**17020

[16] G. Ellingsrud and S.A. Strømme, *Towards the Chow ring of the Hilbert scheme of $\mathbb{P}^2$*, J. reine angew. Math. **441** (1993), 33–44. MR **94i:**14004

[17] E. Frenkel and E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, preprint, math.QA/9911112.

[18] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras*, preprint, math.QA/9810055.

[19] W. Fulton, *Intersection Theory*, A Series of Modern Surveys in Math. 2, Springer-Verlag, 1984. MR **85k:**14004

[20] V. Ginzburg, *$\mathfrak{g}$-modules, Springer's representations and bivariant Chern classes*, Adv. in Math. **61** (1986), 1–48.

[21] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type A_n*, International Math. Research Notices (1993) No.3, 67–85. MR **94j:**17011

[22] V. Ginzburg, M. Kapranov and E. Vasserot, *Langlands reciprocity for algebraic surfaces*, Math. Res. Letters **2** (1995), 147–160. MR **96f:**11086

[23] I. Grojnowski, *Affinizing quantum algebras: From D-modules to K-theory*, preprint, 1994.

[24] ———, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Letters **3** (1996), 275–291. MR **97f:**14041

[25] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, preprint, math.QA/9812022.

[26] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153–215. MR **88d:**11121

[27] A. King, *Moduli of representations of finite dimensional algebras*, Quarterly J. of Math. **45** (1994), 515–530. MR **96a:**16009

[28] M. Kleber, *Finite dimensional representations of quantum affine algebras*, preprint, math.QA/9809087.

[29] P.B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288** (1990), 263–307. MR **92e:**58038

[30] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), 169–178. MR **83c:**20059

[31] ———, *Equivariant K-theory and representations of Hecke algebras*, Proc. Amer. Math. Soc. **94** (1985), 337–342. MR **88f:**22054b

[32] ———, *Cuspidal local systems and graded Hecke algebras. I*, Publ. Math. IHES **67** (1988), 145–202. MR **90e:**16049

[33] ———, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498. MR **90m:**17023

[34] ———, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421. MR **91m:**17018

[35] ———, *Affine quivers and canonical bases*, Publ. Math. IHES **76** (1992), 111–163. MR **94h:**16021

[36] ———, *Introduction to quantum group*, Progress in Math. **110**, Birkhäuser, 1993. MR **94m:**17016

[37] ———, *Cuspidal local systems and graded Hecke algebras. II*, in "Representation of Groups", CMS Conf. Proc., **16**, AMS, 1995, 217–275. MR **96m:**22038

[38] ———, *On quiver varieties*, Adv. in Math. **136** (1998), 141–182. MR **2000c:**16016

[39] ———, *Bases in equivariant K-theory. II*, Representation Theory **3** (1999), 281–353. MR **2000h:**20085

[40] ———, *Quiver varieties and Weyl group actions*, preprint.

[41] I.G. Macdonald, *Symmetric functions and Hall polynomials (2nd ed.)*, Oxford Math. Monographs, Oxford Univ. Press, 1995. MR **96h:**05207

[42] A. Maffei, *Quiver varieties of type A*, preprint, math.AG/9812142.

[43] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Third Enlarged Edition, Springer-Verlag, 1994. MR **95m:**14012

[44] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365–416. MR **95i:**53051

[45] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), 515–560. MR **99b:**17033

[46] ———, *Lectures on Hilbert schemes of points on surfaces*, Univ. Lect. Ser. **18**, AMS, 1999. CMP 2000:02

[47] C.M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–592. MR **91i:**16024

[48] Y. Saito, *Quantum toroidal algebras and their vertex representations*, Publ. RIMS **34** (1998), 155–177. MR **99d:**17022

[49] Y. Saito, K. Takemura, and D. Uglov, *Toroidal actions on level 1 modules of $U_q(\widehat{\mathrm{sl}_n})$*, Transform. Group **3** (1998), 75–102. MR **99b:**17015

[50] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991), 375–422. MR **92g:**58036

[51] R. Sjamaar, *Holomorphic slices, sympletic reduction and multiplicities of representations*, Ann. of Math. **141** (1995), 87–129. MR **96a:**58098

[52] T. Tanisaki, *Hodge modules, equivariant K-theory and Hecke algebras*, Publ. RIMS **23** (1987), 841–879. MR **89c:**20070

[53] R. Thomason, *Algebraic K-theory of group scheme actions*, in "Algebraic Topology and Algebraic K-theory", Ann. of Math. Studies **113** (1987), 539–563. MR **89c:**18016

[54] ______, *Equivariant algebraic vs. topological K-homology Atiyah-Segal style*, Duke Math. J. **56** (1988), 589–636. MR **89f:**14015

[55] ______, *Une formule de Lefschetz en K-théorie équivariante algébrique*, Duke Math. J. **68** (1992), 447–462. MR **93m:**19007

[56] M. Varagnolo and E. Vasserot, *Double-loop algebras and the Fock space*, Invent. Math. **133** (1998), 133–159. MR **99g:**17035

[57] ______, *On the K-theory of the cyclic quiver variety*, preprint, math.AG/9902091.

[58] E. Vasserot, *Affine quantum groups and equivariant K-theory*, Transformation Groups **3** (1998), 269–299. MR **99j:**19007

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: nakajima@kusm.kyoto-u.ac.jp