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Quivers with relations and cluster tilted algebras

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Abstract

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in connection with dual canonical bases. To a cluster algebra of simply laced Dynkin type one can associate the cluster category. Any cluster of the cluster algebra corresponds to a tilting object in the cluster category. The cluster tilted algebra is the algebra of endomorphisms of that tilting object. Viewing the cluster tilted algebra as a path algebra of a quiver with relations, we prove in this paper that the quiver of the cluster tilted algebra is equal to the cluster diagram. We study also the relations. As an application of these results, we answer several conjectures on the connection between cluster algebras and quiver representations.

0 Introduction

Cluster algebras were introduced in the work of S. Fomin and A. Zelevinsky, [FZ02, FZ03a, FZ03b]. This theory appeared in the context of dual canonical basis and more particularly in the study of the Berenstein-Zelevinsky conjecture. Cluster algebras are now connected with many topics: double Bruhat cells, Poisson varieties, total positivity, Teichmüller spaces. The main results on cluster algebras are on the one hand the classification of finite cluster algebras by root systems and on the other hand the realization of algebras of regular functions on double Bruhat cells in terms of cluster algebras.

Recently, many new results have been established relating cluster algebras of simply laced finite type to quiver representations. It has been shown in [CCS] (type A) and [BMR$^+$a] (types A, D, E) that the set of cluster variables is in bijection with the set of indecomposable objects in the so called cluster category $\mathcal{C}$, which is the quotient category $D/\tau^{-1}[1]$ of the bounded derived category D of quiver representations by the inverse Auslander-Reiten translate τ^{-1} composed with the shift [1].

For type A the authors associated in [CCS] a quiver with relations to each cluster in such a way that the indecomposable representations of that quiver with relations are in bijection with all cluster variables outside the cluster. A result of this approach was the description of the denominator of the Laurent polynomial expansion of any cluster variable in the variables of any cluster. In this paper, we generalize this result to the types D and E (Theorem 4.4).

Buan, Marsh, Reineke, Reiten and Todorov [BMR$^+$a] used tilting theory to relate the cluster algebra to the cluster category; each cluster corresponds to a tilting object in $\mathcal{C}$. For several concepts in the theory of cluster algebras, they obtained nice module theoretic interpretations, e.g. exchange pairs, compatibility degree. They called the endomorphism algebra of their tilting object *cluster tilted algebra* and conjectured that this algebra is isomorphic to the path algebra of our quiver with relations [BMR$^+$a, Conj. 9.2]. In this paper we prove this conjecture in type A (Theorem 4.1) and parts of it in types D and E, (Theorem 3.1 and Proposition 3.5). We also prove another of their conjectures [BMR$^+$a, Conj. 9.3] on the module theoretic calculation of the exchange relations in the cluster algebra, (Theorem 4.3). Buan, Marsh and Reiten also announced results on these conjectures. In [BMRb], Buan, Marsh and Reiten studied further the cluster tilted algebra and gave a precise description of its module category.

The paper is organized as follows. In section 1 we recall briefly some facts about cluster tilted algebras. Lemma 1.2 is a new result, but it follows almost immediately from [BMR$^+$a]. In section 2 we list some concepts of cluster algebras that we will need later. Section 3 contains the crucial results. We prove there that the quiver of the cluster tilted algebra is equal to the cluster diagram and that the relations defined in [CCS] are also satisfied in that algebra. In section 4, we prove the conjectures mentioned above. They follow easily from the results in section 3. Finally, in the Appendix we include some general results on embeddings of cluster diagrams in the plane.

1 Cluster tilted algebras

Let k be an algebraically closed field and $\mathcal{Q}_{\rm alt}$ an alternating quiver of simply-laced Dynkin type, D the bounded derived category of finitely generated modules with shift functor [1] and $\mathcal{C} = D/\tau^{-1}[1]$ the cluster category of [BMR$^+$a]. Here τ denotes the Auslander-Reiten translate. By a result of Keller [Kel03], $\mathcal{C}$ is a triangulated category. Let $P_1, \dots, P_n$ be the indecomposable projective $k\mathcal{Q}_{\rm alt}$-modules and $I_1, \dots, I_n$ the injective ones. According to [BMR$^+$a], there is a natural fundamental domain of indecomposable objects for $\mathcal{C} = D/\tau^{-1}[1]$ in D consisting of $\{\text{indecomposable } k\mathcal{Q}_{\rm alt}\text{-modules}\} \cup \{P_i[1] \mid i = 1, \dots, n\}$. We will think of the indecomposable objects of $\mathcal{C}$ as their representatives in this fundamental domain. Thus an indecomposable object M in $\mathcal{C}$ is either a $k\mathcal{Q}_{\rm alt}$-module or $M = P_l[1]$ for some l. An indecomposable object M in $\mathcal{C}$ is called l-free if it satisfies $l \notin \operatorname{Supp} M$ if M is a module and $l \neq i$ if $M = P_i[1]$. Throughout this paper we will use the notation $[M,N]_A = \dim \operatorname{Hom}_A(M,N)$ and $[M,N]^1_A = \dim \operatorname{Ext}_A(M,N)$, for $A = k\mathcal{Q}_{\rm alt}, \mathcal{C}, D$.

Let C be a cluster of the cluster algebra of the same type as $\mathcal{Q}_{\rm alt}$, and let $T = \oplus_{i=1}^n T_i$ be the corresponding tilting object of the cluster category $\mathcal{C}$ [BMR$^+$a]. The following lemma is proved in [BMR$^+$a, Lemma 8.2].

Lemma 1.1 $[T_i, T_j]_{\mathcal{C}} \leq 1$.

Let $(\mathcal{Q}_T, I_T)$ be a quiver with relations such that its path algebra $k\mathcal{Q}_T/\langle I_T\rangle$ is isomorphic to the cluster tilted algebra $\operatorname{End}_{\mathcal{C}}(T)^{op}$ of [BMRb]. Hence the vertices of $\mathcal{Q}_T$ 'are' the indecomposable direct summands $T_1, \dots, T_n$ of T and the lemma implies that there is an arrow $T_j \to T_i$ precisely if $[T_i, T_j]_{\mathcal{C}} = 1$ and no non-zero morphism $f \in \mathrm{Hom}_{\mathcal{C}}(T_i, T_j)$ factors through one of the $T_k, k \neq i, j$.

Following [BMR+a], let $\overline{T} = \oplus_{i=1}^{n-1} T_i$ be an almost complete basic tilting object in $\mathcal{C}$ and let M, M' be the two complements of $\overline{T}$. Then $T = \overline{T} \oplus M$ and $T' = \overline{T} \oplus M'$ are tilting objects, and there are triangles $M' \to B \to M \to M'[1]$ and $M \to B' \to M' \to M[1]$ in $\mathcal{C}$, where $B \to M$ is a minimal right add$\overline{T}$-approximation in $\mathcal{C}$ and $M \to B'$ is a minimal left add$\overline{T}$-approximation in $\mathcal{C}$. Recall that $f : B \to M$ (resp. $f' : M \to B'$) being a minimal right (resp. left) add $\overline{T}$-approximation in $\mathcal{C}$ means

1. B (resp. B') is an object of add $\overline{T}$.
2. The induced map $\mathrm{Hom}_{\mathcal{C}}(X, B) \to \mathrm{Hom}_{\mathcal{C}}(X, M)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(B', X) \to \mathrm{Hom}_{\mathcal{C}}(M, X)$) is surjective for all objects X of add $\overline{T}$.
3. (Minimality) For every map $g : B \to B$ (resp. $g' : B' \to B'$) such that $fg = f$ (resp. $g'f = f$), the map g (resp. g') is an isomorphism.

We have the following

Lemma 1.2

$$B = \oplus_{i \in I} T_i,$$

where $I = \{i \mid M \to T_i \text{ in } \mathcal{Q}_T\}$.

$$B' = \oplus_{i \in I'} T_i,$$

where $I' = \{i \mid T_i \to M \text{ in } \mathcal{Q}_T\}$.

Proof. $B = \oplus_{i \in J} a_i T_i$, $a_i \geq 1$, for some subset $J \subset \{1, 2, \ldots, n-1\}$. We will show first that all a_i are equal to 1. Suppose $a_l > 1$, write $B = X \oplus T_l \oplus \ldots \oplus T_l$ where X has no direct summand isomorphic to T_l. Let h be a generator of $\mathrm{Hom}_{\mathcal{C}}(T_l, M)$. Since $[T_l, M]_{\mathcal{C}} \leq 1$, we can write the minimal right add$T$-approximation $f : B \to M$ in matrix form as $f = [\hat{f}\ b_1 h\ \ldots b_{a_l} h]$, for some scalars $b_1, \ldots, b_{a_l}$ and with $\hat{f}$ the restriction of f to X. If this matrix is $[\hat{f}\ 0 \ldots 0]$ then take $g_0 = \begin{bmatrix} \mathrm{Id}_X & 0 \\ 0 & 0 \end{bmatrix}$ and get $fg_0 = f$. By minimality of f we get that $g_0 : B \to B$ is an isomorphism, contradiction. Otherwise h as well as one of the b_i are non-zero. Say $b_2 \neq 0$. Put

$$g = \begin{bmatrix} \mathrm{Id}_X & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & b_1/b_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Then $fg = f$ and by minimality of f we get that g is an isomorphism, contradiction.

So $B = \oplus_{i \in J} T_i$. To show that $I \subset J$, suppose that there exists i_0 such that $M \to T_{i_0}$ in $\mathcal{Q}_T$ and $i_0 \notin J$; thus T_{i_0} is not a direct summand of B. Since $M \to T_{i_0}$ in $\mathcal{Q}_T$, there is a non-zero element g in $\mathrm{Hom}_{\mathcal{C}}(T_{i_0}, M)$. By property 2 above, there exists $h : T_{i_0} \to B$ such that $g = fh$, and this implies that there is no arrow $M \to T_{i_0}$ in $\mathcal{Q}_{T'}$, contradiction.

To show that $J \subset I$, suppose that there is $i_0 \in J$ such that $M \to T_{i_0}$ is not in $\mathcal{Q}_T$. Suppose first that $[T_{i_0}, M]_{\mathcal{C}} = 0$. Hence the restriction of f to T_{i_0} is zero. Write $B = \oplus_{i \in J \setminus i_0} T_i \oplus T_{i_0}$ and define $g : B \to B$ in matrix block form to be

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $fg = f$ and since $f : B \to M$ is minimal this implies that g is an isomorphism, contradiction. Suppose now that $[T_{i_0}, M]_{\mathcal{C}} = 1$ and let $f_{i_0} : T_{i_0} \to M$ be the restriction of f. Then there exists an arrow $M \to T_l$ in $\mathcal{Q}_T$ such that $[T_{i_0}, T_l]_{\mathcal{C}} = [T_l, M]_{\mathcal{C}} = 1$, because $M \to T_{i_0}$ is not in $\mathcal{Q}_T$. Since we have already shown that $I \subset J$, it is clear that T_l is a direct summand of B. Let $f_l : T_l \to M$ be the restriction of f. If $f_l = 0$, we get a contradiction as above. Thus $f_l \neq 0$ and there exists $h : T_{i_0} \to T_l$ such that $f_{i_0} = f_l h$, since the dimensions of all corresponding Hom-spaces is 1. Write $B = T_{i_0} \oplus T_l \oplus \left(\oplus_{i \in J \setminus \{i_0, l\}} T_i \right)$ and define $g : B \to B$ in matrix block form to be

$$\begin{bmatrix} 0 & 0 & 0 \\ h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $f = [f_{i_0}\, f_l\, f']$. Thus $fg = f$ and by minimality of f we have that g is an isomorphism, contradiction. The proof for B' is similar and left to the reader. ■

2 Cluster algebras

For the proof of Theorem 3.1 we will need some concepts of [FZ03a]. For convenience we recall them here briefly but our exposition is only for simply laced finite types, i.e. A, D, E.

Let I_+ be the set of sinks of $\mathcal{Q}_{\mathrm{alt}}$ and I_- the set of sources. Define the sign function ε on vertices of $\mathcal{Q}_{\mathrm{alt}}$ by $\varepsilon(i) = +1$ if $i \in I_+$ and $\varepsilon(i) = -1$ if $i \in I_-$. Let Q be the root lattice and $\Phi_{\geq -1}$ the set of almost positive roots. Denote the simple (positive) roots by $\alpha_1, \ldots, \alpha_n$ and the corresponding simple reflections by $s_1, \ldots, s_n$. Let τ_+, τ_- be the involutions on $\Phi_{\geq -1}$ given by

$$\tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i,\ i \in I_{-\varepsilon} \\ (\prod_{i \in I_\varepsilon} s_i)\alpha & \text{otherwise} \end{cases} \qquad \varepsilon \in \{+, -\}$$

Let $\langle \tau_+, \tau_- \rangle$ be the group generated by τ_+ and τ_-. Note that the composition $\tau_- \circ \tau_+$ is the Coxeter transformation on positive roots.

There is a bijection $\alpha \mapsto x_\alpha$ between the set of almost positive roots and the set of cluster variables. Two almost positive roots β, β' are called exchangeable if there are two clusters C, C' such that $C' = C \setminus \{x_\beta\} \cup \{x_{\beta'}\}$.

Proposition 2.1 [FZ03a, Prop.3.3] *Given any $\gamma \in \mathrm{Q}$, then there exists a cluster C such that γ can be written as $\gamma = \sum_{x_\alpha \in C} a_{\alpha\gamma} x_\alpha$ with $a_{\alpha\gamma} \geq 0$. The almost positive roots α such that $a_{\alpha\gamma} \neq 0$ in this expansion are called* cluster components *of γ with respect to the cluster C.*

Proposition 2.2 [FZ03a, Prop.3.6] *If $\beta, \beta' \in \Phi_{\geq -1}$ are exchangeable then the set*

$$\{\sigma^{-1}(\sigma(\beta) + \sigma(\beta')) \mid \sigma \in \langle \tau_+, \tau_- \rangle\}$$

consists of two elements of Q, one of which is $\beta + \beta'$, and the other will be denoted by $\beta \uplus \beta'$. In the special case where β' is the negative simple root $-\alpha_l$ we have

$$\beta \uplus (-\alpha_l) = \beta - \alpha_l - \sum_{l - j \in \mathcal{Q}_{\mathrm{alt}}} \alpha_j .$$

Lemma 2.3 [FZ03a, Lemma 4.1] *There exists a sign function $(\beta, \beta') \mapsto \varepsilon(\beta, \beta') \in \{\pm 1\}$ on pairs of exchangeable roots, uniquely determined by the following properties:*

$\varepsilon(-\alpha_j, \beta') = -\varepsilon(j)\,;$
$\varepsilon(\tau\beta, \tau\beta') = -\varepsilon(\beta, \beta')$ *for* $\tau \in \{\tau_+, \tau_-\}$ *and* $\beta, \beta' \notin \{-\alpha_j \mid \tau(-\alpha_j) = -\alpha_j\}$.

Moreover, this function is skew-symmetric:

$$\varepsilon(\beta', \beta) = -\varepsilon(\beta, \beta').$$

3 Quivers and relations

Thinking of the cluster tilted algebra as a path algebra of a quiver with relations, we will prove in this section that the quiver in question is the cluster diagram. Moreover we will show that the relations defined in [CCS] for the cluster diagram are also satisfied in the cluster tilted algebra.

Let $(\mathcal{Q}_C, I_C)$ be the quiver with relations associated to the cluster C in [CCS]. Recall that $\mathcal{Q}_C$ is the cluster diagram of the cluster C as defined in [FZ03a] and that the set of relations I_C can be expressed as follows using the notion of shortest paths. By definition, a *shortest path* in the quiver $\mathcal{Q}_C$ is an oriented path (with no repeated arrow) contained in an induced subgraph of $\mathcal{Q}_C$ which is a cycle. For any arrow $i \to j$ in $\mathcal{Q}_C$, let P_{ji} be the set of shortest paths from j to i in $\mathcal{Q}_C$. We will show in the Appendix, that for any arrow $i \to j$ the set P_{ji} has at most 2 elements. Define

$$p(j,i) = \begin{cases} p & \text{if } P_{ji} = \{p\} \\ p_1 - p_2 & \text{if } P_{ji} = \{p_1, p_2\}. \end{cases}$$

Then

$$I_C = \bigcup_{i \to j} \{p(j,i)\}.$$

Let $\langle I_C \rangle$ be the ideal generated by I_C.

It has been conjectured in [BMR$^+$a, Conj. 9.2] that $k\mathcal{Q}_C/\langle I_C\rangle$ is isomorphic to the cluster tilted algebra $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. We will show that $\mathcal{Q}_C = \mathcal{Q}_T$ and $\langle I_C\rangle \subset \langle I_T\rangle$. In type A, we can then deduce the conjecture using the fact that the number of indecomposable modules over both algebras is equal.

Theorem 3.1 *Let C be any cluster of a cluster algebra of type A, D or E and let $\mathcal{Q}_C$ be its cluster diagram. Let T be a corresponding tilting object in the cluster category and $(\mathcal{Q}_T, I_T)$ the quiver with relations of the cluster tilted algebra* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$*. Then*

$$\mathcal{Q}_T = \mathcal{Q}_C.$$

Proof. The vertices of $\mathcal{Q}_C$ are almost positive roots and will be denoted by greek letters. It has been shown in [FZ03a, sect.3] that there is an arrow $\alpha \to \beta$ in $\mathcal{Q}_C$ if and only if

$$\left\{ \begin{array}{l} \text{Either } \varepsilon(\beta,\beta') = -1 \text{ and } \alpha \text{ is a cluster component of } \beta \uplus \beta' \\ \text{or } \varepsilon(\beta,\beta') = +1 \text{ and } \alpha \text{ is a cluster component of } \beta + \beta', \end{array} \right\} \qquad (1)$$

where β' is the unique almost positive root such that $C \setminus \{\beta\} \cup \{\beta'\}$ is a cluster.

According to [BMR$^+$a], to each almost positive root α corresponds an indecomposable object M_α in $\mathcal{C}$. Let $T = \overline{T} \oplus M_\beta$ and let $M_{\beta'}$ be the other complement of the almost complete basic tilting object $\overline{T}$. The indecomposable object $M_{\beta'}$ in $\mathcal{C}$ corresponds to β'. We may suppose without loss of generality that $M_{\beta'}$ is the first shift of the l-th indecomposable projective module, $M_{\beta'} = P_l[1]$, for some l. That means that $\beta' = -\alpha_l$. Thus

$$\varepsilon(\beta,\beta') = \varepsilon(\beta,-\alpha_l) = -\varepsilon(-\alpha_l,\beta) = \left\{ \begin{array}{ll} 1 & \text{if } l \text{ is a sink} \\ -1 & \text{if } l \text{ is a source} \end{array} \right.$$

where the last two identities follow from Lemma 2.3 and our choice of the sign function ε. Let us suppose first that M_β is different from $\tau M_{\beta'}$ and $\tau^{-1}M_{\beta'}$, that is $M_\beta \neq I_l, P_l$. Then by a result of [BMR$^+$a] we have the following two triangles in $\mathcal{C}$

$$\begin{array}{ccccccc} M_{\beta'} & \to & B & \to & M_\beta & \to & M_{\beta'}[1] \\ M_\beta & \to & B' & \to & M_{\beta'} & \to & M_\beta[1] \end{array}$$

and $B = \oplus_{\gamma\in I} M_\gamma$ and $B' = \oplus_{\gamma\in I'} M_\gamma$ with $I = \{\gamma \mid M_\beta \to M_\gamma \text{ in } \mathcal{Q}_T\}$ and $I' = \{\gamma \mid M_\beta \leftarrow M_\gamma \text{ in } \mathcal{Q}_T\}$, by Lemma 1.2. Now using $M_{\beta'} = P_l[1]$ and $M_{\beta'}[1] = I_l$ these two triangles give

$$\sum_{\gamma\in I} \gamma = \left\{ \begin{array}{ll} \beta - \dim I_l & \text{if } l \text{ is a sink in } \mathcal{Q}_{\mathrm{alt}} \\ \beta - \alpha_l & \text{if } l \text{ is a source in } \mathcal{Q}_{\mathrm{alt}} \end{array} \right. \qquad (2)$$

$$\sum_{\gamma\in I'}\gamma = \left\{\begin{array}{ll}\beta-\alpha_l & \text{if } l \text{ is a sink in } \mathcal{Q}_{\mathrm{alt}}\\ \beta-\dim P_l & \text{if } l \text{ is a source in } \mathcal{Q}_{\mathrm{alt}}\end{array}\right. \tag{3}$$

Indeed, let $B = B_0 \oplus B_1$ with B_0 a $k\mathcal{Q}_{\mathrm{alt}}$-module and $B_1 = \oplus_{j\in J} P_j[1]$. Note that B_1 is zero if l is a source in $\mathcal{Q}_{\mathrm{alt}}$ by Lemma 1.2, and if l is a sink then J is a subset of the set of neighbours of l in $\mathcal{Q}_{\mathrm{alt}}$. In particular, all elements of J are sources in $\mathcal{Q}_{\mathrm{alt}}$ and thus the indecomposable injective I_j is a simple module for $j \in J$. Note also that M_β is a $k\mathcal{Q}_{\mathrm{alt}}$-module since $[P_l[1], M_\beta]^1_{\mathcal{C}} \neq 0$. With this notation, the first triangle gives the following triangle in D:

$$I_l[-1] \to \bigoplus_{j\in J} I_j[-1] \bigoplus B_0 \to M_\beta \to I_l.$$

We apply the functor $\mathrm{Hom}_D(P_i, -)$ to this triangle and get the following exact sequence

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_D(P_i, B_0) & \to & \mathrm{Hom}_D(P_i, M_\beta) & & \\ & \to & \mathrm{Hom}_D(P_i, I_l) & \to & \mathrm{Hom}_D(P_i, \oplus_{j\in J} I_j) & \to & 0\end{array}$$

whence $\underline{\dim}(B_0)_i = \underline{\dim}(M_\beta)_i - \underline{\dim}(I_l)_i + \sum_{j\in J}\delta_{ij}$ since I_j is a simple module for all $j \in J$. Thus $\sum_{\gamma\in I}\gamma = \beta - \dim(I_l)$. This implies equation (2). Now let $B' = B'_0 \oplus B'_1$ with B'_0 a $k\mathcal{Q}_{\mathrm{alt}}$-module and $B'_1 = \oplus_{j\in J'} P_j[1]$. Note that B'_1 is zero if l is a sink in $\mathcal{Q}_{\mathrm{alt}}$, and if l is a source then J' is a subset of the set of neighbours of l in $\mathcal{Q}_{\mathrm{alt}}$. In particular, all elements of J' are sinks in $\mathcal{Q}_{\mathrm{alt}}$ and thus P_j is a simple module for $j \in J'$. The second triangle gives the following triangle in D:

$$M_\beta \to \bigoplus_{j\in J'} P_j[1] \bigoplus B'_0 \to P_l[1] \to M_\beta[1].$$

We apply the functor $\mathrm{Hom}_D(P_i, -)$ to this triangle and get the following exact sequence

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_D(P_i, \bigoplus_{j\in J'} P_j) & \to & \mathrm{Hom}_D(P_i, P_l) & & \\ & \to & \mathrm{Hom}_D(P_i, M_\beta) & \to & \mathrm{Hom}_D(P_i, B'_0) & \to & 0\end{array}$$

whence $\underline{\dim}(B'_0)_i = \underline{\dim}(M_\beta)_i - \underline{\dim}(P_l)_i + \sum_{j\in J'}\delta_{ij}$ since P_j is a simple module for all $j \in J'$. Thus $\sum_{\gamma\in I'}\gamma = \beta - \dim(P_l)$. This implies equation (3).

By Proposition 2.2,

$$\beta \uplus (-\alpha_l) = \beta - \alpha_l - \sum_{l - j\in\mathcal{Q}_{\mathrm{alt}}} \alpha_j$$

and $\alpha_l + \sum_{l-j\in\mathcal{Q}_{\mathrm{alt}}}\alpha_j$ is $\dim P_l$ if l is a source and $\dim I_l$ if l is a sink. Thus

$$\sum_{\gamma\in I}\gamma = \left\{\begin{array}{ll}\beta\uplus\beta' & \text{if } l \text{ is a sink in } \mathcal{Q}_{\mathrm{alt}}\\ \beta+\beta' & \text{if } l \text{ is a source in } \mathcal{Q}_{\mathrm{alt}}\end{array}\right. \tag{4}$$

$$\sum_{\gamma\in I'}\gamma = \begin{cases} \beta+\beta' & \text{if } l \text{ is a sink in } \mathcal{Q}_{\text{alt}} \\ \beta\uplus\beta' & \text{if } l \text{ is a source in } \mathcal{Q}_{\text{alt}}. \end{cases} \tag{5}$$

Hence if l is a sink we have $\varepsilon(\beta,\beta')=1$ and then

$$\begin{array}{rcl} \alpha\to\beta \text{ in } \mathcal{Q}_C & \overset{(1)}{\Longleftrightarrow} & \alpha \text{ is a cluster component of } \beta+\beta' \\ & \overset{(5)}{\Longleftrightarrow} & M_\alpha \text{ is a direct summand of } B' \\ & \overset{\text{Lemma 1.2}}{\Longleftrightarrow} & M_\alpha\to M_\beta \text{ in } \mathcal{Q}_T \end{array}$$

and

$$\begin{array}{rcl} \alpha\leftarrow\beta \text{ in } \mathcal{Q}_C & \overset{(1)}{\Longleftrightarrow} & \alpha \text{ is a cluster component of } \beta\uplus\beta' \\ & \overset{(4)}{\Longleftrightarrow} & M_\alpha \text{ is a direct summand of } B \\ & \overset{\text{Lemma 1.2}}{\Longleftrightarrow} & M_\alpha\leftarrow M_\beta \text{ in } \mathcal{Q}_T \end{array}$$

and if l is a source we have $\varepsilon(\beta,\beta')=-1$ and then

$$\begin{array}{rcl} \alpha\to\beta \text{ in } \mathcal{Q}_C & \overset{(1)}{\Longleftrightarrow} & \alpha \text{ is a cluster component of } \beta\uplus\beta' \\ & \overset{(5)}{\Longleftrightarrow} & M_\alpha \text{ is a direct summand of } B' \\ & \overset{\text{Lemma 1.2}}{\Longleftrightarrow} & M_\alpha\to M_\beta \text{ in } \mathcal{Q}_T \end{array}$$

and

$$\begin{array}{rcl} \alpha\leftarrow\beta \text{ in } \mathcal{Q}_C & \overset{(1)}{\Longleftrightarrow} & \alpha \text{ is a cluster component of } \beta+\beta' \\ & \overset{(4)}{\Longleftrightarrow} & M_\alpha \text{ is a direct summand of } B \\ & \overset{\text{Lemma 1.2}}{\Longleftrightarrow} & M_\alpha\leftarrow M_\beta \text{ in } \mathcal{Q}_T \end{array}$$

We still need to consider $M_\beta\in\{P_l,I_l\}$. These two cases are similar and we will only treat the case $M_\beta=P_l$. Thus $M_{\beta'}=\tau M_\beta$ in $\mathcal{C}$. Then there is no minimal left add$\overline{T}$-approximation $f':B'\to M_{\beta'}$ in $\mathcal{C}$ and we only have one triangle $M_{\beta'}\to B\to M_\beta\to M_{\beta'}[1]$. Furthermore, there is no arrow $M_\gamma\to M_\beta$ in $\mathcal{Q}_T$ because otherwise $1=[M_\beta,M_\gamma]_{\mathcal{C}}=[M_\gamma,\tau M_\beta]^1_{\mathcal{C}}=[M_\gamma,M_{\beta'}]^1_{\mathcal{C}}$ which contradicts the fact that $\overline{T}\oplus M_{\beta'}$ is a tilting object. Thus M_β is a sink in $\mathcal{Q}_T$. Note that $\beta+\beta'=0$ if l is a sink and $\beta\uplus\beta'=0$ if l is a source in $\mathcal{Q}_{\text{alt}}$. Therefore there is no arrow $\alpha\to\beta$ in $\mathcal{Q}_C$ by (1). On the other hand, Lemma 1.2 still gives $B=\oplus_{\gamma\in I}M_\gamma$ and equations (2) and (4) as well as the proof of the equivalence $\alpha\leftarrow\beta\Leftrightarrow M_\alpha\leftarrow M_\beta$ still hold as before. This proves the theorem. ∎

Now we want to study the relations I_T. First we need to investigate shortest paths. Let us write F for the composition $\tau^{-1}[1]$. Given an indecomposable object T in our fundamental domain of $\mathcal{C}$, we say that an indecomposable object $\widetilde{T}$ in D is over T if it lies in the F-orbit of T.

Lemma 3.2 *Let $T \to T'$ be a non-zero morphism between indecomposable objects in $\mathcal{C}$ and let $\widetilde{T}$ be an indecomposable object in D over T. Then there exists a unique indecomposable object $\widetilde{T}'$ in D over T' such that there is a non-zero morphism $\widetilde{T} \to \widetilde{T}'$.*

Proof. The existence of $\widetilde{T}'$ is clear since the $\mathcal{C}$-morphism $T \to T'$ is non-zero. Uniqueness follows easily from the well known fact that for any two indecomposable objects M, N in D we have

$$[M,N]_D \neq 0 \quad \Rightarrow \quad [M,N[a]]_D = 0 \text{ for all } a \neq 0.$$

■

Let $p : T_1 \stackrel{p_{12}}{\to} T_2 \stackrel{p_{23}}{\to} \ldots \stackrel{p_{(k-1)k}}{\to} T_k$ be a path in $\mathcal{Q}_T$ from T_1 to T_k, that is a composition of arrows $T_i \stackrel{p_{i(i+1)}}{\to} T_{i+1}$. Denote by $p^{\mathcal{C}}$ the corresponding element of $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ under the isomorphism $k\mathcal{Q}_T/\langle I_T\rangle \cong \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. Then $p^{\mathcal{C}} \in \mathrm{Hom}_{\mathcal{C}}(T_k, T_1)$ is the composition of morphisms $p^{\mathcal{C}} = p^{\mathcal{C}}_{12} \circ p^{\mathcal{C}}_{23} \circ \ldots \circ p^{\mathcal{C}}_{(k-1)k}$ with each $p^{\mathcal{C}}_{i(i+1)} \in \mathrm{Hom}_{\mathcal{C}}(T_i, T_{i+1})$ non-zero. Let us construct a *lift* p^D of p as follows. Consider first the case where p is a single arrow $T_1 \to T_2$. Then by the lemma, given an indecomposable object $\widetilde{T}_2$ in D over T_2 there exists a unique indecomposable object $\widetilde{T}_1$ over T_1 such that there is a non-zero morphism $\widetilde{T}_2 \to \widetilde{T}_1$. Any such non-zero morphism is called a *lift of $T_1 \to T_2$ starting at $\widetilde{T}_2$*. Note that this lift is unique up to multiplication by a scalar. Now let p be any path. We choose an indecomposable object $\widetilde{T}_k$ over T_k. Using the lemma on each morphism $p^{\mathcal{C}}_{i(i+1)}$, there is a unique family of indecomposable objects $(\widetilde{T}_i)_{i=k-1,\ldots,1}$, with $\widetilde{T}_i$ over T_i, and such that $\mathrm{Hom}_D(\widetilde{T}_{i+1}, \widetilde{T}_i) \neq 0$. For each arrow $T_i \to T_{i+1}$ let $p^D_{i(i+1)}$ be a lift of $p_{i(i+1)}$ starting at $\widetilde{T}_{i+1}$. Then the composition of morphisms $p^D = p^D_{12} \circ p^D_{23} \circ \ldots p^D_{(k-1)k}$ is called a *lift of p starting at T_k*. Note that $p^D \in \mathrm{Hom}_D(\widetilde{T}_k, \widetilde{T}_1)$ is unique up to multiplication by scalar. Note also that p^D may be zero although each $p^D_{i(i+1)}$ is non-zero.

Definition 1 *If p is a closed path in the situation above, that is $T_1 = T_k$, then $p^D \in \mathrm{Hom}_D(\widetilde{T}_1, F^a\widetilde{T}_1)$ and a is called the* winding number *of the path p.*

Proposition 3.3 *Let $p : T_1 \to T_2 \to \ldots \to T_k = T_1$ be a closed path and let p_{21} be the subpath $T_2 \to T_3 \to \ldots \to T_k = T_1$.*

(1) The winding number a is zero if and only if p is a constant path.

(2) Suppose that p is not constant. Then p_{21} is a shortest path if the winding number a is equal to 1.

(3) If p_{21} is not a shortest path then p_{21} is zero in $(\mathcal{Q}_T, I_T)$.

Proof. (1) is obvious. To show (2), suppose $a = 1$. If p_{21} is not a shortest path, then the subquiver of $\mathcal{Q}_T$ induced by the vertices on p_{21} is not a cycle. Now since cycles in cluster diagrams are always oriented, there exist 2 vertices

T_i, T_j on the path p_{21} such that $p_{21} = p_{2i}\, p_{ij}\, p_{j1}$, where p_{kl} is a path from T_k to T_l, and such that there is an arrow $T_i \xrightarrow{\kappa} T_j$ in $\mathcal{Q}_T$ and the subpath p_{ij} of p is not an arrow. Then the path $\kappa\, p_{j1}\, p_{1i}$ is a non-constant closed path in $\mathcal{Q}_T$, thus its winding number is at least 1. Since p_{ij} is not an arrow, it follows that the winding number of the path $p_{ij}\, p_{j1}\, p_{1i}$ is at least 2, contradiction. This proves (2).

Suppose now that (3) is not true. That is, p_{21} is a non-zero non-shortest path. Suppose without loss of generality that T_2 is an indecomposable projective $k\mathcal{Q}_{\mathrm{alt}}$-module. Since p_{12} is an arrow in $\mathcal{Q}_T$, its lift p_{12}^D is non-zero and thus T_1 is an indecomposable $k\mathcal{Q}_{\mathrm{alt}}$-module too. On the other hand, $p_{21}^D \in \mathrm{Hom}_D(T_1, \tau^{-a} T_2[a])$ is non-zero, so $a = 1$ and p_{21} is a shortest path by (2). This proves (3). ■

Conjecture 3.4 *Suppose the situation of Proposition 3.3(2). Then the winding number a is equal to 1 if p_{21} is a shortest path.*

Proposition 3.5 *Let C be any cluster of a cluster algebra of type A, D or E and let $(\mathcal{Q}_C, I_C)$ be the associated quiver with relations. Let T be a corresponding tilting object in the cluster category and $(\mathcal{Q}_T, I_T)$ the quiver with relations of the cluster tilted algebra $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. Then*

$$\langle I_C \rangle \subset \langle I_T \rangle.$$

Proof. Let $T_j \to T_i$ be an arrow in $\mathcal{Q}_T$ and $\mathcal{P}_{ij} = \{p_1, p_2, \ldots, p_m\}$ be the set of shortest paths from T_i to T_j in $\mathcal{Q}_T$. We have to show that

$$\begin{array}{ll} p_1 = 0 & \text{if } m = 1 \\ p_1 = p_2 & \text{if } m = 2 \end{array}$$

The proof is by induction on the rank n. The smallest case is $n = 3$. In this case $\mathcal{Q}_T$ is either a Dynkin quiver of type A_3 and then $m = 0$ or $\mathcal{Q}_T$ is the cyclic quiver of rank 3. In the latter case we may suppose without loss of generality that T_j is the l-th indecomposable projective $k\mathcal{Q}_{\mathrm{alt}}$-module P_l. Then l is a leaf of $\mathcal{Q}_{\mathrm{alt}}$, $T_k = I_l$ the l-th indecomposable injective module and $T_i = P_{l'}[1]$, where l' is the other leaf of $\mathcal{Q}_{\mathrm{alt}}$. We illustrate this situation in the Auslander-Reiten quiver of $\mathcal{C}$ in Figure 1. Obviously $[T_j, T_i]_{\mathcal{C}} = 0$, whence $p_1 : T_i \to T_k \to T_j$ is a zero path in $\mathcal{Q}_T$.

From now on let $n > 3$. We will show the case $m = 1$ first. For convenience, let us relabel the vertices of $\mathcal{Q}_T$ such that $p = p_1 : T_1 \to T_2 \to \ldots \to T_k$. Suppose there exists a T_{i_0} such that the path p does not pass through T_{i_0}, i.e. $k < n$. We may suppose without loss of generality that $T_{i_0} = P_l[1]$ for some l. Let $\overline{C} = C \setminus \{-\alpha_l\}$. By a result of [FZ03a], $\overline{C}$ is a cluster of a cluster algebra of rank $n-1$ and its quiver with relation $(\mathcal{Q}_{\overline{C}}, I_{\overline{C}})$ is the full subquiver $\mathcal{Q}_{\overline{C}}$ of $\mathcal{Q}_C$ with vertices $\mathcal{Q}_C^0 \setminus \{-\alpha_l\}$ and $I_{\overline{C}}$ its usual set of relations. Note that $\mathcal{Q}_{\overline{C}}$ may be disconnected. Let $\overline{\mathcal{C}}$ be the cluster category of the quiver $\mathcal{Q}_{\mathrm{alt}} \setminus l$ and denote by $\overline{T_h}$ $(h \neq i_0)$ the restriction of the l-free object T_h to $\overline{\mathcal{C}}$. Let $\overline{T} = \oplus_{h \neq i_0} \overline{T_h}$. Then $\overline{T}$ is the tilting object in $\overline{\mathcal{C}}$ that corresponds to the cluster $\overline{C}$. We have already

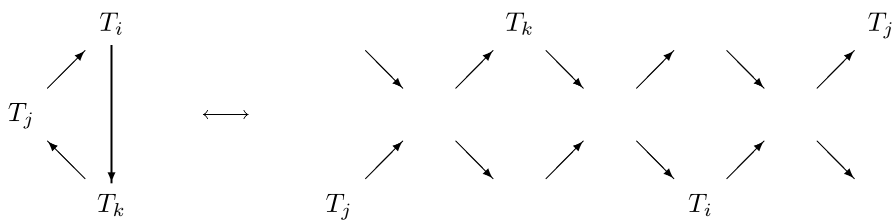

Figure 1: Cyclic quiver of rank 3 and corresponding Auslander-Reiten quiver with tilting object

shown in Theorem 3.1 that $\mathcal{Q}_{\overline{\mathcal{C}}} = \mathcal{Q}_{\overline{T}}$ and by induction we conclude that the path $\overline{p}$, which is the "restriction" of the path p to $\mathcal{Q}_{\overline{T}}$, is zero in $(\mathcal{Q}_{\overline{T}}, I_{\overline{T}})$. We want to show that p is zero in $(\mathcal{Q}_T, I_T)$. Suppose the contrary. That is

$$[\overline{T_k}, \overline{T_1}]_{\overline{\mathcal{C}}} = 0 \quad \text{and} \quad [T_k, T_1]_{\mathcal{C}} = 1. \tag{6}$$

Let us show first that $[T_k, T_1]_D = 0$. We will proceed using a case by case analysis. T_1 (resp. T_k) may be either an indecomposable $k\mathcal{Q}_{\mathrm{alt}}$-module or the first shift of an indecomposable projective $k\mathcal{Q}_{\mathrm{alt}}$-module different from $T_{i_0} = P_l[1]$, hence there are 4 different cases to consider.

1. T_1 and T_k are $k\mathcal{Q}_{\mathrm{alt}}$-modules. Then by definition $[T_k, T_1]_D = [T_k, T_1]_{k\mathcal{Q}_{\mathrm{alt}}}$ and moreover $[T_k, T_1]_{k\mathcal{Q}_{\mathrm{alt}}} = [\overline{T_k}, \overline{T_1}]_{k\mathcal{Q}_{\mathrm{alt}}\setminus\{l\}}$ because l is not in the support of T_1 nor T_k. But this is zero since $[\overline{T_k}, \overline{T_1}]_{\overline{\mathcal{C}}} = 0$.

2. T_k is a $k\mathcal{Q}_{\mathrm{alt}}$-module and $T_1 = P_h[1]$ for some $h \neq l$. Then by definition $[T_k, T_1]_D = [T_k, P_h]^1_{k\mathcal{Q}_{\mathrm{alt}}}$ and $[T_k, P_h]^1_{k\mathcal{Q}_{\mathrm{alt}}} = [\overline{T_k}, \overline{P_h}]^1_{k\mathcal{Q}_{\mathrm{alt}}\setminus\{l\}}$ because l is not in the support of T_1 nor T_k. But this is zero since $[\overline{T_k}, \overline{T_1}]_{\overline{\mathcal{C}}} = 0$.

3. T_k is the first shift of a projective and T_1 is a $k\mathcal{Q}_{\mathrm{alt}}$-module then by definition $[T_k, T_1]_D = 0$.

4. T_1 and T_k are both shifts of projectives, say $T_1 = P_h[1]$ and $T_k = P_{h'}[1]$, then $[T_k, T_1]_{\mathcal{C}} = 1$ implies that $h \to h'$ is an arrow in $\mathcal{Q}_{\mathrm{alt}}$, since $\mathcal{Q}_{\mathrm{alt}}$ is alternating. Moreover $T_k \to T_1$ is an arrow in the Auslander-Reiten quiver of $\mathcal{C}$ and hence $T_1 \to T_k$ is an arrow in $\mathcal{Q}_T$. By hypothesis, we also have an arrow $T_1 \leftarrow T_k$ in $\mathcal{Q}_T$. This is impossible, since $\mathcal{Q}_T$ is a cluster diagram by Theorem 3.1.

Thus $[T_k, T_1]_D = 0$. Moreover, it follows from the fact that T_1, T_k are indecomposable modules or first shifts of projectives, that $[T_k, \tau T_1[-1]]_D = 0$. Hence

$$\begin{aligned} 1 = [T_k, T_1]_{\mathcal{C}} &= [T_k, \tau T_1[-1]]_D + [T_k, T_1]_D + [T_k, \tau^{-1} T_1[1]]_D \\ &= [T_k, \tau^{-1} T_1[1]]_D \qquad (7) \\ &= [T_k, \tau^{-1} T_1]^1_D, \qquad (8) \end{aligned}$$

and also

$$1 = [\tau T_k, T_1]^1_D.$$

By calculations similar to those preceding (7,8), one can show that

$$0 = [\overline{T_k}, \overline{T_1}]_{\overline{\mathcal{C}}} = [\overline{\tau T_k}, \overline{T_1}]^1_D$$

Since T_1 is l-free, the restriction does not change T_1. Thus the restriction must change τT_k and hence τT_k is not l-free. Here we use the fact that if two indecomposable objects M, N are l-free then $[\overline{M}, \overline{N}]^1_D = [M, N]^1_D$. A similar argument shows that $\tau^{-1}T_1$ is not l-free. That is $1 = [P_l[1], \tau T_k]^1_{\mathcal{C}} = [T_k, P_l[1]]_{\mathcal{C}}$ and $1 = [\tau^{-1}T_1, P_l[1]]^1_{\mathcal{C}} = [P_l[1], T_1]_{\mathcal{C}}$. But then there are two paths q_1, q_2 in $\mathcal{Q}_T$, q_1 going from T_1 to $P_l[1]$ and q_2 from $P_l[1]$ to T_k, and q_1, q_2 are both non-zero in $(\mathcal{Q}_T, I_T)$. The composition $q = q_1 q_2$ is a path from T_1 to T_k. This path q is not a shortest path because of the hypothesis $m = 1$ and by Proposition 3.3(3), we have $q = 0$ in $(\mathcal{Q}_T, I_T)$. Consider the lifts $q_2^D \in \mathrm{Hom}_D(T_k, P_l[1])$ and $q_1^D \in \mathrm{Hom}(P_l[1], \tau^{-1}T_1[1])$. Recall our convention that T_k, T_1 are $k\mathcal{Q}_{\mathrm{alt}}$-modules or first shifts of projectives. Now q being zero in $(\mathcal{Q}_T, I_T)$ means that the composition $q_1^D \circ q_2^D = 0$, whence $[T_k, \tau^{-1}T_1[1]]_D = 0$, contradiction to (7).

We have shown that the path $p : T_1 \to T_2 \to \ldots \to T_k$ passes through all vertices of $\mathcal{Q}_T$, that is $k = n$. Since p is a shortest path, the underlying graph of the quiver $\mathcal{Q}_T$ is a cycle. Thus $\mathcal{Q}_C = \mathcal{Q}_T = T_1 \to T_2 \to \ldots \to T_n \to T_1$ with $n > 3$ and by a result of [FZ03a] this implies that the cluster algebra (and hence $\mathcal{Q}_{\mathrm{alt}}$) is of type D_n. Let us label the vertices of $\mathcal{Q}_{\mathrm{alt}}$ as follows.

$$\begin{tikzcd} & & & & & n-1 \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dotted, dash] & (n-3) \arrow[r, dash] & (n-2) \arrow[ur, dash] \arrow[dr, dash] & \\ & & & & & n \end{tikzcd}$$

Note that if one removes any vertex T_i of the quiver $\mathcal{Q}_T$ then the induced subquiver $\mathcal{Q}_T - \{T_i\}$ is a Dynkin quiver of type A_{n-1}. Then the corresponding cluster category $\mathcal{C}_{\mathcal{Q}_T - \{T_i\}}$ is of type A_{n-1} too. Therefore the position of T_i in the Auslander-Reiten quiver of $\mathcal{C}$ must be at level n or $n-1$; that is, $T_i = \tau^{-k}(P_l)$ with $l \in \{n-1, n\}$ and $k \geq 0$. Suppose without loss of generality that $T_1 = P_l[1]$. Now since T is a tilting object and since there are arrows $T_i \to T_{i+1}$ in $\mathcal{Q}_T$, we have for all $2 \leq i \leq n-1$

$$T_i = \begin{cases} \tau^{i-1}P_l[1] & \text{if } i \text{ is odd} \\ \tau^{i-1}P_{l'}[1] & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad T_n = P_{l'}$$

where l' is such that $\{l, l'\} = \{n-1, n\}$. Thus $[T_n, T_1]_{\mathcal{C}} = 0$ and consequently p is zero in $(\mathcal{Q}_T, I_T)$.

Suppose now that $m = 2$. By Lemma 1.1 we have $[T_j, T_i]_{\mathcal{C}} \leq 1$ and therefore either $p_1 = p_2$ in $(\mathcal{Q}_T, I_T)$ (and in this case we are done) or one of p_1, p_2, say

p_1 is zero in $(\mathcal{Q}_T, I_T)$ and p_2 is not zero. Thus $[T_j, T_i]_\mathcal{C} = 1$ and then p_1 being zero means that there is a vertex T_h on the path p_1 such that

$$[T_j, T_h]_\mathcal{C} = 0. \tag{9}$$

We may suppose that $T_h = P_l[1]$ for some l. As we did before in the case $m = 1$, we remove that vertex T_h so that we get a quiver $(\mathcal{Q}_{\overline{T}}, I_{\overline{T}})$ of rank $n-1$. In this quiver, the induced path $\overline{p_2}$ is zero by case $m = 1$. We have seen in the case $m = 1$ that if p_2 is non-zero in $(\mathcal{Q}_T, I_T)$ then $[T_j, P_l[1]]_\mathcal{C} = 1$, contradiction to (9). ■

4 Applications

In this section, we use the results in section 3 to answer conjectures of [BMR$^+$a] and [CCS]. We keep the setup of the previous section. Denote by ν the number of positive roots of the root system corresponding to the type of the cluster algebra. By a result of [BMRb], the number of indecomposable $\mathrm{End}_\mathcal{C}(T)^{\mathrm{op}}$-modules is equal to ν. On the other hand, in [CCS] it has been shown for type A (and conjectured for types D and E) that the number of indecomposable $k\mathcal{Q}_C/\langle I_C\rangle$-modules is also equal to ν. Using this, we can prove in type A the following theorem, which has been conjectured in [BMR$^+$a] for types A, D, E.

Theorem 4.1 *Let C be any cluster of a cluster algebra of type A and let $(\mathcal{Q}_C, I_C)$ be its quiver with relations. Let T be a corresponding tilting object in the cluster category. Then the cluster tilted algebra $\mathrm{End}_\mathcal{C}(T)^{\mathrm{op}}$ is isomorphic to the algebra $k\mathcal{Q}_C/\langle I_C\rangle$.*

Using Theorem 3.1, Proposition 3.5 and the considerations above, the result follows from

Lemma 4.2 *Let A be an algebra and let I be an ideal of A. Suppose that the category mod$A$ of finitely generated A-modules has a finite number of indecomposable modules. Suppose also that the category modA/I has the same number of indecomposable modules. Then, I is zero.*

Proof. We denote by $\overline{\mathrm{mod}}A$ the category of isoclasses of A modules. Let j be the natural map from $\mathrm{mod}A/I$ to mod A. It is clear that j gives a quotient map from $\overline{\mathrm{mod}}A/I$ to $\overline{\mathrm{mod}}A$. We still denote this map by j. The image of j is the subcategory of isoclasses of A-modules on which I vanishes. Moreover, j commutes with direct sums, hence, it sends indecomposable modules on indecomposable ones. It is easily seen that j is injective, so j embeds the set of isoclasses of indecomposable A/I modules in the set of isoclasses of indecomposable A modules. By the hypothesis of the lemma, this restriction of j is bijective. Hence, by the Krull-Schmidt theorem, j is bijective. This implies that I vanishes on all finitely generated A-modules. Considering A as an A-module then gives $I = 0$. ■

Next, we describe the exchange relations of the cluster algebra in terms of the cluster category. Let M be an indecomposable summand of T and $T = \overline{T} \oplus M$ with $\overline{T} = \oplus_{i=1}^{n-1} T_i$. Suppose that M' is another indecomposable object of the cluster category such that M and M' form an exchange pair, that is $T' = \overline{T} \oplus M'$ is another tilting object. By [BMR$^+$a], there are two triangles in $\mathcal{C}$

$$\begin{array}{ccccccc} M & \to & \oplus_{i\in I'} T_i & \to & M' & \to & M[1] \\ M' & \to & \oplus_{i\in I} T_i & \to & M & \to & M'[1] \end{array}$$

Let z, z', x_i be the cluster variables corresponding to M, M', T_i respectively and let C, C' the clusters corresponding to T, T'. Let $\mathcal{B} = (b_{xy})_{x,y\in C}$ be the sign-skew-symmetric matrix associated to the cluster C, see [FZ03a]. Then $C' = C \setminus \{z\} \cup \{z'\}$, $C \cap C' = \{x_1, \ldots, x_{n-1}\}$ and z, z' satisfy the so called exchange relation:

$$zz' = \prod_{x\in C: b_{xz}>0} x^{b_{xz}} + \prod_{x\in C: b_{xz}<0} x^{-b_{xz}}. \tag{10}$$

The following theorem has been conjectured in [BMR$^+$a].

Theorem 4.3 *For any cluster algebra of type A, D, E, in the situation above the exchange relation can be written as*

$$zz' = \prod_{i\in I} x_i + \prod_{i\in I'} x_i.$$

Proof. By definition of the cluster diagram $\mathcal{Q}_C$, there is a vertex for each cluster variable x in C and there is an arrow $x \to y$ precisely if $b_{xy} > 0$. Since the cluster algebra is of type A, D or E, we have $b_{xy} \in \{-1, 0, 1\}$. Thus (10) becomes

$$zz' = \prod_{x\to z \text{ in } \mathcal{Q}_C} x + \prod_{x\leftarrow z \text{ in } \mathcal{Q}_C} x.$$

Now the result follows from Theorem 3.1 and Lemma 1.2. ■

Finally, we generalize a result of [CCS] on denominators of Laurent polynomials. This theorem has been conjectured in [CCS] in a slightly different form using the quiver with relations $(\mathcal{Q}_C, I_C)$ instead of the cluster category.

Theorem 4.4 *Let $C = \{x_1, \ldots, x_n\}$ be any cluster of a cluster algebra of type A, D or E and let $T = \oplus_{i=1}^{n} T_i$ the corresponding tilting object in the cluster category $\mathcal{C}$. Then there is a bijection*

$$\begin{array}{ccc} \{\text{indecomposable objects of } \mathcal{C}\} & \to & \{\text{cluster variables}\} \\ M & \mapsto & x_M \end{array}$$

such that

$$x_M = \frac{P(x_1, \ldots, x_n)}{\prod_{i=1}^{n} x_i^{[T_i, M]^1_{\mathcal{C}}}} \tag{11}$$

where P is a polynomial prime to x_i for all i and $[T_i, M]^1_{\mathcal{C}} = \dim \operatorname{Ext}_{\mathcal{C}}(T_i, M)$.

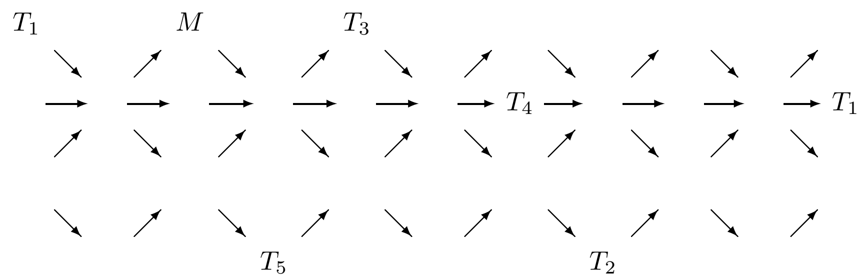

Figure 2: Auslander-Reiten quiver with tilting object

Remark 4.5 *It has been shown in [BMRb] that* $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T,\ )$ *induces an equivalence of categories* $\mathcal{C}/\mathrm{add}T \to \mathrm{mod}\,\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$*. Under this equivalence, the object* M *of* $\mathcal{C}$ *gets mapped to the indecomposable* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$*-module with dimension vector* $(d_1,\dots,d_n)$*,* $d_i = \dim \mathrm{Ext}(T_i, M)$*. Thus the exponent of* x_i *in the denominator is the multiplicity of the simple* $\mathrm{End}_{\mathcal{C}}\,T^{\mathrm{op}}$*-module* S_i *in the image of* M*.*

Proof. The existence of the bijection between the two sets is proved in [BMR$^+$a]. The fact that x_M can be written in terms of the $x_1,\dots,x_n$ as a Laurent polynomial is the Laurent phenomenon proved in [FZ02]. We have to show that the exponents in the denominator are as stated. It has been shown in [FZ03a] that cluster variables are in bijection with almost positive roots. Let α_M, α_i be the almost positive root corresponding to x_M, x_i respectively. By a result of [CCS, Prop. 6.5], the exponent of x_i in the denominator of (11) is equal to the compatibility degree $(\alpha_i \mid\mid \alpha_M)$ of the almost positive roots. Finally, the identity $(\alpha_i \mid\mid \alpha_M) = \dim \mathrm{Ext}_{\mathcal{C}}(T_i, M)$ has been shown in [BMR$^+$a]. ■

Example 4.6 *We give an example of type* D_5*. Let* C *be a cluster having the following diagram* $\mathcal{Q}_C$

$$\begin{tikzcd}
5 \arrow[rr] & & 1 \arrow[rr] \arrow[dl] & & 2 \arrow[dl] \\
& 4 \arrow[ul] \arrow[rr] & & 3 \arrow[ul] &
\end{tikzcd}$$

Performing a mutation at vertex 1 *followed by a mutation at vertex* 2 *shows that* $\mathcal{Q}_C$ *is mutation equivalent to a quiver with underlying graph the Dynkin diagram* D_5*. A corresponding tilting object* $T = \oplus_{i=1}^{n} T_i$ *is illustrated in the Auslander-Reiten quiver of* $\mathcal{C}$ *in figure 2, where* T_i *corresponds to the vertex* i *of* $\mathcal{Q}_C$*. Let* M *be the indecomposable object shown in the same figure. By Theorem 4.4 we have*

$$x_M = \frac{P(x_1,\dots,x_n)}{x_1x_2x_3x_4}.$$

Note that the shape of the Auslander-Reiten quiver of the cluster tilted algebra is obtained from figure 2 by deleting the vertices $T_1, \ldots, T_5$*. At the position of* M *we find the indecomposable* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$*-module* $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T, M)$*, by [BMRb]. It is the first projective and the third injective indecomposable of* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$*. Its dimension vector* $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5)$ *is given by* $d_i = [\tau^{-1}T_i, M]_{\mathcal{C}} = [M, T_i]^1_{\mathcal{C}}$*, thus* $\mathbf{d} = (1, 1, 1, 1, 0)$*.*

A Appendix

In this section, we give some general results about cluster quivers of simply-laced finite type.

Theorem A.1 *For each cluster quiver* Q *of simply-laced finite type, there exists an embedding of* Q *in the plane such that all vertices of* Q *belong to the closure of the unique unbounded connected component of the complement.*

Let us call this a *nice embedding*. It is clear that each bounded connected component is a topological cell.

Corollary A.2 *In a nice embedding, the boundary of each bounded cell is an oriented cycle in the quiver. There are no other edges between its vertices.*

Proof. It is enough to prove that there are no other edges between the boundary vertices of the fixed cell, as this is known to imply the orientation property [FZ03a, Prop. 9.7]. As there are no edges inside the cell, other edges must be outside. Would they exist, they would contradict the property of the embedding that all vertices border the unbounded component. ■

Corollary A.3 *For each arrow* $i \to j$ *in* Q*, there are at most two shortest paths from* j *to* i*. The shortest paths from* j *to* i *are given by the boundaries of the bounded cells which are adjacent to the arrow* $i \to j$ *in any nice embedding.*

Proof. Either side of the arrow $i \to j$ is either a bounded cell or an unbounded component. If one of those sides is a bounded cell, it gives a shortest path from j to i.

Conversely, pick a shortest path from j to i. By definition, there is no other edge between its set of vertices. As there can be no other vertex inside the loop drawn by the shortest path (this would contradict the nice embedding property), one deduces that this loop bounds a cell, which is of course adjacent to the arrow $i \to j$. ■

Let us now prove the theorem. This could be done by inspection of all possible cluster quivers of simply-laced finite type but we use another proof.

Proof. The strategy of proof is the following one. First the Theorem is clearly true for oriented Dynkin diagrams, which are trees. Any plane embedding of a tree is nice. We are going to prove that the statement of the Theorem is stable by mutation of quivers. This is clearly true if the mutation does not change the shape of the quiver.

To go further, it is necessary to have a precise description of what can happen during the mutation process. We need to describe all possible configurations around a vertex of a cluster quiver of simply-laced finite type.

Recall that the *link* of a vertex v in a graph is the graph induced on the set of vertices which are adjacent to v.

Let us start with some simple remarks on the link of a vertex in a cluster quiver of simply-laced finite type.

First, the link of any vertex has at most 3 connected components. This follows from the fact that no orientation of the affine D_4 diagram is of finite type.

Next, as each triangle with vertex v must be oriented, each vertex of the link is either a sink or a source in the link.

We claim that each connected component of the link is a tree. Indeed, there can not be any odd cycle, because sources and sinks must alternate. It is also easy to check that the existence of a 4-cycle or a 6-cycle would imply that the quiver is not of finite type. Any even cycle of length at least 8 would imply that the quiver contains an affine D_4 quiver, which is not of finite type.

For similar reasons, each connected component of the link is a linear tree. It is enough to prove that the existence of a fork would contradict the assumption that the quiver is of finite type. This is readily checked by a sequence of mutations.

Hence we know that each connected component of the link is an alternating linear tree. So we can describe each connected component by its cardinality, up to reversal, and a link can be described by the set of cardinalities of its connected components.

Here is a list of all possible links:

- One connected component: (1);(2);(3);(4);(5);(6),
- Two connected components: (1,1);(1,2);(2,2);(1,3);(1,4);(2,3);(2,4),
- Three connected components: (1,1,1);(1,1,2);(1,2,2).

Indeed the links (7);$(3,3)$;$(1,5)$ and $(1,1,3)$ are not possible as they would give that an affine D_4 quiver is of finite type. Similarly the link $(2,2,2)$ contains an affine E_6 quiver.

For each link, there is only one possible orientation up to global change of orientation, except for $(1,1,2)$ and $(1,3)$ where there are two really different orientations.

Conversely each of those links are realized in a cluster quiver of finite type.

Here comes now the list of all possible shape-changing mutations:

$$(2) \leftrightarrow (1,1), \tag{12}$$
$$(3) \leftrightarrow (1,1,1), \tag{13}$$
$$(5) \leftrightarrow (1,2,2), \tag{14}$$
$$(6) \leftrightarrow (2,4), \tag{15}$$
$$(4) \leftrightarrow (1,1,2), \tag{16}$$
$$(1,3) \leftrightarrow (1,1,2), \tag{17}$$
$$(1,2) \leftrightarrow (1,2), \tag{18}$$
$$(2,2) \leftrightarrow (2,2), \tag{19}$$
$$(1,3) \leftrightarrow (1,3), \tag{20}$$
$$(2,3) \leftrightarrow (2,3). \tag{21}$$

To prove the Theorem, one now has to check in both directions for each of these cases that the existence of a nice embedding before mutation permits to build a nice embedding after mutation.

The principle is the same for all cases. Pick one of these links and assume it is part of a nice embedding. The finiteness assumption and the nice embedding hypothesis together allow to give restrictions on the local picture of the embedding near the fixed vertex v. Then using these restrictions, one concludes that the quiver after mutation still has a nice embedding.

In general, one knows that at least one of the components near v not enclosed by the link of v has to be unbounded.

This is enough to solve the cases $(1,1) \leftrightarrow (2)$, $(1,1,1) \leftrightarrow (3)$ and $(1,2) \leftrightarrow (1,2)$.

CLAIM : let $i \to j$ be an arrow in the link of v. Assume that there is a bounded cell containing this arrow but not v. Then the only edges between the vertices of this cycle (but i and j) and a vertex of the link of v are the edges from i or j to their neighbour in the cycle.

Indeed, the existence of such a vertex and edge would contradict the nice embedding property.

CLAIM : replacing each cell containing an arrow of the link of v and not containing v by a triangle gives a quiver of finite type.

Indeed one can show that all these cycles can only meet or be related by an edge inside the link of v. Hence one can use mutation to shorten the cycles independently until they become triangles.

In some cases, it is necessary to show that at least one of some arrows in the link has an unbounded side. By the claim above, this is done by checking that adding triangles on all these arrows can not give a quiver of finite type. Then one can repeat this argument to get more information on the local configuration.

Combined with the argument on the unbounded cell near v, this is enough to solve all the remaining cases.

Figure 3 displays all possible plane configurations around a vertex v in a nice embedding. The unbounded component is shaded and the vertex v is marked.

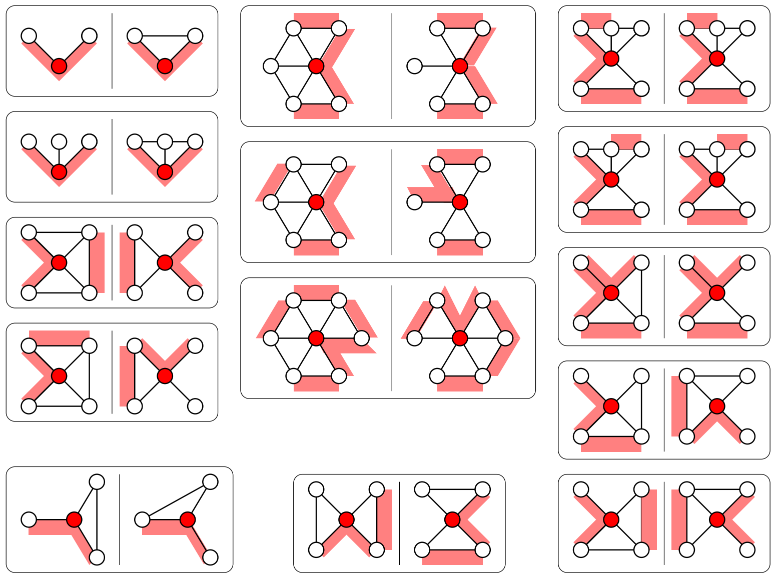

Figure 3: List of all local mutation configurations

Each configuration is paired with the configuration obtained after mutation. ■

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