and mappings $\Phi(F)\to\Phi(E)$, $\Phi(F')\to\Phi(E')$, $\Psi(F)\to\Psi(E)$ and $\Psi(F')\to\Psi(E')$ building up this diagram to a commutative cube. The square

$$\begin{array}{ccc}\Phi_1(E/F) & \to & \Phi_1(E'/F')\\ \downarrow & & \downarrow\\ \Psi_1(E/F) & \to & \Psi_1(E'/F')\end{array}$$

is the cokernel of this mapping of commutative squares. It is known that this cokernel is again a commutative square. Proposition 7 is proved.

Added in proof. Since March 1978, on the same subject, the author completed: *Les espaces de Banach plats sont ultraplats*, Bulletin de la Société Mathématique de Belgique; *Fonctions à valeurs dans les quotients banachiques*, Bulletin de l'Académie Belge, Classe des Sciences; *Holomorphic functional calculus*, Studia Math. vol. 75; and *Quasi-Banach algebras, ideals, and holomorphic functional calculus*, ibid., vol. 75, all four at present in publication.

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QUOTIENT BANACH SPACES; MULTILINEAR THEORY

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The multilinear structure of the category qB is defined by putting a qB-structure on the vector space $qB(E/F, E'/F')$. Multilinear mappings $E_1/F_1\times\ldots\times E_k/F_k\to E'/F'$ are defined by induction.

Strict multilinear mappings are specially interesting. These are induced by bounded multilinear mappings $u_1\colon E_1\times\ldots\times E_k\to E'$ such that $u(x_1,\ldots,x_k)\in F'$ as soon as one of the x_i belongs to the corresponding F_i. All qB-multilinear maps $E_1/F_1\times\ldots\times E_k/F_k\to E'/F'$ are strict if the E_i/F_i are standard qB-spaces.

The tensor product which can be defined in qB is a right-exact functor as it should be. It is unfortunately not an extension of the tensor product which is defined in the category of Banach spaces. If $\bar{F}$ is the closure of F, $E/\bar{F}$ is the "Banachization" of E/F. The projective tensor product of two Banach spaces is the Banachization of their qB-tensor product.

We are interested in qB-algebras. These are qB-spaces $\mathscr{A}$ with a bilinear multiplication belonging to $q_2(\mathscr{A},\mathscr{A};\mathscr{A})$. The qB-algebra is strict if its multiplication is a strict bilinear mapping. It is commutative, or associative if its multiplication is commutative, or associative. The structure of a qB-subalgebra can be put on the center of a qB-algebra.

Every qB-algebra is isomorphic with a strict qB-algebra. A strict qB-algebra is the quotient of a Banach algebra by a two-sided Banach ideal. An associative qB-algebra is isomorphic with the quotient A/α of an associative Banach algebra by a Banach ideal. The isomorphic A/α can even be chosen in such a way that $Z(A/\alpha)=\big(Z(A)+\alpha\big)/\alpha$ where $Z(A/\alpha)$ and $Z(A)$ are the centers of A/α and of A. Every commutative and associative qB-algebra is isomorphic with the quotient of a commutative and associative Banach algebra by a Banach ideal.

This paper is a sequel of [2].

1

Let E/F and E'/F' be qB-spaces. Call $\tilde{q}B^1(E/F, E'/F')$ the space of bounded linear mappings $E\to E'$ which map F into F', and $\tilde{q}B^0(E/F, E'/F')$ the space of bounded linear maps $E \to F'$. We know that $\tilde{q}B = \tilde{q}B^1/\tilde{q}B^0$. The spaces $\tilde{q}B^1$, $\tilde{q}B^0$ become Banach spaces, when we norm them by

$$\|u_1\|_{\tilde{q}B^1} = \sup\{\|u_1x\|_{E'}, \|u_1y\|_{F'} \mid \|x\|_E \leqslant 1, \|y\|_F \leqslant 1\},$$
$$\|v_1\|_{\tilde{q}B^0} = \sup\{\|v_1x\|_{F'} \mid \|x\|_E \leqslant 1\}.$$

DEFINITION 1. The *quotient Banach structure of* $\tilde{q}B(E/F, E'/F')$ is given by the isomorphism of this space with

$$\tilde{q}B^1/\tilde{q}B^0(E/F, E'/F').$$

In this way, $\tilde{q}B$ becomes a functor $\tilde{q}B^* \times \tilde{q}B \to \tilde{q}B$. (If K is a category, K^* is the opposite category, functors from K^* are contravariant functors from K).

DEFINITION 2. Let E_i/F_i $(i = 1, \ldots, k)$ and E'/F' be Banach quotients. Then

$$\tilde{q}B_k(E_1/F_1, \ldots, E_k/F_k; E'/F') = \tilde{q}B(E_1/F_1, \tilde{q}B_{k-1}(E_2/F_2, \ldots, E_k/F_k; E'/F')),$$

where $\tilde{q}B_1 = \tilde{q}B$.

$\tilde{q}B_k$ is a functor $(\tilde{q}B^*)^k \times \tilde{q}B \to \tilde{q}B$. We observe that this functor is symmetric in $E_1/F_1, \ldots, E_k/F_k$, i.e. that permutation of the arguments induce functor isomorphisms. As a matter of fact

PROPOSITION 1. *$\tilde{q}B_k$ is naturally isomorphic with $\tilde{q}B_k^1/\tilde{q}B_k^0$, where $\tilde{q}B_k^0(E_1/F_1, \ldots$ $\ldots, E_k/F_k; E'/F')$ is the space of bounded multilinear mappings $E_1 \times \ldots \times E_k \to F'$, while $\tilde{q}B_k^1(E_1/F_1, \ldots, E_k/F_k; E'/F')$ is the space of bounded multilinear mappings u_1: $: E_1 \times \ldots \times E_k \to E'$ such that $u_1(x_1, \ldots, x_k) \in F'$ as soon as one of the x_i is in the corresponding F_i. Both $\tilde{q}B_k^0$ and $\tilde{q}B_k^1$ is equipped with a norm which makes it a Banach subspace of the space of bounded multilinear mappings $E_1 \times \ldots \times E_k \to E'$.*

We can let

$$\|u_1\|_{\tilde{q}B^0} = \sup\{\|u_1(x_1, \ldots, x_k)\|_{F'} \mid \|x_i\|_{E_i} \leqslant 1\},$$
$$\|u_1\|_{\tilde{q}B^1} = \sup\{\|u_1(x_1, \ldots, x_k)\|_{E'}, \|u_1(y_1, \ldots, y_k)\|_{F'} \mid \|x_i\|_{E_i} \leqslant 1, \|y_i\|_{E_i} \leqslant 1, \text{ one of the } \|y_j\|_{F_j} \leqslant 1\}.$$

We want to define a qB-structure on $qB(E/F, E'/F')$. A quotient Banach structure is defined on a vector space X by an isomorphism φ: $X \to U/V$ where U/V is a Banach quotient. Two such isomorphisms φ_i: $X \to U_i/V_i$ define the same qB-structure if $\varphi_1 \circ \varphi_2^{-1}$: $U_2/V_2 \to U_1/V_1$ is an isomorphism of the category qB.

Let E/F and E'/F' be Banach quotients. Let s: $E_1/F_1 \to E/F$, s': $E_1'/F_1' \to E'/F'$ be isomorphisms of the category qB (e.g. pseudo-isomorphisms). Assume that E_1/F_1 is standard. The linear mapping $u \to s'^{-1} \circ u \circ s$ is a linear bijection

$$qB(E/F, E'/F') \to qB(E_1/F_1, E_1'/F_1') = \tilde{q}B(E_1/F_1, E_1'/F_1').$$

The space $\tilde{q}B$ is a Banach quotient, this bijection defines a Banach quotient structure on $qB(E/F, E'/F')$.

PROPOSITION 2. *This structure does not depend on the choice of the isomorphisms s, s' (with E_1/F_1 standard).*

The fact that the qB-structure defined on $qB(E/F, E'/F')$ does not depend on the choice of s: $E_1/F_1 \to E/F$ is easy. We assume that E_1/F_1 is standard. Let E_2/F_2 be a new standard Banach quotient and t: $E_2/F_2 \to E/F$ be a pseudo-isomorphism. $s \circ t^{-1}$: $E_2/F_2 \to E_1/F_1$ is a strict isomorphism, since it is an isomorphism of the category qB and both E_1/F_1 and E_2/F_2 are standard. The mapping $u \to u \circ s \circ t^{-1}$, $\tilde{q}B(E_1/F_1, E_1'/F_1') \to \tilde{q}B(E_2/F_2, E_1'/F_1')$ is therefore a strict isomorphism, hence an isomorphism.

The proof that this qB-structure does not depend on the choice of s': $E_1'/F_1' \to$ $\to E'/F'$ is a little bit trickier because we do not assume that E_1'/F_1' is standard. Let t': $E_2'/F_2' \to E'/F'$ be a new isomorphism. The mapping $u \to t' \circ s'^{-1} \circ u$ is a bijection $\tilde{q}B(E_1/F_1, E_1'/F_1') \to \tilde{q}B(E_1/F_1, E_2'/F_2')$. We must show that it is a morphism.

The isomorphism $t' \circ s'^{-1}$: $E_1'/F_1' \to E_2'/F_2'$ can be factored $t' \circ s'^{-1} = \tau \circ \sigma^{-1}$ where σ is a pseudo-isomorphism $U/V \to E_1'/F_1'$ and τ is a strict morphism. The mapping $v \to \sigma \circ v$ is a strict morphism and a bijection $\tilde{q}B(E_1/F_1, U/V) \to \tilde{q}B(E_1/F_1, E_1'/F_1')$. This mapping is therefore an isomorphism of the category qB. Its inverse, the mapping $u \to \sigma^{-1} \circ u$ is a morphism. So is the composition $u \to \tau \circ \sigma^{-1} \circ u$ $= t' \circ s'^{-1} \circ u$.

DEFINITION 3. The *quotient Banach structure of* $qB(E/F, E'/F')$ is the quotient Banach structure described in Proposition 2.

DEFINITION 4. We define by induction

$$qB_1(E/F, E'/F') = qB(E/F, E'/F'),$$
$$qB_k(E_1/F_1, \ldots, E_k/F_k; E'/F') = qB\big(E_1/F_1, qB_{k-1}(E_2/F_2, \ldots, E_k/F_k; E'/F')\big).$$

Elements of this space are qB-multilinear maps $E_1/F_1 \times \ldots \times E_k/F_k \to E'/F'$. Elements of $\tilde{q}B_k(E_1/F_1, \ldots, E_k/F_k;\ E'/F')$ are strict multilinear maps. Note that all qB-multilinear maps are strict if the Banach quotients E_i/F_i are standard.

PROPOSITION 3. *The mapping $(u, v) \to v \circ u$ belongs to*

$$qB_2\big(qB(E/F, E'/F'), qB(E'/F', E''/F''); qB(E/F, E''/F'')\big).$$

This is clearly the case when E/F, and E'/F' are standard. But all Banach quotients are isomorphic to standard ones.

2

We must now discuss the tensor products of Banach quotients. Their existence is not difficult to prove.

PROPOSITION 4. *Let E_1/F_1 and E_2/F_2 be Banach quotients. It is possible to find a Banach quotients $E_1/F_1 \otimes_q E_2/F_2$ and an element*

$$\otimes \in qB_2(E_1/F_1, E_2/F_2; E_1/F_1 \otimes_q E_2/F_2)$$

in such a way that every $u \in qB_2(E_1/F_1, E_2/F_2; E/F)$ factors in a unique way $u = u_1 \circ \otimes$ with

$$u_1 \in qB(E_1/F_1 \otimes_q E_2/F_2; E/F).$$

Construction of the tensor product of standard quotients will be sufficient. Let $E_1 = l_1(X_1)$, $E_2 = l_1(X_2)$ and consider the quotients E_1/F_1, E_2/F_2. Every qB-bilinear mapping u: $E_1/F_1 \times E_2/F_2 \to E/F$ is strict, is induced by a bilinear mapping $E_1 \times E_2 \to E$, whose restrictions to $E_1 \times F_2$ and to $F_1 \times E_2$ have their images in F. The bilinear mapping extends to $E_1 \hat{\otimes} E_2$. The spaces E_1 and E_2 having the approximation property, $E_1 \hat{\otimes} F_2$ and $F_1 \hat{\otimes} E_2$ are Banach subspaces of $E_1 \hat{\otimes} E_2$. So is $E_1 \hat{\otimes} F_2 + F_1 \hat{\otimes} E_2$.

Let $U = E_1 \hat{\otimes} E_2$, $V = E_1 \hat{\otimes} F_2 + F_1 \hat{\otimes} E_2$, and let

$$E_1/F_1 \otimes_q E_2/F_2 = U/V.$$

The tensor product mapping $E_1 \times E_2 \to E_1 \hat{\otimes} E_2$ induces a (strict) bilinear mapping $E_1/F_1 \times E_2/F_2 \to U/V$. We shall call this bilinear mapping $\otimes$. The extension to $E_1 \hat{\otimes} E_2$ of a bilinear mapping $E_1 \times E_2 \to E$ inducing u: $E_1/F_1 \times E_2/F_2 \to E/F$ induces a morphism u_1: $U/V \to E/F$. Clearly, u is the composition of $\otimes$ and of u_1. And u_1 is the only morphism $U/V \to E/F$ having this property.

PROPOSITION 5. *If A is free, the functor $A/0 \otimes_q$ is exact. In general, $E_1/F_1 \otimes_q$ is a right-exact functor.*

The right-exactness of the tensor product follows from general categorical principles. The statement: $A \to B \to C \to 0$ is exact, i.e. C is the cokernel of the mapping $A \to B$ means that a mapping $B \to U$ factors through the mapping $B \to C$ if and only if its composition with the mapping $A \to B$ is zero, and the factorization is unique. This again is equivalent with the statement: the sequence $0 \to qB(C, U) \to qB(B, U) \to qB(A, U)$ is exact for all U.

Let now $A \to B \to C \to 0$ be exact, and let D be any object of the category qB. The sequence

$$0 \to qB(C, qB(D, U)) \to qB(B, qB(D, U)) \to qB(A, qB(D, U))$$

is exact, hence also

$$0 \to qB(C \otimes_q D, U) \to qB(B \otimes_q D, U) \to qB(C \otimes_q D, U)$$

and $A \otimes_q D \to B \otimes_q D \to C \otimes_q D \to 0$ is exact.

Let now $A = l_1(X)$ be a free object. To prove that the functor $A \otimes_q$ is exact we shall use the fact that

$$l_1(X)/0 \otimes_q E/F = l_1(X, E/F) = l_1(X, E)/l_1(X, F)$$

if $l_1(X, E)$ is the Banach space of φ: $X \to E$ such that $\|\varphi\| = \sum \|\varphi(x)\| < \infty$. This is verified by looking at the construction of the tensor product (proof of Proposition 4) when E/F is standard. If u: $E/F \to E'/F'$ is a strict morphism induced by u_1: $E \to E'$, the mapping $\varphi \to u_1 \circ \varphi$ induces a strict morphism $l_1(X, E/F) \to l_1(X, E'/F')$. We may call $l_1(X, u)$ this morphism. If u is a pseudo-isomorphism, $l_1(X, u)$ is a pseudo-isomorphism.

This allows us to interpret $l_1(X, \cdot)$ as a functor $qB \to qB$. This functor agrees with $l_1(X)/0 \otimes_q$ when the objects E/F, E'/F' are standard and the morphisms u: $E/F \to E'/F'$ are strict. It agrees therefore with $l_1(X) \otimes_q$ whatever the objects and whatever the morphisms we consider.

To show that $l_1(X, \cdot)$ is an exact functor, it is sufficient to prove that it maps a short exact sequence onto a short exact sequence. And this is the case. Remember that every short exact sequence of q has the form

$$0 \to E'/F \to E/F \to E/F' \to 0$$

(modulo an isomorphism). It is clear that the image sequence is exact.

3

Something must be said about the projective tensor product of two Banach spaces, and the tensor product of these spaces in the category qB. We shall systematically identify a Banach space E and the quotient $E/0$. A Banach quotient E/F is "isomorphic with a Banach space" when F is a closed subspace of E.

DEFINITION 5. The *Banachization of a Banach quotient E/F* is the Banach space $E/\bar{F}$ where $\bar{F}$ is the closure of F in the Banach space E. We call $b(E/F)$ this Banachization. The morphism $E/F \to E/\bar{F}$ induced by the identity $E \to E$ is the canonical mapping.

The following is trivial.

PROPOSITION 6. *Every morphism u: $E/F \to X$ of a Banach quotient E/F into a Banach space X factors in a unique way $u = u_1 \circ \sigma$ where σ: $E/F \to b(E/F)$ is the canonical mapping and u_1: $b(E/F) \to X$ is a bounded linear mapping of Banach spaces.*

The next proposition is a corollary of this triviality.

PROPOSITION 7. *The projective tensor product of two Banach spaces E_1 and E_2 is the Banachization of their tensor product in the category qB.*

Life would be very nice indeed if the qB-tensor product of two Banach spaces were isomorphic to a Banach space. This is unfortunately not the case in general, as was shown by G. Noël [1]. Noël shows that this property is related to the flatness of a Banach space.

DEFINITION 6. A Banach quotient E/F is *flat* if the functor $E/F \otimes_q$ is an exact functor.

PROPOSITION 8. *A flat Banach space U has the approximation property. Let U be a Banach space with the approximation property. The following properties are equivalent*

(i) *U is flat,*

(ii) *$U \otimes_q E$ is a Banach space whenever E is a separable Banach space,*

(iii) *$U \hat{\otimes} F$ is a closed subspace of $U \hat{\otimes} l_1$ whenever F is a closed subspace of l_1,*

(iv) *$U \hat{\otimes} F$ is a closed subspace of $U \hat{\otimes} l_1(M)$ whenever M is a set, and F is a closed subspace of $l_1(M)$,*

(v) *$U \otimes_q E$ is a Banach space whenever E is a Banach space.*

The implication (iv) ⇒ (v) ⇒ (i) ⇒ (ii) ⇒ (iii) are either given by Noël (loc. cit.) or very soft analytic results. Implication (iii) ⇒ (iv) is not difficult once the reader realizes that the problem is fundamentally countable. A complete proof of Proposition 8 will be published elsewhere.

PROPOSITION 9. *An infinite-dimensional reflexive Banach space is not flat.*

This is essentially Proposition 9.6 of G. Noël (loc. cit.) once an obvious misprint has been corrected.

It appears (private conversation with Noël) that $l_2 \otimes_q l_2$ is not a Banach space. The result seems likely, but I do not find the result in Noël's published results. In any case, we can associate a Banach space F to every infinite-dimensional, reflexive Banach space with the approximation property E, in such a way that $E \otimes_q F$ is not a Banach space.

4

DEFINITION 7. A *quotient Banach algebra* $(\mathscr{A}, \cdot)$ is a quotient Banach space $\mathscr{A}$ on which a multiplication is defined by an element of $qB_2(\mathscr{A}, \mathscr{A}; \mathscr{A})$. The quotient Banach algebra is *strict* if its multiplication belongs to $\tilde{q}B_2(\mathscr{A}, \mathscr{A}; \mathscr{A})$.

The left regular representation $a \to (x \to ax)$ and the left regular representation $a \to (x \to xa)$ are elements, m', m'' respectively of $qB(\mathscr{A}, qB(\mathscr{A}, \mathscr{A}))$.

DEFINITION 8. The quotient Banach algebra $(\mathscr{A}, \cdot)$ is *associative* if its multiplication is an associative operation. It is *commutative* if its multiplication is a commutative operation. The center of an associative quotient Banach algebra is $\operatorname{Ker}(m'' - m')$ where m' is the left regular representation and m'' is the right regular representation.

We note that the center of a qB-algebra is a qB-subspace and a subalgebra ... it is the center of the algebra $(\mathscr{A}, \cdot)$.

DEFINITION 9. A *qB-algebra morphism* $(\mathscr{A}_1, \cdot) \to (\mathscr{A}_2, \cdot)$ is a linear mapping which is a morphism both for the qB-space and for the algebra structure.

PROPOSITION 10. *Every quotient Banach algebra is isomorphic to a strict quotient Banach algebra. A strict quotient Banach algebra is isomorphic to the quotient of a Banach algebra by a Banach ideal.*

Every quotient Banach space is isomorphic to a standard one. An algebra structure on a standard space is strict.

Let $\mathscr{A} = E/F$ and $m \in \sigma\tilde{q}_2(\mathscr{A}, \mathscr{A}; \mathscr{A})$. Then m is induced by $m_1\colon E \times E \to E$ where m_1 is a continuous bilinear mapping. And F is a Banach subspace of E, m_1 maps $E \times F$ and $F \times E$ into F, hence F is a Banach ideal of E.

Of course, if A is a Banach algebra, if α is a Banach ideal, multiplication is a bilinear mapping $A \times A \to A$ which induces an element of $\sigma\tilde{q}_2(A/\alpha, A/\alpha; A/\alpha)$, i.e. A/α is a strict quotient Banach algebra.

It may be worth saying explicitly that a Banach ideal α of a Banach algebra A is an ideal on which a Banach space norm exists which is stronger than the norm induced by that of A. This norm is an ideal norm on α, i.e. if α is a left ideal, multiplication $(a, x) \to a \cdot x$ is a continuous bilinear mapping $A \times \alpha \to \alpha$.

Let A_1/α_1 and A_2/α_2 be two strict quotient Banach algebras, let $u\colon A_1/\alpha_1 \to A_2/\alpha_2$ be a strict morphism of quotient Banach spaces, induced by $u_1\colon A_1 \to A_2$.

PROPOSITION 11. *u is a strict morphism of quotient Banach algebras iff u_1 is a homomorphism of A_1 into A_2 modulo α_2, i.e. iff*

$$u_1(xy) - u_1(x)u_1(y) \in \alpha_2$$

for all $x, y \in A$.

This is obvious. The regular representation allows us now to prove

PROPOSITION 12. *An associative qB-algebra is isomorphic to the quotient of an associative Banach algebra by a two-sided Banach ideal.*

We assume that the given algebra A/α is a standard quotient Banach space. The multiplicative structure of A/α is induced by a Banach algebra structure on A. The fact that A/α is associative means that A is an associative modulo α, i.e. that the associator mapping

$$(x, y, z) \to x(yz) - (xy)z$$

maps $A \times A \times A$ into α. Of course, this is a bounded trilinear mapping of $A \times A \times A$ into α (apply the closed graph and Banach–Steinhaus theorems).

A_1 will be the algebra of linear transformations of $A \oplus C$ which leave $\alpha \oplus 0$ invariant, with the norm

$$\|u_1\|_{A_1} = \sup\{\|u_1 z\|_{A\oplus C}, \|u_1 y\|_\alpha \mid \|z\|_{A\oplus C} \leqslant 1, \|y\|_\alpha \leqslant 1\}$$

and α_1 will be the space of bounded linear mappings $A \oplus C \to \alpha$, with the norm

$$\|u_1\|_{\alpha_1} = \sup\{\|u_1 z\|_\alpha \mid \|z\|_{A\oplus C} \leqslant 1\}.$$

Clearly, A_1 is an associative Banach algebra, α_1 is a two-sided ideal in A_1.

We map $\varphi\colon A \to A_1$, mapping $a \in A$ onto the mapping $x \oplus t \to (ax + ta) \oplus 0$, $A \oplus C \to A \oplus C$. This is clearly an injective mapping, and on A, the norms $\|a\|_A$ and $\|\varphi a\|_{A_1}$ are equivalent.

The mapping $\varphi\colon A \to A_1$ is also a homomorphism of A into A_1 mod α_1. This is a direct consequence of the fact that the multiplication in A is associative modulo α. We have

$$(\varphi(ab) - \varphi a \cdot \varphi b)(x \oplus t) = (ab)x - a(bx) \oplus 0.$$

The linear mapping $\varphi(ab) - \varphi a \cdot \varphi b$ belongs to α_1.

Clearly, φ maps α into α_1. Let $a \in \alpha$. Then φa maps $x \oplus t \in A \oplus C$ onto $ax + ta \in \alpha$. And the only elements of A which are mapped by φ into α_1 are the elements of α. If $\varphi a \in \alpha_1$, then $\varphi a(0 \oplus 1) = a \in \alpha$.

Let then $\varphi A + \alpha_1 = A_2$, put on A_2 the norm

$$\|x\| = \inf\{\|a\|_A + \|b\|_{\alpha_1} \mid x = \varphi a + b\}$$

then A_2 is a Banach subalgebra of A_1 (up to norm-equivalence), α_1 is a Banach ideal of A_2, and $\varphi\colon A\to A_2$ induces an isomorphism $A/\alpha\to A_2/\alpha_2$.

5

We now want to investigate the center of an associative quotient Banach algebra. Proposition 12 shows that it is sufficient to consider the case where $\mathscr{A}$ is the quotient of an associative Banach algebra A by a two-sided Banach ideal α. Let A/α be such a quotient.

The center of A/α is clearly $Z_\alpha(A)/\alpha$ where

$$Z_\alpha(A)=\{a\in A|\ \forall x\in A\colon ax-xa\in\alpha\}$$

with the norm

$$\|a\|_{Z_\alpha(A)}=\|a\|_A+\sup\{\|ax-xa\|_\alpha|\ \|x\|_A\leqslant 1\}.$$

This contains $Z(A)+\alpha$, if $Z(A)$ is the center of A, but can obviously be larger than $Z(A)+\alpha$.

PROPOSITION 13. *An associative qB-algebra is isomorphic with A_1/α_1 where A_1 is an associative Banach algebra, α_1 is a two-sided Banach ideal, and*

$$Z_{\alpha_1}(A_1)=Z(A_1)+\alpha_1.$$

We shall assume, as we may, that the given algebra has the form A/α where A is an associative Banach algebra, where α is a Banach ideal of A, and that

$$\|ab\|_A\leqslant\|a\|_A\|b\|_A,$$

$$\|ax\|_\alpha\leqslant\|a\|_A\|x\|_\alpha,$$

$$\|xa\|_\alpha\leqslant\|a\|_A\|x\|_\alpha.$$

For each $a\in Z_\alpha(A)$, we adjoin an indeterminate z_a to A and consider the polynomial algebra $A[\{z_a\}]$ in all these indeterminates. We define the norm of a monomial by

$$\|uz_{a_1}\dots z_{a_k}\|=(k+1)\|u\|_A\|a_1\|_{Z_\alpha(A)}\dots\|a_k\|_{Z_\alpha(A)}.$$

Clearly, if $m=uz_{a_1}\dots z_{a_k}$ and $m'=u'z_{a'_1}\dots z_{a'_k}$ are two monomials

$$\|m\cdot m'\|\leqslant\|m\|\cdot\|m'\|$$

because $k+k'+1\leqslant(k+1)(k'+1)$.

A polynomial can be decomposed in a unique way as a sum of monomials. The norm of a polynomial is the sum of the norms of its terms. The polynomial algebra becomes in this way a normed algebra, A_1 is the completion of A.

To define a linear mapping $\varphi\colon A_1\to A$, we shall use a total order $\prec$ on $Z_\alpha(A)$ and let

$$\varphi(uz_{a_1}\dots z_{a_k})=ua_1\dots a_k$$

if $a_1\prec\dots\prec a_k$. The linear mapping thus defined on the set of monomials has clearly a bounded linear extension $\mathscr{A}_1\to\mathscr{A}$.

Let $m=uz_{a_1}\dots z_{a_k}$ and $n=vz_{b_1}\dots z_{b_l}$ be two monomials. Then

$$\varphi(m)\varphi(n)=ua_1\dots a_kvb_1\dots b_l$$

while

$$\varphi(mn)=uvc_1\dots c_{k+l},$$

where $(c_1,\dots,c_{k+l})$ is the sequence $(a_1,\dots,a_k,\ b_1,\dots,b_l)$, rearranged in increasing order.

We observe that

$$\varphi(mn)-\varphi(m)\varphi(n)$$

is a sum of at most $k(l+1)$ terms, each of which is a product of $k+l+1$ factors, $k+l$ among the elements $u,a_1,\dots,a_k,v,b_1,\dots,b_l$, the remaining one being a commutator a_iv-va_i or $a_ib_j-b_ja_i$.

We use the estimates

$$\|va_i-a_iv\|\leqslant\|v\|_A\|a_i\|_{Z_\alpha(A)},$$

$$\|a_ib_j-b_ja_i\|\leqslant\|a_i\|_A\|b_j\|_{Z_\alpha(A)}\leqslant\|a_i\|_{Z_\alpha(A)}\|b_j\|_{Z_\alpha(A)}$$

and obtain

$$\begin{aligned}\|\varphi(m\cdot n)-\varphi(m)\varphi(n)\|_\alpha\\ &\leqslant k(l+1)\|u\|_A\|a_1\|_{Z_\alpha(A)}\dots\|a_k\|_{Z_\alpha(A)}\|v\|_A\|b_1\|_{Z_\alpha(A)}\dots\|b_l\|_{Z_\alpha(A)}\\ &\leqslant\|m\|_\alpha\|n\|_\alpha.\end{aligned}$$

This estimate shows that the mapping $(m,n)\to\varphi(m\cdot n)-\varphi(m)\varphi(n)$ extends to a bounded linear mapping $A_1\times A_1\to\alpha$, i.e. that φ is a homomorphism $A_1\to\to A\bmod\alpha$. Let $\alpha_1=\varphi^{-1}\alpha$, φ induces a pseudo-isomorphism for the structures of quotient Banach spaces $A_1/\alpha_1\to A/\alpha$. It induces therefore also an isomorphism for the structure of quotient Banach algebra (because $\varphi\colon A_1\to A$ is a homomorphism modulo α).

A_1/α_1 has the announced property. Every equivalence class of its center contains an indeterminate z_a. The indeterminate is in the center of A_1.

PROPOSITION 14. *A commutative and associative quotient Banach algebra is isomorphic to the quotient of a commutative and associative Banach algebra by a Banach ideal.*

Let A/α be the given quotient Banach algebra. We may assume that A is associative and that $Z_\alpha(A)=Z(A)+\alpha$. But A/α is commutative, $Z_\alpha(A)=A_1$.

The inclusion map $Z(A)\to A$ induces an isomorphism $Z(A)/Z(A)\cap\alpha\to A/\alpha$ and $Z(A)$ is commutative.

References

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