QUOTIENT OF LAPLACE AND GUMBEL RANDOM VARIABLES

SARALEES NADARAJAH

Received 5 July 2005; Revised 13 December 2005; Accepted 13 March 2006

The Gumbel and the Laplace distributions are perhaps two of the most applied distributions in engineering. Motivated by engineering issues, the exact distribution of the quotient $|X/Y|$ is derived when X and Y are independent Gumbel and Laplace random variables. Tabulations of the associated percentage points and a computer program for generating them are also given.

1. Introduction

The Gumbel and the Laplace distributions are perhaps two of the most applied distributions in engineering. The Gumbel distribution is also known as the extreme value distribution of type I. Some of its recent application areas in engineering include flood frequency analysis, network engineering, nuclear engineering, offshore engineering, risk-based engineering, space engineering, software reliability engineering, structural engineering, and wind engineering. A recent book by Kotz and Nadarajah [4], which describes this distribution, lists over fifty applications ranging from accelerated life testing to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, and wind speeds (to mention just a few).

The Laplace distribution has found applications in a variety of engineering areas that range from image and speech recognition to ocean engineering. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data.

Since Gumbel and Laplace distributions are popular in engineering, there are many real situations where measurements could be modeled by these distributions. Some examples are:

(1) in communication theory, X and Y could represent the random noise corresponding to two signals;

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2006, Article ID 90598, Pages 1–7
DOI 10.1155/MPE/2006/90598

(2) in ocean engineering, X and Y could represent distributions of navigation errors;
(3) in image and speech recognition, X and Y could represent "input" distributions;
(4) in chemical engineering, X and Y could represent the remission times of two chemicals when they are administered to two kinds of mechanical systems;
(5) in civil engineering, X and Y could represent future observations on the strength of an engineering design (e.g., the strength of a bridge);
(6) in hydrology, X and Y could represent the extreme rainfall at two stations.

In each of the examples above, it will be of interest to study the distribution of the quotient $|X/Y|$. For example, in communication theory, $|X/Y|$ could represent the relative strength of the two different signals. In ocean engineering, $|X/Y|$ could represent the relative safety of navigation. In mechanical engineering, $|X/Y|$ could represent the relative effectiveness of the two chemicals. In civil engineering, $|X/Y|$ could represent some measure of reliability of the engineering design. In hydrology, $|X/Y|$ could represent the relative extremity of rainfall at the two stations.

The distribution of the quotient X/Y has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Marsaglia [5] and Korhonen and Narula [3] for normal family, Press [7] for Student's t family, Basu and Lochner [1] for Weibull family, Shcolnick [9] for stable family, Hawkins and Han [2] for noncentral chi-squared family, Provost [8] for gamma family, and Pham-Gia [6] for beta family.

However, there is relatively little work of this kind when X and Y belong to different families. In this note, we study the exact distribution of $|X/Y|$ when X and Y are independent Gumbel and Laplace random variables with pdf's

$$f_X(x) = \exp\left(-\frac{x-\mu}{\sigma}\right)\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \tag{1.1}$$

$$f_Y(y) = \frac{\lambda}{2}\exp\{-\lambda|y-\theta|\}, \tag{1.2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu < \infty$, $-\infty < \theta < \infty$, $\sigma > 0$, and $\lambda > 0$. Tabulations of the associated percentage points and a computer program for generating them are also provided. The calculations involve several special functions, including the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x t^{a-1}\exp(-t)dt \tag{1.3}$$

and the complementary incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1}\exp(-t)dt. \tag{1.4}$$

The properties of these special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2. Cumulative distribution function

Theorem 2.1 derives explicit expressions for the cdf of $|X/Y|$ in terms of the incomplete gamma functions.

Theorem 2.1. *Suppose X and Y are distributed according to (1.1) and (1.2), respectively. The cdf of $Z = |X/Y|$ can be expressed as one of the following:*

(1) *if $\theta < 0$, then*

$$F(z) = \frac{\lambda\sigma}{2z}\left[\frac{1}{ab}\gamma\left(\frac{\lambda\sigma}{z}, cd\right) + ab\left\{\Gamma\left(-\frac{\lambda\sigma}{z}, cd\right) - \Gamma\left(-\frac{\lambda\sigma}{z}, c\right)\right\} + \frac{a}{b}\gamma\left(\frac{\lambda\sigma}{z}, c\right) - \frac{b}{a}\Gamma\left(-\frac{\lambda\sigma}{z}, \frac{c}{d}\right) - \frac{a}{b}\left\{\gamma\left(\frac{\lambda\sigma}{z}, \frac{c}{d}\right) - \gamma\left(\frac{\lambda\sigma}{z}, c\right)\right\} - ab\Gamma\left(-\frac{\lambda\sigma}{z}, c\right)\right]; \tag{2.1}$$

(2) *if $\theta = 0$, then*

$$F(z) = \frac{\lambda\sigma}{z}\left[\frac{1}{b}\gamma\left(\frac{\lambda\sigma}{z}, c\right) - b\Gamma\left(-\frac{\lambda\sigma}{z}, c\right)\right]; \tag{2.2}$$

(3) *if $\theta < 0$, then*

$$F(z) = \frac{\lambda\sigma}{2z}\left[\frac{a}{b}\gamma\left(\frac{\lambda\sigma}{z}, \frac{c}{d}\right) + \frac{b}{a}\left\{\Gamma\left(-\frac{\lambda\sigma}{z}, \frac{c}{d}\right) - \Gamma\left(-\frac{\lambda\sigma}{z}, c\right)\right\} + \frac{1}{ab}\gamma\left(\frac{\lambda\sigma}{z}, c\right) - ab\Gamma\left(-\frac{\lambda\sigma}{z}, cd\right) - \frac{1}{ab}\left\{\gamma\left(\frac{\lambda\sigma}{z}, cd\right) - \gamma\left(\frac{\lambda\sigma}{z}, c\right)\right\} - \frac{b}{a}\Gamma\left(-\frac{\lambda\sigma}{z}, c\right)\right] \tag{2.3}$$

for $z > 0$, where $a = \exp(\lambda\theta)$, $b = \exp(\lambda\mu/z)$, $c = \exp(\mu/\sigma)$, and $d = \exp(\theta z/\sigma)$.

Proof. One can write

$$\begin{aligned}
\Pr(|X/Y| \leq z) &= \int_{-\infty}^{\infty}\left\{F_X(|y|z) - F_X(-|y|z)\right\}f_Y(y)dy \\
&= \frac{\lambda}{2}\left[\int_{-\infty}^{\theta}\exp\left\{\lambda(y-\theta) - \exp\left(\frac{\mu+yz}{\sigma}\right)\right\}dy\right. \\
&\quad + \int_{\theta}^{0}\exp\left\{\lambda(\theta-y) - \exp\left(\frac{\mu+yz}{\sigma}\right)\right\}dy \\
&\quad + \int_{0}^{\infty}\exp\left\{\lambda(\theta-y) - \exp\left(\frac{\mu-yz}{\sigma}\right)\right\}dy
\end{aligned}$$

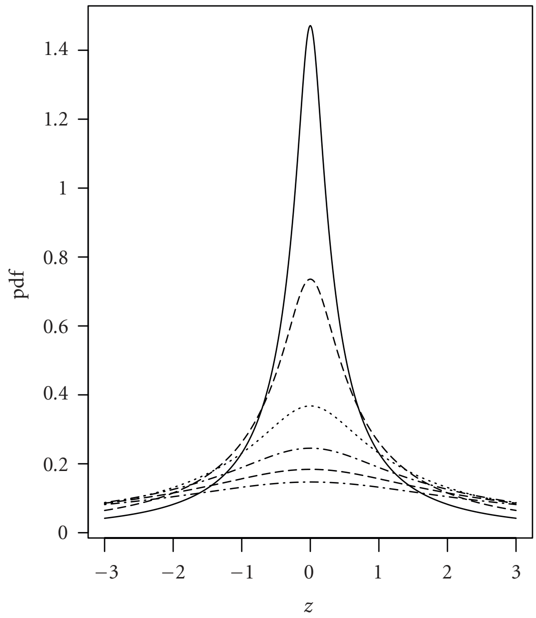

Figure 2.1. Plots of the pdf of (2.5) for $\lambda\sigma = 0.5, 1, 2, 3, 4, 5$. The curves from the top to the bottom correspond to increasing values of $\lambda\sigma$.

$$
\begin{aligned}
&- \int_{-\infty}^{\theta} \exp\left\{\lambda(y-\theta) - \exp\left(\frac{\mu - yz}{\sigma}\right)\right\} dy \\
&- \int_{\theta}^{0} \exp\left\{\lambda(\theta-y) - \exp\left(\frac{\mu - yz}{\sigma}\right)\right\} dy \\
&- \int_{0}^{\infty} \exp\left\{\lambda(\theta-y) - \exp\left(\frac{\mu + yz}{\sigma}\right)\right\} dy \Bigg] \\
= \frac{\lambda\sigma}{2z} \Bigg[&\frac{1}{ab} \int_0^{cd} v^{\lambda\sigma/z-1} \exp(-v) dv + ab \int_{cd}^{c} v^{-\lambda\sigma/z-1} \exp(-v) dv \\
&+ \frac{a}{b} \int_0^{c} u^{\lambda\sigma/z-1} \exp(-u) du - \frac{b}{a} \int_{c/d}^{\infty} u^{-\lambda\sigma/z-1} \exp(-u) du \\
&- \frac{a}{b} \int_{c}^{c/d} u^{\lambda\sigma/z-1} \exp(-u) du - ab \int_{c}^{\infty} v^{-\lambda\sigma/z-1} \exp(-v) dv \Bigg],
\end{aligned}
\tag{2.4}
$$

where the last step follows by substituting $u = \exp\{(\mu - yz)/\sigma\}$ and $v = \exp\{(\mu + yz)/\sigma\}$. The result in (2.1) follows from (2.4) by using the definitions of the incomplete gamma functions defined in Section 1. Setting $\theta = 0$ gives the result in (2.2). Replace θ by $-\theta$ in (2.1) to obtain the result in (2.3). □

Consider the standard Gumbel and Laplace distributions with $\mu = 0$ and $\theta = 0$. In this case, (2.2) reduces to the simpler form

$$F(z) = \frac{\lambda\sigma}{z}\left\{\gamma\left(\frac{\lambda\sigma}{z},1\right) - \Gamma\left(-\frac{\lambda\sigma}{z},1\right)\right\}. \tag{2.5}$$

Note that (2.5) depends only on $\lambda\sigma$, the product of the scale parameters. Furthermore, (2.5) reduces to degenerate distributions in the limiting cases $\lambda\sigma \to 0$ and $\lambda\sigma \to \infty$.

Figure 2.1 above illustrates possible shapes of the pdf of (2.5) for a range of values of $\lambda\theta$. The effect of the parameter is evident. The parameters μ and θ simply control the location of the pdf of $Z = |X/Y|$.

3. Percentiles

In this section, we provide tabulations of percentage points z_p associated with the cdf (2.5) of $Z = |X/Y|$. These values are obtained by numerically solving the equation

$$\frac{\lambda\sigma}{z_p}\left\{\gamma\left(\frac{\lambda\sigma}{z_p},1\right) - \Gamma\left(-\frac{\lambda\sigma}{z_p},1\right)\right\} = p. \tag{3.1}$$

Evidently, this involves computation of the incomplete gamma functions and routines for this are widely available. We used the function GAMMA $(\cdot,\cdot)$ in the algebraic manipulation package, MAPLE. Table 3.1 provides the numerical values of z_p for $\lambda\sigma = 0.1, 0.2, \ldots, 5$.

We hope these numbers will be of use to the practitioners mentioned in the introduction. Similar tabulations could be easily derived for other values of $\lambda\sigma$ and p by using the GAMMA $(\cdot)$ function in MAPLE. A sample program is shown in the appendix below.

Appendix

The following program in MAPLE can be used to generate tables similar to that presented in Section 3.

```
p:=lambda*sigma:
ff:=(p/z)*(GAMMA(p/z)-GAMMA(p/z,1)-GAMMA(-p/z,1)):
p1:=fsolve(ff=0.01,z=0..1000):
p2:=fsolve(ff=0.05,z=0..1000):
p3:=fsolve(ff=0.1,z=0..1000):
p4:=fsolve(ff=0.90,z=0..1000):
p5:=fsolve(ff=0.95,z=0..1000):
p6:=fsolve(ff=0.99,z=0..1000):
print(p,p1,p2,p3,p4,p5,p6).
```

Table 3.1. Percentage points of $Z = |X/Y|$.

$\lambda\sigma$	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	0.001351115	0.006812125	0.01381703	0.9186777	1.942657	10.00132
0.2	0.002712925	0.01359741	0.02769946	1.832344	3.841862	20.18182
0.3	0.004148245	0.02036646	0.04128311	2.755540	5.816555	30.3209
0.4	0.005349738	0.02722697	0.05517113	3.673644	7.737383	40.45507
0.5	0.00674141	0.0341019	0.0692158	4.597857	9.638007	49.83905
0.6	0.008188162	0.04105924	0.08344926	5.515774	11.61478	60.44436
0.7	0.009696907	0.04762376	0.0966465	6.3995	13.44198	70.90835
0.8	0.01068737	0.05465819	0.1110102	7.373572	15.57044	81.81201
0.9	0.01209260	0.06075592	0.1239814	8.212888	17.25587	89.18965
1	0.01367985	0.06800081	0.1383829	9.189068	19.33069	100.2222
1.1	0.01486944	0.07500787	0.1519422	10.15826	21.43559	111.9211
1.2	0.01600875	0.08152697	0.1657749	11.04029	23.12398	120.1613
1.3	0.0175391	0.08907592	0.1799106	11.99019	25.08107	129.6253
1.4	0.01926416	0.09644032	0.1949094	12.90918	27.32613	142.1615
1.5	0.02046504	0.1026064	0.2077930	13.74593	29.08487	150.0171
1.6	0.02182045	0.1097083	0.2217979	14.71370	30.85967	160.568
1.7	0.02300527	0.1157000	0.2350247	15.66953	32.95361	172.6839
1.8	0.02429262	0.1222268	0.2482944	16.48039	34.86024	175.9620
1.9	0.02525058	0.1286303	0.2629522	17.48526	36.72677	191.7719
2	0.02725426	0.1356990	0.275934	18.44431	38.85844	205.2303
2.1	0.02857707	0.1440879	0.2913955	19.22595	40.47796	212.9673
2.2	0.03012728	0.1499321	0.3045255	20.19328	42.291	219.4435
2.3	0.03129156	0.1566246	0.3178748	21.20539	44.73074	235.1875
2.4	0.03281368	0.1638796	0.3324096	22.00995	46.21768	239.447
2.5	0.03387920	0.1712344	0.3475101	23.07892	48.56283	250.0943
2.6	0.03593847	0.1789796	0.3625134	24.01946	50.68632	264.2627
2.7	0.03653574	0.1842649	0.3737424	24.89275	52.17294	269.2083
2.8	0.03844623	0.1916820	0.3879259	25.66422	54.3857	284.8752
2.9	0.03888073	0.1967611	0.4003512	26.68252	56.19697	297.0643
3	0.04132923	0.2046606	0.415163	27.57299	58.0705	303.4964
3.1	0.04208519	0.21221	0.4283762	28.48251	59.96408	313.4921
3.2	0.04330353	0.2184272	0.4438443	29.36639	61.54506	315.8305
3.3	0.0444831	0.2250572	0.4574535	30.28838	63.70593	334.9964
3.4	0.04575357	0.2309297	0.4685282	31.27244	66.05638	342.3006
3.5	0.04712695	0.2398521	0.4842837	32.50126	68.54356	357.3495
3.6	0.04930681	0.2469442	0.5021143	33.12675	69.79813	361.953
3.7	0.05030483	0.2544659	0.5145381	34.06142	71.63831	372.2156
3.8	0.05092874	0.2587068	0.5269603	34.93049	73.13104	385.4243
3.9	0.0530573	0.2653743	0.5400184	35.68201	74.82779	391.4862
4	0.05417395	0.2727553	0.5535613	36.68836	76.95862	406.8885
4.1	0.05503099	0.280475	0.5696706	37.69073	79.4004	411.0571
4.2	0.05783188	0.2876157	0.5805482	38.61597	81.48548	426.8407
4.3	0.05874114	0.2939004	0.5946736	39.53981	83.52469	437.0593
4.4	0.05921993	0.3006934	0.6121295	40.38349	85.50906	449.4799
4.5	0.06238164	0.3073551	0.6228517	41.57249	87.40623	454.776
4.6	0.06285294	0.3153175	0.638604	42.31586	88.88288	461.6626
4.7	0.06354897	0.3213952	0.651258	43.36174	91.12971	467.8765
4.8	0.0654103	0.3283733	0.66615	44.29262	92.92313	478.6359
4.9	0.0677629	0.3356406	0.6803458	44.89927	94.93208	492.1779
5	0.06880682	0.3424891	0.6953104	46.10605	96.7116	497.5622

Acknowledgments

The author would like to thank the referee and the editor for carefully reading the paper and for their help in improving the paper.

References

[1] A. P. Basu and R. H. Lochner, *On the distribution of the ratio of two random variables having generalized life distributions*, Technometrics **13** (1971), 281–287.

[2] D. L. Hawkins and C.-P. Han, *Bivariate distributions of some ratios of independent noncentral chi-square random variables*, Communications in Statistics. A. Theory and Methods **15** (1986), no. 1, 261–277.

[3] P. J. Korhonen and S. C. Narula, *The probability distribution of the ratio of the absolute values of two normal variables*, Journal of Statistical Computation and Simulation **33** (1989), no. 3, 173–182.

[4] S. Kotz and S. Nadarajah, *Extreme Value Distributions: Theory and Applications*, Imperial College Press, London, 2000.

[5] G. Marsaglia, *Ratios of normal variables and ratios of sums of uniform variables*, Journal of the American Statistical Association **60** (1965), 193–204.

[6] T. Pham-Gia, *Distributions of the ratios of independent beta variables and applications*, Communications in Statistics. Theory and Methods **29** (2000), no. 12, 2693–2715.

[7] S. J. Press, *The t ratio distribution*, Journal of the American Statistical Association **64** (1969), 242–252.

[8] S. B. Provost, *On the distribution of the ratio of powers of sums of gamma random variables*, Pakistan Journal of Statistics **5** (1989), no. 2, 157–174.

[9] S. M. Shcolnick, *On the ratio of independent stable random variables*, Stability Problems for Stochastic Models (Uzhgorod, 1984), Lecture Notes in Math., vol. 1155, Springer, Berlin, 1985, pp. 349–354.

Saralees Nadarajah: Department of Statistics, University of Nebraska, Lincoln, NE 68583, USA
E-mail address: snadaraj@unlserve.unl.edu