QUOTIENTS OF GAUSSIAN PRIMES

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Abstract. It has been observed many times, both in the Monthly and elsewhere, that the set of all quotients of prime numbers is dense in the positive real numbers. In this short note we answer the related question: "Is the set of all quotients of Gaussian primes dense in the complex plane?"

Quotient sets $\{s/t : s,t \in \mathbb{S}\}$ corresponding to subsets $\mathbb{S}$ of the natural numbers have been intensely studied in the Monthly over the years [1,4,7,8,10,13]. Moreover, it has been observed many times in the Monthly and elsewhere that the set of all quotients of prime numbers is dense in the positive reals (e.g., [2, Ex. 218], [3, Ex. 4.19], [8, Thm. 4], [4, Cor. 5], [11, Ex. 7, p. 107], [12, Thm. 4], [13, Cor. 2]).

In this short note we answer the related question: *"Is the set of all quotients of Gaussian primes dense in the complex plane?"* The author became convinced of the nontriviality of this problem after consulting several respected number theorists who each admitted not seeing a simple solution.

In the following, we refer to the traditional primes $2,3,5,7,\ldots$ as *rational primes*, remarking that a rational prime p is a *Gaussian prime* (i.e., a prime in the ring $\mathbb{Z}[i] := \{a+bi : a,b \in \mathbb{Z}\}$ of *Gaussian integers*) if and only if $p \equiv 3 \pmod 4$. In general, a nonzero Gaussian integer is prime if and only if it is of the form $\pm p$ or $\pm pi$ where p is a rational prime congruent to $3 \pmod 4$ or if it is of the form $a+bi$ where a^2+b^2 is a rational prime (see Figure 1). We refer the reader to [5] for complete details.

Theorem. *The set of quotients of Gaussian primes is dense in the complex plane.*

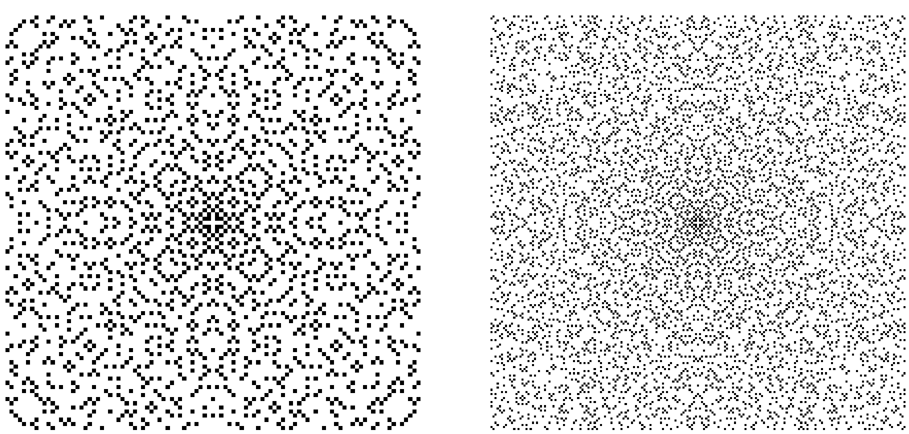

Figure 1. Gaussian primes $a+bi$ satisfying $|a|,|b| \leq 50$ and $|a|,|b| \leq 100$, respectively.

Proof. It suffices to show that each region of the form

$$\{z \in \mathbb{C} : \alpha < \arg z < \beta,\ r < |z| < R\}. \tag{1}$$

contains a quotient of Gaussian primes.

We first claim that if $0 < a < b$, then for sufficiently large real x, the open interval (xa, xb) contains a rational prime congruent to $3 \pmod 4$. Let $\pi_3(x)$ denote the number of rational primes congruent to $3 \pmod 4$ which are $\leq x$. By the Prime Number Theorem for Arithmetic Progressions [5, Thm. 4.7.4],

$$\lim_{x\to\infty} \frac{\pi_3(x)}{x/\log x} = \frac{1}{2},$$

whence

$$\begin{aligned}
\lim_{x\to\infty} [\pi_3(xb) - \pi_3(xa)] &= \lim_{x\to\infty} \pi_3(xb)\left[1 - \frac{\pi_3(xa)}{\pi_3(xb)}\right] \\
&= \lim_{x\to\infty} \pi_3(xb)\left[1 - \frac{xa\log xb}{xb\log xa}\right] \\
&= \left(1 - \frac{a}{b}\right)\lim_{x\to\infty} \pi_3(xb) \\
&= \infty,
\end{aligned}$$

which establishes the claim.

Next observe that the sector $\alpha < \arg z < \beta$ contains Gaussian primes of arbitrarily large magnitude. This follows from an old result of I. Kubilyus (illustrated in Figure 2) which states that the number of Gaussian primes γ satisfying $0 \leq \alpha \leq \arg\gamma \leq \beta \leq 2\pi$ and $|\gamma|^2 \leq u$ is

$$\frac{2}{\pi}(\beta - \alpha)\int_2^u \frac{dx}{\log x} + O\left(u\exp(-b\sqrt{\log u})\right) \tag{2}$$

where $b > 0$ is an absolute constant [9] (see also [6, Thms. 2,3]).

ρ	N	K
100	50	53
500	946	940
1,000	3,327	3,346
5,000	66,712	66,651
10,000	245,085	245,200
25,000	1,384,746	1,385,602
50,000	5,168,740	5,167,941

(A) $\frac{\pi}{24} \leq \arg z \leq \frac{2\pi}{47}$

ρ	N	K
1,000	0	5
5,000	0	100
10,000	369	367
50,000	7,823	7,732
100,000	28,964	28,971
250,000	167,197	167,099
500,000	632,781	631,552

(B) $\frac{\pi}{31415} \leq \arg z \leq \frac{2\pi}{31415}$

FIGURE 2. The number N of Gaussian primes in the specified sector with $|z| < \rho$, along with the corresponding estimate K (rounded to the nearest whole number) provided by (2)

Putting this all together, we conclude that there exists a Gaussian prime γ in the sector $\alpha < \arg z < \beta$ whose magnitude is large enough to ensure that

$$\pi_3\left(\frac{|\gamma|}{r}\right) - \pi_3\left(\frac{|\gamma|}{R}\right) \geq 2.$$

This yields a rational prime $q \equiv 3 \pmod 4$ such that

$$\frac{|\gamma|}{R} < q < \frac{|\gamma|}{r}.$$

Since q is real and positive, it follows that $r < |\frac{\gamma}{q}| < R$ and $\alpha < \arg \frac{\gamma}{q} < \beta$ so that γ/q is a quotient of Gaussian primes which belongs to the desired region (1). $\square$

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References

[1] J. Bukor and J. T. Tóth, On accumulation points of ratio sets of positive integers, *Amer. Math. Monthly* **103** (1996) 502–504.
[2] J.-M. De Koninck and A. Mercier, *1001 Problems in Classical Number Theory*, American Mathematical Society, Providence, RI, 2007.
[3] B. Fine and G. Rosenberger, *Number Theory: An Introduction via the Distribution of Primes*, Birkhäuser, Boston, 2007.
[4] S. R. Garcia, V. Selhorst-Jones, D. E. Poore, and N. Simon, Quotient sets and Diophantine equations, *Amer. Math. Monthly* **118** (2011) 704–711.
[5] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, sixth ed., Oxford University Press, Oxford, 2008, Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.
[6] G. Harman and P. Lewis, Gaussian primes in narrow sectors, *Mathematika* **48** (2001) 119–135 (2003).
[7] S. Hedman and D. Rose, Light subsets of $\mathbb{N}$ with dense quotient sets, *Amer. Math. Monthly* **116** (2009) 635–641.
[8] D. Hobby and D. M. Silberger, Quotients of primes, *Amer. Math. Monthly* **100** (1993) 50–52.
[9] I. Kubilyus, The distribution of Gaussian primes in sectors and contours, *Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk* **137(19)** (1950) 40–52.
[10] A. Nowicki, Editor's endnotes, *Amer. Math. Monthly* **117** (2010) 755–756.
[11] P. Pollack, *Not Always Buried Deep: A Second Course in Elementary Number Theory*, American Mathematical Society, Providence, RI, 2009.
[12] P. Ribenboim, *The Book of Prime Number Records*, 2nd ed., Springer-Verlag, New York, 1989.
[13] P. Starni, Answers to two questions concerning quotients of primes, *Amer. Math. Monthly* **102** (1995) 347–349.

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