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Quotients of reflexive modules

by

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Abstract. Ist M ein R-Rechtsmodul und S ein fester R-Bimodul, so heißt M in Verallgemeinerung einer wohlbekannten Begriffsbildung S-reflexiv, wenn die kanonische Abbildung $\sigma\colon M \to M^{**}$ ein Isomorphismus ist, wobei $M^* = \mathrm{Hom}_R(M, S)$ ist. Beispiele hierfür sind endlich erzeugte K-Vektorräume mit $R = S = K$ oder $M = S^I$ (I eine nicht meßbare Indexmenge) mit $S \subsetneq \boldsymbol{Q}$ und $R = \boldsymbol{Z}$. Es werden Moduln G der Form $0 \to K \to Y \to X \to G \to 0$ mit S-reflexiven X, Y untersucht. Ist ferner jeder Teilmodul von X^* projektiv, so gilt der allgemeine Satz: $G \cong D \oplus \mathrm{Ext}^1_R(A, S)$, wobei D ein direkter Summand von Y ist und $K \cong \mathrm{Hom}_R(A, S)$. Aus diesem Resultat lassen sich viele interessante Spezialfälle herleiten: Ist R ein schlanker Ring (z.B. ein abzählbarer Dedekindring, der kein Körper ist), so erfüllen die cartesischen Potenzen von R die Voraussetzungen des Satzes. Sind X, Y, S abelsche Gruppen, so heiße (X, Y) BEŁZ-Paar bezüglich S, falls X und Y S-reflexiv sind, $\mathrm{Ext}(X^*, S) = 0$ ist und ferner für jeden Homomorphismus $f\colon X \to Y$ der Annihilator $f(X)^\perp$ von $f(X)$ in Y^* ein direkter Summand von Y^* ist. Obiges Resultat läßt sich nun sofort auf BEŁZ-Paare anwenden. Ist G schlank und $\mathrm{End}(G) \cong \boldsymbol{Z}$, so ist (G^I, G^J) ein BEŁZ-Paar bezüglich G. Ebenso ist $(\boldsymbol{Z}^I, R)$ ein BEŁZ-Paar bezüglich $\boldsymbol{Z}$, falls R aus der Reid-Klasse (Kleinste Klasse, die $\boldsymbol{Z}$ enthält und unter $\oplus$ und $\prod$ abgeschlossen ist) ist.

Ferner wird gezeigt: Ist R ein schlanker Dedekindring und $A \subset R^J$, so sind im Universum V = L äquivalent: (1) $A \cong R^I$ oder A projektiv und endlich erzeugt. (2) A ist direkter Summand von R^J (3) $R^J/A \cong R^I$ oder R^J/A ist projektiv und endlich erzeugt.

§ 1. Introduction. Let M^I be the cartesian power (over I) of some right R-module $M = M_R$, i.e. the set of all functions on the set I with values in M and the scalar multiplication and addition defined by components. The aim of this paper is, to obtain informations about the structure of quotient R-modules G of the form

$$(*) \qquad 0 \to K \to X \to Y \to G \to 0$$

with certain conditions posed on X and Y as explained in the following. In M. Dugas and R. Göbel [6; Satz p. 15] the structure of G was determined explicitly in a model of ZFC+V = L where $K = 0$, $R = \boldsymbol{Z}$ and X, Y are isomorphic to cartesian powers of $\boldsymbol{Z}$ over any non measurable sets; cf. Remark after (3.5). In particular we get

$$(**) \qquad G \cong D \oplus \mathrm{Ext}_R(A, S) \text{ where } D \text{ is a direct summand of } Y \text{ and } K = \mathrm{Hom}_R(A, S)$$
for a R-bimodule S and some left R-module A.

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In this case we have $S = Z$, and D is a cartesian power of Z as follows from R. J. Nunke [25; p. 69, Theorem 5]. Using the methods in [6] and a result of B. Charles [3; p. 35, Theorem 7], which actually goes back to S. Balcerzyk, A. Białynicki-Birula and J. Łoś [2; p. 453, Theorem 1], M. Huber [17; Theorem 2.3] extended [6]: In the abelian case $R = Z$ condition (*) implies (**) if X, Y are isomorphic to cartesian powers over non measurable sets of a subgroup $S \subsetneqq Q$. During an Oberwolfach-conference R. Baer asked in which *other* cases (*) implies the splitting theorem (**). Hence we will consider the appropriate objects X and Y, which are S-reflexive modules with respect to some R-bimodule S; cf. J. Dieudonné [4] and in particular F. Kasch [21; § 12] if $S = R$. Recall that X is S-reflexive, if the canonical homomorphism σ_X from X into the double-dual X^{**S} is an isomorphism, where $X^{*S} = \mathrm{Hom}_R(X, S)$; cf. § 2. Notice that X^{**S} is a right R-module if $X = X_R$. *Then (*) implies (**) if X and Y are S-reflexive such that each submodule of X^{*S} is projective*; Theorem 3.3. Using results of O. Gerstner, L. Kaup, H. G. Weidner [11], G. Heinlein [13], H. L. Hiller, M. Huber and S. Shelah [15], I. Kaplansky [20] and E. L. Lady [23], the structure of G in (**) can be determined for Dedekind domains in many cases; cf. (3.5) and remark after (3.5). It seems to be interesting to remark, that *the quotient field $Q(R)$ of any countable Dedekind domain $R \neq Q(R)$ can never be obtained as quotient "R^J/R^I" for any non measurable cardinals $|I|$ and $|J|$ and arbitrarily (wild) embeddings $0 \to R^J \to R^I$*. This holds in particular for $R = Z$, *without* using $V = L$ or any other peculiar set theory. The countability of R is necessary, as follows from an example due to C. U. Jensen [19; p. 217]. Along this line, following some ideas and problems of R. J. Nunke [25], we will give a new characterization of direct summands of R^J for slender Dedekind domains R in the model of ZFC+V = L, which extends and sharpens R. J. Nunke [25; p. 69, Theorem 5]: *A is a direct summand of R^J if and only if $R^I/A \cong R^I$ for some I or R^J/A is finitely generated and projective*; cf. (3.9). The set theoretic assumption cannot be abolished as shown already in R. Göbel and M. Dugas [6] in the case $R = Z$.

Next we will apply our Theorem 3.3 in the case $R = Z$ of abelian groups and will show that (3.3) includes the results mentioned above and has further interesting "abelian consequences". Hence we will introduce the useful notion of a *BEŁZ-pair (X, Y) with respect to some abelian group S, characterized by the conditions*

(1) *X and Y are S-reflexive,*

(2) $\mathrm{Ext}_Z(X^{*S}, S) = 0$,

(3) *if $f\colon X \to Y$ is a homomorphism, then the annihilator $f(X)^{\perp}$ of $f(X)$ in Y^{*S} is a direct summand of Y^{*S}.*

The letters "BEŁZ" stand for the initials of the authors Balcerzyk, Białynicki-Birula, Ehrenfeucht, Łoś and Zeeman of [2], [7] and [33] who showed first that (Z^I, Z^J) and more generally (G^I, G^J) with $G \subsetneqq Q$ are BEŁZ-pairs with respect to G for non measurable cardinals $|I|$, $|J|$. *Then (*) implies (**) for BEŁZ-pairs (X, Y) with respect to some group S*; cf. (4.2).

Hence we are left to determine BEŁZ-pairs with respect to some groups S: *Since (S^I, S^J) for $S \subsetneqq Q$ and non measurable $|I|$, $|J|$ are BEŁZ-pairs*, we get the known splitting theorems of [6] and [17]. If $\mathfrak{R}$ *is the Reid-class*, i.e. the smallest class of abelian groups, containing Z and being closed under taking direct sums and cartesian products, *then (Z^I, R) for $R \in \mathfrak{R}$ and non measurable cardinals $|I|$ are BEŁZ-pairs, with respect to Z. There are many further examples G of any rank, in particular of rank 2: (G^I, G^J) for non measurable cardinals with $\mathrm{End}_Z(G) = Z$ and slender G are BEŁZ-pairs*, cf. (4.5). In (4.2) epimorphic images of Z^J modulo subgroups which are epimorphic images of cartesian products of Z are determined. Hence it is natural to ask for the structure of arbitrary epimorphic images of cartesian powers of Z. *We will give an example, which shows that such groups in general do not split into a direct sum $U \oplus V$ with $U^* = 0$ and V torsion-less.* Therefore any obvious generalization of the famous theorem of R. J. Nunke [26; p. 70 Theorem 5] that epimorphic images of Z^N are of this form unfortunately never holds.

§ 2. Reflexive and slender modules. In this section we will give our basic definitions and state some of the known results which will be applied in § 3 and § 4. We will use the convention ${}_RM$, M_R, ${}_RM_R$ for left, right or bi-R-modules M and put M in one of these cases if there is no ambiguity. In general we will start with M_R and obtain a left R-module $M^* = M^{S^*} = \mathrm{Hom}_R(M, S)$ if $S = {}_RS_R$ is a fixed R-bimodule. If $M = {}_RM$ is a left R-module, put $M^* = M^{S^*} = \mathrm{Hom}_R(M, S)$ for the right R-module and $X^{*SS*} = X^{**S} = X^{**}$ for $X = X_R$. If $\varrho\colon X_R \to Y_R$ is an R-homomorphism, the canonical homomorphisms $\varrho^*\colon Y^* \to X^*$ and $\sigma_Y\colon Y \to Y^{**}$ are defined by components:

$$\varrho^*(\varphi) = \varphi \cdot \varrho \quad \text{and} \quad \sigma_Y(y)(\varphi) = \varphi(y) \quad \text{for all } y \in Y \text{ and } \varphi \in Y^* .$$

Then the exact sequence $0 \to A \to B \to C \to 0$ implies the contravariant Cartan–Eilenberg-sequence

$$0 \to C^* \to B^* \to A^* \to \mathrm{Ext}_R(C, S) \to \mathrm{Ext}_R(B, S) \to \mathrm{Ext}_R(A, S)$$

and the corresponding covariant sequence, where $\mathrm{Ext}_R = \mathrm{Ext}_R^1$; cf. P. Hilton and U. Stambach [16; p. 100, 102]. If $X \subseteq Y$, we define $X^{\perp S} = X^{\perp} = \{\varphi \in Y^{*S};\ \varphi(X) = 0\}$ which is the *annihilator* of X in Y and $X^{\perp\perp S} = X^{\perp\perp} = \bigcap_{\varphi \in X^{\perp S}} \ker(\varphi)$. The module Y will be called *S-reflexive* if σ_Y is an isomorphism. The investigation of rings R where all finitely generated modules are R-reflexive goes back to J. Dieudonné [4] and they are characterized by F. Kasch; cf. F. Kasch [21; § 12]. Reflexivity for modules which are not finitely generated comes in effectively first with investigations of E. Specker [30], E. C. Zeeman [33] and A. Ehrenfeucht and J. Łoś [7]; cf. L. Fuchs [10; § 94]. Hence we call $M = M_R$ an *R-slender* module if for any homomorphism $\varrho\colon R^N \to M$ there is a cofinite subset C of N such that $(R^C)^{\varrho} = 0$ (where $R^C \subseteq R^N$ in the canonical way). In particular, R is slender if R_R is slender. *In the following all index sets I and J will be of cardinality less than the first measurable cardinality $\aleph_\beta$.* A cardinal $|W|$ is measurable if there is a countably additive measure on W with the values 0 and 1 which is 0 on the elements of W and 1 on W. Measurable cardinals are very large cardinals; cf. F. R. Drake [5; pp. 173–199]. They do not even exist in a constructible universe (with V = L), as shown by D. S. Scott [31]. Following J. Łoś, *S is R-slender if the canonical monomorphism* $\bigoplus_{i\in I} M_i^{*S} \to (\prod_{i\in I} M_i)^{*S}$ *is bijective for any set I* (of non measurable cardinality) and any modules M_i $(i \in I)$; cf. O. Gerstner, L. Kaup and H. G. Weidner [11; p. 506, Satz 3] (= GKW in the following) or D. Allouch [1; p. 13, Theorem 2.2]. For slender modules S we will use without any further reference the properties

(i) $$(\prod_{i\in I} M_i)^{**} \cong \prod_{i\in I} M_i^{**S},$$

(ii) $$(\bigoplus_{i\in I} M_i)^{**S} \cong \bigoplus_{i\in I} M_i^{**S},$$

cf. D. Allouch [1; p. 13, Proposition 2.5], G. Heinlein [13, p. 5, Satz], E. L. Lady [23; p. 403, Theorem 4] or GKW [11; p. 508, Korollar 5]. Hence S-reflexivity for slender S is hereditary with respect to taking direct sums, cartesian products and (obviously) direct summands.

In the case of Dedekind domain R, slender rings are characterized by G. Heinlein [13; p. 79, Satz 7.34]: *For a Dedekind domain R are equivalent*:

(1) *R is not slender,*

(2) *R is a local ring and complete with respect to the* Jac(R)*-adic topology,*

(3) $\mathrm{Ext}_R(Q(R), R) = 0$ *for the quotient field $Q(R)$ of R.*

Furthermore there is a result which generalizes E. Sąsiada [28]:

THEOREM. *A countable R-module S is slender if there is a set $\mathfrak{T}$ of ideals of R such that*

(a) $X^N = X \cdot R^N$ *for all finite intersection X of ideals in $\mathfrak{T}$,*

(b) *S is a Hausdorff space with respect to the $\mathfrak{T}$-adic topology.*

Remark. Condition (a) is equivalent with

(c) *If $U \in \mathfrak{T}$ and V is an ideal of R generated by at most countably many elements from U, there is a finitely generated ideal F of R with $V \subseteq F \subseteq U$.*

A countable Dedekind domain R is slender if there are $\aleph_0$ ideals of R with $X^N = X \cdot R^N$ for all finite intersections X and 0 for infinite intersections. The proofs of this theorem and the remark may be found in G. Wittkamp [32]. They are modifications of E. L. Lady [23; p. 399, Theorem 1, p. 400, Theorem 2], where the proofs rest on the wrong statement [23; p. 398] that the "$\mathfrak{M}$-strong topology" is complete. There is a counterexample $K[X]$ of all polynomials over a field K with infinitely many variables X. Take $\mathfrak{M} = \{V_0\}$ where V_0 is the ideal of all polynomials f with $f(0) = 0$. Furthermore, there are very nice and unusual slender rings constructed in G. Heinlein [13], which are all real-valued C^r-functions on certain manifolds.

In the following we will use the notations $X \subseteq Y$ and $X \sqsubseteq Y$ for submodules and direct summands respectively.

§ 3. Splitting for reflexive modules. Using the notations of § 1 and § 2 we will show the following

LEMMA 3.1. *Let R be any ring and S an R-bimodule. For a submodule A of an S-reflexive module X are equivalent*:

(1) $A^{\perp S} \sqsubset X^{*S}$,

(2) $A^{\perp\perp S} \sqsubset X$.

Furthermore, the quotient $X^{*S}/A^{\perp S}$ is always a submodule of A^{*S}.

Proof. (1) → (2): There is a submodule B of X^{*S} such that $X^{*S} = A^{\perp S} \oplus B$. Hence X^{**S} splits into $U \oplus V$ with

$$U = \{\varphi \in X^{**S};\ \varphi(B) = 0\} \quad \text{and} \quad V = \{\varphi \in X^{**S};\ \varphi(A^{\perp S}) = 0\}.$$

Since $\sigma_X(A^{\perp\perp S})(A^{\perp S}) = A^{\perp S}(A^{\perp\perp S}) = 0$, we get $\sigma_X(A) \subseteq \sigma_X(A^{\perp\perp S}) \subseteq V$. Conversely, if $\sigma_X(x)(A^{\perp S}) = 0$ for some $x \in X$, then $A^{\perp S}(x) = 0$. Therefore $V = \sigma_X(A^{\perp\perp S})$ and we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & X & \to & X/A & \to & 0 \\ & & \downarrow{\scriptstyle \sigma_{X|A}} & & \downarrow{\scriptstyle \sigma_X} & & \downarrow & & \\ 0 & \to & V & \to & X^{**S} & \to & U & \to & 0 \end{array}$$

where $\sigma_{X|A}$ is the restriction of σ_X to A. The module X is reflexive, hence σ_X is bijective and $A^{\perp\perp S} = \sigma_X^{-1}(V) \sqsubset X$.

(2) → (1): There is a submodule C of X such that $X = A^{\perp\perp S} \oplus C$. Hence X^{*S} splits into $U \oplus V$ with $U = \{\varphi \in X^{*S};\ \varphi(A^{\perp\perp S})\} = 0$ and $V = \{\varphi \in X^{*S};\ \varphi(C)\} = 0$. Therefore $A^{\perp S}(A^{\perp\perp S}) = 0$ by definition. Conversely, if $\varphi \in U$ then $\varphi(A) = 0$ from $A \subset A^{\perp\perp S}$. Therefore $A^{\perp S} = U$. The module $X^{*S}/A^{\perp S}$ is a submodule of A^{*S}, as follows from our construction.

LEMMA 3.2. *Let S be an R-bimodule. If $Y \subseteq X$ are S-reflexive such that $Y^{\perp\perp S} = X$, there is an exact sequence*

$$0 \to X/Y \to \mathrm{Ext}_R(B, S) \to \mathrm{Ext}_R(Y^{*S}, S) \to \mathrm{Ext}_R(X^{*S}, S)$$

where $B = Y^{*S}/X^{*S}$ is an S^*-module and $B^{S^*} = 0$.

Proof. There is an exact sequence $0 \to X^{*S} \to Y^{*S} \to B \to 0$ because of $Y^{\perp\perp S} = X$. Now we obtain the following commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \to & Y & \to & X & \to & X/Y & \to & 0 & & & & \\ & & \downarrow{\scriptstyle \sigma_Y} & & \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \pi} & & & & & & \\ 0 \to B^{S^*} & \to & Y^{**S} & \to & X^{**S} & \to & \mathrm{Ext}_R(B, S) & \to & \mathrm{Ext}_R(Y^{*S}, S) & \to & \mathrm{Ext}_R(X^{*S}, S) & & \end{array}$$

where π is induced from σ_Y and σ_X. Since X and Y are S-reflexive, σ_X and σ_Y are isomorphisms, $B^{S^*} = 0$ and π is a monomorphism.

THEOREM 3.3. *Let X and Y be S-reflexive right R-modules such that each submodule of X^{*S} is projective. If $0 \to K \to X \to Y \to G \to 0$ is an exact sequence, there is a left R-module A which is an epimorphic image of X^{*S} and a direct summand D of Y, such that $G \cong D \oplus \mathrm{Ext}_R(A, S)$ and $K = \mathrm{Hom}_R(A, S) = A^{S^*}$.*

Remark. The assumption "each submodule of X^{*S} is projective" can be replaced by $f(X)^{\perp} \sqsubset Y^{*S}$ and $\mathrm{Ext}_R(X^{*S}, S) = 0$, if f is the given homomorphism from X into Y.

Proof. Let $f^*\colon Y^* \to X^*$ be the homomorphism induced from $f\colon X \to Y$ of the given exact sequence. Furthermore put $U = \operatorname{im}(f) \subseteq Y$ and $B = \operatorname{im}(f^*) \subseteq X^*$. Hence B is projective by our assumption on X^*. If $i\colon U \to Y$ is the identity of Y restricted to U, we get

$$\operatorname{im}(i^*) = \{\sigma \in U^*;\ \exists \tau \in Y^* \text{ with } \tau|_U = \sigma\}.$$

Next we define a map $\varrho\colon \operatorname{im}(i^*) \to B$ by $\varrho(\sigma) = f\cdot\tau$ if and only if $\tau_{|U} = \sigma$. One easily shows that ϱ is a well defined isomorphism and $\operatorname{im}(i^*)$ is projective, since B is projective. If $j\colon B \to X^*$ is the inclusion, we get a short exact sequence $0 \to B \xrightarrow{j} X^* \to A \to 0$ with $A = X^*/B$. Dualizing this sequence we obtain the diagram

$$\begin{array}{ccccccccc} 0 \to & B^\perp & \to & X^{**} & \to & B^* & \to \operatorname{Ext}_R(A, S) \to \operatorname{Ext}_R(X^*, S) = 0 \\ & \uparrow \alpha & & \uparrow \sigma_X & & \uparrow \gamma & \\ 0 \to & K & \to & X & \xrightarrow{f} & U & \to 0 \end{array}$$

where α is the restriction of the homomorphism σ_X and γ is the homomorphism induced by α and σ_X. First we show that $\alpha\colon K \to B^\perp$ is an isomorphism:

If $k \in K$ we get $k^\alpha(B) = \sigma_X(k)(B) = B(k) = k^{f\cdot Y^*} = 0$ since $k \in K = \ker(f)$. Hence we get $K^\alpha \subseteq B^\perp$. Conversely, let $x \in X$ such that $\sigma_X(x) \in B^\perp$. Then we get $0 = \sigma_X(x)(B) = B(x) = (x^f)^{Y^*}$ and $x^f = 0$ since σ_Y is injective, i.e. $x \in \ker(f) = K$.

Therefore σ_X and α in our diagram are isomorphism and we get $\operatorname{Ext}_R(A, S) \cong B^*/U^\gamma$ from the diagram.

Next we consider the exact sequence $0 \to U \xrightarrow{i} Y \to G \to 0$ which induces $0 \to U^\perp \to Y^* \to \operatorname{im}(i^*) \to 0$. Since $\operatorname{im}(i^*)$ is projective as shown above, the last sequence splits and $Y^* = U^\perp \oplus V$ and there is a natural isomorphism ξ from V onto $\operatorname{im}(i^*)$. Now let be $V^* = \{\varphi \in Y^{**};\ \varphi(U^\perp) = 0\}$ (which is canonically isomorphic with $\operatorname{Hom}_R(V, S)$ because of the splitting) and identify $(U^\perp)^* \subseteq Y^*$ in the same way. Then $Y^{**} = (U^\perp)^* \oplus V^*$ and we have $\sigma_Y(y)(U^\perp) = U^\perp(y) = 0$ if and only if $y \in U^{\perp\perp}$. Therefore $(U^{\perp\perp})^{\sigma_Y} = V^*$, and we get the following diagram

$$(*) \qquad \begin{array}{ccccccc} 0 \to & U & \xrightarrow{\gamma} & B^* & \to & B^*/U & \to 0 \\ & \| & & \uparrow \eta & & \uparrow \bar\eta & \\ 0 \to & U & \xrightarrow{i} & U^{\perp\perp} & \to & U^{\perp\perp}/U & \to 0 \end{array}$$

where $\eta = \sigma_Y \cdot \xi^* \cdot \varrho^{*-1}$ and $\xi^*\colon V^* \to (\operatorname{im}(i^*))^*$ and $\varrho^{*-1}\colon (\operatorname{im}(i^*))^* \to B^*$ are the isomorphisms induced from ξ and ϱ.

Next we show that $(*)$ commutes and hence induces the isomorphism $\bar\eta$ coming from the isomorphism η:

If $u \in U$, then $u^\eta\colon B \to S$ is the map $(fu \to u^v)$ for all $v \in V$, since $B = fY^* = fV$. Now let be $x \in X$ such that $x^f = u$. Then we get

$$\sigma_X(x)(fv) = x^{fv} = u^v = (fv)^{u^\eta} \text{ and } u^\gamma = \sigma_X(x)|_B.$$

Consequently we have $u^\eta = u^\gamma$ for all $u \in U$ and $(*)$ commutes.

Since $\operatorname{Ext}_R(A, S) \cong B^*/U$ from investigating the first injection j and the first diagram and $U^{\perp\perp}/U \cong B^*/U$ from the second injection i and the second diagram $(*)$, we derive with Lemma 3.1 that $G = Y/U \cong U^{\perp\perp}/U \oplus D$ and D is a direct summand of Y. Q.E.D.

Lemma 3.4. *If D is a direct summand of R^J for some slender Dedekind domain R, then D is finitely generated or $D = R^I$ for some subset I of J.*

Proof. Projective modules over Dedekind domains which are not finitely generated are free, as follows from a well-known result of I. Kaplansky [20; p. 331, Theorem 2(b)]. Therefore a result of E. L. Lady [23, p. 403, Corollary] proves the lemma.

Corollary 3.5. *Let $0 \to K \to R^J \to R^I \to M \to 0$ be an exact sequence.*

(a) *If R is a slender and hereditary ring, M decomposes into $M \cong D \oplus \operatorname{Ext}_R(A, R)$, where $D \sqsubset R^I$ and $K = A^{R^*} (= \operatorname{Hom}_R(A, R))$ for some left R-module A with $|A| \leqslant |R| \cdot |J|$.*

(b) *If R is a Dedekind domain with $\operatorname{Ext}_R(Q(R), R) \neq 0$ for the quotient field $Q(R)$ of R, then* (a) *holds and $D \cong R^T$ for some set T with $|T| \leqslant |I|$ or D is finitely generated and projective.*

Proof. The theorem of G. Heinlein [12; p. 79, (7.34) Satz] mentioned in § 2 shows that R is slender in cases (a) and (b). From Theorem 3.3 follows $M \cong D \oplus \operatorname{Ext}_R(A, R)$ with $A^* = K$ and $D \sqsubset R^I$. Lemma 3.4 now determines the structure of D in case (b).

Remark. The structure theorem that $\operatorname{Ext}_R(A, R)$ is compact for $A^* = 0$ and $R = Z$ in the model ZFC+V = L of H. L. Hiller, M. Huber and S. Shelah [15; p. 47, Theorem B] carries over to countable Dedekind domains. Hence the structure of $R^I/R^J \cong D \oplus \operatorname{Ext}(B, R)$ is determined explicitly in this case; compare M. Dugas and R. Göbel [6; Korollar p. 16].

If R is a slender Dedekind domain, there is a well known result describing the arguments of Ext from outside:

Lemma 3.6. *Let R be a slender Dedekind domain and A be an R-module with $A^* = \operatorname{Hom}_R(A, R) = 0$. If $\operatorname{Ext}_R(A, R)$ is divisible and torsion free, then A is divisible and torsion free.*

For a proof which follows immediately by application of the Cartan–Eilenberg–sequence we refer to O. Gerstner [12; Hilfssatz 9]. Therefore we get for *any* model of ZFC the following

Corollary 3.7. *If R is a countable Dedekind domain $\neq Q(R)$, the quotient field $Q(R)$ cannot be represented as R^I/R^J.*

Proof. We assume $R^I/R^J \cong Q(R)$. From Corollary 3.5(a) follows $R^I/R^J \cong \operatorname{Ext}_R(B, R) = Q(R)$ and $B^* = 0$. Hence B is divisible and torsion free, i.e. B is a $Q(R)$-vector space. Therefore $Q(R) \cong \operatorname{Ext}_R(B, R) \cong \operatorname{Ext}_R(\bigoplus_T Q(R), R) \cong \prod_T \operatorname{Ext}_R(Q(R), R)$ for some set T. Since R is slender, we get that $\operatorname{Ext}_R(Q(R), R) \neq 0$, and hence uncountable by C. U. Jensen [19; p. 217, Proposition 1]. Therefore $Q(R)$ is uncountable, which is a contradiction.

The countability of R in Corollary 3.7 is necessary as shown by C. U. Jensen [19; p. 222], who gives an example of an uncountable discrete valuation ring R with $\mathrm{Ext}_R(Q(R), R) = Q(R)$. Corollary 3.7 is an analogue of H. L. Hiller and S. Shelah [14; p. 316, Corollary]. Next we will give a characterization of direct summands of R^J, hence the following lemma will be needed several times:

LEMMA 3.8. *Let A be a finitely generated and projective R-module and R be a Dedekind domain. Then A is reflexive and A^* is finitely generated and projective.*

Proof. The module A is a direct summand of a free R-module, hence reflexive. There are R-submodules $J_1, \ldots, J_n$ of the quotient field of R such that $A = \bigoplus_{i=1}^{n} J_i$. Hence $A^* = \bigoplus_{i=1}^{n} \mathrm{Hom}(J_i, R) = \bigoplus_{i=1}^{n} J_i^{-1}$ is projective, since R is a Dedekind domain.

An R-module W is called *R-Whitehead-module*, if $\mathrm{Ext}_R(W, R) = 0$. Following an idea of S. Shelah [29], P. C. Eklof [8; p. 42, Theorem 10.8] showed that any R-Whitehead-module over a countable Dedekind domain R is projective in a model for ZFC+V = L. Furthermore there are no measurable cardinals in ZFC+V = L by D. S. Scott [31]. Hence our next theorem may be interpreted in ZFC+V = L although it is formulated more general:

THEOREM 3.9. *Let R be a slender Dedekind domain such that all R-Whitehead-modules of non measurable cardinality are projective. For a submodule A of $P = R^J$ are equivalent:*

(1) *A^* is free or finitely generated and projective and $A^{\perp\perp} = A$.*

(2) *A^* is free or finitely generated and projective and* $\bigcap_{\varphi \in (P/A)^*} \ker(\varphi) = 0$.

(3) *$A \cong R^I$ or A is finitely generated and projective and $A^{\perp\perp} = A$.*

(4) *A is a direct summand of P.*

(5) *$P/A \cong R^I$ or P/A is finitely generated and projective.*

Proof. (1) → (2): follows from the definition of $A^{\perp\perp}$.

(2) → (3): If A^* is free, we get $A = A^{\perp\perp} \sqsubset P$ from Lemma 3.1 and (3) follows from Corollary 3.7. If A^* is finitely generated and projective, $P^*/A^{\perp} \subseteq A^*$ follows from Lemma 3.1. Hence P^*/A is projective, since R is a Dedekind domain. Therefore $A^{\perp\perp} \sqsubset P$ and $A = A^{\perp\perp} \sqsubset P$ follows from Lemma 3.1. Application of Lemma 3.4 shows (3) in this case.

(3) → (4): If $A \cong R^I$, it follows from the slenderness of R that A^* is free. From Lemma 3.1 we get $A = A^{\perp\perp} \sqsubset P$. If A is finitely generated and projective, then A^* is finitely generated and projective by Lemma 3.8. Hence $A = A^{\perp\perp} \sqsubset P$ by Lemma 3.1.

(4) → (5): follows from Lemma 3.4.

(5) → (1): Let be $U = P^*/A^{\perp}$, then the diagram with the following canonical maps commutes:

$$\begin{array}{ccccccccc} 0 \to & A & \to & P & \to & P/A & \to 0 & & \\ & \downarrow{\scriptstyle \sigma_{P/A}} & & \downarrow{\scriptstyle \sigma_P} & & \downarrow{\scriptstyle \sigma_{P/A}} & & & \\ 0 \to & U^* & \to & P^{**} & \to & (A^{\perp})^* & \to & \mathrm{Ext}_R(U, R) & \to 0 \end{array}$$

Since $(A^{\perp})^* = (P/A)^{**}$ and $P, P/A$ are reflexive by (5), the maps σ_P and $\sigma_{P/A}$ are isomorphisms. Hence $\mathrm{Ext}_R(U, R) = 0$, i.e. U is an R-Whitehead-module and therefore projective by assumption. Therefore $A^{\perp\perp} \sqsubset P$ follows from Lemma 3.1. In particular we get $P/A = A^{\perp\perp}/A \oplus D$ for some submodule D of P. Since P/A is reflexive, $A^{\perp\perp}/A$ is reflexive. However, $(A^{\perp\perp}/A)^* = 0$ and therefore $A = A^{\perp\perp} \sqsubset P$. Finally we apply Lemma 3.4 to obtain (1). Q.E.D.

§ 4. The special case $R = Z$ of abelian groups. In order to apply our Theorem 3.3 in the case of abelian groups, we introduce the following useful notion for a given abelian group S:

DEFINITION 4.1. A pair (A, B) of abelian groups will be called a *BEŁZ-pair (with respect to S)* if

(1) A and B are S-reflexive,

(2) $\mathrm{Ext}_Z(A^{*S}, S) = 0$,

(3) if $f\colon A \to B$ is a homomorphism, then $f(A)^{\perp S}$ is a direct summand of B^{*S}.

The pair (Z^I, Z^J) for non mesurable $|I|$, $|J|$ is a BEŁZ-pair with respect to Z as shown independently by A. E. Ehrenfeucht and J. Łoś [7] and E. C. Zeeman [33], c.f. Fuchs [10; § 94]. This was generalized by S. Balcerzyk, A. Białynicki–Birula and J. Łoś [2] who showed that Z may be replaced by any proper subgroup of Q. Some years later, B. Charles [3] obtained the same result.

Application of (3.3) shows

COROLLARY 4.2 *Let (A, B) be a BEŁZ-pair for some S and $0 \to K \to A \to B \to G \to 0$ an exact sequence. Then we get $G \cong D \oplus \mathrm{Ext}_Z(C, S)$ with $D \sqsubset B$ and $C^{*S} \cong K$ for some group C.*

Examples for BEŁZ-pairs:

(4.3) *(G^I, G^J) with respect to G for $G \subsetneq Q$*; c.f. above.

(4.4) *(Z^I, G) with respect to Z and $G \in \mathfrak{R}$ = Reid-class*, which is defined as follows:

It is an open question of G. R. Reid [27, p. 153] whether the Reid-class $\mathfrak{R}$, which is the smallest class $\mathfrak{R}$ of abelian groups containing Z and is closed under $\bigoplus_{<\aleph_\beta}$, $\prod_{<\aleph_\beta}$, is the class of all kernel groups.

$\mathfrak{R}$-groups are Z-reflexive as follows by transfinite induction using the slenderness of Z if we introduce the following hierarchy in $\mathfrak{R}$:

(0) Z is of type 0.

(α+1) $\prod_{<\aleph_\beta} A_i$, $\bigoplus_{<\aleph_\beta} A_i$ are of type $\alpha+1$ if all A_i are of type $\leqslant \alpha$,

(λ) $\prod_{<\aleph_\beta} A_i$, $\bigoplus_{<\aleph_\beta} A_i$ are of type λ for limit ordinals λ, if all A_i are of type $<\lambda$.

(4.5) (G^I, G^J) *with respect to* G, *where* G *satisfies the conditions*

(a) $\mathrm{End}_Z(G) \cong Z$,

(b) G *is slender*,

Besides the case (4.3) there are many groups G satisfying (4.5) even if the rank of G is 2:

In particular:

$$G = \langle u, v, 5^{-\infty}(v+\sqrt[3]{3}\cdot u)\rangle \text{ or}$$
$$G = \langle u, v, 2^{-\infty}u, 3^{-\infty}v, 5^{-\infty}(u+v)\rangle \text{ or}$$
$$G = \langle u, v, 2^{-\infty}u, 7^{-\infty}(v+\sqrt{2}u)\rangle \text{ satisfy (4.5)}$$

since $\mathrm{End}(G) \cong Z$. Further examples can be obtained from Król [22]. We would like to thank Otto Mutzbauer (Universität Würzburg) for telling us these examples of rank 2, which follow easily from O. Mutzbauer [24].

Further examples of arbitrary rank are to be found in the area of "rigid systems"; c.f. L. Fuchs [10, p. 124ff].

Hence for our examples (4.3), (4.4) and (4.5) there is the splitting Theorem 4.2. In the case (4.3), D is again a product R^M; c.f. S. Balcerzyk, A. Białynicki-Birula and J. Łoś [2]. Since $(G^I)^{*G}$ is free in case (4.5), again D is of the form G^M. If $K = 0$ in (4.2), then the structure of $\mathrm{Ext}(C, S)$ in (4.2) is known in $V = L$; c.f. M. Huber [18], who shows that $\mathrm{Ext}(C, S)$ is compact in the universe $V = L$.

After determining the epimorphic images of Z^J with kernel isomorphic to an epimorphic image of Z^I (Corollary 4.2), we will consider *arbitrary* epimorphic images of Z^J:

If $|J| = \aleph_0$, epimorphic images of Z^J are isomorphic to a direct sum of some Z^M with $|M| \leqslant \aleph_0$ and a cotorsion group. This is a well-known theorem of R. J. Nunke [26, p. 70, Theorem 5]. This does not hold, if $|J| > \aleph_0$: There are epimorphic images E of Z^J such that E is $\aleph_1$-free and $E^* = 0$, as shown in G. A. Reid [27, p. 37]. On the other hand there are epimorphic images of Z^J which are torsionless (i.e. subgroups of some product Z^M) but not a product Z^K, as constructed in R. J. Nunke [25; p. 70/71]. Hence one might think that epimorphic images of products Z^J will always split in the sense of Stein into a direct sum of a summand A with dual $A^* = 0$ and a torsionless complement; c.f. Fuchs [9; p. 94; Corollary 19.3 (K. Stein)]. Unfortunately there is the following

PROPOSITION 4.6. *Let* $|I| = 2^{\aleph}$ *be a non mesurable cardinal and* H *an abelian group with* $H^* = 0$ *and* $1 < |H| \leqslant \aleph$. *There is an epimorphic image* G *of* Z^I *such that* $\mathrm{Ker}(\sigma_G) \cong H$. *If* H *is slender*, $\mathrm{Ker}(\sigma_G)$ *is not a direct summand of* G.

Proof. Following an idea of R. J. Nunke [25; pp. 70–71] one easily shows, that there is a free subgroup $F \subset Z^I$ of rank $\aleph_p$ such that $F^{\perp\perp} = F$. Take any subgroup $A \subseteq F$ such that $F/A \cong H$. Then $A^{\perp\perp} \subseteq F^{\perp\perp}$ in general and $F \subseteq A^{\perp\perp}$ since $(F/A)^* = 0$. Hence $A^{\perp\perp}/A \cong H$, which shows $H = \ker(\sigma_G)$ with $G = Z^I/A$.

If H is slender and $H \sqsubset G$, then H is a slender epimorphic image of Z^I. Therefore $0 \neq H$ is finitely generated and free, which contradicts $H^* = 0$.

COROLLARY 4.7. *Let* I *be a set of cardinality* $\geqslant 2^{\aleph_0}$. *There are epimorphic images* G *of* Z^I *such that there is no decomposition* $G = H \oplus F$ *with* $H^* = 0$ *and* σ_F *injective.*

Remark. "σ_F injective" is equivalent to F torsionless in the sense of H. Bass or to F is e.h. group in the sense of R. J. Nunke [25] or F is subgroup of some Product Z^K.

Proof of 4.7. Choose H slender with $H^* = 0$, e.g. $H = \bigoplus Q_p$ and apply Proposition 4.6.

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On pointed 1-movability and related notions

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Abstract. In this paper we discuss several problems which arose in a study of pointed 1-movability. We also prove some new theorems.

1. Introduction. The main aim of this paper is to summarize several problems in continua theory which arose in a study of pointed 1-movability and related notions. Some new results are also obtained. All spaces under discussion are at least metrizable. Terminology used is standard. The definitions of undefined terms from shape theory may be found in the book [3]. By a continuum is meant a nonvoid, compact, connected space. A one-dimensional continuum is called a curve. If N is a manifold, then $\dot{N}$ denotes its boundary and $\mathring{N}$ its interior.

Let X be a continuum lying in an ANR($\mathfrak{M}$)-space M and let x_0 be a point of X. We shall be dealing with the following properties of X:

(MOV*) (*pointed movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X which can be deformed rel. x_0 within U into any neighborhood of X [3].

(MOV) (*movability*). The same definition as above with no restriction on x_0 [3].

(1 MOV*) (*pointed 1-movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X such that each loop in (V, x_0) can be deformed within (U, x_0) into any neighborhood of X [comp. 3, 18 and 26].

(1 MOV) (1-*movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X such that for each neighborhood W of X and for each mapping $f\colon Y\to V$, where Y is a curve, there is a mapping $g\colon Y\to W$ homotopic to f in U [3].

(n1 MOV) (*nearly* 1-*movability*). For each neighborhood U of X in M there is a neighborhood $V\subset U$ of X such that for each mapping $f\colon D\to V$, where D is a 2-disk, and for each neighborhood W of X there exist a sequence $D_1, \ldots, D_k$ of disjoint disks in $\mathring{D}$ and an extension $\bar{f}\colon D\setminus\bigcup_{i=1}^{k}\mathring{D}_i\to U$ of f such that $\bar{f}(\bigcup_{i=1}^{k}\dot{D}_i)\subset W$ [26].

To define the next property recall that an inverse sequence $\underline{X}$ of ANR-sets