

QUOTIENTS OF TORIC VARIETIES BY THE ACTION OF A SUBTORUS

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Abstract. We consider the action of a subtorus of the big torus on a toric variety. The aim of the paper is to define a natural notion of a quotient for this setting and to give an explicit algorithm for the construction of this quotient from the combinatorial data corresponding to the pair consisting of the subtorus and the toric variety. Moreover, we study the relations of such quotients with good quotients. We construct a good model, i.e. a dominant toric morphism from the given toric variety to some “maximal” toric variety having a good quotient by the induced action of the given subtorus.

Introduction. Let X be an algebraic variety with a regular action of an algebraic group G . A categorical quotient is a morphism $p: X \rightarrow Y$ which is G -invariant (i.e. constant on G -orbits) and satisfies the following universal property: every G -invariant morphism $f: X \rightarrow Z$ factors uniquely through p (see [Mu; Fo; Ki]).

Though this universal property seems to be a minimal requirement for a quotient, there is no hope for the general existence of categorical quotients. (See e.g. [AC; Ha] for an explicit example of a \mathbf{C}^* -action on a smooth four-dimensional toric variety which does not have a categorical quotient, even if one allows the quotient space Y to be an algebraic or analytic space.)

In the present article we consider toric varieties X with an action of an algebraic torus H ; we refer to these varieties as toric H -varieties. The specialization of the definition of the categorical quotient to the category of toric varieties leads to the following notion: We call a toric morphism $p: X \rightarrow Y$ a *toric quotient*, if it is H -invariant and every H -invariant toric morphism factors uniquely through p . For this kind of quotient we can actually prove the existence (see Theorem 1.4):

For every toric H -variety X there exists a toric quotient.

Our proof of this result is constructive. In fact, we introduce the notion of a *quotient fan* of a fan by some sublattice (see Section 2) and give an algorithm for the calculation of this quotient fan. We obtain the existence of toric quotients by applying this algorithm to the fan Δ of X and the lattice L of one-parameter subgroups of the acting torus T of X factoring through H .

A particularly important notion of quotient is the so-called good quotient (see [Se]) generalizing the quotients occurring in Mumford's geometric invariant theory for projective varieties. Unfortunately, good quotients exist only under very special

circumstances. However, for any toric H -variety X we can construct a *good model* \bar{X} . More precisely, we show (see Theorem 3.5):

There exists a “maximal” toric H -variety \bar{X} with a good quotient such that there is a dominant H -equivariant toric morphism from X to \bar{X} . The good quotient of \bar{X} by H coincides with the toric quotient of X by H .

In fact, the good model defines an adjoint functor to the forgetful functor from the category of toric H -varieties with good quotients into the category of toric H -varieties. Again our proof of the existence of the good model is constructive and works in terms of fans. The good model can be used to understand the obstructions for the existence of a good quotient.

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1. Toric quotients. First we briefly recall some of the basic definitions. A normal algebraic variety X is called a *toric variety* if there is an algebraic action of a torus T on X with an open orbit. We always assume the action to be effective and refer to T as the *acting torus* of X . For every toric variety X we fix a point x_0 in its open orbit which we call the base point of X .

Let X, X' be toric varieties with acting tori T, T' and base points x_0 and x'_0 , respectively. A regular map $f: X \rightarrow X'$ is called a *toric morphism* if $f(x_0) = x'_0$ and there is a homomorphism $\varphi: T \rightarrow T'$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $(t, x) \in T \times X$.

Now let H be any algebraic torus. We call a given toric variety X with acting torus T a *toric H -variety*, if H acts on X by means of a morphism $H \times X \rightarrow X$, $(h, x) \mapsto h * x$ of algebraic varieties such that the actions of H and T on X commute.

1.1 REMARK. If X is a toric H -variety, then there is a homomorphism ψ from H into the acting torus T of X such that the action of H on X is given by $h * x = \psi(h) \cdot x$.

PROOF. The action of H permutes the T -orbits since it commutes with the T -action. The open orbit $T \cdot x_0$ is even H -stable because there is only one T -orbit of maximal dimension. Since the action of T is effective, for every $h \in H$, there is a unique element $\psi(h)$ in T such that $h * x_0 = \psi(h) \cdot x_0$. Now it is straightforward to check that the map $H \rightarrow T$, $h \mapsto \psi(h)$ has the required properties. \square

An H -equivariant toric morphism $f: X \rightarrow X'$ of two toric H -varieties will be called a *toric H -morphism*. If the action of H on X' is trivial, which means that f is constant on H -orbits, then we will say that f is *H -invariant*.

1.2 DEFINITION. We call an H -invariant toric morphism $p: X \rightarrow Y$ a *toric quotient* for the toric H -variety X , if it has the following universal property: for every H -invariant toric morphism $f: X \rightarrow Z$ there is a unique toric morphism $\tilde{f}: Y \rightarrow Z$ such that the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Z \\
 p \searrow & & \nearrow \tilde{f} \\
 & Y &
 \end{array}$$

is commutative. Note that the toric quotient Y is uniquely determined by this property. We denote the quotient space Y also by $X/\text{tor}H$.

1.3 EXAMPLE. Let $H := \mathbf{C}^*$ act on the toric variety $X := \mathbf{C}^2$ by the homomorphism $t \mapsto (t^a, t^b)$, where a and b are relatively prime integers and $a > 0$. Then one can verify directly that the toric quotient of X is the constant map $\mathbf{C}^2 \rightarrow \{0\}$ if $b > 0$, and that otherwise it is the morphism

$$p: \mathbf{C}^2 \rightarrow \mathbf{C}, \quad (z, w) \mapsto \begin{cases} w & \text{if } b=0, \\ z^{-b}w^a & \text{if } b < 0. \end{cases}$$

So p corresponds to the inclusion $\mathbf{C}[X]^H \subset \mathbf{C}[X]$, i.e. p equals the categorical quotient for the action of H on the affine variety X . In fact, this holds generally for affine toric H -varieties (see Example 3.1).

Further basic examples of toric quotients are invariant toric fibrations, e.g. line bundles on toric varieties. For the toric quotient defined here we have not only uniqueness but also the existence.

1.4 THEOREM. *Every toric H -variety X has a toric quotient $p: X \rightarrow X/\text{tor}H$.*

For the proof of this theorem we use the description of toric varieties by means of fans. Let us first fix some notation. For an algebraic torus T , denote by N_T the lattice $\text{Hom}(\mathbf{C}^*, T)$ of its one parameter subgroups. A fan Δ in N_T is a finite set of strictly convex rational polyhedral cones in $N_T^{\mathbf{R}} := N_T \otimes_{\mathbf{Z}} \mathbf{R}$ satisfying the following two conditions: any two cones of Δ intersect in a common face, and if $\sigma \in \Delta$, then Δ also contains all the faces of σ . We denote a fan Δ in N_T also as a pair (N_T, Δ) .

For every fan Δ in N_T , there is a corresponding toric variety X_Δ with the acting torus T (as basic references for this construction, see e.g. [Fu] and [Od]). The assignment $\Delta \mapsto X_\Delta$ yields an equivalence between the category of fans and the category of toric varieties (with fixed base point), where maps of fans correspond to toric morphisms. Recall that a map of fans $F: (N, \Delta) \rightarrow (N', \Delta')$ by definition is a \mathbf{Z} -linear homomorphism from N to N' , also denoted by F , such that for every cone $\sigma \in \Delta$ there is a cone $\tau \in \Delta'$ with $F^{\mathbf{R}}(\sigma) \subset \tau$ (where $F^{\mathbf{R}}: N^{\mathbf{R}} \rightarrow N'^{\mathbf{R}}$ is the scalar extension of F).

Now, if a torus H acts on a toric variety X_Δ by a homomorphism φ from H to the acting torus T of X_Δ , let L denote the (primitive) sublattice of N_T corresponding to the subtorus $\varphi(H)$ of T . Then a toric morphism $f: X_\Delta \rightarrow X_{\Delta'}$ is H -invariant if and only if the corresponding map of fans $F: (N, \Delta) \rightarrow (N', \Delta')$ satisfies $L \subset \ker(F)$. So in the language of fans, Theorem 1.4 reads as follows:

1.5 THEOREM. *Let Δ be a fan in a lattice N and let L be a primitive sublattice of N . Then there is a map of fans $P: (N, \Delta) \rightarrow (\tilde{N}, \tilde{\Delta})$ with $L \subset \ker(P)$ such that the following universal property is satisfied: for every map of fans $F: (N, \Delta) \rightarrow (N', \Delta')$ with $L \subset \ker F$ there is a unique map of fans $\tilde{F}: (\tilde{N}, \tilde{\Delta}) \rightarrow (N', \Delta')$ with $F = \tilde{F} \circ P$.*

The fan $\tilde{\Delta}$ occurring in the above theorem will be called the *quotient fan* of Δ by L . Note that our concept of a quotient fan differs from the notion introduced in [Ka; St; Ze], since we require the existence of a map of fans from Δ to $\tilde{\Delta}$. We will prove Theorem 1.5 in the next section by describing an explicit algorithm to construct the quotient fan. The algorithm starts with projecting cones of Δ to N/L . But then two types of difficulties occur:

Firstly, the projected cones in general are no longer strictly convex. Secondly, it can happen that the projected cones do not intersect in a common face. Therefore the construction requires an iteration of steps refining the first naïve approach. The first of the above-mentioned difficulties already occurs in Example 1.3. Here is its fan-theoretic version:

1.6 EXAMPLE. The fan Δ of the toric variety C^2 consists of the faces of the cone $\sigma \in \mathbf{R}^2$ spanned by the canonical basis vectors e_1 and e_2 . The action of $H = C^*$ on X_Δ considered in Example 1.3 corresponds to the line L through the point (a, b) .

Let $P: \mathbf{Z}^2 \rightarrow \mathbf{Z}^2/L$ denote the projection. If $b \leq 0$, the quotient fan $\tilde{\Delta}$ of Δ by L is the fan of faces of $P^{\mathbf{R}}(\sigma)$ in $\tilde{N} := \mathbf{Z}^2/L$. If $b > 0$, then $P^{\mathbf{R}}(\sigma)$ fails to be strictly convex and the quotient fan is just the zero fan in $\tilde{N} = \{0\}$.

2. Computation of the quotient Fan. Let N be a lattice, i.e. a free \mathbf{Z} -module of finite rank. In this section we construct the quotient fan of a fan Δ in N by a primitive sublattice L of N and thereby prove Theorem 1.5. In fact our construction is done in a more general framework. We will not only consider fans but also sets of convex rational polyhedral cones which are not required to be strictly convex nor to intersect pairwise in a common face.

More precisely, we will speak of a *system S of N -cones* if S is a finite set of convex cones in the space $N^{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ such that every $\sigma \in S$ is generated by finitely many vectors of N . A *map $F: (N, S) \rightarrow (N', S')$* of a system S of N -cones to a system S' of N' -cones is a lattice homomorphism from N to N' , also denoted by F , such that for every $\sigma \in S$ there is a cone $\tau \in S'$ with $F^{\mathbf{R}}(\sigma) \subset \tau$. This notion generalizes the concept of a map of fans.

We also need the following “intermediate” notion: A system Σ of N -cones is called a *quasifan* in N , if for each $\sigma \in \Sigma$ the faces of σ also belong to Σ and for any two cones σ and σ' of Σ the intersection $\sigma \cap \sigma'$ is a face of σ . So a quasifan is a fan if all its cones are strictly convex. A map of two quasifans is just a map of the underlying systems of cones.

2.1 DEFINITION. Let N be a lattice and let S be a system of N -cones. If $L \subset \hat{L} \subset N$ are primitive sublattices, then we call a (quasi-) fan $\tilde{\Delta}$ in $\tilde{N} := N/\hat{L}$ a *quotient (quasi-) fan* of Δ by L if it has the following properties:

- (i) The projection $P: N \rightarrow \tilde{N}$ defines a map of the systems S and $\tilde{\Delta}$ of cones.
- (ii) For every map $F: (N, S) \rightarrow (N', \Delta')$ from S to a (quasi-) fan Δ' in a lattice N' with $F(L)=0$, there is a map $\tilde{F}: (\tilde{N}, \tilde{\Delta}) \rightarrow (N', \Delta')$ of (quasi-) fans such that the following diagram is commutative:

$$\begin{array}{ccc} (N, S) & \xrightarrow{F} & (N', \Delta') \\ & \searrow P & \nearrow \tilde{F} \\ & & (\tilde{N}, \tilde{\Delta}) \end{array}$$

By definition, quotient fans and quotient quasifans are uniquely determined. These two notions are related to each other by the following:

2.2 REMARK. Let Σ be a quasifan with maximal cones $\sigma_1, \dots, \sigma_r$ in a lattice N . For the maximal sublattice L of N contained in $\bigcap_{i=1}^r \sigma_i$ let $P: N \rightarrow \tilde{N} := N/L$ denote the projection. Then the cones $P^{\mathbf{R}}(\sigma_1), \dots, P^{\mathbf{R}}(\sigma_r)$ are the maximal cones of the quotient fan $\tilde{\Delta}$ of Σ by L .

PROOF. Set $\sigma_0 := \bigcap_{i=1}^r \sigma_i$. Then σ_0 is a cone with $V := L^{\mathbf{R}}$ as the smallest face. Since σ_0 is a face of each σ_i , it follows that $V = \ker(P^{\mathbf{R}})$ is also the smallest face of every σ_i . This implies $(P^{\mathbf{R}})^{-1}(P^{\mathbf{R}}(\sigma_i)) = \sigma_i$ for every i .

As a consequence we obtain that every cone $P^{\mathbf{R}}(\sigma_i)$ is strictly convex. Now we check that for any two i and j the cones $P^{\mathbf{R}}(\sigma_i)$ and $P^{\mathbf{R}}(\sigma_j)$ intersect in a common face. Note that

$$P^{\mathbf{R}}(\sigma_i) \cap P^{\mathbf{R}}(\sigma_j) = P^{\mathbf{R}}(\sigma_i \cap \sigma_j).$$

Choose a supporting hyperplane W of σ_i defining the face $\sigma_i \cap \sigma_j$. Since W contains V , its projection $P^{\mathbf{R}}(W) = W/V$ is a supporting hyperplane of $P^{\mathbf{R}}(\sigma_i)$ that cuts out $P^{\mathbf{R}}(\sigma_i \cap \sigma_j)$. Therefore $P^{\mathbf{R}}(\sigma_i) \cap P^{\mathbf{R}}(\sigma_j)$ is a face of $P^{\mathbf{R}}(\sigma_i)$.

So the cones $P^{\mathbf{R}}(\sigma_1), \dots, P^{\mathbf{R}}(\sigma_r)$ together with their faces define a fan $\tilde{\Delta}$ in \tilde{N} . By construction, $\tilde{\Delta}$ satisfies the properties of a quotient fan of Σ by L . \square

The main result of this section is the following:

2.3 THEOREM. For a given system S of N -cones and a primitive sublattice L of N , there is an algorithm to construct the quotient fan $\tilde{\Delta}$ of S by L .

PROOF. Set $N_1 := N/L$ and let $P_1: N \rightarrow N_1$ denote the projection. We first construct a quotient quasifan Σ in N_1 of Δ by L by means of the following procedure:

Initialization. Define S_1 to be the system of N_1 -cones consisting of those of the $P_1^{\mathbf{R}}(\sigma)$, $\sigma \in \Delta$, that are maximal with respect to set-theoretic inclusion.

Loop. While there exist cones τ_1 and τ_2 in S_1 such that $\tau_1 \cap \tau_2$ is not a face of τ_1 do the following: Let ρ_2 be the minimal face of τ_2 that contains $\tau_1 \cap \tau_2$. If $\rho_2 \not\subset \tau_1$, replace

τ_1 by the convex hull $\text{conv}(\tau_1 \cup \rho_2)$ of $\tau_1 \cup \rho_2$. Otherwise let ρ_1 be the minimal face of τ_1 that contains $\tau_1 \cap \tau_2$ and replace τ_2 by $\text{conv}(\tau_2 \cup \rho_1)$. Omit all cones of S_1 that are properly contained in the new one.

Output. Let Σ be the system of N_1 -cones consisting of all the faces of the cones of S_1 .

The above loop is finite: passing through the loop does not increase the number $|S_1|$ of cones of S_1 . So, after finitely many, say K , steps $|S_1|$ stays fixed. For each iteration, there is a cone τ of S_1 that is replaced by a strictly larger cone of the form $\text{conv}(\tau \cup \rho)$ with a face ρ of some other cone of S_1 . According to Lemma 2.4 below we obtain

$$|\tau \cap P_1^{\mathbf{R}}(S^{(1)})| < |\text{conv}(\tau \cup \rho) \cap P_1^{\mathbf{R}}(S^{(1)})|,$$

where $S^{(1)}$ denotes a minimal set of generators of the cones of S . Thus in every step after the first K steps the number $\sum_{\tau \in S_1} |\tau \cap P_1^{\mathbf{R}}(S^{(1)})|$ strictly increases. But this can only happen a finite number of times. So the loop is indeed finite.

Now by construction Σ is a quasifan. We have to verify that it fulfills Property (ii) of Definition 2.1. So let $F: (N, S) \rightarrow (N', \Sigma')$ be a map of quasifans with $L \subset \ker(F)$. Then there is a lattice homomorphism $F_1: N_1 \rightarrow N'$ with $F = F_1 \circ P_1$. Clearly F_1 defines a map from the system S_1 of cones defined in the initialization to the system Σ' of cones.

Assume that after n iterations of the loop, F_1 still defines a map of the systems of cones S_1 and Σ' , and that in the next step we replaced the cone τ_1 by $\text{conv}(\tau_1 \cup \rho_2)$, where ρ_2 is the minimal face of τ_2 such that $\tau_1 \cap \tau_2 \subset \rho$. We have to check that there is a cone in Σ' containing $F_1^{\mathbf{R}}(\text{conv}(\tau_1 \cup \rho_2))$. Let τ'_1 and τ'_2 be cones of Σ' such that $F_1^{\mathbf{R}}(\tau_1) \subset \tau'_1$ and $F_1^{\mathbf{R}}(\tau_2) \subset \tau'_2$. Then

$$F_1^{\mathbf{R}}(\rho_2) \cap (\tau'_1 \cap \tau'_2) \neq \emptyset.$$

Since $\tau'_1 \cap \tau'_2$ is a face of τ'_2 and $F_1^{\mathbf{R}}(\rho_2) \subset \tau'_2$, we obtain $F_1^{\mathbf{R}}(\rho_2) \subset \tau'_1 \cap \tau'_2$. This implies $F_1^{\mathbf{R}}(\rho_2) \subset \tau'_1$. In particular, it follows that

$$F_1^{\mathbf{R}}(\text{conv}(\tau_1 \cup \rho_2)) \subset \tau'_1.$$

Thus after τ_1 is replaced by $\text{conv}(\tau_1 \cup \rho_2)$ the map F_1 still defines a map of the systems S_1 and Σ' of cones.

Repeating this argument we obtain that F_1 defines also a map of the quasifans Σ and Σ' . Thus Σ fulfills the desired universal mapping property and hence it is the quotient quasifan of S by L .

Now let V denote the maximal linear subspace contained in the intersection of all maximal cones of the quasifan Σ . Set $L_1 := N_1 \cap V$. Then, according to Remark 2.2, the quotient fan $\tilde{\Sigma}$ in N_1/L_1 of Σ by L_1 is obtained by projecting the maximal cones of Σ to $N_1^{\mathbf{R}}/L_1^{\mathbf{R}}$. It follows that the fan $\tilde{\Sigma}$ in $N_1/L_1 = N/\hat{L}$ with $\hat{L} := P_1^{-1}(L_1)$ is also the quotient fan of S by L . \square

We have used the following elementary fact about cones:

2.4 LEMMA. *Let $\sigma = \text{cone}(v_1, \dots, v_r)$ be the (not necessarily strictly) convex cone spanned by v_1, \dots, v_r in a real vector space V . Then every face τ of σ is generated as a cone by the vectors in $\tau \cap \{v_1, \dots, v_r\}$.*

As a consequence of the construction of the quotient fan we note:

2.5 REMARK. Let τ_1, \dots, τ_k be the maximal cones of the quotient fan of S by L , and let $F(S)$ denote the set of all faces of the cones of S . Then

$$\tau_i = \text{conv} \left(\bigcup_{\sigma \in F(S): P^R(\sigma) \subset \tau_i} P^R(\sigma) \right).$$

PROOF OF THEOREMS 1.4 AND 1.5. It suffices to verify Theorem 1.5. So, let $\tilde{\Delta}$ be the quotient fan of Δ by L . Then, if P denotes the projection from N onto the lattice of $\tilde{\Delta}$, we have only to check that the factorization of every L -invariant map of fans $F: (N, \Delta) \rightarrow (N', \Delta')$ through P is unique. But this follows from the fact that P is surjective by construction. \square

In the case of small codimension of L in N , there is an easy explicit description of the quotient fan:

2.6 EXAMPLE. Let Δ be a fan in a lattice N , and let $L \subset N$ be a primitive sublattice of codimension 2. Denote by P the canonical projection $N \rightarrow N/L$ and define an equivalence relation on the set of maximal cones of Δ as follows:

Set $\sigma \sim \tau$ if there is a sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_r = \tau$ of cones $\sigma_i \in \Delta$ such that $P^R(\sigma_i^\circ) \cap P^R(\sigma_{i+1}^\circ) \neq \emptyset$. For each maximal cone $\sigma \in \Delta$ denote by $\bar{\sigma}$ the convex hull of the union of all maximal cones $\tau \sim \sigma$.

Let V denote the sum of all linear subspaces of the cones $P^R(\bar{\sigma})$, $\sigma \in \Delta$ and set $\hat{L} := N \cap P^{-1}(V)$. Moreover, let $Q: N \rightarrow N/\hat{L}$ be the projection. Then the faces of the cones $Q^R(\bar{\sigma})$, where σ varies over the maximal cones of Δ , form the quotient fan of Δ by L .

In [Ew] a special case of our notion of the quotient fan is introduced for the abstract description of orbit closures of the acting torus of a toric variety. In fact these orbit closures are toric quotients of certain neighbourhoods:

2.7 EXAMPLE. Let Δ be a fan in a lattice N . For a cone $\tau \in \Delta$, let x_τ be the corresponding distinguished point in the toric variety X_Δ (see [Fu, p. 27]). Let B_τ be the orbit of the acting torus T of X_Δ through x_τ . Denoting by $\text{star}(\tau)$ the set of all cones $\sigma \in \Delta$ that contain τ as a face, we obtain the closure of the orbit B_τ as

$$V(\tau) := \overline{B_\tau} = \bigcup_{\sigma \in \text{star}(\tau)} B_\sigma.$$

The union $U(\tau)$ of the affine charts X_σ , $\sigma \in \text{star}(\tau)$, is an open T -invariant neighbourhood of the orbit closure $V(\tau)$. For the set of maximal cones of the fan $\Delta(\tau)$

corresponding to $U(\tau)$ we have

$$\Delta(\tau)^{\max} = \Delta^{\max} \cap \text{star}(\tau).$$

Let L be the intersection of the linear hull $\text{Lin}(\tau)$ of τ in $N^{\mathbf{R}}$ with the lattice N , and let $P: N \rightarrow N/L$ denote the projection. Then the cones $P^{\mathbf{R}}(\sigma)$, $\sigma \in \Delta(\tau)^{\max}$ are the maximal cones of the quotient fan $\tilde{\Delta}(\tau)$ of $\Delta(\tau)$ by L . Moreover $\tilde{\Delta}(\tau)$ is the fan of $V(\tau)$, viewed as a toric variety with acting torus T/T_{x_τ} (see e.g. [Fu, p. 52]). In other words, the toric morphism $p: U(\tau) \rightarrow V(\tau)$ associated to P is the toric quotient of $U(\tau)$ by T_{x_τ} .

3. Good models. Let X be an algebraic variety with a regular action of a reductive group G . If X is affine, then the categorical quotient for this action always exists, and is given by the morphism corresponding to the inclusion of the algebra $\mathbf{C}[X]^G$ of G -invariant regular functions on X into $\mathbf{C}[X]$. For general X , the idea of glueing affine quotients of G -stable affine charts leads to the following definition (see [Se]):

A G -invariant morphism $p: X \rightarrow Y$ of algebraic varieties is called a *good quotient*, if there exists a covering $(U_i)_{i \in I}$ of Y by affine open sets such that every $W_i := p^{-1}(U_i)$ is affine and the restriction $p|_{W_i}: W_i \rightarrow U_i$ is the categorical quotient for the action of G restricted to W_i . If in addition the morphism p separates orbits, it is called a *geometric quotient*.

Now, coming back to the setting of toric H -varieties, we will first give the description of the affine case in terms of fans:

3.1 EXAMPLE. Let T be an algebraic torus and let σ be a rational strictly convex cone in $N_T^{\mathbf{R}}$. Denote by X_σ the associated affine toric variety. For a given subtorus $H \subset T$ let L be the sublattice of N_T corresponding to H . Define τ to be the maximal face of σ with $L \cap \tau^\circ \neq \emptyset$ and set

$$\hat{L} := (L^{\mathbf{R}} + \text{Lin}(\tau)) \cap N_T.$$

Denote by $P: N_T \rightarrow N_T/\hat{L}$ the canonical projection. Then $P^{\mathbf{R}}(\sigma)$ is a rational strictly convex cone in $N_T^{\mathbf{R}}/\hat{L}^{\mathbf{R}}$, and the toric morphism $p: X_\sigma \rightarrow X_{P^{\mathbf{R}}(\sigma)}$ associated to P is the toric quotient for the action of H on X .

The coordinate algebra of $X_{P^{\mathbf{R}}(\sigma)}$ can be identified with the algebra $\mathbf{C}[X]^H$ of H -invariant regular functions on X , since every H -invariant character of T extending to a regular function on X factors through p . This shows that p is also the categorical quotient.

If a toric H -variety X has a good quotient $p: X \rightarrow Y$, then it follows that Y is a toric variety and p is a toric morphism. Moreover, we can conclude that if a good quotient exists, it coincides with the toric quotient. Conversely, as a consequence of Example 3.1, our procedure for the calculation of the quotient fan yields a good quotient if and only if it produces an affine map. So we can characterize fan-theoretically when a given toric quotient is good (see also [Sw] and [Hm]):

3.2 PROPOSITION. *Suppose $p: X_\Delta \rightarrow X_{\tilde{\Delta}}$ is the toric quotient of a toric H -variety X_Δ . Let $P: (N_T, \Delta) \rightarrow (N_{\tilde{T}}, \tilde{\Delta})$ be the associated map of fans. Then p is good if and only if the following two conditions are satisfied:*

(i) *For every maximal cone $\tau_i \in \tilde{\Delta}$ there is a maximal cone $\sigma_i \in \Delta$ such that $P^R(\sigma_i) = \tau_i$.*

(ii) *Every ray $\rho \in \Delta^{(1)}$ with $P^R(\rho) \subset \tau_i$ is contained in σ_i .*

Moreover, p is geometric, if in addition $\dim \tau_i = \dim \sigma_i$ for all i .

Good quotients have excellent properties, but unfortunately they only rarely exist. Białynicki-Birula and Świącicka [BB; Sw] give a complete description of all open subsets of X having a good quotient. Instead of looking at subsets one can also try to modify X to obtain a toric H -variety having a good quotient. This approach leads to the following notion:

3.3 DEFINITION. Let $p: X \rightarrow X/\text{tor}H$ denote the toric quotient of the action of H on X . Suppose that $g: X \rightarrow \bar{X}$ is a dominant toric H -morphism to a toric H -variety \bar{X} having a good quotient. Then we call g a *good model* for the toric H -variety X , if it has the following universal property: If $f: X \rightarrow Z$ is a toric H -morphism and the toric H -variety Z has a good quotient, then there is a unique toric H -morphism $\bar{f}: \bar{X} \rightarrow Z$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & \bar{X} \\ f \searrow & & \swarrow \bar{f} \\ & Z & \end{array}$$

Being defined by a universal property, a good model is unique up to isomorphism. If g is a good model, then there is a unique toric morphism $\bar{p}: \bar{X} \rightarrow X/\text{tor}H$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & \bar{X} \\ p \searrow & & \swarrow \bar{p} \\ & X/H & \\ & \text{tor} & \end{array}$$

is commutative. It follows that \bar{p} is in fact the toric and hence the good quotient for the action of H on \bar{X} . Before proceeding to the general construction of the good model we give some elementary examples.

3.4 EXAMPLES. (a) Let X be $\mathbb{C}^2 \setminus \{0\}$ and let H be the subtorus $\{(t, t^{-1}); t \in \mathbb{C}^*\}$ of the acting torus $(\mathbb{C}^*)^2$ of X . Then the toric quotient is the map $p: X \rightarrow \mathbb{C}$ defined by $p(z, w) = zw$ and the good model is just the inclusion of X in \mathbb{C}^2 (compare Example 1.3). So in this case the “missing” fixed point 0 has to be added to X .

(b) Let X be the blow-up of \mathbb{C}^2 at the point 0. Then the action of the torus H in (a) as well as the toric quotient map extend naturally to X . The good model of X is

the blow-down map $g: X \rightarrow \mathbb{C}^2$ contracting the exceptional curve to a point.

(c) If X is complete, then the toric quotient space is also complete and the good model equals the toric quotient.

The main result of this section is the following:

3.5 THEOREM. *Every toric H -variety X has a good model. If $X = X_\Delta$, then the good model is obtained as follows: Let $P: (N, \Delta) \rightarrow (\bar{N}, \bar{\Delta})$ be the map of fans corresponding to the toric quotient $p: X_\Delta \rightarrow X_{\bar{\Delta}}$ of the action of H on X_Δ . For every maximal cone τ_i , $i = 1, \dots, r$, of $\bar{\Delta}$, set*

$$\sigma_i := \text{conv}\{\rho \in \Delta^{(1)}; P^{\mathbf{R}}(\rho) \subset \tau_i\}.$$

Moreover, let V be the maximal linear subspace contained in $\bigcap_{i=1}^r \sigma_i$, set $L := V \cap N$ and let $G: N \rightarrow \bar{N} := N/L$ denote the projection. Then $G(\sigma_1), \dots, G(\sigma_r)$ are the maximal cones of a fan $\bar{\Delta}$ in \bar{N} , the projection G defines a map of fans from Δ to $\bar{\Delta}$ and the associated toric morphism $g: X_\Delta \rightarrow X_{\bar{\Delta}}$ is the good model for X_Δ .

The assignment $X \mapsto \bar{X}$ is even functorial. More precisely, if X and X' are toric H -varieties with good models $g: X \rightarrow \bar{X}$ and $g': X' \rightarrow \bar{X}'$, then for every toric H -morphism $f: X \rightarrow X'$, there is a unique toric H -morphism $\bar{f}: \bar{X} \rightarrow \bar{X}'$ such that $\bar{f} \circ g = g' \circ f$. A fancy formulation of the properties of the good model in the language of categories is the following:

3.6 COROLLARY. *The assignment $X \mapsto \bar{X}$ is adjoint to the forgetful functor from the category of toric H -varieties with good quotients into the category of toric H -varieties.*

PROOF OF THEOREM 3.5. First we prove that $\sigma_1, \dots, \sigma_r$ are the maximal cones of a quasifan S . Let σ_i and σ_j be two cones of S and let σ denote the minimal face of σ_i containing the intersection $\sigma_i \cap \sigma_j$. Then there is a vector $v \in (\sigma_i \cap \sigma_j)^\circ \cap \sigma^\circ$. Moreover, for this v we have

$$P^{\mathbf{R}}(v) \in P^{\mathbf{R}}((\sigma_i \cap \sigma_j)^\circ \cap \sigma^\circ) \subset P^{\mathbf{R}}((\sigma_i \cap \sigma_j)^\circ) \cap P^{\mathbf{R}}(\sigma^\circ) \subset (\tau_i \cap \tau_j) \cap P^{\mathbf{R}}(\sigma)^\circ.$$

In particular, the intersection of $P^{\mathbf{R}}(\sigma)^\circ$ with $\tau_i \cap \tau_j$ is not empty. Since $\tau_i \cap \tau_j$ is a face of τ_i and the cone $P^{\mathbf{R}}(\sigma)$ is contained in τ_i we obtain

$$P^{\mathbf{R}}(\sigma) \subset \tau_i \cap \tau_j \subset \tau_j.$$

By Lemma 2.4, σ is the convex hull of some rays ρ_1, \dots, ρ_r of Δ . For each of these rays we have $P^{\mathbf{R}}(\rho_i) \subset \tau_j$. By the definition of σ_j all the rays ρ_i are contained in σ_j . This implies $\sigma \subset \sigma_j$ and hence $\sigma = \sigma_i \cap \sigma_j$. So σ_i and σ_j intersect in a face of σ_i . That means S is indeed a quasifan. Now we can apply Remark 2.2 to conclude that $G^{\mathbf{R}}(\sigma_1), \dots, G^{\mathbf{R}}(\sigma_r)$ in \bar{N} are the maximal cones of the quotient fan $\bar{\Delta}$ of S by L . Moreover, G defines a map from the fan Δ to the fan $\bar{\Delta}$.

In the next step of the proof we show that $X_{\bar{\Delta}}$ has a good quotient by H . Since L is contained in some σ_i and $P^{\mathbf{R}}$ maps σ_i to the strictly convex cone τ_i , we have $P(L) = 0$.

It follows that P defines an L -invariant map of systems of cones from S to $\tilde{\Delta}$. Consequently, there is a unique map of fans $\bar{P}: (\bar{N}, \bar{\Delta}) \rightarrow (\tilde{N}, \tilde{\Delta})$ with $P = \bar{P} \circ G$. Note that the associated toric morphism $\bar{p}: X_{\bar{\Delta}} \rightarrow X_{\tilde{\Delta}}/_{\text{tor}} H$ is the toric quotient for $X_{\bar{\Delta}}$ by H .

To check that \bar{p} is a good quotient, we use Proposition 3.2. The first condition of 3.2 is fulfilled since by Remark 2.5, $P^{\mathbf{R}}(\sigma_i) = \tau_i$ for every i and hence the maximal cone τ_i of $\tilde{\Delta}$ is the image under $\bar{P}^{\mathbf{R}}$ of the maximal cone $G^{\mathbf{R}}(\sigma_i)$ of $\bar{\Delta}$. For the verification of the second condition, let $\bar{\rho}$ be a ray in $\tilde{\Delta}$ with $\bar{P}^{\mathbf{R}}(\bar{\rho}) \subset \tau_i$. Then by Lemma 2.4 there is a ray $\rho \in \Delta$ with $G^{\mathbf{R}}(\rho) = \bar{\rho}$. Since $P^{\mathbf{R}}(\rho) \subset \tau_i$, by definition ρ is contained in σ_i and hence $\bar{\rho} = G^{\mathbf{R}}(\rho) \subset G^{\mathbf{R}}(\sigma_i)$.

To complete the proof we have to verify the universal property of good models for g . So let X' be a toric H -variety with a good quotient $p': X' \rightarrow X'/_{\text{tor}} H$ and let $f: X \rightarrow X'$ be a toric H -morphism. Denote the fans associated to X' and $X'/_{\text{tor}} H$ by Δ' and $\tilde{\Delta}'$ respectively, and let $F: (N, \Delta) \rightarrow (N', \Delta')$ be the map of fans associated to f .

Now suppose for the moment that the linear map $F: N \rightarrow N'$ also defines an L -invariant map from the system of cones S to Δ' . Then, since $\bar{\Delta}$ is the quotient fan of S by L , there is a unique map of fans $\bar{F}: (\bar{N}, \bar{\Delta}) \rightarrow (N', \Delta')$ with $F = \bar{F} \circ G$. Clearly, the toric morphism $\bar{f}: X_{\bar{\Delta}} \rightarrow X_{\Delta'}$ associated to \bar{F} provides us with the required factorization of f through g .

So it remains to show that for a given cone $\sigma_i \in S$ there is a cone $\sigma'_i \in \Delta'$ with $F^{\mathbf{R}}(\sigma_i) \subset \sigma'_i$. (Since L is contained in σ_i and σ'_i is strictly convex, this also implies that $F(L) = 0$.) Consider the following commutative diagrams of toric morphisms and of the associated maps of fans:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ p \downarrow & & \downarrow p' \\ X/H & \xrightarrow{f'} & X'/H, \\ \text{tor} & & \text{tor} \end{array} \quad \begin{array}{ccc} (N, \Delta) & \xrightarrow{F} & (N', \Delta') \\ P \downarrow & & \downarrow P' \\ (\tilde{N}, \tilde{\Delta}) & \xrightarrow{F'} & (\tilde{N}', \tilde{\Delta}'). \end{array}$$

Let ρ be any ray of Δ which is contained in σ_i . Since $P^{\mathbf{R}}(\sigma_i) = \tau_i$, there is a maximal cone τ'_i in $\tilde{\Delta}'$ containing $F'^{\mathbf{R}}(P^{\mathbf{R}}(\sigma_i)) = P'^{\mathbf{R}}(F^{\mathbf{R}}(\sigma_i))$. So in particular, $P'^{\mathbf{R}}(F^{\mathbf{R}}(\rho)) \subset \tau'_i$. Suppose that σ' is the minimal cone of Δ' containing $F^{\mathbf{R}}(\rho)$. Then $P'^{\mathbf{R}}(\sigma)^\circ$ intersects τ'_i and therefore $P'^{\mathbf{R}}(\sigma)$ is contained in τ'_i .

Now p' is a good quotient and therefore by Proposition 3.2 there is a maximal cone $\sigma'_i \in \Delta'$ with $P'^{\mathbf{R}}(\sigma'_i) = \tau'_i$. Moreover, any cone of Δ' which is mapped into τ'_i by $P'^{\mathbf{R}}$ is a face of σ'_i . So in particular, $\sigma \subset \sigma'_i$ and hence $F^{\mathbf{R}}(\rho) \subset \sigma'_i$. Since σ_i is generated by the rays of Δ that it contains, we finally obtain $F^{\mathbf{R}}(\sigma_i) \subset \sigma'_i$. \square

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