

# R-current three-point functions in 4d $\mathcal{N} = 1$ superconformal theories

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**ABSTRACT:** In 4d  $\mathcal{N} = 1$  superconformal field theories (SCFTs) the R-symmetry current, the stress-energy tensor, and the supersymmetry currents are grouped into a single object, the Ferrara-Zumino multiplet. In this work we study the most general form of three-point functions involving two Ferrara-Zumino multiplets and a third generic multiplet. We solve the constraints imposed by conservation in superspace and show that non-trivial solutions can only be found if the third multiplet is R-neutral and transforms in suitable Lorentz representations. In the process we give a prescription for counting independent tensor structures in superconformal three-point functions. Finally, we set the Grassmann coordinates of the Ferrara-Zumino multiplets to zero and extract all three-point functions involving two R-currents and a third conformal primary. Our results pave the way for bootstrapping the correlation function of four R-currents in 4d  $\mathcal{N} = 1$  SCFTs.

**KEYWORDS:** Conformal and W Symmetry, Conformal Field Theory, Superspaces

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## 1 Introduction

In the last decade the conformal bootstrap has been widely used to explore the space of conformal field theories (CFTs), both from a numerical perspective [1] but also from an analytical one [2, 3]. Important results have been obtained for 3d condensed matter systems, but also for higher-dimensional theories in presence of supersymmetry (see [4] for a review and a summary of important results).

Most of these techniques heavily rely on the computation of conformal blocks, or superconformal blocks in the case of superconformal field theories (SCFTs). While for CFTs this problem has now been completely solved in 3d [5, 6] and 4d [7, 8], it is still an open question in several supersymmetric cases. Superconformal blocks can be expressed as finite linear combinations of ordinary blocks. Nevertheless, finding the exact expression is technically challenging. A notable example are the superconformal blocks of the stress-energy tensor multiplet four point function in 4d  $\mathcal{N} = 2$  theories, see e.g. [9].

In this work we focus on 4d  $\mathcal{N} = 1$  theories. Computations of superconformal blocks have already been performed in the literature for four-point functions of scalar operators, e.g. chiral or antichiral [10, 11], linear [12, 13], and general [14–16]. Here we take the next logical step and address a more complicated case, i.e. we compute all necessary ingredients for the calculation of superconformal blocks of four-point functions involving the R-current  $J_\mu$ . The vector operator  $J_\mu$  is in the same multiplet as the supersymmetry current and the stress-energy tensor, called the Ferrara-Zumino multiplet,  $\mathcal{J}_\mu$  [17]. Our results are obtained by explicitly working out the projection of the superconformal three-point function with the Ferrara-Zumino multiplet at the first two points and a general allowed operator at the third. The main motivation is to use these results to bootstrap the four-point function of  $J_\mu$ . This will provide a new way to explore the space of SCFTs, and hopefully shed more light on the “minimal” 4d  $\mathcal{N} = 1$  SCFT studied with bootstrap techniques in [16, 18, 19] and attempted to be identified by analytical means in [20, 21].

Unlike the case of extended supersymmetry in 4d, in  $\mathcal{N} = 1$  SCFTs the supermultiplet containing the stress-energy tensor does not contain a scalar primary operator. Dealing with spinning operators raises the complication of the computations significantly, although the general procedure remains the same.

Let us outline the logic we follow here. Our starting point is the superconformal three-point function in superspace with the Ferrara-Zumino multiplet at the first two points and a general allowed superconformal multiplet at the third.<sup>1</sup> The superspace expression for the three-point function, constructed following the constraints laid out in [22, 23], involves many structures with *a priori* independent coefficients. The first step is to work out the relations among these coefficients due to the shortening condition satisfied by the Ferrara-Zumino multiplet. Typically, this reduces the number of independent coefficients drastically, and in some cases sets the whole three-point function to zero. Subsequently, we perform an expansion in the fermionic coordinates  $\theta_3$  and  $\bar{\theta}_3$ , after setting  $\theta_1 = \theta_2 = 0$  and

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<sup>1</sup>There are various types of allowed operators at the third point, belonging to different Lorentz representations and having appropriate R-charges.

$\bar{\theta}_1 = \bar{\theta}_2 = 0$  in order to focus on the R-current at the first two points. With this expansion we are able to identify three-point functions of conformal primary operators.

This last step is the most complicated and delicate one: any given order of the expansion contains a combination of conformal primaries and descendants which must be disentangled. For example, at order  $\theta_3\bar{\theta}_3$  the expansion of the superconformal three-point function contains not only terms belonging to three-point functions of the schematic form  $\langle JJ(Q\bar{Q}\mathcal{O})_p \rangle$ , where  $Q$  is the supersymmetric charge and “ $p$ ” denotes that the operator is conformal primary, but also terms belonging to three-point functions of the schematic form  $\langle JJ(P\mathcal{O}) \rangle$ , where  $P$  is the generator of translations. The latter contributions can be subtracted away using the results of [24], where the specific way contamination from conformal descendants can happen was worked out in generality. To carry out our calculations we have expanded the *Mathematica* package<sup>2</sup> developed for the purposes of [24]. For the structures associated with three-point functions of conformal primary operators we have used the *Mathematica* package `CFTs4D` [25].

A non-trivial check on our computations is supplied by the fact that when the third operator in the three-point function satisfies a shortening condition, then a unitarity bound is saturated and the corresponding three-point function should vanish. This typically happens automatically after the Ward identities for conservation at the first two points have been solved, i.e. the solution for the independent three-point function coefficients involves explicit factors of  $\Delta - \Delta_u$ , where  $\Delta_u$  is the dimension at the unitarity bound. While in some of our cases this story is repeated, we have also encountered situations where solving the Ward identities at the first two points is not enough to guarantee vanishing of the three-point function when the third operator saturates its unitarity bound. The Ward identity at the third point needs to be imposed in those cases, something that results in the proper vanishing of the three-point function. In some cases the resulting requirement is non-trivial, i.e. it does not set all independent (after satisfying the Ward identity at the first two points) coefficients to zero at the unitarity bound, but rather it relates them in the appropriate way.

In section 2 we review known results about the structure of three-point functions with two conserved spin-one currents at the first two points. In section 3 we explore general constraints on our three-point functions of interest in superspace. We introduce an index-free notation and we provide counting arguments for the number of independent structures in superconformal three-point functions with two Ferrara-Zumino and a general multiplet in  $\mathcal{N} = 1$  superspace. The analysis of the Ward identity constraints is contained in section 4, while our final results for the various three-point function coefficients can be found in section 5. Appendix A contains the superconformal three-point function structures in the various cases of interest. Appendix B contains structures using Lorentz vector indices for the case where the third operator in the three-point function is an integer-spin representation of the Lorentz group. Some special cases regarding the solutions of the Ward identity constraints are treated in appendix C, while conventions for the supersymmetric derivatives used for the implementation of the Ward identities are included in appendix D. A *Mathematica* file with a summary of all results is also attached to this paper as supplementary material.

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<sup>2</sup>The package can be made available upon request.

## 2 Warming up: $\langle J_\mu J_\nu \mathcal{O}_{\ell+k,\ell} \rangle$ three-point function in CFTs

Before diving into the complications of supersymmetric CFTs, let us briefly review the structure of the three-point function involving two identical conserved spin-one currents  $J_\mu$  and a third conformal primary operator  $\mathcal{O}_{\ell+k,\ell}$ , with scaling dimension  $\Delta$ , in a generic CFT in 4d. The integers  $\ell$  and  $k$  determine the transformation properties of  $\mathcal{O}_{\ell+k,\ell}$  under the Lorentz group  $\text{SO}(1,3)$ . The most general correlation function consistent with conformal symmetry can be compactly written in embedding space; following [26] we lift the fields to 6d and we introduce 6d spinors  $S_{i\alpha}$  and  $\bar{S}_i^{\dot{\alpha}}$ ,  $i = 1, 2, 3$ , to contract the spacetime indices. The three-point function is non-zero in three cases:

$$\langle J(X_1)J(X_2)\mathcal{O}_{\ell+k,\ell}(X_3) \rangle = \mathcal{K}_{\Delta,\ell,k} \begin{cases} \sum_{i=1}^5 \lambda^{(i)} \mathcal{S}^{(i)}(X_i, S_i, \bar{S}_i) + \lambda^{(-)} \mathcal{S}^{(-)}(X_i, S_i, \bar{S}_i) & \text{for } k=0, \\ \sum_{i=1}^4 \lambda^{(i)} \mathcal{T}^{(i)}(X_i, S_i, \bar{S}_i) & \text{for } k=2, \\ \lambda \mathcal{R}(X_i, S_i, \bar{S}_i) & \text{for } k=4, \end{cases} \quad (2.1)$$

where the prefactor is

$$\mathcal{K}_{\Delta,\ell,k} = \frac{J_3^{\ell-2}}{X_{12}^{\Delta+\ell-8+\frac{1}{2}k} X_{13}^{\Delta-\ell-\frac{1}{2}k} X_{23}^{\Delta-\ell-\frac{1}{2}k}}, \quad (2.2)$$

and we have defined the tensor structures

$$\begin{aligned} \mathcal{S}^{(1)} &= J_1 J_2 J_3^2, & \mathcal{T}^{(1)} &= I_{13} I_{23} I_{32} K_2 J_3, \\ \mathcal{S}^{(2)} &= I_{23} I_{32} J_1 J_3, & \mathcal{T}^{(2)} &= I_{13} I_{23} I_{31} K_1 J_3, \\ \mathcal{S}^{(3)} &= I_{13} I_{31} J_2 J_3, & \mathcal{T}^{(3)} &= I_{13} I_{21} K_1 J_3^2, \\ \mathcal{S}^{(4)} &= I_{13} I_{31} I_{23} I_{32}, & \mathcal{T}^{(4)} &= I_{12} I_{23} K_2 J_3^2, \\ \mathcal{S}^{(5)} &= I_{12} I_{21} J_3^2, & \mathcal{R} &= I_{13} I_{23} K_1 K_2 J_3^2, \\ \mathcal{S}^{(-)} &= I_{12} I_{23} I_{31} + I_{13} I_{12} I_{32}. \end{aligned} \quad (2.3)$$

The quantities appearing in (2.2) and (2.3) are combinations of the 6d coordinates  $X_i$  and spinors  $S_{i\alpha}$  and  $\bar{S}_i^{\dot{\alpha}}$ . Their definitions can be found in [26].<sup>3</sup> Notice that for the special cases  $\ell = 0, 1$ , not all of the above tensor structures are allowed. The case of  $\mathcal{O}_{\ell,\ell+k}$  is similar to (2.1): one can define tensor structures  $\bar{\mathcal{T}}^{(i)}$  and  $\bar{\mathcal{R}}$  analogous to (2.3) with the replacement  $I_{ij} K_m \rightarrow I_{ji} \bar{K}_m$ .

Permutation symmetry and conservation of the current  $J_\mu$  impose a set of conditions summarized in table 1. We found two special cases. First, when  $k = 0$ ,  $\ell = 0$ , then  $\lambda^{(4)} = 0$  since the associated tensor structure does not exist. As a consequence,  $\lambda^{(2)}$  vanishes as well and there is only one degree of freedom. A second exception is for  $k = 2$ ,  $\ell = 0$ ; in this

<sup>3</sup>Schematically we have:  $X_{ij} = X_i \cdot X_j$ ,  $I_{ij} = \bar{S}_i S_j$ ,  $J_{i,jk} = \bar{S}_i \mathbf{X}_j \bar{\mathbf{X}}_k S_i / X_{jk}$ ,  $K_{i,jk} = S_j \bar{\mathbf{X}}_i S_k \sqrt{X_{jk} / (X_{ij} X_{ik})}$ . In addition,  $\mathbf{X}_{ab}$  is obtained contracting  $X^M$  with the 6d gamma matrices. Finally  $J_1 \equiv J_{1,23}$ ,  $J_2 \equiv J_{2,13}$ ,  $J_3 \equiv J_{3,12}$  and similarly for  $K_i$ . In **CFTs4D** the structures  $I, J, K$  are denoted  $\hat{I}, \hat{J}, \hat{K}$ .

$k$	$\ell$	$1 \leftrightarrow 2$	Conservation	d.o.f
0	Even	$\lambda^{(3)} = -\lambda^{(2)}, \lambda^{(-)} = 0$	$\sigma\lambda^{(2)} = 4\ell(\Delta - 3)\lambda^{(1)} - (\Delta - \ell - 4)(\Delta + \ell)\lambda^{(4)},$ $\sigma\lambda^{(5)} = 4(\Delta - 3)(\Delta - 2)\lambda^{(1)} + (\Delta - \ell - 4)(\Delta - \ell - 6)\lambda^{(4)}$	$2^+$
0	Odd	$\lambda^{(3)} = \lambda^{(2)}, \lambda^{(1,4,5)} = 0$	$\lambda^{(2)} = 0$	$1^-$
2	Even	$\lambda^{(2)} = -\lambda^{(1)}, \lambda^{(4)} = \lambda^{(3)}$	$\lambda^{(3)} = -\frac{\Delta - \ell - 6}{2(\Delta - 2)}\lambda^{(1)}$	1
2	Odd	$\lambda^{(2)} = \lambda^{(1)}, \lambda^{(4)} = -\lambda^{(3)}$	$\lambda^{(3)} = -\frac{\Delta - \ell - 6}{2(\ell + 2)}\lambda^{(1)}$	1
4	Even	-	-	1
4	Odd	$\lambda = 0$	-	0

**Table 1.** Constraints imposed by permutation symmetry and current conservation. The last column shows the number of independent coefficients after all conditions are imposed. The case  $\ell = 0$  is special. Here we defined  $\sigma = 2\ell(\ell + 8) - 4(\ell - 1)\Delta - 2\Delta^2$ .

case permutation symmetry and current conservation sets the three-point function to zero, except for the special case  $\Delta = 2$ , when  $\lambda^{(3)} = \lambda^{(4)}$ , while all the rest vanishes. However, in SCFTs this operator is below the unitarity bounds (see section 4). Besides these special cases, we stress that all the denominators in table 1 are non-zero whenever the dimension of  $\mathcal{O}_{\ell+k,\ell}$  satisfies the unitarity bounds [27, 28]

$$\begin{aligned} \Delta &\geq \ell + \frac{1}{2}k + 2 && \text{for } \ell > 0, \\ \Delta &\geq \frac{1}{2}k + 1 && \text{for } \ell = 0. \end{aligned} \tag{2.4}$$

### 3 Three-point function of two Ferrara-Zumino and a general multiplet

Let us now move on to the supersymmetric case. For our analysis we will mostly follow the formalism developed in [22, 23] and the conventions of Wess and Bagger [29]. With  $\mathcal{N} = 1$  supersymmetry the conserved currents arising from superconformal transformations are contained in a single superconformal multiplet, the Ferrara-Zumino multiplet  $\mathcal{J}_{\alpha\dot{\alpha}}$  [17]. This satisfies the shortening condition<sup>4</sup>

$$D^\alpha \mathcal{J}_{\alpha\dot{\alpha}} = \bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = 0. \tag{3.1}$$

The expansion of  $\mathcal{J}_{\alpha\dot{\alpha}}$  in components reads (see, e.g. [30])

$$\begin{aligned} -\frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \mathcal{J}_{\alpha\dot{\alpha}}(z) &= J_\mu(x) + \frac{i}{2}\theta^\alpha S_{\mu\alpha}(x) - \frac{i}{2}\bar{S}_{\mu\dot{\alpha}}(x)\bar{\theta}^{\dot{\alpha}} + \theta^\alpha\sigma_{\alpha\dot{\alpha}}^\nu\bar{\theta}^{\dot{\alpha}} \left( T_{\mu\nu}(x) - \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\partial^\rho J^\lambda(x) \right) \\ &\quad - \frac{1}{8}\theta^2\partial_\nu S_\mu(x)\sigma^\nu\bar{\theta} - \frac{1}{8}\bar{\theta}^2\theta\sigma^\nu\partial_\nu\bar{S}_\mu(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 J_\mu(x), \end{aligned} \tag{3.2}$$

where  $z = (x, \theta, \bar{\theta})$  is a point in superspace,  $J_\mu$  is the R-symmetry U(1) current,  $S_{\mu\alpha}$  the supersymmetry current and  $T_{\mu\nu}$  the stress-energy tensor. The condition (3.1) implies the following conservation and irreducibility conditions:

$$\partial_\mu J^\mu = \partial_\mu T^{\mu\nu} = T^\mu{}_\mu = T_{[\mu\nu]} = \partial_\mu S_\alpha^\mu = \partial_\mu \bar{S}_{\dot{\alpha}}^\mu = \bar{\sigma}_\mu^{\dot{\alpha}\alpha} S_\alpha^\mu = \bar{S}_{\dot{\alpha}}^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} = 0. \tag{3.3}$$

<sup>4</sup>See appendix D for the definitions of superspace derivatives  $D^\alpha$  and  $\bar{D}^{\dot{\alpha}}$ .

### 3.1 General properties

In this section we study the most general form of the three-point function of two Ferrara-Zumino multiplets and a third general superconformal multiplet  $\mathcal{O}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}} = \mathcal{O}_{(\gamma_1 \dots \gamma_j); (\dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}})}$  consistent with 4d  $\mathcal{N} = 1$  superconformal symmetry. We recall that superconformal multiplets are labelled by two integers,  $j$  and  $\bar{j}$ , indicating that the superconformal primary in the multiplet transforms in the  $(\frac{1}{2}j, \frac{1}{2}\bar{j})$  representation of the Lorentz group,<sup>5</sup> and two reals,  $q$  and  $\bar{q}$ , which give the scaling dimension and R-charge of the superconformal primary operator via

$$\Delta = q + \bar{q}, \quad R = \frac{2}{3}(q - \bar{q}). \quad (3.4)$$

While the supercurrent satisfies  $q_{\mathcal{J}} = \bar{q}_{\mathcal{J}} = \frac{3}{2}$ , for a general supermultiplet  $\mathcal{O}$  the values  $q, \bar{q}$  can assume any value consistent with the unitarity bounds<sup>6</sup> [31, 32]:

$$\Delta \geq |q - \bar{q} - \frac{1}{2}(j - \bar{j})| + \frac{1}{2}(j + \bar{j}) + 2. \quad (3.5)$$

Our goal is to start from the superspace expression of the three-point function

$$\langle \mathcal{J}_{\alpha\dot{\alpha}}(z_1) \mathcal{J}_{\beta\dot{\beta}}(z_2) \mathcal{O}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(z_3) \rangle, \quad (3.6)$$

and then solve the constraints imposed by the shortening condition<sup>7</sup> (3.1). From the results reviewed in section 2 we know that if we set  $\theta_i = \bar{\theta}_i = 0$  in (3.6), the only nonzero contributions come from superprimaries satisfying

- (A)  $R = 0, j = \bar{j},$
- (B)  $R = 0, j = \bar{j} + 2,$
- (C)  $R = 0, j = \bar{j} + 4,$

and their conjugates. In superspace, however, we would expect the correlator (3.6) to be non-vanishing whenever there is a non-zero three-point function between a component of  $\mathcal{O}$  and any pairs of the fields appearing in the expansion (3.2). For instance, when only  $\theta_{1,2} = \bar{\theta}_{1,2} = 0$ , we could get a non-supersymmetric three-point function between two currents  $J$  and a superconformal descendant, i.e. the representation corresponding to cases (A)–(C) may only arise after the action of  $Q$ 's and/or  $\bar{Q}$ 's on some superconformal primary.<sup>8</sup> Similarly, since in CFTs the OPE of two stress-energy tensors  $T_{\mu\nu}$  can contain primaries with  $j - \bar{j}$  up to  $\pm 8$ , we could expect that these cases should be considered too.

One of the most important results of this work is showing that cases (A)–(C) are instead the only relevant ones: although it is possible to construct other three-point functions in superspace, the shortening condition (3.1) sets all of them to zero. In section 3.4 we give a group theoretic argument for this fact. In addition, we have verified that the conservation

<sup>5</sup>In section 2 we considered operators with  $j = \ell + k, \bar{j} = \ell$ .

<sup>6</sup>Chiral (antichiral) representations are special cases and correspond to  $\bar{q} = \bar{j} = 0$  ( $q = j = 0$ ).

<sup>7</sup>In this work we use interchangeably the terminology “shortening condition” and “Ward identity”.

<sup>8</sup>Examples of this have appeared before in the literature [13, 16].

conditions admit non-trivial solutions only in the cases (A)–(C). In this paper we will present in detail only the non-vanishing cases.

Before analyzing each of the cases (A)–(C) individually, let us discuss the general properties of the correlator (3.6). As shown in [23], the most general three-point function consistent with superconformal symmetry can be written as

$$\langle \mathcal{J}_{\alpha\dot{\alpha}}(z_1) \mathcal{J}_{\beta\dot{\beta}}(z_2) \mathcal{O}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(z_3) \rangle = \frac{x_{1\bar{3}} \alpha\dot{\alpha}' x_{3\bar{1}} \alpha'\dot{\alpha} x_{2\bar{3}} \beta\dot{\beta}' x_{3\bar{2}} \beta'\dot{\beta}}{(x_{3\bar{1}}^2 x_{1\bar{3}}^2 x_{3\bar{2}}^2 x_{2\bar{3}}^2)^2} t^{\dot{\alpha}\alpha'; \dot{\beta}'\beta'}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(X, \Theta, \bar{\Theta}), \quad (3.7)$$

where<sup>9</sup>

$$\begin{aligned} X &= \frac{x_{3\bar{1}} \tilde{x}_{1\bar{2}} x_{2\bar{3}}}{x_{1\bar{3}}^2 x_{3\bar{2}}^2}, & x_{\alpha\dot{\alpha}} &= \sigma_{\alpha\dot{\alpha}}^\mu x_\mu, & \tilde{x}^{\dot{\alpha}\alpha} &= \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} x_{\beta\dot{\beta}}, \\ \Theta &= i \left( \frac{1}{x_{1\bar{3}}^2} x_{3\bar{1}} \bar{\theta}_{31} - \frac{1}{x_{2\bar{3}}^2} x_{3\bar{2}} \bar{\theta}_{32} \right), & \bar{\Theta} &= \Theta^*, \end{aligned} \quad (3.8)$$

with  $\bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j$  and the supersymmetric interval between  $x_i$  and  $x_j$  defined by

$$x_{\bar{i}j} = -x_{j\bar{i}} \equiv x_{ij} - i\theta_i \sigma \bar{\theta}_i - i\theta_j \sigma \bar{\theta}_j + 2i\theta_j \sigma \bar{\theta}_i. \quad (3.9)$$

In addition, the tensor  $t$  must satisfy the homogeneity property [23]

$$t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(\lambda \bar{\lambda} X, \lambda \Theta, \bar{\lambda} \bar{\Theta}) = \lambda^{\frac{2}{3}(2q+\bar{q}-9)} \bar{\lambda}^{\frac{2}{3}(q+2\bar{q}-9)} t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(X, \Theta, \bar{\Theta}). \quad (3.10)$$

Let us pause for a moment to appreciate the importance of the result (3.7). The arbitrariness of the three-point function is now entirely contained in the tensor  $t$ , which is only a function of the coordinates  $X, \Theta, \bar{\Theta}$ , while the prefactor takes care of reproducing the correct covariance properties at the first two points. Moreover, since  $\Theta, \bar{\Theta}$  are two component Grassmann spinors with R-charge  $-1, +1$ , respectively, while  $X$  and  $x_{\bar{i}j}$  are neutral, it follows that

$$R = 0, \pm 1, \pm 2, \quad (3.11)$$

with  $R$  the R-charge of  $\mathcal{O}$ . As we have already mentioned, and will prove in section 3.4, the shortening condition (3.1) only allows neutral supermultiplets,  $R = 0$ , to have a non-vanishing correlation function. In that case, the tensor  $t$  is only allowed to depend on the Grassmann variables through the combination

$$t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(X, \Theta, \bar{\Theta}) = t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(X, \bar{X}), \quad \bar{X}^\mu = X^\mu + 2i\Theta \sigma^\mu \bar{\Theta}. \quad (3.12)$$

Moreover, from the invariance of the three-point function under  $z_1 \leftrightarrow z_2$  and  $\alpha\dot{\alpha} \leftrightarrow \beta\dot{\beta}$ , still for  $R = 0$ , we get

$$t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(X, \bar{X}) = t^{\dot{\beta}\beta; \dot{\alpha}\alpha}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_{\bar{j}}}(-\bar{X}, -X). \quad (3.13)$$

Finally, in the case of real operators, for instance spin- $\ell$  supermultiplets with zero R-charge, we have the condition

$$t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_\ell; \dot{\gamma}_1 \dots \dot{\gamma}_\ell}(X, \bar{X})^* = t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_\ell; \dot{\gamma}_1 \dots \dot{\gamma}_\ell}(\bar{X}, X). \quad (3.14)$$

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<sup>9</sup>In [23]  $X, \Theta$  and  $\bar{\Theta}$  correspond to, respectively,  $X_3, \Theta_3$  and  $\bar{\Theta}_3$ .



### 3.2 Index-free notation

For practical computations it is very convenient to contract all free indices with auxiliary commuting spinors  $\eta_i, \bar{\eta}_i$ . In this notation the three-point function reads

$$\begin{aligned} & \langle \mathcal{J}(\eta'_1, \bar{\eta}'_1, z_1) \mathcal{J}(\eta'_2, \bar{\eta}'_2, z_2) \mathcal{O}_{j,\bar{j}}(\eta_3, \bar{\eta}_3, z_3) \rangle \\ &= \frac{\eta'_1{}^{\alpha_1} x_{1\bar{3}} \partial_{\bar{\eta}_1} \partial_{\eta_1} x_{3\bar{1}} \bar{\eta}'_1{}^{\dot{\alpha}_1} \eta'_2{}^{\alpha_2} x_{2\bar{3}} \partial_{\bar{\eta}_2} \partial_{\eta_2} x_{3\bar{2}} \bar{\eta}'_2{}^{\dot{\alpha}_2}}{(x_{\bar{3}1}{}^2 x_{1\bar{3}}{}^2 x_{\bar{3}2}{}^2 x_{2\bar{3}}{}^2)^2} t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta}), \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} \mathcal{J}(\eta, \bar{\eta}, z) &= \eta^\alpha \bar{\eta}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}}(z), & \mathcal{O}_{j,\bar{j}}(\eta, \bar{\eta}, z) &= \eta^{\alpha_1} \dots \eta^{\alpha_j} \bar{\eta}^{\dot{\alpha}_1} \dots \bar{\eta}^{\dot{\alpha}_j} \bar{\mathcal{O}}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_j}(z), \\ t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta}) &= \eta_{1\alpha} \bar{\eta}_{1\dot{\alpha}} \eta_{2\beta} \bar{\eta}_{2\dot{\beta}} \eta_3^{\gamma_1} \dots \eta_3^{\gamma_j} \bar{\eta}_3^{\dot{\gamma}_1} \dots \bar{\eta}_3^{\dot{\gamma}_j} t^{\dot{\alpha}\alpha; \dot{\beta}\beta}_{\gamma_1 \dots \gamma_j; \dot{\gamma}_1 \dots \dot{\gamma}_j}(X, \Theta, \bar{\Theta}). \end{aligned} \quad (3.16)$$

and derivatives with respect to  $\eta$  obey  $\partial_{\eta^\alpha} \eta^\beta = \delta_\alpha^\beta$ ,  $\partial_{\eta^\alpha} = -\epsilon_{\alpha\beta} \partial_{\eta_\beta}$ .

The problem is now reduced to finding the most general form of  $t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta})$ . This can be achieved using the building blocks<sup>10</sup>

$$\begin{aligned} [i\bar{j}] &= \frac{\eta_i X \bar{\eta}_j}{|X|}, & [\Theta\bar{\Theta}] &= \frac{\Theta X \bar{\Theta}}{X^2}, & [ij] &= \eta_i \eta_j, & [\bar{i}\bar{j}] &= \bar{\eta}_i \bar{\eta}_j, \\ [\Theta j] &= \frac{\Theta \eta_j}{|X|^{1/2}}, & [\bar{\Theta} \bar{j}] &= \frac{\bar{\Theta} \bar{\eta}_j}{|X|^{1/2}}, & [j\bar{\Theta}] &= \frac{\eta_j X \bar{\Theta}}{|X|^{3/2}}, & [\Theta \bar{j}] &= \frac{\Theta X \bar{\eta}_j}{|X|^{3/2}}. \end{aligned} \quad (3.17)$$

In addition, the homogeneity property (3.10) can now be written as

$$t(\eta_{1,2}, \kappa \eta_3, \bar{\eta}_{1,2}, \bar{\kappa} \bar{\eta}_3, \lambda \bar{\lambda} X, \lambda \Theta, \bar{\lambda} \bar{\Theta}) = \lambda^{\frac{2}{3}(2q+\bar{q}-9)} \bar{\lambda}^{\frac{2}{3}(q+2\bar{q}-9)} \kappa^j \bar{\kappa}^{\bar{j}} t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta}), \quad (3.18)$$

while the symmetry property (3.13) for the first two points reads

$$t(\eta_{1,2,3}, \bar{\eta}_{1,2,3}, X, \bar{X}) = t(\eta_{2,1,3}, \bar{\eta}_{2,1,3}, -\bar{X}, -X). \quad (3.19)$$

Lastly, recalling that complex conjugation swaps the order of fermions, the reality condition (3.14) becomes

$$t(\eta_i, \bar{\eta}_i, X, \bar{X})^* = t(\bar{\eta}_i, \eta_i, \bar{X}, X). \quad (3.20)$$

Even though we have drastically simplified the problem, finding a complete basis of tensor structures is still a non-trivial task. In order to circumvent the issue of dealing with spinor identities but, at the same time, be sure we do not miss any contributions, it is important to derive separately the expected number of independent tensor structures appearing in (3.15). This counting is performed in the following section. After that we construct a complete basis for the cases of interest — namely three-point functions of operators with vanishing R-charge — by providing an equal number of independent tensor structures. Their independence can be easily proven by setting to zero the Grassmann coordinates  $\theta_{1,2}$  and  $\bar{\theta}_{1,2}$  and matching with the non-supersymmetric three-point functions

<sup>10</sup>This definition for the notation with square brackets differs from the one used in appendix A by a replacement  $X \rightarrow U$ , defined later in (4.3). The choice made here is less convenient for writing down the structures explicitly.

reviewed in section 2. The tensor structures associated to three-point functions of operators with non-zero R-charge can be read from the *Mathematica* notebook attached to this paper as supplementary material. They are constructed in the same way, and their number agrees with the counting of the next section as well. We also checked their linear independence by replacing numerical values for the various quantities that appear.

### 3.3 Counting supersymmetric tensor structures

In this section we obtain a group theoretical counting of the independent number of tensor structures appearing in (3.15) along the same lines as [33].

Let us analyze first the case of an operator  $\mathcal{O}_{j,\bar{j}}$  with zero R-charge. We can start by dividing the function  $t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta})$  into three parts. The first part contains neither  $\Theta$  nor  $\bar{\Theta}$ ; it is thus built with  $[ij]$ ,  $[i\bar{j}]$ ,  $[\bar{i}j]$  only. The second part is analogous to the previous one but with an overall  $\Theta^2\bar{\Theta}^2$  factor. The third part is instead built with exactly one  $\Theta$  and one  $\bar{\Theta}$ . In order to enumerate the structures in the first part we can simply follow a standard approach for non-supersymmetric CFTs. One possible way is to choose a conformal frame [33–35] that fixes all bosonic coordinates and breaks  $\text{Spin}(2,4) \rightarrow \text{Sp}(2,\mathbb{R})$ . After restricting the polarizations  $\eta_i$  and  $\bar{\eta}_i$  to this subgroup,  $\eta_{i\alpha}$  and  $\tilde{\eta}_{i\alpha} \equiv X_{\alpha\dot{\alpha}}\bar{\eta}_i^{\dot{\alpha}}$  transform in the same representation. Therefore we are allowed to make the contractions<sup>11</sup>

$$\eta_i\eta_j, \quad \eta_i\tilde{\eta}_j, \quad \tilde{\eta}_i\tilde{\eta}_j, \quad (3.21)$$

which can be easily lifted in a one-to-one way to  $[ij]$ ,  $[i\bar{j}]$  and  $[\bar{i}j]$ . For the purpose of the subsequent arguments it is better to state this reasoning backwards. The problem we need to address is the counting of independent structures built out of products of  $[ij]$ ,  $[i\bar{j}]$  and  $[\bar{i}j]$ , modded by all identities stemming from  $\epsilon_{\alpha[\beta}\epsilon_{\gamma\delta]} = 0$ . This is mathematically equivalent to counting  $\text{Sp}(2,\mathbb{R})$  invariant tensors built out of  $\eta_i$  and  $\tilde{\eta}_i$ , which are in one-to-one correspondence with the tensor structures in  $\langle \mathcal{J}_{1,1}(\eta_1, \bar{\eta}_1, x_1)\mathcal{J}_{1,1}(\eta_2, \bar{\eta}_2, x_2)\mathcal{O}_{j,\bar{j}}(\eta_3, \bar{\eta}_3, x_3) \rangle$ .

The second part presents no difference apart from the trivial  $\Theta^2\bar{\Theta}^2$  overall factor. The third part, instead, can be interpreted in the following way: since there is only one  $\Theta$  and only one  $\bar{\Theta}$  we can ignore the fact that they anticommute and replace them by a fourth pair of polarizations  $\eta_4, \bar{\eta}_4$ . In the same way as we argued before, the enumeration of all structures is equivalent to the enumeration of  $\text{Sp}(2,\mathbb{R})$  invariants built out of  $\eta_i$  and  $\tilde{\eta}_i$ , where now  $i = 1, \dots, 4$ . The claim is that these are in one-to-one correspondence with a three-point function where the third operator transforms under the reducible representation of  $\text{SL}(2,\mathbb{C})$  given by  $(\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (\frac{1}{2}, \frac{1}{2})$ .<sup>12</sup> This follows because irreducible representations are tensors of the form  $\mathcal{O}_{(\alpha_1\dots\alpha_j);(\dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}})}(x)$ , where parentheses denote symmetrization. In the index-free notation this condition is enforced by contracting all indices with the same  $\eta$ . Tensors not corresponding to irreducible representations must be contracted with

<sup>11</sup>The contractions are made with  $\epsilon_{\alpha\beta}$  and  $\epsilon^{\alpha\beta}$ . All indices are undotted at this point.

<sup>12</sup>The choice of attaching the polarizations  $\eta_4, \bar{\eta}_4$  to the third operator is arbitrary and does not affect the result. In this case it is convenient because we want to keep manifest the permutation symmetry in the first two points.

independent polarizations. For example, the following operator belongs to  $(\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (\frac{1}{2}, \frac{1}{2})$ ,

$$\begin{aligned} \mathcal{O}_{(\alpha_1 \dots \alpha_j)\beta; (\dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})\dot{\beta}}(x) &\rightarrow \mathcal{O}(x, \eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2) \\ &\equiv \eta_1^{\alpha_1} \dots \eta_1^{\alpha_j} \bar{\eta}_1^{\dot{\alpha}_1} \dots \bar{\eta}_1^{\dot{\alpha}_{\bar{j}}} \eta_2^\beta \bar{\eta}_2^{\dot{\beta}} \mathcal{O}_{(\alpha_1 \dots \alpha_j)\beta; (\dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})\dot{\beta}}(x). \end{aligned}$$

Now we are ready to perform the actual counting. In order to do so we will use the main formula derived in [33]

$$N = \left( \text{Res}_{\text{SO}(1,2)}^{\text{SO}(1,3)} \rho_1 \otimes \rho_2 \otimes \rho_3 \right)^{\text{SO}(1,2)}. \quad (3.22)$$

Here  $\rho_1 = \rho_2 = (\frac{1}{2}, \frac{1}{2})$  and  $\rho_3$  is the spin representation of the third operator. The notation  $\text{Res}_H^G$  indicates the restriction of a representation of  $G$  to a representation of  $H \subseteq G$ , the superscript  $(\rho)^H$  denotes the  $H$ -singlets in  $\rho$ . We assume that this formula generalizes for  $\rho_3$  not irreducible. Moreover, as remarked in [33],  $\text{Res}_H^G$  commutes with the tensor product. A last ingredient is necessary, namely the permutation symmetry of the first two points. This is taken care of in [33] as well. It is sufficient to replace  $\rho_1 \otimes \rho_2$  by  $S^2 \rho_1$  if  $j$  is even and by  $\wedge^2 \rho_1$  if  $j$  is odd, where  $S^2$  and  $\wedge^2$  denote respectively the symmetrized square and the exterior square of representations.

Assuming, now,  $j - \bar{j}$  even,<sup>13</sup> we can write down the formulae

$$\begin{aligned} N(j, \bar{j})^{(j \text{ even})} &= 2 \text{Res} \left( S^2 \left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) \right) + \text{Res} \left( \wedge^2 \left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) \otimes \left( \frac{1}{2}, \frac{1}{2} \right) \right), \\ N(j, \bar{j})^{(j \text{ odd})} &= 2 \text{Res} \left( \wedge^2 \left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) \right) + \text{Res} \left( S^2 \left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) \otimes \left( \frac{1}{2}, \frac{1}{2} \right) \right), \end{aligned} \quad (3.23)$$

where we abbreviated  $\text{Res}_{\text{SO}(1,2)}^{\text{SO}(1,3)}$  with  $\text{Res}$  and a superscript  $\text{SO}(1,2)$  in all terms is understood. We denote the number of independent structures in the three-point function  $\langle \mathcal{J}_{1,1} \mathcal{J}_{1,1} \mathcal{O}_{j,\bar{j}} \rangle$  with  $N(j, \bar{j})$ . The factor of “2” counts the first and second part. The second term comes from putting  $\rho_3 = (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}j, \frac{1}{2}\bar{j})$  and corresponds to the third part. Notice that when the first term has the  $S^2$  product the second has the  $\wedge^2$  product and vice versa. This is because the tensor product  $(\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (\frac{1}{2}, \frac{1}{2})$  contains representations with different parity w.r.t.  $(\frac{1}{2}j, \frac{1}{2}\bar{j})$ . The result can be computed with the well known relations

$$\text{Res} \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) = \bigoplus_{\ell=\frac{1}{2}|j-\bar{j}|}^{\frac{1}{2}(j+\bar{j})} \underline{\ell}, \quad \underline{\ell} \otimes \underline{\ell}' = \bigoplus_{k=|\ell-\ell'|}^{\ell+\ell'} \underline{k}, \quad (3.24)$$

where  $\underline{\ell}$  indicates the spin- $\ell$  representation of  $\text{SO}(1,2)$ . Finally the (anti)symmetrized products are given by

$$\begin{aligned} S^2 \underline{\ell} &= \bigoplus_{k=2\ell \bmod 2}^{2\ell} \underline{k}, & \wedge^2 \underline{\ell} &= \bigoplus_{k=2\ell+1 \bmod 2}^{2\ell} \underline{k}, \\ S^2 \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) &= (S^2 \frac{1}{2}j, S^2 \frac{1}{2}\bar{j}) \oplus (\wedge^2 \frac{1}{2}j, \wedge^2 \frac{1}{2}\bar{j}), \\ \wedge^2 \left( \frac{1}{2}j, \frac{1}{2}\bar{j} \right) &= (S^2 \frac{1}{2}j, \wedge^2 \frac{1}{2}\bar{j}) \oplus (\wedge^2 \frac{1}{2}j, S^2 \frac{1}{2}\bar{j}), \end{aligned} \quad (3.25)$$

<sup>13</sup>If  $j - \bar{j}$  is odd the result is trivially zero.

where  $S^2$  and  $\wedge^2$  inside the parenthesis  $(\frac{1}{2}j, \frac{1}{2}\bar{j})$ , stand for the direct sum of all possible pairs of the resulting irreps.

Collecting all the above results, the numbers of independent tensors structures consistent with permutation symmetry read

$$\begin{aligned}
 N(\ell, \ell)^{(\ell \text{ even})} &= 16, & N(\ell, \ell)^{(\ell \text{ odd})} &= 16, \\
 N(\ell + 2, \ell)^{(\ell \text{ even})} &= 10, & N(\ell + 2, \ell)^{(\ell \text{ odd})} &= 13, \\
 N(\ell + 4, \ell)^{(\ell \text{ even})} &= 4, & N(\ell + 4, \ell)^{(\ell \text{ odd})} &= 4.
 \end{aligned}
 \tag{3.26}$$

Let us now consider the case of a supermultiplet  $\mathcal{O}_{j, \bar{j}}$  with non-zero R-charge. In this case the superconformal primary does not contribute to the three-point function. As a consequence the structures in  $t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta})$  contain an overall  $\Theta, \bar{\Theta}, \Theta^2$  or  $\bar{\Theta}^2$ . If the R-charge is  $\pm 2$  the problem is readily solved by multiplying the non-supersymmetric three-point functions by  $\Theta^2$  or  $\bar{\Theta}^2$ . The counting is therefore the same as in section 2.

When instead the R-charge is  $\pm 1$  (say 1 for simplicity) we can derive a similar formula as in eq. (3.23). Here the structures can be divided into two parts, the first proportional to  $\Theta$  and the second proportional to  $\Theta^2 \bar{\Theta}$ . In both cases the free fermionic variable can be interpreted as an extra  $\eta_4$  or  $\bar{\eta}_4$  contracting a reducible operator  $\mathcal{O}$  belonging to, respectively,  $(\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (\frac{1}{2}, 0)$  or  $(\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (0, \frac{1}{2})$ . Therefore, taking  $j - \bar{j}$  odd<sup>14</sup> one finds

$$\begin{aligned}
 N(j, \bar{j})^{(j \text{ even})} &= \text{Res}(S^2(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (0, \frac{1}{2})) + \text{Res}(\wedge^2(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (\frac{1}{2}, 0)), \\
 N(j, \bar{j})^{(j \text{ odd})} &= \text{Res}(\wedge^2(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (0, \frac{1}{2})) + \text{Res}(S^2(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}j, \frac{1}{2}\bar{j}) \otimes (\frac{1}{2}, 0)).
 \end{aligned}
 \tag{3.27}$$

Again notice that the products  $S^2$  and  $\wedge^2$  are inverted in the two terms. The counting now gives

$$\begin{aligned}
 N(\ell \pm 1, \ell)^{(\ell \text{ even})} &= 10, & N(\ell \pm 1, \ell)^{(\ell \text{ odd})} &= 10, \\
 N(\ell \pm 3, \ell)^{(\ell \text{ even})} &= 5, & N(\ell \pm 3, \ell)^{(\ell \text{ odd})} &= 5, \\
 N(\ell \pm 5, \ell)^{(\ell \text{ even})} &= 1, & N(\ell \pm 5, \ell)^{(\ell \text{ odd})} &= 1.
 \end{aligned}
 \tag{3.28}$$

From this analysis we can deduce a recipe for counting structures of more general superconformal three-point functions. Let us assume that all operators are different and belong to  $SO(1, 3)$  representations  $\rho_1, \rho_2$  and  $\rho_3$  respectively. The cases with non-trivial permutation symmetries can be treated similarly as above. Every function  $t(\eta_i, \bar{\eta}_i, X, \Theta, \bar{\Theta})$  can contain a subset of the monomials

$$\Theta^0 \bar{\Theta}^0, \quad \Theta^\alpha, \quad \bar{\Theta}^2 \Theta^\alpha, \quad \bar{\Theta}^{\dot{\alpha}}, \quad \Theta^\alpha \bar{\Theta}^{\dot{\alpha}}, \quad \Theta^2 \bar{\Theta}^{\dot{\alpha}}, \quad \Theta^2, \quad \bar{\Theta}^2, \quad \Theta^2 \bar{\Theta}^2.
 \tag{3.29}$$

Which ones are present depends on the R-charges of the operators  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ , which we will call  $r_1, r_2, r_3$ . Define

$$\delta = r_3 - r_1 - r_2.
 \tag{3.30}$$

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<sup>14</sup>The even case can be treated similarly and it trivially gives zero.

The possible values are  $\delta = \pm 2, \pm 1, 0$ . Let us denote as  $N_X(\rho_1, \rho_2, \rho_3)$  the number of structures of a given order  $X$  in  $\Theta, \bar{\Theta}$ , where  $X$  is any monomial in (3.29). Following the analysis above we have

$$\begin{aligned}
 N_1(\rho_i) &= N_{\Theta^2\bar{\Theta}^2}(\rho_i) = N_{\Theta^2}(\rho_i) = N_{\bar{\Theta}^2}(\rho_i) = \text{Res } \rho_1 \otimes \rho_2 \otimes \rho_3, \\
 N_{\Theta}(\rho_i) &= N_{\Theta\bar{\Theta}^2}(\rho_i) = \text{Res } \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \left(\frac{1}{2}, 0\right), \\
 N_{\bar{\Theta}}(\rho_i) &= N_{\Theta^2\bar{\Theta}}(\rho_i) = \text{Res } \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \left(0, \frac{1}{2}\right), \\
 N_{\Theta\bar{\Theta}}(\rho_i) &= \text{Res } \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \left(\frac{1}{2}, \frac{1}{2}\right).
 \end{aligned} \tag{3.31}$$

where again  $\text{Res} \equiv \text{Res}_{\text{SO}(1,2)}^{\text{SO}(1,3)}$  and a superscript  $\text{SO}(1,2)$  in all terms is understood. Then the general formula for the number  $N(\rho_1, \rho_2, \rho_3; \delta)$  of tensor structures in the three-point function  $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$  may be written as

$$N(\rho_1, \rho_2, \rho_3; \delta) = \begin{cases} 2N_1(\rho_1, \rho_2, \rho_3) + N_{\Theta\bar{\Theta}}(\rho_1, \rho_2, \rho_3) & \delta = 0, \\ N_1(\rho_1, \rho_2, \rho_3) & \delta = \pm 2, \\ N_{\Theta}(\rho_1, \rho_2, \rho_3) + N_{\bar{\Theta}}(\rho_1, \rho_2, \rho_3) & \delta = \pm 1. \end{cases} \tag{3.32}$$

### 3.4 Conserved tensor structures

To conclude this section we will address the issue of conservation. In particular we will study the consequence of (3.1) on a general three-point function  $\langle \mathcal{J} \mathcal{J} \mathcal{O} \rangle$ . For simplicity we will omit the superspace coordinate dependence. The constraints we need to impose are

$$\langle D^\alpha \mathcal{J}_{\alpha\dot{\alpha}} \mathcal{J}_{\beta\dot{\beta}} \mathcal{O}_{j,\bar{j}} \rangle = 0, \tag{3.33a}$$

$$\langle \bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} \mathcal{J}_{\beta\dot{\beta}} \mathcal{O}_{j,\bar{j}} \rangle = 0. \tag{3.33b}$$

These conditions are not independent; in fact there are linear relations between them. First we can observe that taking the derivative  $\bar{D}$  at the second point of (3.33a) and the derivative  $D$  at the second point of (3.33b) give the same result, modulo permuting the first two operators,

$$\langle D^\alpha \mathcal{J}_{\alpha\dot{\alpha}} \bar{D}^{\dot{\beta}} \mathcal{J}_{\beta\dot{\beta}} \mathcal{O}_{j,\bar{j}} \rangle = \langle \bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} D^\beta \mathcal{J}_{\beta\dot{\beta}} \mathcal{O}_{j,\bar{j}} \rangle \Big|_{1 \leftrightarrow 2}. \tag{3.34}$$

Moreover, by taking  $D$  of (3.33a) and permuting points  $z_1$  and  $z_2$  we obtain identically zero. The same holds if we take  $\bar{D}$  of (3.33b). The prescription to count the number of conserved tensor structures [33] is to take the number of non-conserved tensor structures, subtract all degrees of freedom contained in the equations (3.33) and add back all linear relations between such equations. The complication with supersymmetry is that a superspace equation decomposes into a certain number of ordinary bosonic equations by projecting on the various terms in (3.29). This depends on the R-charge of  $\mathcal{O}$ . Let us start assuming that  $\mathcal{O}$  is real. The conservation conditions impose a number of constraints equal to the number of tensor structures present in (3.33). This number is given by<sup>15</sup>

$$N_{\bar{\Theta}}(D\mathcal{J}, \mathcal{J}, \mathcal{O}) + N_{\Theta\bar{\Theta}^2}(D\mathcal{J}, \mathcal{J}, \mathcal{O}) + N_{\Theta}(\bar{D}\mathcal{J}, \mathcal{J}, \mathcal{O}) + N_{\Theta^2\bar{\Theta}}(\bar{D}\mathcal{J}, \mathcal{J}, \mathcal{O}). \tag{3.36}$$

<sup>15</sup>We denote the various representations in  $N_X(\dots)$  in the following way:

$$\mathcal{J} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad D\mathcal{J} = \left(0, \frac{1}{2}\right), \quad \bar{D}\mathcal{J} = \left(\frac{1}{2}, 0\right), \quad \mathcal{O} = \left(\frac{1}{2}j, \frac{1}{2}\bar{j}\right). \tag{3.35}$$

Even though  $N_{\bar{\Theta}} = N_{\Theta^2\bar{\Theta}}$ , etc., we keep them distinct to track down the various contributions. As anticipated, however, not all the tensor structures in (3.33) give a non trivial constraint. This is a consequence of the fact that the three-point functions  $D$  (3.33a) and  $\bar{D}$  (3.33b) are made of identical operators. To keep this into account one must subtract from (3.36) the numbers

$$N_{\Theta^2}(D\mathcal{J}, D\mathcal{J}, \mathcal{O}), \quad (3.37a)$$

$$N_{\Theta^2}(\bar{D}\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}). \quad (3.37b)$$

Similarly, given the relation  $\bar{D}$  (3.33a)  $\sim D$  (3.33b), we should naïvely subtract from (3.36) the number

$$N_1(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}) + N_{\Theta\bar{\Theta}}(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}) + N_{\Theta^2\bar{\Theta}^2}(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}). \quad (3.38)$$

However, the above expression would give rise to an over-counting: the conditions given by  $N_{\Theta^2\bar{\Theta}^2}(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O})$  and by  $N_1(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O})$  are dependent. Indeed, by using a suitable representation of the differential operators,<sup>16</sup> one can show that the terms  $\Theta^2\bar{\Theta}^2$  cannot be generated by applying  $D$  and  $\bar{D}$  on  $\langle \mathcal{J}\mathcal{J}\mathcal{O} \rangle$ . The correct counting is then

$$\begin{aligned} N(\mathcal{J}, \mathcal{J}, \mathcal{O}; 0)^{(\text{cons.})} &= N(\mathcal{J}, \mathcal{J}, \mathcal{O}; 0) - N(D\mathcal{J}, \mathcal{J}, \mathcal{O}; 1) - N(\bar{D}\mathcal{J}, \mathcal{J}, \mathcal{O}; -1) \\ &\quad + N(D\mathcal{J}, D\mathcal{J}, \mathcal{O}; 2) + N(\bar{D}\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}; -2) \\ &\quad + N_1(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}) + N_{\Theta\bar{\Theta}}(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}). \end{aligned} \quad (3.39)$$

In addition, since the currents  $\mathcal{J}$  are identical, we need to take into account the permutation symmetry as we explained in the previous section by replacing the product  $\rho_1 \otimes \rho_2$  by either  $S^2\rho_1$  or  $\wedge^2\rho_1$ . There is a subtlety in the (anti)symmetrization of two  $D\mathcal{J}$ 's or two  $\bar{D}\mathcal{J}$ 's: these operators get an extra minus due to their fermionic nature. Thus for  $\ell$  even (odd) we must take the  $S^2$  ( $\wedge^2$ ) product in  $N(D\mathcal{J}, D\mathcal{J}, \mathcal{O}; -2)$  and  $N(\bar{D}\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}; 2)$ . Explicitly this formula yields

$$\begin{aligned} N(\ell, \ell)^{(\ell \text{ even, cons.})} &= 2, & N(\ell, \ell)^{(\ell \text{ odd, cons.})} &= 2, \\ N(\ell + 2, \ell)^{(\ell \text{ even, cons.})} &= 1, & N(\ell + 2, \ell)^{(\ell \text{ odd, cons.})} &= 2, \\ N(\ell + 4, \ell)^{(\ell \text{ even, cons.})} &= 1, & N(\ell + 4, \ell)^{(\ell \text{ odd, cons.})} &= 1, \end{aligned} \quad (3.40)$$

where, as before,  $N(j, \bar{j})$  is a shorthand for  $N(\mathcal{J}, \mathcal{J}, (\frac{1}{2}j, \frac{1}{2}\bar{j}); 0)$ .

We can similarly obtain the respective formulas when  $\mathcal{O}$  has non-zero R-charge. Without loss of generality we take the R-charge to be negative.<sup>17</sup> Skipping the details of the derivation we show the answer for  $R = -1$ ,

$$\begin{aligned} N(\mathcal{J}, \mathcal{J}, \mathcal{O}; -1)^{(\text{cons.})} &= N(\mathcal{J}, \mathcal{J}, \mathcal{O}; 1) - N(\bar{D}\mathcal{J}, \mathcal{J}, \mathcal{O}; -2) - N_1(D\mathcal{J}, \mathcal{J}, \mathcal{O}) \\ &\quad - N_{\Theta\bar{\Theta}}(D\mathcal{J}, \mathcal{J}, \mathcal{O}) + N_{\Theta}(D\mathcal{J}, D\mathcal{J}, \mathcal{O}) + N_{\bar{\Theta}}(D\mathcal{J}, \bar{D}\mathcal{J}, \mathcal{O}), \end{aligned} \quad (3.41)$$

<sup>16</sup>See appendix D, specifically eq. (D.3a).

<sup>17</sup>The other case can be obtained by complex conjugation. In order to prove the same formula for  $R = 1$  we would need a representation of the differential operators where  $\bar{D}_{\dot{\alpha}} \rightarrow \partial/\partial\Theta^{\alpha}$ , namely (D.3b).

and for  $R = -2$ ,

$$N(\mathcal{J}, \mathcal{J}, \mathcal{O}; -2)^{(\text{cons.})} = N(\mathcal{J}, \mathcal{J}, \mathcal{O}; -2) - N_{\bar{\Theta}}(D\mathcal{J}, \mathcal{J}, \mathcal{O}) + N_1(D\mathcal{J}, D\mathcal{J}, \mathcal{O}). \quad (3.42)$$

In all cases with non-zero R-charge (3.41) and (3.42) yield non-positive results. Therefore we conclude that there are no structures allowed after conservation, as mentioned in section 3.1.

## 4 Ward identities and their solution

In this section we explore the consequences of the Ferrara-Zumino shortening conditions, eq. (3.1). For each of the cases (A)–(C) we write the correlator in superspace in terms of tensor structures satisfying the conditions (3.10), (3.13) and eventually (3.14). One can straightforwardly check their independence, and since they match in number the predictions of section 3.3, they form a basis of superconformal three-point functions.

### 4.1 Case A: $(\frac{1}{2}\ell, \frac{1}{2}\ell)$ operators

We begin by considering the case where  $\mathcal{O}$  is a spin- $\ell$  Lorentz representation with zero R-charge. This means that  $j = \bar{j} = \ell$ , i.e.  $\mathcal{O}$  has  $\ell$  dotted and  $\ell$  undotted indices, which are symmetrized separately, and  $q = \bar{q} = \frac{1}{2}\Delta$ . The unitarity bound (3.5) simply becomes

$$\Delta \geq \ell + 2, \quad (4.1)$$

which agrees with the usual non-supersymmetric unitarity bounds, eq. (2.4).

#### 4.1.1 Even $\ell$

When all Grassmann variables are set to zero, then there are four possible parity-even structures in  $t_A$  for  $\ell$  even [7]. The Ward identity that follows from (3.1) will relate these structures, leaving, in the end, at most two independent parity-even structures [7]. In our case there may also be structures that become identically zero when all Grassmann variables are set to zero.

It is easier to perform the Ward-identity analysis by first introducing auxiliary commuting spinors  $\eta_i, \bar{\eta}_i$ ,  $i = 1, 2, 3$ , as in (3.16). We can use these spinors to write the three-point function in the form<sup>18</sup>

$$\begin{aligned} & \langle \mathcal{J}(\eta'_1, \bar{\eta}'_1, z_1) \mathcal{J}(\eta'_2, \bar{\eta}'_2, z_2) \mathcal{O}_\ell(\eta_3, \bar{\eta}_3, z_3) \rangle \\ &= \frac{\eta'_1 x_{1\bar{3}} \partial_{\bar{\eta}_1} \partial_{\eta_1} x_{3\bar{1}} \bar{\eta}'_1 \eta'_2 x_{2\bar{3}} \partial_{\bar{\eta}_2} \partial_{\eta_2} x_{3\bar{2}} \bar{\eta}'_2}{(x_{\bar{3}1}^2 x_{\bar{1}3}^2 x_{\bar{3}2}^2 x_{\bar{2}3}^2)^2} t_A(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}), \end{aligned} \quad (4.2)$$

with the definition

$$U = \frac{1}{2}(X + \bar{X}). \quad (4.3)$$

---

<sup>18</sup>In order to have a consistent treatment of all cases we have chosen to express  $t_A^{(\ell \text{ even})}$  as a function of  $U, \Theta \bar{\Theta}$  instead of  $X, \bar{X}$ .

As anticipated in section 3.3, we can parametrize  $t_A$  in terms of 16 coefficients,

$$t_A^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta\bar{\Theta}) = \frac{(\eta_3 U \bar{\eta}_3)^{\ell-2}}{U^{4-\Delta+\ell}} \left( \sum_{i=1}^4 \mathcal{P}_A^{(i)} (A_i + B_i \xi^2) + \sum_{i=1}^8 \mathcal{P}'_A^{(i)} C_i \right), \quad \xi^2 = \frac{\Theta^2 \bar{\Theta}^2}{U^2}, \quad (4.4)$$

where the tensor structures  $\mathcal{P}_A^{(i)}$  and  $\mathcal{P}'_A^{(i)}$  are given in appendix A. When all the variables  $\theta_i$  and  $\bar{\theta}_i$  are set to zero, only the structures  $\mathcal{P}_A^{(i)}$  survive, while all the others vanish. Hence, the coefficients  $A_i$  must be related to the coefficients  $\lambda^{(j)}$  introduced in section 2 for the case of traceless symmetric tensors ( $k = 0$ ). It is straightforward to find

$$\lambda^{(1)} = -(A_1 + A_3 - A_4), \quad \lambda^{(2)} = -\lambda^{(3)} = -2A_4, \quad \lambda^{(4)} = A_2, \quad \lambda^{(5)} = A_1 - A_3 - A_4. \quad (4.5)$$

The Ward identities following from (3.1) are satisfied if

$$\partial_{\eta_1} \mathcal{D} t_A^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta\bar{\Theta}) = 0. \quad (4.6)$$

Equation (4.6) decomposes into fourteen independent equations for the sixteen *a priori* independent coefficients  $A_1, \dots, C_8$ .<sup>19</sup> We may express all coefficients in terms of  $A_1$  and  $A_2$  to find

$$\begin{aligned} A_3 &= -\frac{(\Delta-\ell-6)(\Delta-\ell-2)}{3(\Delta-2)^2-\ell(\ell+2)} A_1 + \frac{(\Delta-\ell-6)(\Delta-\ell-4)}{4(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ A_4 &= -\frac{2(\Delta-3)\ell}{3(\Delta-2)^2-\ell(\ell+2)} A_1 - \frac{(\Delta-\ell-4)(3\Delta+\ell-6)}{4(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ B_1 &= -\frac{1}{4}(\Delta-\ell-6)(\Delta+\ell-4) A_1, \\ B_2 &= \frac{8(\Delta-3)(\ell-1)\ell}{3(\Delta-2)^2-\ell(\ell+2)} A_1 - \frac{(\Delta-\ell-4)(\Delta+\ell-2)(3(\Delta-4)^2-\ell(\ell+2))}{4(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ B_3 &= \frac{(\Delta-\ell-8)(\Delta-\ell-6)(\Delta-\ell-2)(\Delta+\ell-2)}{4(3(\Delta-2)^2-\ell(\ell+2))} A_1 - \frac{(\Delta-\ell-8)(\Delta-\ell-6)(\Delta-\ell-4)(\Delta+\ell-2)}{16(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ B_4 &= \frac{\ell(\Delta-\ell-6)((\Delta-5)(\Delta+\ell)+8)}{2(3(\Delta-2)^2-\ell(\ell+2))} A_1 + \frac{(\Delta-\ell-6)(\Delta-\ell-4)(\Delta+\ell-2)(3\Delta+\ell-12)}{16(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ C_1 &= \frac{(\Delta-\ell-6)(\Delta-\ell-2)(\Delta+\ell-2)}{3(\Delta-2)^2-\ell(\ell+2)} A_1 - \frac{(\Delta-\ell-6)(\Delta-\ell-4)(\Delta+\ell-2)}{4(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ C_2 &= \frac{4(\Delta-3)(\ell-1)\ell}{3(\Delta-2)^2-\ell(\ell+2)} A_1 + \frac{3(\Delta-3)(\Delta-\ell-4)(\Delta+\ell-2)}{2(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ C_3 &= C_6 = C_7 = C_8 = 0, \\ C_4 &= \frac{2\ell(\Delta^2-4\Delta-\ell^2-2\ell+6)}{3(\Delta-2)^2-\ell(\ell+2)} A_1 - \frac{(\ell+3)(\Delta-\ell-4)(\Delta+\ell-2)}{2(3(\Delta-2)^2-\ell(\ell+2))} A_2, \\ C_5 &= -\frac{2\ell(\ell-1)(\Delta-\ell-6)}{3(\Delta-2)^2-\ell(\ell+2)} A_1 - \frac{3(\Delta-\ell-6)(\Delta-\ell-4)(\Delta+\ell-2)}{4(3(\Delta-2)^2-\ell(\ell+2))} A_2. \end{aligned} \quad (4.7)$$

The above expressions are divergent if we set  $\Delta = 2$  and  $\ell = 0$ . The special value  $\ell = 0$  is discussed separately in appendix C. In that case there is then only one undetermined coefficient, consistently with the results of [23].

<sup>19</sup>For this it is crucial to use identities that stem from  $\epsilon_{\alpha[\beta\epsilon\gamma\delta]} = 0$  and the corresponding identity with dotted indices. Examples include the identities in [25, appendix E]. In this work we circumvent the need to impose such identities by substituting numerical values for the various quantities that appear. This is equivalent to working in the superconformal frame.



### 4.1.2 Odd $\ell$

For odd spins we may write

$$t_A^{(\ell \text{ odd})}(\eta_i, \bar{\eta}_i, U, \Theta\bar{\Theta}) = \frac{(\eta_3 U \bar{\eta}_3)^{\ell-3}}{U^{3-\Delta+\ell}} \left( \sum_{i=1}^2 \tilde{\mathcal{P}}_A^{(i)} (A_i + B_i \xi^2) + \sum_{i=1}^3 \tilde{\mathcal{P}}_A^{\prime(i)} (C_i \zeta + D_i \zeta') + \sum_{i=1}^6 \tilde{\mathcal{P}}_A^{\prime\prime(i)} E_i \right),$$

$$\xi^2 = \frac{\Theta^2 \bar{\Theta}^2}{U^2}, \quad \zeta = \frac{\Theta \eta_3 \bar{\Theta} \bar{\eta}_3}{|U|}, \quad \zeta' = \frac{\Theta U \bar{\Theta}}{|U|^{3/2}} \eta_3 U \bar{\eta}_3, \quad (4.8)$$

where  $A_i, B_i, C_i, D_i, E_i$  are real constants and the tensor structures  $\tilde{\mathcal{P}}_A^{(i)}, \tilde{\mathcal{P}}_A^{\prime(i)}$  and  $\tilde{\mathcal{P}}_A^{\prime\prime(i)}$  are given in appendix A.

In the lowest component of the three-point function (4.2) for general odd spin  $\ell$  there is a parity-odd and a parity-even structure, with respective coefficients denoted by  $\lambda^{(-)}$  and  $\lambda^{(2)}$  in section 2. In terms of  $A_1$  and  $A_2$  of (4.8) we find

$$\lambda^{(-)} = 2iA_1, \quad \lambda^{(2)} = -2A_2. \quad (4.9)$$

Again, the shortening condition (3.1) implies

$$\partial_{\eta_1} \mathcal{D} t_A^{(\ell \text{ odd})}(\eta_i, \bar{\eta}_i, U, \Theta\bar{\Theta}) = 0. \quad (4.10)$$

For generic  $\Delta, \ell$  this gives fourteen independent equations for the sixteen unknowns  $A_1, \dots, E_6$ , and thus there are two undetermined coefficients, just as in the even-spin case. If we choose  $A_1$  and  $C_1$  as independent we have

$$\begin{aligned} A_2 &= B_2 = E_5 = E_6 = 0, \\ B_1 &= -\frac{1}{4}(\Delta - \ell - 6)(\Delta + \ell - 4)A_1, \\ C_2 &= \frac{2(\Delta - \ell - 6)(\Delta - \ell - 2)(2\Delta + \ell - 4)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1 + \frac{(\Delta - \ell - 6)(\Delta - \ell - 4)(2\Delta + \ell - 4)}{4(\Delta - 2)(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1, \\ C_3 &= -2(\Delta - 4)A_1, \\ D_1 &= -\frac{8(\ell - 1)(\Delta - \ell - 2)(\Delta + \ell)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1 - \frac{(\Delta - \ell - 4)(5\Delta^2 + 2\Delta\ell - 22\Delta - \ell^2 - 8\ell + 24)}{2(\Delta - 2)(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1, \\ D_2 &= -\frac{(\Delta - \ell - 8)(\Delta - \ell - 6)(\Delta - \ell - 2)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1 - \frac{(\Delta - \ell - 8)(\Delta - \ell - 6)(\Delta - \ell - 4)}{8(\Delta - 2)(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1, \\ D_3 &= (\Delta - \ell - 6)A_1, \\ E_1 &= \frac{(\Delta - \ell - 4)(\Delta + \ell)}{4(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1 \\ &\quad + \frac{2(\Delta^3 - 9\Delta^2 - \Delta\ell^2 - 2\Delta\ell + 24\Delta + 5\ell^2 + 10\ell - 24)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1, \\ E_2 &= \frac{(\Delta - \ell - 6)(\Delta - \ell - 4)(2\Delta + \ell - 4)}{4(\Delta - 2)(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1 \\ &\quad - \frac{(\Delta - \ell - 6)(\Delta^2 + 2\Delta\ell - 4\Delta - \ell^2 - 10\ell + 8)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1, \\ E_3 &= -\frac{(\Delta - \ell - 6)(\Delta - \ell - 2)(\Delta + \ell)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1 - \frac{(\Delta - \ell - 6)(\Delta - \ell - 4)(\Delta + \ell)}{8(\Delta - 2)(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1, \\ E_4 &= \frac{2(\ell - 1)(3\Delta^2 - 16\Delta - \ell^2 - 2\ell + 24)}{5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3)}A_1 \\ &\quad - \frac{(\Delta - \ell - 4)(5\Delta^2 + 2\Delta\ell - 22\Delta - \ell^2 - 8\ell + 24)}{4(\Delta - 2)(5(\Delta - 3)(\Delta - 1) - 3(\ell - 1)(\ell + 3))}C_1. \end{aligned} \quad (4.11)$$

The result  $A_2 = 0$  is expected because for conserved currents there is only one parity-odd structure in  $t_A^{(\ell \text{ odd})}$  if all Grassmann variables are set to zero [7]. If we set  $\ell = 1$  and  $\Delta = 3$ , corresponding to the third operator in the three-point function being the supercurrent, we find a singularity in the above expression. This is just an artifact of the choice of variables and is resolved in appendix C. In the same appendix we also show the relation between the coefficients defined here and the anomaly coefficients  $a$  and  $c$ .

## 4.2 Case B: $(\frac{1}{2}(\ell + 2), \frac{1}{2}\ell)$ operators

Here we start with

$$\begin{aligned} & \langle \mathcal{J}(\eta_1, \bar{\eta}_1, z_1) \mathcal{J}(\eta_2, \bar{\eta}_2, z_2) \mathcal{O}_{\ell+2, \ell}(\eta_3, \bar{\eta}_3, z_3) \rangle \\ &= \frac{\eta_{1x_{1\bar{3}}} \partial_{\bar{\eta}'_1} \partial_{\eta'_1 x_{3\bar{1}}} \bar{\eta}_1 \eta_{2x_{2\bar{3}}} \partial_{\bar{\eta}'_2} \partial_{\eta'_2 x_{3\bar{2}}} \bar{\eta}_2}{(x_{\bar{3}1}{}^2 x_{1\bar{3}}{}^2 x_{\bar{3}2}{}^2 x_{2\bar{3}}{}^2)^2} t_B(\eta'_{1,2}, \eta_3, \bar{\eta}'_{1,2}, \bar{\eta}_3, U, \Theta \bar{\Theta}), \end{aligned} \quad (4.12)$$

where  $\mathcal{O}$  has zero R-charge. The unitarity bound is

$$\Delta \geq \ell + 4. \quad (4.13)$$

Similarly to (3.18) and (3.19) we must impose the homogeneity property

$$t_B(\eta_{1,2}, \kappa \eta_3, \bar{\eta}_{1,2}, \bar{\kappa} \bar{\eta}_3, \lambda \bar{\lambda} U, \lambda \Theta \bar{\lambda} \bar{\Theta}) = (\lambda \bar{\lambda})^{\Delta-6} \kappa^{\ell+2} \bar{\kappa}^\ell t_B(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}). \quad (4.14)$$

The symmetry properties for the first two points is identical to (3.19). However, unlike in case (A), we do not have a reality condition as in (3.20).

### 4.2.1 Even $\ell$

For general even  $\ell$ , we can parametrize the correlator as

$$t_B^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = \frac{(\eta_3 U \bar{\eta}_3)^{\ell-1}}{U^{5-\Delta+\ell}} \left( \sum_{i=1}^2 \mathcal{P}_B^{(i)} (A_i + B_i \xi^2) + \sum_{i=1}^6 \mathcal{P}'_B^{(i)} C_i \right), \quad \xi^2 = \frac{\Theta^2 \bar{\Theta}^2}{U^2}, \quad (4.15)$$

where the structures  $\mathcal{P}_B^{(i)}$  and  $\mathcal{P}'_B^{(i)}$  are defined in appendix A. At the lowest order in  $\theta_i, \bar{\theta}_i$  only the structures  $\mathcal{P}_B^{(i)}$  survive and they must reproduce the tensor structures  $\mathcal{T}^{(i)}$  introduced in section 2. The relations between  $A_1$  and  $A_2$  of (4.15) and the  $\lambda^{(i)}$  in (2.1) is

$$\lambda^{(1)} = -\lambda^{(2)} = A_1, \quad \lambda^{(3)} = \lambda^{(4)} = -2A_2. \quad (4.16)$$

The Ward identity this time requires two independent conditions,

$$\partial_{\eta_i} \mathcal{D} t_B^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = 0, \quad \partial_{\bar{\eta}_i} \bar{\mathcal{D}} t_B^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = 0, \quad (4.17)$$

due to the lack of a reality condition on  $t_B$ . This leads, for generic  $\Delta$  and  $\ell$ , to nine linear constraints for the ten unknowns  $A_1, \dots, C_6$ . Solving in terms of  $A_1$  we find

$$\begin{aligned}
 A_2 &= -\frac{\Delta-\ell-6}{4(\Delta-2)}A_1, & B_1 &= -\frac{(\Delta-4)(\Delta-\ell-6)(\Delta+\ell-2)}{4(\Delta-2)}A_1, \\
 B_2 &= \frac{(\Delta-\ell-8)(\Delta-\ell-6)(\Delta+\ell-2)}{16(\Delta-2)}A_1, & C_1 &= \frac{i(\Delta-\ell-6)(\Delta+\ell-2)}{4(\Delta-2)}A_1, \\
 C_2 &= -\frac{3i(\Delta-\ell-6)(\Delta+\ell-2)}{4(\Delta-2)}A_1, & C_3 &= \frac{i(\Delta-\ell-6)(2\Delta+\ell-4)}{\Delta-2}A_1, \\
 C_4 &= \frac{i(2\Delta-\ell-8)(\Delta+\ell-2)}{\Delta-2}A_1, & C_5 &= -\frac{i(\Delta-\ell-6)(3\Delta+\ell-8)}{4(\Delta-2)}A_1, \\
 C_6 &= -\frac{i(\Delta-\ell-8)(\Delta-\ell-6)}{4(\Delta-2)}A_1.
 \end{aligned} \tag{4.18}$$

#### 4.2.2 Odd $\ell$

For general odd  $\ell$  we find the independent structures

$$t_B^{(\ell \text{ odd})}(\eta_i, \bar{\eta}_i, U, \Theta\bar{\Theta}) = \frac{(\eta_3 U \bar{\eta}_3)^{\ell-2}}{U^{4-\Delta+\ell}} \left( \sum_{i=1}^2 \tilde{\mathcal{P}}_B^{(i)} (A_i + B_i \xi^2) + \sum_{i=1}^9 \tilde{\mathcal{P}}_B^{\prime(i)} C_i \right), \quad \xi^2 = \frac{\Theta^2 \bar{\Theta}^2}{U^2}, \tag{4.19}$$

where the structures  $\tilde{\mathcal{P}}_B^{(i)}$  and  $\tilde{\mathcal{P}}_B^{\prime(i)}$  are defined in appendix A. Similarly to the even- $\ell$  case, in the lowest component of the three-point function (4.12) for general odd  $\ell$  there are two structures, which are related to the  $\lambda^{(i)}$  in (2.1) by

$$\lambda^{(1)} = \lambda^{(2)} = A_1 - A_2, \quad \lambda^{(3)} = -\lambda^{(4)} = -A_2. \tag{4.20}$$

We now need to impose the conservation at the first two points. For generic  $\Delta$  and  $\ell$  this gives eleven independent linear constraints for the thirteen unknowns  $A_1, \dots, C_9$ . Therefore there are two undetermined coefficients, which we choose to be  $A_1$  and  $C_1$ . The result is

$$\begin{aligned}
 A_2 &= \frac{\Delta-\ell-6}{\Delta+\ell-2}A_1, & B_1 &= -\frac{1}{4}(\Delta-\ell-6)(\Delta+\ell-4)A_1, \\
 B_2 &= -\frac{1}{4}(\Delta-\ell-8)(\Delta-\ell-6)A_1, \\
 C_2 &= -\frac{i(\Delta-\ell-8)(\Delta-\ell-6)(\Delta-\ell-2)}{2(5(\Delta-2)^2-\ell(\ell+4))}A_1 + \frac{(\Delta-\ell-8)(\Delta-\ell-6)}{2(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_3 &= -\frac{i(\Delta-\ell-6)(\Delta+\ell+2)(3(\Delta-2)^2+\ell(\ell-4))}{6(\Delta+\ell-2)(5(\Delta-2)^2-\ell(\ell+4))}A_1 - \frac{(\Delta-\ell-6)(\Delta+\ell+2)}{3(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_4 &= \frac{2i\ell(3\Delta-\ell-8)(\Delta+\ell+2)}{3(5(\Delta-2)^2-\ell(\ell+4))}A_1 - \frac{2(5(\Delta-4)(\Delta-2)-\ell(2\Delta+\ell))}{3(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_5 &= -\frac{2i\ell(\Delta-\ell-6)(\Delta-\ell-2)(3\Delta+\ell-4)}{3(\Delta+\ell-2)(5(\Delta-2)^2-\ell(\ell+4))}A_1 + \frac{(\Delta-\ell-6)(5\Delta+\ell-10)}{3(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_6 &= -\frac{2i\ell(\Delta-3)(\Delta-\ell-6)}{5(\Delta-2)^2-\ell(\ell+4)}A_1 - \frac{(\Delta-\ell-6)(5\Delta+\ell-10)}{2(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_7 &= -\frac{i(\Delta-\ell-8)(\Delta-\ell-6)(3(\Delta-2)^2+\ell(\ell-4))}{3(\Delta+\ell-2)(5(\Delta-2)^2-\ell(\ell+4))}A_1 - \frac{2(\Delta-\ell-8)(\Delta-\ell-6)}{3(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_8 &= \frac{i(\Delta-3)(\Delta-\ell-6)(\Delta+\ell-2)}{5(\Delta-2)^2-\ell(\ell+4)}A_1 + \frac{(\Delta-\ell-6)(3\Delta+\ell-4)}{2(5(\Delta-2)^2-\ell(\ell+4))}C_1, \\
 C_9 &= i(\Delta-\ell-6)A_1.
 \end{aligned} \tag{4.21}$$

In the previous cases, studied in section 4.1 and section 4.2.1, imposing conservation at points  $z_1$  and  $z_2$  automatically implied conservation at point  $z_3$  when the dimension of the multiplet  $\mathcal{O}(z_3)$  saturates the unitarity bound ((4.1) or (4.13) respectively). In the present case this is not true: the conservation condition  $D_3^{\gamma_1} \mathcal{O}_{\gamma_1 \dots \gamma_{\ell+2}; \dot{\gamma}_1 \dots \dot{\gamma}_\ell} = 0$  when  $\Delta = \ell + 4$ , or equivalently

$$\partial_{\eta_3} D_3 \langle \mathcal{J}(\eta_1, \bar{\eta}_1, z_1) \mathcal{J}(\eta_2, \bar{\eta}_2, z_2) \mathcal{O}_{\ell+2, \ell}(\eta_3, \bar{\eta}_3, z_3) \rangle \Big|_{\Delta=\ell+4} = 0, \quad (4.22)$$

gives rise to a further non-trivial constraint, which is solved if  $C_1$  takes the form

$$C_1 = \frac{2i\ell(\ell^2 + \ell - 2)}{(\ell + 4)(\ell + 5)} A_1 + (\Delta - \ell - 4) C_1', \quad (4.23)$$

with  $C_1'$  another arbitrary constant.

### 4.3 Case C: $(\frac{1}{2}(\ell + 4), \frac{1}{2}\ell)$ operators

Here we start with

$$\begin{aligned} & \langle \mathcal{J}(\eta_1, \bar{\eta}_1, z_1) \mathcal{J}(\eta_2, \bar{\eta}_2, z_2) \mathcal{O}_{\ell+4, \ell}(\eta_3, \bar{\eta}_3, z_3) \rangle \\ &= \frac{\eta_1 x_{1\bar{3}} \partial_{\bar{\eta}_1'} \partial_{\eta_1' x_{3\bar{1}}} \bar{\eta}_1 \eta_2 x_{2\bar{3}} \partial_{\bar{\eta}_2'} \partial_{\eta_2' x_{3\bar{2}}} \bar{\eta}_2}{(x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2)^2} t_C(\eta_{1,2}, \eta_3, \bar{\eta}_{1,2}, \bar{\eta}_3, U, \Theta \bar{\Theta}), \end{aligned} \quad (4.24)$$

where  $\mathcal{O}$  has zero R-charge. The unitarity bound is

$$\Delta \geq \ell + 6, \quad (4.25)$$

and homogeneity property now takes the form

$$t_C(\eta_{1,2}, \kappa \eta_3, \bar{\eta}_{1,2}, \bar{\kappa} \bar{\eta}_3, \lambda \bar{\lambda} U, \lambda \Theta \bar{\lambda} \bar{\Theta}) = (\lambda \bar{\lambda})^{\Delta-6} \kappa^{\ell+4} \bar{\kappa}^\ell t_C(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}). \quad (4.26)$$

#### 4.3.1 Even $\ell$

In this case we parametrize the correlator in terms of four tensor structures

$$t_C^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = \frac{(\eta_3 U \bar{\eta}_3)^\ell}{U^{6-\Delta+\ell}} \left( \mathcal{P}_C^{(1)} (A + B \xi^2) + \sum_{i=2}^3 \mathcal{P}_C^{(i)} C_i \right), \quad \xi^2 = \frac{\Theta^2 \bar{\Theta}^2}{U^2}, \quad (4.27)$$

where as usual the quantities  $\mathcal{P}_C^{(i)}$  are defined in appendix A. At the lowest order in the Grassmann variables there is only one structure, and the associated parameter is related to the coefficient  $\lambda$  in (2.1) by

$$\lambda = -A. \quad (4.28)$$

Similarly to (4.17) we need to require

$$\partial_{\eta_1} \mathcal{D} t_C^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = 0, \quad \partial_{\bar{\eta}_1} \bar{\mathcal{D}} t_C^{(\ell \text{ even})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = 0. \quad (4.29)$$

This leads, for generic  $\Delta$  and  $\ell$ , to three linear constraints for the four unknowns  $A, \dots, C_2$ . Solving in terms of  $A$  we find

$$B = -\frac{1}{4}(\Delta - \ell - 8)(\Delta + \ell - 2)A, \quad C_1 = i(\Delta - \ell - 8)A, \quad C_2 = i(\Delta + \ell - 2)A. \quad (4.30)$$

Analogously to section 4.2.2, conservation at points  $z_{1,2}$  alone is not sufficient to ensure conservation at point  $z_3$  when  $\Delta$  saturates the bound (4.25). Instead, the constraint

$$\partial_{\eta_3} D_3 \langle \mathcal{J}(\eta_1, \bar{\eta}_1, z_1) \mathcal{J}(\eta_2, \bar{\eta}_2, z_2) \mathcal{O}_{\ell+4, \ell}(\eta_3, \bar{\eta}_3, z_3) \rangle = 0 \quad (4.31)$$

implies that  $A$  must be of the form

$$A = (\Delta - \ell - 6)A', \quad (4.32)$$

where  $A'$  is an arbitrary constant.

### 4.3.2 Odd $\ell$

Similarly to the even- $\ell$  case we have the four structures

$$t_C^{(\ell \text{ odd})}(\eta_i, \bar{\eta}_i, U, \Theta \bar{\Theta}) = \frac{(\eta_3 U \bar{\eta}_3)^{\ell-1}}{U^{5-\Delta+\ell}} \sum_{i=1}^4 \tilde{\mathcal{P}}_C^{(i)} C_i, \quad (4.33)$$

where the structures  $\tilde{\mathcal{P}}_C^{(i)}$  are defined in appendix A. This three-point function vanishes in the limit  $\theta_3, \bar{\theta}_3 \rightarrow 0$ , consistently with the fact that conformal invariance does not allow any structure for a three-point function of the form  $\langle J(x_1) J(x_2) \mathcal{O}_{\ell+4, \ell}(x_3) \rangle$  when  $\ell$  is odd.

The Ward identities for conservation at points  $z_{1,2}$  impose three constraints on these four constants. Thus, choosing  $C_1$  as independent, we obtain

$$C_2 = -\frac{\Delta - \ell - 8}{2(\Delta + \ell + 4)} C_1, \quad C_3 = \frac{3(\Delta - \ell - 8)}{2(\Delta + \ell + 4)} C_1, \quad C_4 = -\frac{6(\Delta - 2)}{\Delta + \ell + 4} C_1. \quad (4.34)$$

As in the even- $\ell$  case conservation at point  $z_3$  is not automatic. We need to impose the constraint in (4.31) which amounts to requiring  $C_1$  to be of the form

$$C_1 = (\Delta - \ell - 6)C_1'. \quad (4.35)$$

## 5 Three-point function coefficients

In this section we will set  $\theta_{1,2} = \bar{\theta}_{1,2} = 0$  and perform an expansion of the three-point functions presented in section 4 in  $\theta_3, \bar{\theta}_3$ . The results will allow us to extract the OPE coefficients of the various operators inside the superconformal multiplet  $\mathcal{O}_{j, \bar{j}}(z)$  in terms of the coefficients  $A_i, B_i, C_i, D_i, E_i$  defined before. Clearly the zeroth order in  $\theta_3, \bar{\theta}_3$  corresponds to the contribution of the superconformal primary. Similarly the order  $\theta_3 \bar{\theta}_3$  contains contributions from operators of the form  $(Q\bar{Q}\mathcal{O})_{j\pm 1, \bar{j}\pm 1; p}(x)$ , and the order  $\theta_3^2 \bar{\theta}_3^2$  contains contributions from  $(Q^2 \bar{Q}^2 \mathcal{O})_{j, \bar{j}; p}(x)$ , where “ $p$ ” stands for primary. However, we also expect contributions from descendant operators of the schematic form  $(\partial\mathcal{O})(x)$  at the first order and of the form  $(\partial^2\mathcal{O})(x), (\partial Q\bar{Q}\mathcal{O})(x)$  at the second order. The precise way in which this mixing takes place is described in [24]. Following those results we are able to isolate the contributions of each primary and compute the associated OPE coefficients in the basis of non-supersymmetric three-point functions adopted in [25] and reviewed in section 2. The expansion of (4.2) in  $\theta_3$  can be performed using an extension of the

*Mathematica* package developed in [24]. Notice that the operators  $(Q\bar{Q}\mathcal{O})_{j\pm 1, \bar{j}\pm 1; p}(x)$  and  $(Q^2\bar{Q}^2\mathcal{O})_{j, \bar{j}; p}(x)$  are not normalized in a standard way, but rather according to [24, table 1].

In the following we will go through each of the cases presented in section 4. In each subsection we explicitly write the non-vanishing three-point functions between the currents  $J$  and the various primary superconformal descendants in terms of the tensor structures introduced in section 2. In order not to overload the notation we use the same symbols for the OPE coefficients multiple times — since each subsection corresponds to a different superprimary, we are confident that this will not create any confusion. Schematically, we indicate with  $\lambda^{(i)}$  the OPE coefficients associated to the superconformal primary,  $\lambda_{\pm\pm}^{(i)}$  or  $\lambda_{\pm\mp}^{(i)}$  those associated to the superdescendants at order  $\theta_3, \bar{\theta}_3$ , and finally  $\hat{\lambda}^{(i)}$  the coefficients of the order  $\theta_3^2\bar{\theta}_3^2$ . The expression for  $\lambda^{(i)}$  were given in the previous section, eqs. (4.5), (4.9), (4.16), (4.20), (4.28) and will not be repeated.

Here we only report the expression of the  $\lambda$ 's after imposing the conservation conditions discussed in the previous section. The coefficients before the use of conservation conditions can be found in the *Mathematica* notebook attached to this paper as supplementary material. Because the conservation condition in superspace (3.1) enforces the conservation of  $J$ , the  $\lambda$ 's satisfy the relations summarized in table 1. Thus, we do not show the value of all of them, but only the independent ones and a few non-trivial relations. We have checked explicitly that all the constraints of table 1 are verified.

## 5.1 Case A

### 5.1.1 Even $\ell$

At order  $\theta_3\bar{\theta}_3$  we anticipate four different three-point functions, involving the four different conformal primary operators one can obtain by acting with  $Q\bar{Q}$  on  $\mathcal{O}$  [24]. We denote them by  $(Q\bar{Q}\mathcal{O})_{\ell\pm 1, \ell\pm 1; p}$  and  $(Q\bar{Q}\mathcal{O})_{\ell\pm 1, \ell\mp 1; p}$ . In the former case the resulting primary is still a traceless symmetric operator, but with odd spin; according to section 2, we expect only one tensor structure. The latter case instead corresponds to a primary with  $j - \bar{j} = \pm 2$ , and we thus expect four tensor structures parametrized by only one independent coefficient. In more detail we have

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell\pm 1, \ell\pm 1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \lambda_{\pm\pm}^{(-)} \mathcal{K}_{\Delta+1, \ell\pm 1, 0} \mathcal{S}^{(-)}(\eta_i, \bar{\eta}_i, x_i), \tag{5.1}$$

where  $\lambda_{\pm\pm}^{(-)}$  are coefficients while the remaining quantities are defined in section 2. The  $\pm\pm$  subscript refers to the addition of unity to the  $\ell$  labels of  $Q\bar{Q}\mathcal{O}$  in the left-hand side, while the sign in the superscript indicates the parity of the corresponding structure. Similarly, we have the three-point functions

$$\begin{aligned} \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+1, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell-1, 2} \sum_{j=1}^4 \lambda_{+-}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \\ \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell-1, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell-1, 2} \sum_{j=1}^4 \lambda_{-+}^{(j)} \bar{\mathcal{T}}^{(j)}(\eta_i, \bar{\eta}_i, x_i). \end{aligned} \tag{5.2}$$

Using (4.7), we obtain the following relations:

$$\begin{aligned}
 \lambda_{++}^{(-)} &= -\frac{2i(\Delta+\ell)(\Delta^2-6\Delta+\ell^2+8)}{(\ell+1)^2(3(\Delta-2)^2-\ell(\ell+2))}A_1 - \frac{i(\Delta-\ell-4)(\Delta+\ell-1)(\Delta+\ell)}{(\ell+1)^2(3(\Delta-2)^2-\ell(\ell+2))}A_2, \\
 \lambda_{--}^{(-)} &= -\frac{2i(\ell-1)(\Delta-\ell-2)(\Delta^2-6\Delta+\ell^2+4\ell+12)}{\ell(3(\Delta-2)^2-\ell(\ell+2))}A_1 \\
 &\quad - \frac{i(\ell+2)(\ell+3)(\Delta-\ell-2)(\Delta-\ell-3)(\Delta+\ell-2)}{\ell^2(3(\Delta-2)^2-\ell(\ell+2))}A_2, \\
 \lambda_{-+}^{(1)} = -\lambda_{+-}^{(1)} &= \frac{4i(\Delta-3)(\Delta-\ell-2)(\Delta+\ell)}{(\Delta-1)(3(\Delta-2)^2-\ell(\ell+2))}A_1 - \frac{i(\ell+3)(\Delta-2)(\Delta-\ell-2)(\Delta+\ell)}{\ell(\Delta-1)(3(\Delta-2)^2-\ell(\ell+2))}A_2, \\
 \lambda_{-+}^{(3)} = \lambda_{+-}^{(3)} &= -\frac{\Delta-\ell-4}{2(\ell+1)}\lambda_{+-}^{(1)}. \tag{5.3}
 \end{aligned}$$

When the unitarity bound (4.1) is saturated we get  $\lambda_{\mp\pm}^{(1,3)} = \lambda_{--}^{(-)} = 0$ . This reflects the shortening of the multiplet of  $\mathcal{O}$  [24].

As expected in (5.1) we do not get any contribution associated with a parity-even tensor structure. Also, the relation between  $\lambda_{\pm\mp}^{(1)}$  and  $\lambda_{\pm\mp}^{(3)}$  is exactly the one expected from table 1 with the proper shifts in the dimension  $\Delta$  and spin  $\ell$  of the primary.

At order  $\theta_3^2\bar{\theta}_3^2$  the three-point function (4.2) gives rise to a unique three-point function, after subtraction of three-point functions involving conformal descendants of  $\mathcal{O}$ . The final three-point function involves five parity-even structures,

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q^2\bar{Q}^2\mathcal{O})_{\ell, \ell; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+2, \ell, 0} \sum_{j=1}^5 \hat{\lambda}^{(j)} \mathcal{S}^{(j)}(\eta_i, \bar{\eta}_i, x_i). \tag{5.4}$$

After imposing the Ward identity constraints (4.7) we obtain

$$\begin{aligned}
 \hat{\lambda}^{(1)} &= \frac{8(\Delta-2)(\Delta-\ell-2)(\Delta+\ell)(\Delta^3-3\Delta^2-6\Delta-\Delta\ell^2+9\ell^2+2\Delta^2\ell-14\Delta\ell+28\ell+8)}{(\Delta-1)^2(3(\Delta-2)^2-\ell(\ell+2))}A_1 \\
 &\quad + \frac{4(\Delta-\ell-2)(\Delta+\ell)(\Delta^4-10\Delta^3+39\Delta^2-82\Delta-4\Delta\ell^2+6\ell^2-\Delta^3\ell+8\Delta^2\ell-35\Delta\ell+38\ell+64)}{(\Delta-1)^2(3(\Delta-2)^2-\ell(\ell+2))}A_2, \\
 \hat{\lambda}^{(4)} &= -\frac{64\ell(\ell-1)(\Delta-2)(\Delta-\ell-2)(\Delta+\ell)}{(\Delta-1)^2(3(\Delta-2)^2-\ell(\ell+2))}A_1 \\
 &\quad - \frac{4(\Delta-\ell-2)(\Delta+\ell)(3\Delta^4-21\Delta^3-\Delta^2\ell^2-2\Delta^2\ell+48\Delta^2-\Delta\ell^2-2\Delta\ell-36\Delta+4\ell^2+8\ell)}{(\Delta-1)^2(3(\Delta-2)^2-\ell(\ell+2))}A_2, \\
 \hat{\lambda}^{(2)} = -\hat{\lambda}^{(3)} &= -\frac{4\ell(\Delta-1)\hat{\lambda}^{(1)} - (\Delta-\ell-2)(\Delta+\ell+2)\hat{\lambda}^{(4)}}{2\Delta^2+4\Delta(\ell+1)-2\ell(\ell+4)}, \\
 \hat{\lambda}^{(5)} &= -\frac{4\Delta(\Delta-1)\hat{\lambda}^{(1)} + (\Delta-\ell-2)(\Delta-\ell-4)\hat{\lambda}^{(4)}}{2\Delta^2+4\Delta(\ell+1)-2\ell(\ell+4)}, \tag{5.5}
 \end{aligned}$$

consistently with the relations of table 1. All coefficients vanish at the unitarity bound (4.1), as a consequence of the multiplet shortening.

### 5.1.2 Odd $\ell$

Just as in the even-spin case, at order  $\theta_3\bar{\theta}_3$  we have four different three-point functions. Two of these three-point functions involve five structures (because they contain even-spin operators at the third point) parametrized by two independent coefficients, while the other two involve four structures parametrized by one coefficient [25]. The first three-point function takes the form

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell\pm 1, \ell\pm 1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+1, \ell\pm 1, 0} \sum_{i=1}^5 \lambda_{\pm\pm}^{(i)} \mathcal{S}^{(i)}(\eta_i, \bar{\eta}_i, x_i), \quad (5.6)$$

and we also have

$$\begin{aligned} \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+1, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell-1, 2} \sum_{j=1}^4 \lambda_{+-}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \\ \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell-1, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell-1, 2} \sum_{j=1}^4 \lambda_{-+}^{(j)} \bar{\mathcal{T}}^{(j)}(\eta_i, \bar{\eta}_i, x_i). \end{aligned} \quad (5.7)$$

The coefficients of these three-point functions are given by, after use of (4.11),

$$\begin{aligned} \lambda_{++}^{(1)} &= -\frac{2(\Delta+\ell)(3\Delta^2+2\Delta\ell-12\Delta-3\ell^2-10\ell+16)}{(\ell+1)^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} A_1 + \frac{(\Delta-\ell-4)(\Delta+\ell)}{2(\ell+1)^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} C_1, \\ \lambda_{++}^{(4)} &= \frac{8(\Delta-\ell-2)(\Delta+\ell-3)(\Delta+\ell)}{(\ell+1)^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} A_1 - \frac{(\Delta+\ell)(3\Delta^2-6\Delta-\ell^2-4\ell)}{2(\Delta-2)(\ell+1)^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} C_1, \\ \lambda_{++}^{(2)} &= -\lambda_{++}^{(3)} = \frac{4(\Delta-2)(\ell+1)\lambda_{++}^{(1)} - (\Delta-\ell-4)(\Delta+\ell+2)\lambda_{++}^{(4)}}{2(\ell+1)(\ell+9) - 2(\Delta+1)^2 - 4(\Delta+1)\ell}, \\ \lambda_{++}^{(5)} &= \frac{4(\Delta-2)(\Delta-1)\lambda_{++}^{(1)} + (\Delta-\ell-4)(\Delta-\ell-6)\lambda_{++}^{(4)}}{2(\ell+1)(\ell+9) - 2(\Delta+1)^2 - 4(\Delta+1)\ell}, \\ \lambda_{--}^{(1)} &= -\frac{2(\Delta-\ell-2)(3\Delta^2\ell^2+\Delta^2\ell+6\Delta^2+2\Delta\ell^3-2\Delta\ell^2+12\Delta\ell-32\Delta-3\ell^4-19\ell^3-28\ell^2-28\ell+48)}{\ell^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} A_1 \\ &\quad + \frac{(\ell+3)(\ell+4)(\Delta-\ell-2)^2}{2\ell^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} C_1, \\ \lambda_{--}^{(4)} &= \frac{8(\ell-2)(\ell-1)(\Delta-\ell-2)(\Delta-\ell-5)(\Delta+\ell)}{\ell^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} A_1 - \frac{(\ell+3)(\ell+4)(\Delta-\ell-2)(3\Delta^2-6\Delta-\ell^2+4)}{2\ell^2(\Delta-2)(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} C_1, \\ \lambda_{--}^{(2)} &= -\lambda_{--}^{(3)} = \frac{4(\ell-1)(\Delta-2)\lambda_{--}^{(1)} - (\Delta-\ell-2)(\Delta+\ell)\lambda_{--}^{(4)}}{2(\ell^2 - (\Delta-2)(\Delta+2\ell) - 4)}, \\ \lambda_{--}^{(5)} &= \frac{4(\Delta-2)(\Delta-1)\lambda_{--}^{(1)} + (\Delta-\ell-2)(\Delta-\ell-4)\lambda_{--}^{(4)}}{2(\ell^2 - (\Delta-2)(\Delta+2\ell) - 4)}, \\ \lambda_{-+}^{(1)} &= \lambda_{+-}^{(1)} = \frac{4(\ell-1)(\Delta-4)(\Delta-\ell-2)(\Delta+\ell)}{\ell(\ell+1)(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} A_1 \\ &\quad + \frac{(\ell+4)(\Delta-1)(\Delta-\ell-2)(\Delta+\ell)}{2(\Delta-2)\ell(\ell+1)(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3))} C_1, \\ \lambda_{-+}^{(3)} &= -\lambda_{+-}^{(3)} = \frac{\Delta-\ell-4}{2(\Delta-1)} \lambda_{-+}^{(1)}. \end{aligned} \quad (5.8)$$



When the unitarity bound (4.1) is saturated,  $\lambda_{\mp\pm}^{(1,2)} = \lambda_{--}^{(j)} = 0$ , as necessary due to multiplet shortening.

At order  $\theta_3^2 \bar{\theta}_3^2$  and after subtraction of three-point functions involving descendants of  $\mathcal{O}$  we are left with a single three-point function. This involves one independent coefficient  $\hat{\lambda}^{(-)}$ , multiplying the structure  $\mathcal{S}^{(-)}$ , exactly as in (4.9) but with  $\Delta \rightarrow \Delta + 2$ . We find

$$\begin{aligned} \hat{\lambda}^{(-)} &= -\frac{8i}{\delta_A}(\Delta-2)(\Delta-\ell-2)(\Delta+\ell)\left(5\Delta^4-35\Delta^3-3\Delta^2\ell^2-6\Delta^2\ell+84\Delta^2+13\Delta\ell^2+26\Delta\ell\right. \\ &\quad \left.-84\Delta-16\ell^2-32\ell+48\right)A_1-\frac{4i}{\delta_A}(\Delta-\ell-2)(\Delta+\ell)(\ell+3)(\ell+4)(\Delta-1)C_1, \\ \delta_A &= (\Delta-2)(\Delta-1)^2(5(\Delta-3)(\Delta-1)-3(\ell-1)(\ell+3)), \end{aligned} \quad (5.9)$$

consistently both with the unitarity bound and the fact that there is no parity-even structure in the three-point function involving two conserved currents in the first two points when  $\ell$  is odd.

## 5.2 Case B

### 5.2.1 Even $\ell$

At order  $\theta_3 \bar{\theta}_3$  we have four different three-point functions; one of them is identically zero by conservation (see table 1), another one involves a traceless symmetric tensor with odd spin, while the other two have one independent coefficient each. More specifically, we have

$$\begin{aligned} \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+3, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell+1, 2} \sum_{j=1}^4 \lambda_{++}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \\ \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+1, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell-1, 2} \sum_{j=1}^4 \lambda_{--}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \end{aligned} \quad (5.10)$$

as well as

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+3, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = 0, \quad (5.11)$$

and

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+1, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+1, \ell+1, 0} \lambda_{-+}^{(-)} \mathcal{S}^{(-)}(\eta_i, \bar{\eta}_i, x_i). \quad (5.12)$$

Using eq. (4.18), the above coefficients are given by

$$\begin{aligned} \lambda_{++}^{(1)} &= -\frac{i(\Delta+\ell)(\Delta+\ell+2)}{(\ell+1)(\Delta-2)(\Delta+\ell+1)} A_1, & \lambda_{++}^{(3)} &= -\frac{\Delta-\ell-6}{2(\ell+3)} \lambda_{++}^{(1)}, \\ \lambda_{--}^{(1)} &= -\frac{i(\ell+1)(\ell+3)(\ell+4)(\Delta-\ell-4)(\Delta-\ell-2)}{\ell(\ell+2)(\Delta-2)(\Delta-\ell-3)} A_1, & \lambda_{--}^{(3)} &= -\frac{\Delta-\ell-4}{2(\ell+1)} \lambda_{--}^{(1)}, \\ \lambda_{-+}^{(-)} &= \frac{i(\ell+4)(\Delta-3)(\Delta-\ell-4)(\Delta+\ell)}{(\ell+1)(\ell+2)(\Delta-2)^2} A_1. \end{aligned} \quad (5.13)$$

When the unitarity bound (4.13) is saturated,  $\lambda_{-+}^{(-)} = \lambda_{--}^{(1,2)} = 0$ , as required by multiplet shortening.

At order  $\theta_3^2 \bar{\theta}_3^2$  we expect four tensor structures parametrized by one independent coefficient. The structures are the same as the ones in (4.16) but with  $\Delta \rightarrow \Delta + 2$ . If we denote

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q^2 \bar{Q}^2 \mathcal{O})_{\ell+2, \ell, p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+2, \ell, 2} \sum_{j=1}^4 \hat{\lambda}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \quad (5.14)$$

then, after using (4.18), we obtain

$$\hat{\lambda}^{(1)} = -\frac{4(\Delta-3)(\Delta-\ell-4)(\Delta-\ell-2)(\Delta+\ell)(\Delta+\ell+2)}{(\Delta-2)(\Delta-\ell-3)(\Delta+\ell+1)} A_1, \quad \hat{\lambda}^{(3)} = -\frac{\Delta-\ell-4}{2\Delta} \hat{\lambda}^{(1)}. \quad (5.15)$$

As expected all  $\hat{\lambda}^{(j)}$  go to zero when the bound (4.13) is saturated.

### 5.2.2 Odd $\ell$

At order  $\theta_3 \bar{\theta}_3$  we have four different three-point functions. One of them is parametrized by only one coefficient, another is the three-point function of a symmetric traceless operator with even spin and contains therefore two independent coefficients, while the other two contain one coefficient each. We have

$$\begin{aligned} \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q \bar{Q} \mathcal{O})_{\ell+3, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell+1, 2} \sum_{j=1}^4 \lambda_{++}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \\ \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q \bar{Q} \mathcal{O})_{\ell+1, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \mathcal{K}_{\Delta+1, \ell-1, 2} \sum_{j=1}^4 \lambda_{--}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i), \end{aligned} \quad (5.16)$$

and

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q \bar{Q} \mathcal{O})_{\ell+3, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \lambda_{+-} \mathcal{K}_{\Delta+1, \ell-1, 4} \mathcal{R}(\eta_i, \bar{\eta}_i, x_i), \quad (5.17)$$

while the symmetric traceless one is

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q \bar{Q} \mathcal{O})_{\ell+1, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+1, \ell+1, 0} \sum_{i=1}^5 \lambda_{-+}^{(i)} \mathcal{S}^{(i)}(\eta_i, \bar{\eta}_i, x_i), \quad (5.18)$$

With the use of (4.21) and (4.23) we obtain

$$\begin{aligned}
 \lambda_{++}^{(1)} &= -\frac{2i}{\delta_2}(\Delta-1)(\Delta+\ell+2)\left(5\ell^4+4\Delta\ell^3+19\ell^3+3\Delta^2\ell^2-14\Delta\ell^2+58\ell^2+27\Delta^2\ell-170\Delta\ell\right. \\
 &\quad \left.+248\ell+60\Delta^2-360\Delta+480\right)A_1-\frac{4}{\delta_1}(\Delta-1)(\Delta-\ell-4)(\Delta+\ell+2)C_1', \\
 \lambda_{++}^{(3)} &= -\frac{\Delta-\ell-6}{2(\Delta-1)}\lambda_{++}^{(1)}, \\
 \lambda_{--}^{(1)} &= -\frac{2i}{\delta_4}(\ell-1)(\Delta-1)(\Delta-\ell-4)(\Delta-\ell-2)(3\Delta^2+4\ell\Delta-16\Delta+5\ell^2+4\ell+36)A_1 \\
 &\quad -\frac{4}{\ell\delta_3}(\ell+4)(\ell+5)(\Delta-1)(\Delta-\ell-4)(\Delta-\ell-2)C_1', \\
 \lambda_{--}^{(3)} &= -\frac{\Delta-\ell-4}{2(\Delta-1)}\lambda_{--}^{(1)}, \\
 \lambda_{+-} &= \frac{4i}{\Delta\delta_6}(\Delta-\ell-4)(\Delta-\ell-2)(\Delta+\ell+2)(3\ell\Delta^2+12\Delta^2+2\ell^2\Delta-4\ell\Delta-58\Delta-3\ell^2-12\ell+60)A_1 \\
 &\quad -\frac{2}{\ell\delta_5}(\ell+5)(\Delta-\ell-2)(\Delta-\ell-4)(\Delta+\ell+2)C_1', \\
 \lambda_{-+}^{(1)} &= \frac{2i}{\delta_8}(\Delta-\ell-4)\left(3\Delta\ell^5-12\ell^5-5\Delta^2\ell^4+59\Delta\ell^4-176\ell^4-19\Delta^3\ell^3+135\Delta^2\ell^3-132\Delta\ell^3-496\ell^3\right. \\
 &\quad \left.-3\Delta^4\ell^2-115\Delta^3\ell^2+1056\Delta^2\ell^2-2412\Delta\ell^2+800\ell^2-27\Delta^4\ell-136\Delta^3\ell+2324\Delta^2\ell-6608\Delta\ell\right. \\
 &\quad \left.+4864\ell-60\Delta^4+240\Delta^3+720\Delta^2-3840\Delta+3840\right)A_1-\frac{4}{\delta_7}(\ell+5)(\Delta-2)(\Delta-\ell-4)^2C_1', \\
 \lambda_{-+}^{(4)} &= -\frac{4i\ell}{\delta_8}(\Delta-\ell-4)\left(3\ell\Delta^4+12\Delta^4-3\ell^2\Delta^3-21\ell\Delta^3-96\Delta^3-3\ell^3\Delta^2+45\ell^2\Delta^2+168\ell\Delta^2\right. \\
 &\quad \left.+360\Delta^2-\ell^4\Delta+13\ell^3\Delta-128\ell^2\Delta-532\ell\Delta-672\Delta-\ell^4-44\ell^3+4\ell^2+416\ell+480\right)A_1 \\
 &\quad +\frac{4}{\delta_7}(\ell+5)(\Delta-\ell-4)(3\Delta(\Delta-2)-\ell(\ell+4))C_1', \\
 \lambda_{-+}^{(2)} &= \frac{4(\ell+1)(\Delta-2)\lambda_{-+}^{(1)}-(\Delta-\ell-4)(\Delta+\ell+2)\lambda_{-+}^{(4)}}{2(\ell+1)(\ell+9)-2(\Delta+1)^2-4(\Delta+1)\ell}, \\
 \lambda_{-+}^{(5)} &= \frac{4(\Delta-2)(\Delta-1)\lambda_{-+}^{(1)}+(\Delta-\ell-4)(\Delta-\ell-6)\lambda_{-+}^{(4)}}{2(\ell+1)(\ell+9)-2(\Delta+1)^2-4(\Delta+1)\ell}, \tag{5.19}
 \end{aligned}$$

where we have defined

$$\begin{aligned}
 \delta_1 &= 3(\ell+1)(\ell+3)(5(\Delta-2)^2-\ell(\ell+4)), & \delta_2 &= (\ell+4)(\ell+5)(\Delta+\ell+1)\delta_1, \\
 \delta_3 &= 3(\ell+2)(5(\Delta-2)^2-\ell(\ell+4)), & \delta_4 &= (\Delta-\ell-3)(\Delta+\ell-2)\delta_3, \\
 \delta_5 &= 3(\ell+3)(5(\Delta-2)^2-\ell(\ell+4)), & \delta_6 &= (\ell+4)(\Delta+\ell-2)\delta_5, \\
 \delta_7 &= 3(\ell+1)(\ell+2)(5(\Delta-2)^2-\ell(\ell+4)), & \delta_8 &= (\ell+4)(\Delta-2)(\Delta+\ell-2)\delta_7. \tag{5.20}
 \end{aligned}$$

When the unitarity bound (4.13) is saturated,  $\lambda_{-+}^{(1,\dots,4)} = \lambda_{+-} = \lambda_{--}^{(1,2)} = 0$ , as required by multiplet shortening.

At order  $\theta_3^2 \bar{\theta}_3^2$  we expect two structures parametrized by one coefficient. These structures are the same as the ones in (4.20) but with  $\Delta \rightarrow \Delta + 2$ . More specifically, we have

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q^2 \bar{Q}^2 \mathcal{O})_{\ell+2, \ell, p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+2, \ell, 2} \sum_{j=1}^4 \hat{\lambda}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i). \quad (5.21)$$

After using (4.21) and (4.23) we obtain

$$\begin{aligned} \hat{\lambda}^{(1)} &= -\frac{8}{\delta_{10}} (\ell+2)(\Delta-\ell-4)(\Delta-\ell-2)(\Delta+\ell+2) \left( 15\Delta^5 + 15\Delta^4\ell - 135\Delta^4 - 3\Delta^3\ell^2 - 147\Delta^3\ell \right. \\ &\quad \left. + 390\Delta^3 - 3\Delta^2\ell^3 + 27\Delta^2\ell^2 + 546\Delta^2\ell - 300\Delta^2 + 31\Delta\ell^3 - 2\Delta\ell^2 - 884\Delta\ell - 360\Delta - 46\ell^3 - 112\ell^2 \right. \\ &\quad \left. + 488\ell + 480 \right) A_1 - \frac{32i}{\delta_9} (\ell+2)(\ell+4)(\ell+5)(\Delta-1)(\Delta-\ell-4)(\Delta-\ell-2)(\Delta+\ell+2) C_1', \\ \hat{\lambda}^{(3)} &= -\frac{\Delta-\ell-4}{2(\ell+2)} \hat{\lambda}^{(1)} \end{aligned} \quad (5.22)$$

where we defined the denominators

$$\begin{aligned} \delta_9 &= 3\Delta(\Delta-2)(\Delta-\ell-3)(\Delta+\ell+1)(5(\Delta-2)^2 - \ell(\ell+4)), \\ \delta_{10} &= (\Delta+\ell-2)\delta_9. \end{aligned} \quad (5.23)$$

Consistently with multiplet shortening  $\hat{\lambda}^{(j)} = 0$  at the unitarity bound (4.13).

### 5.3 Case C

#### 5.3.1 Even $\ell$

At order  $\theta_3 \bar{\theta}_3$  only one three-point function is non-zero, namely

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+3, \ell+1, p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+1, \ell+1, 2} \sum_{j=1}^4 \lambda_{-+}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i). \quad (5.24)$$

With the use of (4.30) and (4.32) we find

$$\lambda_{-+}^{(1)} = \frac{2i(\ell+3)(\ell+5)(\Delta-4)(\Delta-\ell-6)}{(\ell+1)(\ell+4)(\Delta-3)} A', \quad \lambda_{-+}^{(3)} = -\frac{\Delta-\ell-6}{2(\ell+3)} \lambda_{-+}^{(1)}. \quad (5.25)$$

As expected from multiplet shortening, both coefficients vanish when  $\Delta$  saturates (4.25).

At order  $\theta_3^2 \bar{\theta}_3^2$  we have only one structure, namely the same one multiplied by  $\lambda$  in (4.28) with  $\Delta \rightarrow \Delta + 2$ ,

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q^2 \bar{Q}^2 \mathcal{O})_{\ell+4, \ell, p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \hat{\lambda} \mathcal{K}_{\Delta+2, \ell, 4} \mathcal{R}(\eta_i, \bar{\eta}_i, x_i). \quad (5.26)$$

The coefficient is determined to be

$$\hat{\lambda} = \frac{4(\Delta-4)(\Delta-\ell-6)(\Delta-\ell-2)(\Delta+\ell+4)}{\Delta+1} A', \quad (5.27)$$

which vanishes for  $\Delta$  saturating (4.25) as needed.

### 5.3.2 Odd $\ell$

We have three non-zero three-point functions with one independent coefficient each, i.e.

$$\begin{aligned} \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+5, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \lambda_{++} \mathcal{K}_{\Delta+1, \ell+1, 4} \mathcal{R}(\eta_i, \bar{\eta}_i, x_i), \\ \langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+3, \ell-1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle &= \lambda_{--} \mathcal{K}_{\Delta+1, \ell-1, 4} \mathcal{R}(\eta_i, \bar{\eta}_i, x_i), \end{aligned}$$

and

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q\bar{Q}\mathcal{O})_{\ell+3, \ell+1; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = \mathcal{K}_{\Delta+1, \ell+1, 2} \sum_{j=1}^4 \lambda_{-+}^{(j)} \mathcal{T}^{(j)}(\eta_i, \bar{\eta}_i, x_i). \quad (5.28)$$

With the use of (4.34) and (4.35) we obtain

$$\begin{aligned} \lambda_{++} &= -\frac{\Delta - \ell - 6}{(\ell + 1)(\ell + 5)} C_1', \\ \lambda_{--} &= -\frac{(\ell + 5)(\ell + 6)(\Delta - \ell - 6)(\Delta - \ell - 2)}{\ell(\ell + 4)(\Delta + \ell + 4)} C_1', \\ \lambda_{-+}^{(1)} &= -\frac{2(\ell + 6)(\Delta - 1)(\Delta - \ell - 6)}{(\ell + 1)(\ell + 4)(\Delta + \ell + 4)} C_1', \\ \lambda_{-+}^{(3)} &= -\frac{\Delta - \ell - 6}{2(\Delta - 1)} \lambda_{-+}^{(1)}. \end{aligned} \quad (5.29)$$

As expected  $\lambda_{-+}^{(j)} = \lambda_{--} = 0$  at the unitarity bound (4.25), due to multiplet shortening.

The  $\theta_3^2 \bar{\theta}_3^2$  order is purely a descendant, consistently with the fact that there is no three-point function at the lowest order, i.e. one has

$$\langle J(\eta_1, \bar{\eta}_1, x_1) J(\eta_2, \bar{\eta}_2, x_2) (Q^2 \bar{Q}^2 \mathcal{O})_{\ell+4, \ell; p}(\eta_3, \bar{\eta}_3, x_3) \rangle = 0. \quad (5.30)$$

We checked that this is indeed the case.

## 6 Discussion

In this work we have studied the constraints imposed by  $\mathcal{N} = 1$  superconformal symmetry on the three-point function involving two Ferrara-Zumino supermultiplets  $\mathcal{J}_{\alpha\dot{\alpha}}$  and a third supermultiplet  $\mathcal{O}_{j, \bar{j}}$ . We started from the most general parametrization of such a correlator in superspace and we imposed the shortening conditions  $D^\alpha \mathcal{J}_{\alpha\dot{\alpha}} = \bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = 0$ . Similarly to the non supersymmetric case, reviewed in section 2, these constraints give rise to linear equations among the coefficients of the superspace tensor structures. We found that non-trivial solutions exist only when the superconformal primary contained in  $\mathcal{O}_{j, \bar{j}}$  has a non-vanishing three-point function with two the R-currents  $J_\mu$ .<sup>20</sup> This is equivalent to requiring that  $\mathcal{O}_{j, \bar{j}}$  is neutral under the R-symmetry and satisfies  $j - \bar{j} = 0, \pm 2, \pm 4$ . We also give

<sup>20</sup>To be precise, when  $j = \bar{j} \pm 4$  and  $j$  is odd the non-supersymmetric case vanishes while the supersymmetric one is allowed.

a group theoretical counting of the number of independent superspace tensor structures expected for this correlator.

The results of section 4 are completely general and in principle contain all the information needed in order to extract the three-point function between any two conserved currents and a third operator. A striking consequence of our results is that the OPE of any two conserved currents in 4d  $\mathcal{N} = 1$  SCFTs contains at most the same superconformal primaries entering the OPE  $J_\mu \times J_\nu$ . This represents a big difference compared with the non-supersymmetric case where, for instance, the OPE of two stress-energy tensors contains more general representations. For this reason we expect that a bootstrap study of the correlation function involving four copies of the R-current  $J_\mu$  will be able to capture interesting features of  $\mathcal{N} = 1$  theories.

This last point was the main motivation for the present work: setting the stage for a numerical bootstrap study of the correlation function  $\langle J_\mu J_\nu J_\rho J_\sigma \rangle$ . A key ingredient needed for this analysis is the superconformal block decomposition of the correlator. Introducing a proper basis of tensor structures  $\mathbb{T}_4^{(i)}$  we can schematically write

$$\langle J_\mu J_\nu J_\rho J_\sigma \rangle \propto \sum_{\mathcal{O}} \sum_{s=1}^{n_4} W_{\mathcal{O}}^{(i)} \mathbb{T}_4^{(i)}, \tag{6.1}$$

where we have omitted a kinematic prefactor, and the partial waves  $W_{\mathcal{O}}^{(i)}$  represent the contribution of an entire superconformal multiplet to the four-point function,

$$W_{\mathcal{O}}^{(i)} = \sum_{\mathcal{O}'=\mathcal{O}, Q\bar{Q}\mathcal{O}, \dots} \sum_{a,b=1}^{n_{\mathcal{O}'}} \lambda_{JJ\mathcal{O}'}^a \lambda_{\bar{\mathcal{O}}'JJ}^b g_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}^{(a,b;i)}. \tag{6.2}$$

In the above expression  $g_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}^{(a,b;i)}$  are the non-supersymmetric conformal blocks for spinning correlators computed in [7, 8] and the  $\lambda$ 's are the three-point function coefficients between two currents and the various components of the supermultiplet  $\mathcal{O}$ ; because of supersymmetry they are all determined in terms of the superconformal primary ones. The main result of this work is the exact form of these relations. This was achieved in section 5. Notice that the coefficients presented there do not correspond exactly to the ones in (6.2) since the superconformal descendants do not have a standard normalization. Thus, one has to divide by their norms, which have been already computed [24].

As anticipated, the next step is to bootstrap numerically the correlation function of four R-currents, along the same lines as the 3d global symmetry current bootstrap [36] and the 3d stress-energy tensor bootstrap [37]. In particular one has to remove the redundant crossing symmetry conditions by properly taking into account the conservation constraint. This step could be more involved in the supersymmetric case. Given the universal nature of the Ferrara-Zumino multiplet, this study will open a window on all local 4d  $\mathcal{N} = 1$  SCFTs, potentially leading to discovering new ones and hopefully shedding light on the putative minimal SCFT studied in [16, 18, 19].

The three-point functions computed in this work can also be used in combination with lightcone bootstrap techniques [2, 3] and the recent OPE inversion formula [38, 39] or the methods of [40] to study the behavior of Regge trajectories in supersymmetry.

Finally, starting from our results in superspace, with a bit more technical effort it is possible to compute the three-point functions of other components of  $\mathcal{J}$ , such as the stress-energy tensor. Although at the level of three-point functions one does not get any additional information, it is not excluded that the crossing constraints coming from their four-point functions would impose independent restrictions. It would be interesting to investigate this direction.

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## A Tensor structures in spinor formalism

Here we list the tensor structures appearing in the three-point functions of this paper. Let us introduce the following useful notation:

$$\begin{aligned}
 [i\bar{j}] &= \frac{\eta_i U \bar{\eta}_j}{|U|}, & [\Theta\bar{\Theta}] &= \frac{\Theta U \bar{\Theta}}{U^2}, & [ij] &= \eta_i \eta_j, & [\bar{i}\bar{j}] &= \bar{\eta}_i \bar{\eta}_j, \\
 [\Theta j] &= \frac{\Theta \eta_j}{|U|^{1/2}}, & [\bar{\Theta}\bar{j}] &= \frac{\bar{\Theta} \bar{\eta}_j}{|U|^{1/2}}, & [j\bar{\Theta}] &= \frac{\eta_j U \bar{\Theta}}{|U|^{3/2}}, & [\Theta\bar{j}] &= \frac{\Theta U \bar{\eta}_j}{|U|^{3/2}}.
 \end{aligned} \tag{A.1}$$

Let us start with case (A) for  $\ell$  even. In (4.4) we have the following tensor structures:

$$\begin{aligned}
 \mathcal{P}_A^{(1)} &= [12][\bar{1}\bar{2}][3\bar{3}]^2, \\
 \mathcal{P}_A^{(2)} &= [13][23][\bar{1}\bar{3}][\bar{2}\bar{3}], \\
 \mathcal{P}_A^{(3)} &= ([1\bar{1}][2\bar{2}] + [1\bar{2}][2\bar{1}])[3\bar{3}]^2, \\
 \mathcal{P}_A^{(4)} &= ([13](\bar{1}\bar{3}[2\bar{2}] + \bar{2}\bar{3}[2\bar{1}]) + [23](\bar{1}\bar{3}[1\bar{2}] + \bar{2}\bar{3}[1\bar{1}]))[3\bar{3}], \\
 \mathcal{P}_A^{\prime(1)} &= i([12](\Theta\bar{1}[\bar{\Theta}2] + \Theta\bar{2}[\bar{\Theta}1]) - \bar{1}\bar{2}([\Theta1][2\bar{\Theta}] + [\Theta2][1\bar{\Theta}]))[3\bar{3}]^2, \\
 \mathcal{P}_A^{\prime(2)} &= i([12](\bar{1}\bar{3}[3\bar{2}] + \bar{2}\bar{3}[3\bar{1}]) - \bar{1}\bar{2}([13][2\bar{3}] + [23][1\bar{3}]))[\Theta3][\bar{\Theta}3], \\
 \mathcal{P}_A^{\prime(3)} &= ([12](\bar{1}\bar{3}[3\bar{1}] + \bar{2}\bar{3}[3\bar{1}]) + \bar{1}\bar{2}([13][2\bar{3}] + [23][1\bar{3}]))[\Theta3][\bar{\Theta}3], \\
 \mathcal{P}_A^{\prime(4)} &= i([12][\Theta3](\bar{\Theta}1[\bar{2}\bar{3}] + \bar{\Theta}2[\bar{1}\bar{3}]) - \bar{1}\bar{2}([\Theta2][13] + [\Theta1][23])[\bar{\Theta}3])[3\bar{3}], \\
 \mathcal{P}_A^{\prime(5)} &= i([12](\bar{1}\bar{3}[3\bar{1}] + \bar{2}\bar{3}[3\bar{1}]) - \bar{1}\bar{2}([13][2\bar{3}] + [23][1\bar{3}]))[3\bar{3}][\Theta\bar{\Theta}], \\
 \mathcal{P}_A^{\prime(6)} &= [3\bar{3}]([\bar{1}\bar{2}](\Theta\bar{3}[1\bar{\Theta}3] + [13][\Theta2][\bar{\Theta}3]) + [12](\bar{2}\bar{3}[\Theta3][\bar{\Theta}1] + \bar{1}\bar{3}[\Theta3][\bar{\Theta}2])) \\
 &\quad - 2[12][\bar{1}\bar{3}][\bar{2}\bar{3}][\Theta3][3\bar{\Theta}] - 2[\bar{1}\bar{2}][13][23][\Theta3][\bar{\Theta}3], \\
 \mathcal{P}_A^{\prime(7)} &= ([12](\bar{2}\bar{3}[\Theta\bar{1}] + \bar{1}\bar{3}[\Theta2])[3\bar{\Theta}] + \bar{1}\bar{2}[\Theta\bar{3}](\Theta\bar{3}[1\bar{\Theta}] + [13][2\bar{\Theta}]))[3\bar{3}], \\
 \mathcal{P}_A^{\prime(8)} &= ([12][3\bar{1}][3\bar{2}][\Theta\bar{3}][\bar{\Theta}3] + \bar{1}\bar{2}[13][2\bar{3}][\Theta3][3\bar{\Theta}]).
 \end{aligned} \tag{A.2}$$

If  $\ell = 0$  only the structures  $\mathcal{P}_A^{(1)}$ ,  $\mathcal{P}_A^{(3)}$  and  $\mathcal{P}_A^{\prime(1)}$  are present. The structures for the odd-spin case appearing in (4.8) read

$$\begin{aligned}
 \tilde{\mathcal{P}}_A^{(1)} &= i([12]([\bar{1}\bar{3}][\bar{3}\bar{2}] + [\bar{2}\bar{3}][\bar{3}\bar{1}]) - [\bar{1}\bar{2}]([13][\bar{2}\bar{3}] + [23][\bar{1}\bar{3}])) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_A^{(2)} &= ([12]([\bar{1}\bar{3}][\bar{3}\bar{2}] + [\bar{2}\bar{3}][\bar{3}\bar{1}]) + [\bar{1}\bar{2}]([13][\bar{2}\bar{3}] + [23][\bar{1}\bar{3}])) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_A^{\prime(1)} &= [13][23][\bar{1}\bar{3}][\bar{2}\bar{3}], \\
 \tilde{\mathcal{P}}_A^{\prime(2)} &= ([\bar{1}\bar{1}][\bar{2}\bar{2}] + [\bar{1}\bar{2}][\bar{2}\bar{1}]) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_A^{\prime(3)} &= [12][\bar{1}\bar{2}][3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_A^{\prime\prime(1)} &= ([13][\bar{1}\bar{3}][\Theta 2][\bar{\Theta}\bar{2}] + [13][\bar{2}\bar{3}][\Theta 2][\bar{\Theta}\bar{1}] + [23][\bar{1}\bar{3}][\Theta 1][\bar{\Theta}\bar{2}] + [23][\bar{2}\bar{3}][\Theta 1][\bar{\Theta}\bar{1}]) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_A^{\prime\prime(2)} &= ([13]([\bar{1}\bar{3}][\bar{2}\bar{2}] + [\bar{2}\bar{3}][\bar{2}\bar{1}]) + [23]([\bar{1}\bar{3}][\bar{1}\bar{2}] + [\bar{2}\bar{3}][\bar{1}\bar{1}])) [\Theta\bar{\Theta}] [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_A^{\prime\prime(3)} &= ([\bar{1}\bar{1}][\Theta 2][\bar{\Theta}\bar{2}] + [\bar{1}\bar{2}][\Theta 2][\bar{\Theta}\bar{1}] + [\bar{2}\bar{1}][\Theta 1][\bar{\Theta}\bar{2}] + [\bar{2}\bar{2}][\Theta 1][\bar{\Theta}\bar{1}]) [3\bar{3}]^3, \\
 \tilde{\mathcal{P}}_A^{\prime\prime(4)} &= ([\bar{1}\bar{1}][23][\bar{2}\bar{3}] + [\bar{1}\bar{2}][23][\bar{1}\bar{3}] + [\bar{2}\bar{1}][13][\bar{2}\bar{3}] + [\bar{2}\bar{2}][13][\bar{1}\bar{3}]) [\Theta 3][\bar{\Theta}\bar{3}][3\bar{3}], \\
 \tilde{\mathcal{P}}_A^{\prime\prime(5)} &= i([13][\Theta 2] + [23][\Theta 1]) [\bar{\Theta}\bar{3}][3\bar{1}][3\bar{2}] - [\Theta 3]([\bar{\Theta}\bar{1}][3\bar{2}] + [\bar{\Theta}\bar{2}][3\bar{1}]) [1\bar{3}][\bar{2}\bar{3}] [3\bar{3}], \\
 \tilde{\mathcal{P}}_A^{\prime\prime(6)} &= i([13][\bar{2}\bar{3}] + [23][\bar{1}\bar{3}])([\Theta\bar{1}][\bar{\Theta}\bar{2}] + [\Theta\bar{2}][\bar{\Theta}\bar{1}]) \\
 &\quad - ([\bar{1}\bar{3}][\bar{3}\bar{2}] + [\bar{2}\bar{3}][\bar{3}\bar{1}])([\Theta 1][\bar{2}\bar{\Theta}] + [\Theta 2][\bar{1}\bar{\Theta}]) [3\bar{3}]^2. \tag{A.3}
 \end{aligned}$$

If  $\ell = 1$  the structures  $\tilde{\mathcal{P}}_A^{\prime(1)}$ ,  $\tilde{\mathcal{P}}_A^{\prime\prime(4)}$  and  $\tilde{\mathcal{P}}_A^{\prime\prime(5)}$  are not present.

Next let us consider case (B) for  $\ell$  even. In (4.15) we have the following tensor structures:

$$\begin{aligned}
 \mathcal{P}_B^{(1)} &= [13][23]([\bar{2}\bar{3}][\bar{3}\bar{1}] + [\bar{1}\bar{3}][\bar{3}\bar{2}]), \\
 \mathcal{P}_B^{(2)} &= ([13]([\bar{2}\bar{1}][\bar{3}\bar{2}] + [\bar{3}\bar{1}][\bar{2}\bar{2}]) + [23]([\bar{1}\bar{1}][\bar{3}\bar{1}] + [\bar{3}\bar{1}][\bar{1}\bar{2}])) [3\bar{3}], \\
 \mathcal{P}_B^{\prime(1)} &= [\bar{1}\bar{2}]([13][\Theta 2][\bar{3}\bar{\Theta}] + [23][\Theta 1][\bar{3}\bar{\Theta}]) [3\bar{3}], \\
 \mathcal{P}_B^{\prime(2)} &= [\bar{1}\bar{2}]([13][\Theta 3][\bar{2}\bar{\Theta}] + [23][\Theta 3][\bar{1}\bar{\Theta}]) [3\bar{3}], \\
 \mathcal{P}_B^{\prime(3)} &= [12][\bar{3}\bar{1}][\bar{3}\bar{2}][\Theta 3][\bar{\Theta}\bar{3}], \\
 \mathcal{P}_B^{\prime(4)} &= [13][23][\bar{1}\bar{2}][\Theta 3][\bar{\Theta}\bar{3}], \\
 \mathcal{P}_B^{\prime(5)} &= [12]([\bar{3}\bar{2}][\Theta 3][\bar{\Theta}\bar{1}] + [\bar{3}\bar{1}][\Theta 3][\bar{\Theta}\bar{2}]) [3\bar{3}], \\
 \mathcal{P}_B^{\prime(6)} &= [12]([\bar{3}\bar{2}][\Theta\bar{1}][\bar{3}\bar{\Theta}] + [\bar{3}\bar{1}][\Theta\bar{2}][\bar{3}\bar{\Theta}]) [3\bar{3}]. \tag{A.4}
 \end{aligned}$$

If  $\ell = 0$  the structures  $\mathcal{P}_B^{(1)}$ ,  $\mathcal{P}_B^{\prime(3)}$  and  $\mathcal{P}_B^{\prime(4)}$  are not present. The structures for the odd- $\ell$  case appearing in (4.19) are

$$\begin{aligned}
 \tilde{\mathcal{P}}_B^{(1)} &= [13][\bar{1}\bar{2}][23][3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_B^{(2)} &= [12][\bar{3}\bar{1}][\bar{3}\bar{2}][3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_B^{\prime(1)} &= [13][23]([\bar{2}\bar{3}][\bar{3}\bar{1}] + [\bar{1}\bar{3}][\bar{3}\bar{2}]) [\Theta 3][\bar{\Theta}\bar{3}], \\
 \tilde{\mathcal{P}}_B^{\prime(2)} &= ([[\bar{1}\bar{1}][\bar{3}\bar{2}] + [\bar{3}\bar{1}][\bar{1}\bar{2}])[\Theta 3][\bar{2}\bar{\Theta}] + ([\bar{2}\bar{1}][\bar{3}\bar{2}] + [\bar{3}\bar{1}][\bar{2}\bar{2}])[\Theta 3][\bar{1}\bar{\Theta}]) [3\bar{3}]^2,
 \end{aligned}$$



$$\begin{aligned}
 \tilde{\mathcal{P}}_B'^{(3)} &= ([23]([3\bar{2}][\Theta 1][\bar{\Theta} 1] + [3\bar{1}][\Theta 1][\bar{\Theta} 2]) - [13]([3\bar{2}][\Theta 2][\bar{\Theta} 1] + [3\bar{1}][\Theta 2][\bar{\Theta} 2])) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_B'^{(4)} &= [13][23]([\bar{1}\bar{3}][\Theta 3][\bar{\Theta} 2] + [\bar{2}\bar{3}][\Theta 3][\bar{\Theta} 1]) [3\bar{3}], \\
 \tilde{\mathcal{P}}_B'^{(5)} &= [3\bar{1}][3\bar{2}]([13][\Theta 2][\bar{\Theta} 3] + [23][\Theta 1][\bar{\Theta} 3]) [3\bar{3}], \\
 \tilde{\mathcal{P}}_B'^{(6)} &= ([13]([2\bar{1}][3\bar{2}] + [3\bar{1}][2\bar{2}]) + [23]([1\bar{1}][3\bar{2}] + [3\bar{1}][1\bar{2}])) [\Theta 3][\bar{\Theta} 3][3\bar{3}], \\
 \tilde{\mathcal{P}}_B'^{(7)} &= [3\bar{1}][3\bar{2}]([\Theta 1][2\bar{\Theta}] + [\Theta 2][1\bar{\Theta}]) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_B'^{(8)} &= (([13][2\bar{2}] + [23][1\bar{2}])[\Theta 3][\bar{\Theta} 1] - ([13][2\bar{1}] + [23][1\bar{1}])[\Theta 3][\bar{\Theta} 2]) [3\bar{3}]^2, \\
 \tilde{\mathcal{P}}_B'^{(9)} &= [12][\bar{1}\bar{2}][\Theta 3][3\bar{\Theta}][3\bar{3}]^2.
 \end{aligned} \tag{A.5}$$

If  $\ell = 1$  the structure  $\tilde{\mathcal{P}}_B'^{(1)}$  is not present.

Finally let us consider case (C) for  $\ell$  even. In (4.27) we have the following tensor structures:

$$\begin{aligned}
 \mathcal{P}_C^{(1)} &= [13][23][3\bar{1}][3\bar{2}], \\
 \mathcal{P}_C'^{(1)} &= [12][3\bar{1}][3\bar{2}][\Theta 3][3\bar{\Theta}], \\
 \mathcal{P}_C'^{(2)} &= [13][23][\bar{1}\bar{2}][\Theta 3][3\bar{\Theta}].
 \end{aligned} \tag{A.6}$$

The structures for the odd- $\ell$  case appearing in (4.33) are

$$\begin{aligned}
 \tilde{\mathcal{P}}_C^{(1)} &= [13][23]([3\bar{2}][\Theta 3][\bar{\Theta} 1] + [3\bar{1}][\Theta 3][\bar{\Theta} 2]) [3\bar{3}], \\
 \tilde{\mathcal{P}}_C^{(2)} &= [3\bar{1}][3\bar{2}]([13][\Theta 2][3\bar{\Theta}] + [23][\Theta 1][3\bar{\Theta}]) [3\bar{3}], \\
 \tilde{\mathcal{P}}_C^{(3)} &= [3\bar{1}][3\bar{2}]([13][\Theta 3][2\bar{\Theta}] + [23][\Theta 3][1\bar{\Theta}]) [3\bar{3}], \\
 \tilde{\mathcal{P}}_C^{(4)} &= [13][23][3\bar{1}][3\bar{2}][\Theta 3][\bar{\Theta} 3].
 \end{aligned} \tag{A.7}$$

## B Tensor structures in vector formalism

The structures in  $t_A$  can be found by considering the problem purely in Lorentz vector indices. For explicit computations it is of course more convenient to use spinor indices, which we can contract with auxiliary commuting spinors — this relieves us from the headache of removal of traces. In vector form we have<sup>21</sup>

$$\begin{aligned}
 t_{A\mu\nu;\rho_1\dots\rho_\ell}^{(\ell \text{ even})}(X, \bar{X}) &= \frac{1}{(X \cdot \bar{X})^{3+\frac{1}{2}\ell-q}} \left( \sum_{i=1}^4 \mathcal{P}_{\mu\nu;(\rho_1\rho_2)}^{(i)} (\mathcal{A}_i + \mathcal{B}_i \beta^2) + \sum_{i=5}^{12} \mathcal{P}_{\mu\nu;(\rho_1\rho_2)}^{(i)} \mathcal{C}_i \right) U_{\rho_3} \cdots U_{\rho_\ell}, \\
 \beta^2 &= \frac{V^2}{X \cdot \bar{X}},
 \end{aligned} \tag{B.1}$$

<sup>21</sup>In these vector-form expressions a subtraction of the traces in the indices  $\rho_1, \dots, \rho_\ell$  is understood.

where  $\mathcal{A}_i$ ,  $\mathcal{B}_i$  and  $\mathcal{C}_i$  are real constants. The tensor structures  $\mathcal{P}^{(i)}$  are given by

$$\begin{aligned}
 \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(1)} &= \eta_{\mu\nu} U_{\rho_1} U_{\rho_2}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(7)} &= V_{[\mu}\eta_{\nu]\rho_1} U_{\rho_2}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(2)} &= \frac{X_{(\mu}\bar{X}_{\nu)}}{X \cdot \bar{X}} U_{\rho_1} U_{\rho_2}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(8)} &= i\eta_{\rho_1[\mu}\epsilon_{\nu]\kappa\lambda\rho_2} X^\kappa \bar{X}^\lambda, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(3)} &= U_{(\mu}\eta_{\nu)\rho_1} U_{\rho_2}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(9)} &= i\frac{U \cdot V}{X \cdot \bar{X}} \epsilon_{\mu\nu\kappa\rho_1} U^\kappa U_{\rho_2}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(4)} &= X \cdot \bar{X} \eta_{\rho_1(\mu}\eta_{\nu)\rho_2}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(10)} &= i\epsilon_{\mu\nu\kappa\rho_1} U^\kappa V_{\rho_2}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(5)} &= \frac{i\epsilon_{\mu\nu\kappa\lambda} X^\kappa \bar{X}^\lambda}{X \cdot \bar{X}} U_{\rho_1} U_{\rho_2}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(11)} &= \frac{\eta_{\rho_1[\mu} U_{\nu]} U \cdot V U_{\rho_2}}{X \cdot \bar{X}}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(6)} &= \epsilon_{\mu\nu\lambda\rho_1} V^\lambda U_{\rho_2}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2}^{(12)} &= iU_{[\mu}\epsilon_{\nu]\kappa\lambda\rho_1} \frac{X^\kappa \bar{X}^\lambda}{X \cdot \bar{X}} U_{\rho_2},
 \end{aligned} \tag{B.2}$$

with the definitions<sup>22</sup>

$$U = \frac{1}{2}(X + \bar{X}), \quad V = i(X - \bar{X}). \tag{B.3}$$

The structures  $\mathcal{P}^{(1)}$ ,  $\mathcal{P}^{(2)}$ ,  $\mathcal{P}^{(5)}$  in (B.1) appear for  $\ell = 0$  [23], while the others do not. Since  $V \rightarrow 0$  and  $\bar{X} \rightarrow X$  if the Grassmann variables are set to zero, we have recovered the four structures expected from the results of [7], associated with the coefficients  $\mathcal{A}_1$ ,  $\mathcal{A}_2$ ,  $\mathcal{A}_3$  and  $\mathcal{A}_4$ .

In the case of odd spins and before application of the Ward identity for the supercurrent there is a parity-odd and a parity-even structure if all Grassmann variables are set to zero [7]. Additionally, there are nilpotent structures. We find

$$\begin{aligned}
 t_{\mathcal{A}\mu\nu;\rho_1\dots\rho_\ell}^{(\ell \text{ odd})}(X, \bar{X}) &= \frac{1}{(X \cdot \bar{X})^{3+\frac{1}{2}\ell-q}} \left( \sum_{i=1}^2 \mathcal{P}_{\mu\nu;(\rho_1\rho_2\rho_3)}^{(i)} (\mathcal{A}_i + \mathcal{B}_i \beta^2) + \sum_{i=3}^6 \mathcal{P}_{\mu\nu;(\rho_1\rho_2\rho_3)}^{(i)\lambda} (\mathcal{C}_i V_\lambda + \mathcal{D}_i \zeta_\lambda) \right. \\
 &\quad \left. + \sum_{i=7}^{10} \mathcal{P}_{\mu\nu;(\rho_1\rho_2\rho_3)}^{(i)} \mathcal{E}_i \right) U_{\rho_4} \cdots U_{\rho_\ell}, \\
 \beta^2 &= \frac{V^2}{X \cdot \bar{X}}, \quad \zeta_\mu = \frac{U \cdot V}{X \cdot \bar{X}} U_\mu,
 \end{aligned} \tag{B.4}$$

where  $\mathcal{A}_i$ ,  $\mathcal{B}_i$ ,  $\mathcal{C}_i$ ,  $\mathcal{D}_i$ ,  $\mathcal{E}_i$  are real constants and the tensor structures  $\mathcal{P}^{(i)}$  are given by

$$\begin{aligned}
 \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(1)} &= \epsilon_{\mu\nu\lambda\rho_1} U^\lambda U_{\rho_2} U_{\rho_3}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(6)\lambda} &= X \cdot \bar{X} \eta_{\rho_1(\mu}\eta_{\nu)\rho_2} \delta^\lambda_{\rho_3}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(2)} &= U_{[\mu}\eta_{\nu](\rho_1} U_{\rho_2} U_{\rho_3}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(7)} &= \frac{U_{(\mu} V_{\nu)} U_{\rho_1} U_{\rho_2} U_{\rho_3}}{X \cdot \bar{X}}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(3)\lambda} &= \eta_{\mu\nu} \delta^\lambda_{\rho_1} U_{\rho_2} U_{\rho_3}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(8)} &= \eta_{\rho_1(\mu}\epsilon_{\nu)\kappa\lambda\rho_2} X^\kappa \bar{X}^\lambda U_{\rho_3}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(4)\lambda} &= \delta^\lambda_{(\mu} \eta_{\nu)\rho_1} U_{\rho_2} U_{\rho_3}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(9)} &= U_{(\mu}\eta_{\nu)\rho_1} V_{\rho_2} U_{\rho_3}, \\
 \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(5)\lambda} &= \frac{\delta^\lambda_{\rho_1} U_\mu U_\nu U_{\rho_2} U_{\rho_3}}{X \cdot \bar{X}}, & \mathcal{P}_{\mu\nu;\rho_1\rho_2\rho_3}^{(10)} &= \frac{U_{(\mu}\epsilon_{\nu)\kappa\lambda\rho_1} X^\kappa \bar{X}^\lambda U_{\rho_2} U_{\rho_3}}{X \cdot \bar{X}}.
 \end{aligned} \tag{B.5}$$

When all Grassmann variables are set to zero the parity-odd structure associated with  $\mathcal{A}_1$  and the parity-even structure associated with  $\mathcal{A}_2$  are the only ones that survive.

<sup>22</sup> $U$  and  $V$  are called  $Q$  and  $P$ , respectively, in [23].

## C Special cases in the solutions of the Ward identities

In section 4 we studied the Ward identities arising from equation (3.1) assuming generic values for  $\Delta, \ell$ . As one can see by inspecting equations (4.7), (4.11), (4.18), (4.21), (4.30) and (4.34) all terms are well defined for unitary values of  $\Delta$ , except for the special cases  $\ell = 0$  in (4.7) and  $\ell = 1$  in (4.11) for which we find a pole at  $\Delta = \ell + 2$ , i.e. the unitarity bound. These poles are artifacts of the choice of independent coefficients made when solving the equations arising from conservation and can therefore be removed. A choice which is valid for all unitary values of  $\Delta$  and  $\ell$  does not exist.

Case (A) for  $\ell = 0$  has only the structures associated with the coefficients  $A_1, B_1, A_3, B_3$  and  $C_1$ . If we choose to solve for  $A_3$  we obtain simply

$$A_1 = B_1 = B_3 = C_1 = 0. \quad (\text{C.1})$$

Since we are at the unitarity threshold most of the coefficients of the expansion in  $\theta_3 \bar{\theta}_3$  computed in section 5 vanish. We are left with only

$$\lambda_{++}^{(-)} = 2i A_3. \quad (\text{C.2})$$

Case (A) for  $\Delta = 3, \ell = 1$  corresponds to the third operator being the Ferrara-Zumino multiplet  $\mathcal{J}_{\gamma\dot{\gamma}}$ . The structures associated with  $C_1, D_1, E_4$  and  $E_5$  are not present. Solving for  $A_1$  and  $D_2$  we find

$$\begin{aligned} A_2 = B_1 = B_2 = E_6 &= 0, \\ C_2 = -\frac{3}{2}E_3 = D_2, & \quad C_3 = -\frac{1}{2}D_3 = 2A_1 \\ E_1 = -2A_1 - \frac{1}{3}D_2, & \quad E_2 = 4A_1 + D_2. \end{aligned} \quad (\text{C.3})$$

The coefficients of the expansion in  $\theta_3 \bar{\theta}_3$  are

$$\lambda_{+++}^{(1)} = -2A_1 - \frac{1}{6}D_2, \quad \lambda_{+++}^{(2)} = -\lambda_{+++}^{(3)} = \frac{2}{3}(3A_1 + D_2), \quad \lambda_{+++}^{(4)} = -\frac{1}{3}D_2, \quad \lambda_{+++}^{(5)} = \frac{1}{2}(4A_1 + D_2), \quad (\text{C.4})$$

with all others being zero. This case was already analyzed in [23] with the same results. The relation between our coefficients and the coefficients  $A, \dots, J$  defined in [23] is

$$\begin{aligned} A &= A_1, & D &= \frac{1}{4}(D_3 - D_2 - E_2 - E_3), & G &= \frac{1}{2}C_2, \\ B &= \frac{1}{8}(4A_1 - B_1), & E &= \frac{1}{2}E_1, & H &= D_2, \\ C &= \frac{1}{8}(C_3 - C_2 - E_1), & F &= E_2, & J &= \frac{1}{2}E_3. \end{aligned} \quad (\text{C.5})$$

Taking the spin-two superconformal descendant of each superconformal primary yields the three-point correlator of the stress-energy tensor,  $T_{\mu\nu}$ . It is interesting, thus, to relate the coefficients defined here,  $A_1$  and  $D_2$ , with the anomaly coefficients  $c$  (proportional to the central charge  $C_T$ ) and  $a$  (Euler anomaly). Using [23, eq. (11.7)] together with (C.5), or, equivalently, following [41, appendix C] and using the relation between  $C_J$  and  $C_T$  stemming from supersymmetry [24, eq. (5.5)] one obtains

$$A_1 = -\frac{8}{9\pi^6}(5a - 3c), \quad D_2 = \frac{16}{3\pi^6}(2a - 3c). \quad (\text{C.6})$$

The precise relation between  $c$  and  $C_T$  is [41]  $c = \frac{\pi^4}{40}C_T$ .

## D Conventions for the supersymmetric derivatives

In order to impose (3.1) we follow the formalism of [23] and pass the derivatives  $D_{1\alpha}$  and  $\bar{D}_{1\dot{\alpha}}$  through the prefactor in (3.7). This can be done using the identities

$$\begin{aligned}\bar{D}_1^{\dot{\alpha}} \left( \frac{(x_{3\bar{1}})_{\alpha\dot{\alpha}}}{(x_{1\bar{3}})^2} F^\alpha(X, \Theta, \bar{\Theta}) \right) &= -i \frac{1}{x_{3\bar{1}}^2 x_{1\bar{3}}^2} \mathcal{D}_\alpha F^\alpha(X, \Theta, \bar{\Theta}), \\ D_1^\alpha \left( \frac{(x_{1\bar{3}})_{\alpha\dot{\alpha}}}{(x_{3\bar{1}})^2} F^{\dot{\alpha}}(X, \Theta, \bar{\Theta}) \right) &= -i \frac{1}{x_{3\bar{1}}^2 x_{1\bar{3}}^2} \bar{\mathcal{D}}_{\dot{\alpha}} F^{\dot{\alpha}}(X, \Theta, \bar{\Theta}).\end{aligned}\tag{D.1}$$

The superspace derivatives are defined in the following way

$$D_{i\alpha} = \frac{\partial}{\partial \theta_i^\alpha} + i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}_i^{\dot{\alpha}} \frac{\partial}{\partial x_i^\mu}, \quad \bar{D}_{i\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}_i^{\dot{\alpha}}} - i \theta_i^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \frac{\partial}{\partial x_i^\mu}.\tag{D.2}$$

For the derivatives  $\mathcal{D}$  and  $\bar{\mathcal{D}}$  we can use three different representations according to whether we prefer writing the expressions in terms of  $X$ ,  $\bar{X}$  or  $U = \frac{1}{2}(X + \bar{X})$ .

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \Theta^\alpha} - 2i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\Theta}^{\dot{\alpha}} \frac{\partial}{\partial X^\mu}, \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\Theta}^{\dot{\alpha}}},\tag{D.3a}$$

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \Theta^\alpha}, \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\Theta}^{\dot{\alpha}}} + 2i \Theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \frac{\partial}{\partial \bar{X}^\mu},\tag{D.3b}$$

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \Theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\Theta}^{\dot{\alpha}} \frac{\partial}{\partial U^\mu}, \quad \bar{\mathcal{D}}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\Theta}^{\dot{\alpha}}} + i \Theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \frac{\partial}{\partial U^\mu}.\tag{D.3c}$$

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