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R-SPACES ASSOCIATED WITH A HERMITIAN SYMMETRIC PAIR

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1. Introduction.

The linear isotropy representation of a Riemannian symmetric pair (G, K) is defined as the differential of the left action of K on G/K at the origin. Every orbit of the linear isotropy representation of (G, K) is called an *R-space associated with* (G, K), which is an important example of equivariant homogeneous Riemannian submanifolds in a Euclidean sphere (See Takagi-Takahashi [2] and Takeuchi-Kobayashi [3]).

This paper is concerned with the linear isotropy representation of a Hermitian symmetric pair (G, K). Its restriction to the center of K defines an S^1-action on the associated R-spaces. We determine all R-spaces associated with Hermitian symmetric pairs (G, K) on which the semisimple part of K acts transitively. In particular, we know all irreducible Hermitian symmetric pairs such that each of the associated R-spaces has such a property. This result is utilizable for the classification of orthogonal transformation groups by their cohomogeneity (See the forthcoming paper [4] concerned with this problem in low cohomogeneity).

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2. Statement of the result.

Let (G, K) be an irreducible Hermitian symmetric pair of compact type and $\mathfrak{g}$ [resp. $\mathfrak{k}$] the Lie algebra of G [resp. K]. Then $\mathfrak{g}$ has the canonical direct sum decomposition:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{m},$$

where $\mathfrak{m}$ is the subspace of $\mathfrak{g}$ satisfying

$$[\mathfrak{k}, \mathfrak{m}]\subset\mathfrak{m} \quad \text{and} \quad [\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{k}.$$

The tangent space of G/K at the origin can be naturally identified with $\mathfrak{m}$. Then

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the linear isotropy representation of (G, K) is nothing but the adjoint action Ad of K on $\mathfrak{m}$.

Let K_s be the analytic subgroup of K corresponding to the semisimple part $\mathfrak{k}_s=[\mathfrak{k}, \mathfrak{k}]$ of $\mathfrak{k}$ and $\mathfrak{z}$ be the 1-dimensional center of $\mathfrak{k}$. We can take an element H_0 in $\mathfrak{z}$ such that

$$(\mathrm{ad}\, H_0|_{\mathfrak{m}})^2 = -id_{\mathfrak{m}} ,$$

because (G, K) is a Hermitian symmetric pair.

Take a maximal Abelian subalgebra $\mathfrak{h}$ in $\mathfrak{k}$. Then $\mathfrak{h}$ is also a maximal Abelian subalgebra in $\mathfrak{g}$ and the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let Δ denote the set of all non-zero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$. For each $\alpha \in \Delta$, define a subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}^C$ by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^C ; [H, X] = \alpha(H) X \text{ for all } H \in \mathfrak{h}^C\}$$

and choose a non-zero vector $X_\alpha \in \mathfrak{g}_\alpha$ such that

$$X_\alpha - X_{-\alpha},\ \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{g} \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = \frac{2}{\alpha(H_\alpha)} H_\alpha ,$$

where H_α in $\mathfrak{h}^C$ is the dual vector of α with respect to the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}^C$. The set of all compact [resp. noncompact] roots in Δ is denoted by Δ_c [resp. Δ_n]:

$$\mathfrak{k}^C = \mathfrak{h}^C + \sum_{\alpha \in \Delta_c} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{m}^C = \sum_{\alpha \in \Delta_n} \mathfrak{g}_\alpha .$$

Fix the lexicographic ordering in the dual space of the real vector space $\sqrt{-1}\mathfrak{h}$ with respect to an ordered basis

$$\sqrt{-1} H_0 (= Y_1),\ Y_2, \cdots, Y_m ;\ m = \dim_{\boldsymbol{R}} (\sqrt{-1}\mathfrak{h})$$

in $\sqrt{-1}\mathfrak{h}$. Let Δ^+ [resp. Δ_n^+] denote the set of all positive roots in Δ [resp. Δ_n]. There is a direct sum decomposition of $\mathfrak{m}$:

$$\mathfrak{m} = \sum_{\alpha \in \Delta_n^+} \{\boldsymbol{R}(X_\alpha - X_{-\alpha}) + \boldsymbol{R}\sqrt{-1}(X_\alpha + X_{-\alpha})\} .$$

According to Harish-Chandra [1, § 6], there exists a subset $\Gamma = \{\gamma_1, \cdots, \gamma_r\}$ of Δ_n^+ such that $\gamma_i \pm \gamma_j \notin \Delta$ $(1 \leq i, j \leq r)$ and

$$\mathfrak{a} = \textstyle\sum_{i=1}^r \boldsymbol{R}\sqrt{-1}(X_{\gamma_i} + X_{-\gamma_i})$$

is a maximal Abelian subspace of $\mathfrak{m}$, where r is the rank of the symmetric pair (G, K).

Consider the automorphism, so-called Cayley transformation,

$$\upsilon=\exp\frac{\pi}{4}\operatorname{ad}\left(\sum_{i=1}^{r}(X_{\gamma_i}-X_{-\gamma_i})\right)$$

of the Lie algebra $\mathfrak{g}^C$. We have $\upsilon(\mathfrak{a})\subset\mathfrak{k}$, since

$$\upsilon(\sqrt{-1}(X_{\gamma_i}+X_{-\gamma_i}))=\frac{2\sqrt{-1}}{\gamma_i(H_{\gamma_i})}H_{\gamma_i} \qquad (1\leq i\leq r).$$

Let $^{-}$ denote the restriction of a linear form on $\mathfrak{h}^C$ to $\upsilon(\mathfrak{a}^C)$. The sets of all non-zero elements in $\bar{\Delta}$, $\overline{\Delta^+}$, $\bar{\Delta}_c$, $\bar{\Delta}_n$, and $\overline{\Delta_n^+}$ are denoted by R, R^+, R_c, R_n, and R_n^+ respectively. R is isomorphic to the restricted root system of the Hermitian symmetric pair (G, K). By Harish-Chandra [1, § 6], there are only two possibilities:

Case i) R is of type C;

$$R=\{\pm\bar{\gamma}_i\}\cup\left\{\frac{1}{2}(\pm\bar{\gamma}_i\pm\bar{\gamma}_j);\ i\neq j\right\},$$

$$R_c=\left\{\frac{1}{2}(\bar{\gamma}_i-\bar{\gamma}_j);\ i\neq j\right\},$$

$$R_n=\{\pm\bar{\gamma}_i\}\cup\left\{\pm\frac{1}{2}(\bar{\gamma}_i+\bar{\gamma}_j);\ i\neq j\right\},$$

Case ii) R is of type BC;

$$R=\{\pm\bar{\gamma}_i\}\cup\left\{\frac{1}{2}(\pm\bar{\gamma}_i\pm\bar{\gamma}_j);\ i\neq j\right\}\cup\left\{\pm\frac{1}{2}\bar{\gamma}_i\right\},$$

$$R_c=\left\{\frac{1}{2}(\bar{\gamma}_i-\bar{\gamma}_j);\ i\neq j\right\}\cup\left\{\pm\frac{1}{2}\bar{\gamma}_i\right\},$$

$$R_n=\{\pm\bar{\gamma}_i\}\cup\left\{\pm\frac{1}{2}(\bar{\gamma}_i+\bar{\gamma}_j);\ i\neq j\right\}\cup\left\{\pm\frac{1}{2}\bar{\gamma}_i\right\}.$$

Then our result is the following:

THEOREM. *Let M be an R-space associated with an irreducible Hermitian symmetric pair (G, K). Then the following two conditions are equivalent.*

1) *The action of K_s on M is transitive.*
2) *The restricted root system R of (G, K) is of type BC or there exists a γ_i in Γ such that $\gamma_i(\upsilon(M\cap\mathfrak{a}))=\{0\}$.*

In particular, K_s acts transitively on each of the associated R-spaces if and only if R is of type BC.

REMARK. Suppose that M is an R-space of the highest dimension among those associated with a given irreducible Hermician symmdtric pair (G, K), i.e., M is a maximum dimensional K-orbit of the linear isotropy representation of

(G, K). Then $M\cap\mathfrak{a}$ contains a regular element H, which satisfies $\gamma_i(\upsilon(H))\neq 0$ for all i. Then the transitivety of K_s on M is equivalent to the condition that the restricted root system R is of type BC.

3. Proof of Theorem.

Fix an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, which is a negative multiple of the restriction of the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}^C$ to $\mathfrak{g}$.

Let H be any fixed element of $M\cup\mathfrak{a}$ and $\mathfrak{k}_H$ denote the centralizer of H in $\mathfrak{k}$:

$$\mathfrak{k}_H=\{T\in\mathfrak{k}\,;[T, H]=0\}. \tag{1}$$

The orthogonal complement of $\mathfrak{k}_H$ in $\mathfrak{k}$ is denoted by $\mathfrak{k}_H^\perp$.

Since $\mathfrak{k}_s$ is the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{k}$, the kernel of the orthogonal projection p of $\mathfrak{k}$ to $\mathfrak{z}$ is equal to $\mathfrak{k}_s$.

Since K and K_s are compact and connected, the condition 1) in Theorem is equivalent to

$$\dim\mathfrak{k}-\dim\mathfrak{k}_H=\dim\mathfrak{k}_s-\dim(\mathfrak{k}_H\cap\mathfrak{k}_s),$$

that is,

$$\dim\mathfrak{k}_H=1+\dim(\mathfrak{k}_H\cap\mathfrak{k}_s),$$

which is equivalent to $p(\mathfrak{k}_H)=\mathfrak{z}$, because $\dim\mathfrak{z}=1$.

On the other hand, $p(\mathfrak{k}_H)=\{0\}$ if and only if $\mathfrak{k}_H\subset\mathfrak{k}_s=\mathfrak{z}^\perp$, that is, $\mathfrak{k}_H^\perp\supset\mathfrak{z}$. If we take $H_1\in\mathfrak{k}_H$ and $H_2\in\mathfrak{k}_H^\perp$ such that

$$H_0=H_1+H_2, \tag{2}$$

then $H_1=0$ is equivalent to $\mathfrak{k}_H^\perp\supset\mathfrak{z}$.

So the condition 1) in Theorem is equivalent to $H_1\neq 0$ in the equation (2). Therefore the following lemma completes the proof of our theorem.

LEMMA. *$H_1\neq 0$ if and only if either the restricted root system R of (G, K) is of type BC or there exists a γ_i in Γ such that $\gamma_i(\upsilon(H))=0$.*

PROOF of Lemma. Let $\mathfrak{b}$ be the orthogonal complement of $\upsilon(\mathfrak{a})=\sum_{i=1}^r \boldsymbol{R}\sqrt{-1}H_{\gamma_i}$ in $\mathfrak{h}=\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}H_\alpha$.

Put $\Gamma_H=\{\gamma_i\in\Gamma\,;\gamma_i(\upsilon(H))=0\}$, $\mathfrak{a}_H=\sum_{\gamma_i\in\Gamma_H}\boldsymbol{R}\sqrt{-1}H_{\gamma_i}$, and $\mathfrak{a}_H^\perp=\sum_{\gamma_i\notin\Gamma_H}\boldsymbol{R}\sqrt{-1}H_{\gamma_i}$. Then $\mathfrak{a}_H^\perp$ is the orthogonal comlement of $\mathfrak{a}_H$ in $\upsilon(\mathfrak{a})$. We have an orthogonal direct sum decomposition of $\mathfrak{h}$:

$$\mathfrak{h}=(\mathfrak{b}+\mathfrak{a}_H)+\mathfrak{a}_H^\perp. \tag{3}$$

As the first step, we claim that the decomposition of H_0 with respect to the decomposition (3) is the same as the equation (2). In fact, $\mathfrak{k}_H \supset \mathfrak{b}+\mathfrak{a}_H$, since $[\nu(\mathfrak{b}+\mathfrak{a}_H), \nu(H)]=\{0\}$ by

$$\nu\left[\frac{2\sqrt{-1}}{\gamma_i(H_{\gamma_i})}\right]=-\sqrt{-1}(X_{\gamma_i}+X_{-\gamma_i}) \qquad (1\leqq i\leqq r),$$
$$\nu|_{\mathfrak{b}}=\mathrm{id}_{\mathfrak{b}} \quad \text{and} \quad \nu(\mathfrak{b}+\mathfrak{a})=\mathfrak{h}.$$

We also have $\mathfrak{k}_H^{\perp}\supset\mathfrak{a}_H^{\perp}$, since $\langle\nu(\mathfrak{k}_H), \nu(\mathfrak{a}_H^{\perp})\rangle=0$ by

$$\nu(\mathfrak{k}_H)\subset\mathfrak{h}+\sum_{\substack{\alpha\in\Delta\\ \alpha\in(\nu(H))=0}}(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}), \qquad \nu(\mathfrak{a}_H^{\perp})\subset\sum_{\gamma_i\notin\Gamma_H}(\mathfrak{g}_{\gamma_i}+\mathfrak{g}_{-\gamma_i}).$$

Therefore $\mathfrak{h}\cap\mathfrak{k}_H=\mathfrak{b}+\mathfrak{a}_H$ and $\mathfrak{h}\cap\mathfrak{k}_H^{\perp}=\mathfrak{a}_H^{\perp}$. In particular,

$$H_2=-\sum_{\gamma_i\notin\Gamma_H}\frac{\sqrt{-1}}{\gamma_i(H_{\gamma_i})}H_{\gamma_i}, \tag{4}$$

because we have

$$\gamma(H_0)=-\sqrt{-1} \qquad \text{for all } \gamma\in\Delta_n^+$$

by the definition of Δ_n^+. As a result, we obtain

$$H_1=H_0+\sum_{\gamma\notin\Gamma_H}\frac{\sqrt{-1}}{\gamma_i(H_{\gamma_i})}H_{\gamma_i}. \tag{5}$$

As the second step, we claim that $H_1\neq 0$ in the equation (5) if and only if either R is of type BC or $\Gamma_H\neq\phi$. We may assume that $H\neq 0$. Then there exists $\gamma\in\Gamma-\Gamma_H$.

If R is of type BC, then there is a compact root α such that

$$\bar{\alpha}=\frac{1}{2}\bar{\gamma}.$$

In this case, by the equation (4) and $\alpha(H_0)=0$ for all $\alpha\in\Delta_c$, we have

$$\alpha(H_1)=\alpha(-H_2)=\frac{1}{2}\gamma(-H_2)=\frac{\sqrt{-1}}{2}\neq 0,$$

especially $H_1\neq 0$.

Now suppose that R is of type C. If $\Gamma_H\neq\phi$, we can take $\gamma_j\in\Gamma_H$. There exists a compact root α such that

$$\bar{\alpha}=\frac{1}{2}(\bar{\gamma}-\bar{\gamma}_j).$$

In this case, by the equation (4),

$$\alpha(H_1)=\alpha(-H_2)=\frac{1}{2}\gamma(-H_2)=\frac{1}{2}\sqrt{-1}\neq 0,$$

especially $H_1\neq 0$. Here we have used the fact

$$\gamma_j(\mathfrak{a}_H^{\perp})=\{0\},$$

which follows from the orthogonality of elements in Γ. If $\Gamma_H=\phi$, then

$$\beta(H_1)=\beta(H_0)+\beta(-H_2)=-\sqrt{-1}+\beta(-H_2)=0$$

for all $\beta\in\Delta_n^+$, by the equation (4) and $R_n^+=\left\{\frac{1}{2}(\bar{\gamma}_p+\bar{\gamma}_q);\ 1\leqq p, q\leqq r\right\}$. On the other hand

$$\alpha(H_1)=\alpha(-H_2)=0 \qquad \text{for all } \alpha\in\Delta_c,$$

by $R_c=\left\{\frac{1}{2}(\bar{\gamma}_p-\bar{\gamma}_q);\ p\neq q\right\}$. So $H_1=0$. This completes the proof of Lemma.

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