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Radial conformal motions in Minkowski space–time

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A study of radial conformal Killing fields (RCKF) in Minkowski space–time is carried out, which leads to their classification into three disjointed classes. Their integral curves are straight or hyperbolic lines admitting orthogonal surfaces of constant curvature, whose sign is related to the causal character of the field. Otherwise, the kinematic properties of the timelike RCKF are given and their applications in kinematic cosmology is discussed. [S0022-2488(99)00507-1]

I. INTRODUCTION

The study of vector (and tensor) fields in a Lorentzian metric is a key issue, both from the theoretical and practical points of view. Infinitesimal transformations, fluid flows, eigendirections of a given 2-tensor field, critical points, continuous symmetries, directions attached to coordinate systems, light propagation and polarization in a medium, geodesic and accelerated observers are some examples of basic concepts which are described using vector fields. This work is devoted to analyzing in the Minkowski space–time the main properties of a particular type of fields that we have called *radial* conformal motions.

There are several reasons for carrying out such a study: (i) homothetic and hyperbolic radial motions belong to this kind of fields, (ii) conserved quantities along null geodesics are obtained from conformal Killing vectors, with particular expressions for the radial case, (iii) isotropic distribution functions of photons verifying the Liouville equation can be built from these conserved quantities and, (iv) this study can be easily extended to any conformally flat space–time and used to obtain its conformal factor imposing a given kinematic property of the field (geodesic, homogeneous expansion, etc.); in fact, Infeld–Schild work on kinematic cosmology[1] tacitly involves the concept of timelike radial conformal Killing field (RCKF) in the geodesic case.

Firstly, in Sec. II, we introduce the concept of RCKF and obtain its general expression and the type of subalgebra generated by them; we also present a study of their causal character according to the different domains of the space–time. We continue with a classification of these fields related to the sign of a quantity invariant by internal conformal transformations of the Minkowski metric (Sec. III). The associated integral curves are plotted in Sec. IV and we show that the orthogonal hypersurfaces of the field have constant curvature whose sign is related to its causal character (Sec. V). In Sec. VI, we focus on timelike RCKF and discuss their kinematic properties pointing out their connection with the Milne's interpretation of the cosmological recession velocity in the Minkowski space–time.[2] Finally, in Sec. VII, we comment on several applications of the present study. Some of these results have been presented, without proof, in the E.R.E., annual Spanish relativity meeting.[3]

II. RADIAL CONFORMAL KILLING FIELDS

Let us consider a radial vector field

$$\xi=\alpha(t,r,\theta,\varphi)\frac{\partial}{\partial t}+\beta(t,r,\theta,\varphi)\frac{\partial}{\partial r}$$

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in the Minkowski space–time,

$$\eta = -dt\otimes dt + dr\otimes dr + r^2 h, \tag{1}$$

with $h = d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi$, the metric on the 2-sphere.

The equation $\mathcal{L}_\xi \eta \propto \eta$ expresses that ξ is a conformal Killing field (or conformal motion) of η, where $\mathcal{L}_\xi$ represents the Lie derivative with respect to ξ. This condition leads to that the functions α and β are independent of the angular coordinates θ and φ, resulting in

$$\alpha(t,r) = a(t^2+r^2) + bt + c, \quad \beta(t,r) = r(2at+b), \tag{2}$$

where a, b, and c are arbitrary constants.

Proposition 1: In the Minkowski space–time, the general form of a RCKF is

$$\xi = (a(t^2+r^2)+bt+c)\frac{\partial}{\partial t} + r(2at+b)\frac{\partial}{\partial r}, \tag{3}$$

with a, b, and c as arbitrary constants.

Consequently, ξ can be obtained as linear combination of (the generators of) the timelike translation $\xi_1 = (\partial/\partial t)$, the dilation $\xi_2 = t(\partial/\partial t) + r(\partial/\partial r)$, and the special nonlinear conformal transformation along the t-axis $\xi_3 = (t^2+r^2)(\partial/\partial t) + 2tr(\partial/\partial r)$, that is

$$\xi = a\xi_3 + b\xi_2 + c\xi_1.$$

The Lie brackets of these generators,

$$[\xi_1,\xi_2] = \xi_1, \quad [\xi_1,\xi_3] = 2\xi_2, \quad [\xi_2,\xi_3] = \xi_3$$

give us the type of the Lie algebra generated by RCKF. In fact, if we consider the vector fields

$$e_1 = \xi_1 - \tfrac{1}{4}\xi_3, \quad e_2 = \xi_2, \quad e_3 = -\xi_1 - \tfrac{1}{4}\xi_3,$$

then, their commutation relations are

$$[e_1,e_2] = -e_3, \quad [e_2,e_3] = e_1, \quad [e_3,e_1] = e_2$$

showing that this Lie algebra is isomorphic to the pseudo-orthogonal algebra AO(1,2); then we have the following result:

Proposition 2: RCKF generate a three-dimensional Lie algebra of Bianchi type VIII.

Note that using null coordinates, $u = t+r$ and $v = t-r$, expression (3) is written in a completely symmetric form

$$\xi = (au^2+bu+c)\frac{\partial}{\partial u} + (av^2+bv+c)\frac{\partial}{\partial v}, \tag{4}$$

where each null coordinate appears separately and in the same way in the corresponding component of the field ξ.

From (3) and (4) we have

$$P \equiv -\eta(\xi,\xi) = [a(t^2+r^2)+bt+c]^2 - r^2(2at+b)^2 = (au^2+bu+c)(av^2+bv+c). \tag{5}$$

The discussion of the sign of P will give the causal character of ξ in the different regions of the space–time (domains of causality of ξ). It is convenient to introduce the quantity

$$\Delta \equiv b^2 - 4ac$$

that make this discussion easier. The results are shown in Table I and plotted in Fig. 1.

TABLE I. Causal character of the field ξ given by (3) or (4) for the different values of the constants a and $\Delta=b^2-4ac$.

		Causal character of a radial conformal Killing vector
$a=0$	$b=0$	timelike everywhere
	$b\neq 0$	null on the light cone at the point $(t=-(c/b),\ r=0)$, timelike inside of the light cone and spacelike outside of the light cone. See Fig. 1(i)
$a\neq 0$	$\Delta<0$	timelike everywhere
	$\Delta=0$	timelike everywhere except for the light cone on $(t=-(b/2a),\ r=0)$ where it is null. See Fig. 1(ii)
	$\Delta>0$	null on the light cones at the points $(t_\pm, r=0)$, $t_\pm=(-b\pm\sqrt{\Delta})/2a$, timelike inside or outside of the two light cones and spacelike in other domains. See Fig. 1(iii)

III. A CLASSIFICATION OF THE RADIAL CONFORMAL MOTIONS

In order to classify the RCKF in equivalence classes, it is convenient to take in account the degree of freedom of the null coordinates $\{u,v\}$. Then we consider the coordinates

$$\bar{u}=\bar{u}(u),\quad \bar{v}=\bar{v}(v),\quad \bar{\theta}=\theta,\quad \bar{\varphi}=\varphi$$

verifying the condition

$$\bar{u}_u\bar{v}_v=\left(\frac{\bar{u}-\bar{v}}{u-v}\right)^2 \tag{6}$$

to obtain a conformally flat form of the Minkowski metric (1),

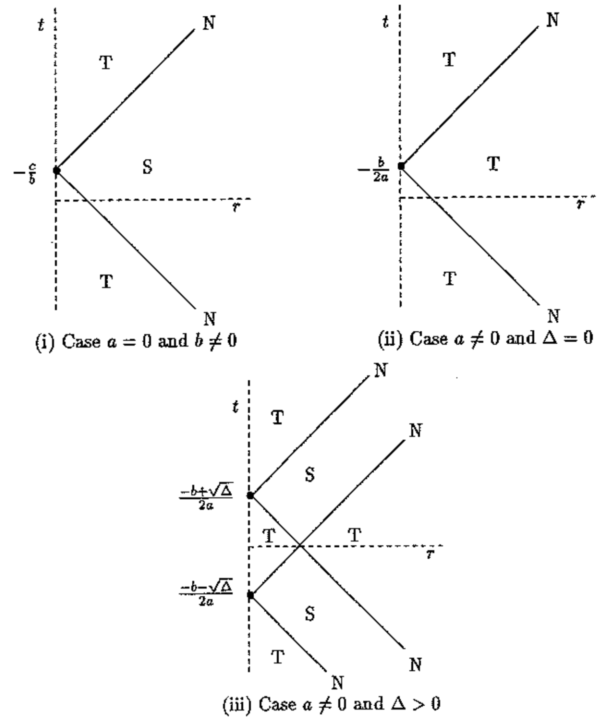

FIG. 1. Domains of causality of a RCKF ξ according to the values of the coefficient a and $\Delta=b^2-4ac$. The different possibilities for the causal character, Timelike, Spacelike or Null, in each domain, are abbreviated with the capital letters T, S, or N, respectively.

$$\eta=F(\bar{u},\bar{v})\left[-\frac{1}{2}(d\bar{u}\otimes d\bar{v}+d\bar{v}\otimes d\bar{u})+\left(\frac{\bar{u}-\bar{v}}{2}\right)^2 h\right],$$

where the *internal conformal factor, F*, is

$$F=\frac{1}{\bar{u}_u\bar{v}_v}=u_{\bar{u}}v_{\bar{v}} \tag{7}$$

with subindexes denoting derivation with respect to the coordinates.

The integration of Eq. (6) with respect to $\bar{u}(u)$, considering the v-coordinate as a parameter, gives

$$\bar{u}(u)=\frac{\bar{v}_v(u-v)}{1+(u-v)A}+\bar{v}, \tag{8}$$

where A is an arbitrary function of v. Now, taking into account that $\bar{u}$ does not depend on v, the derivative of (8) with respect to v gives the following system of equations for A and $\bar{v}$:

$$\begin{cases} A\bar{v}_{vv}-A_v\bar{v}_v+A^2\bar{v}_v=0 \\ 2A\bar{v}_v+\bar{v}_{vv}=0. \end{cases} \tag{9}$$

If $A=0$ the solution of Eqs. (8) and (9) is linear, $\bar{u}(u)=pu+q$ and $\bar{v}(v)=pv+q$, where $p\neq 0$ and q are arbitrary constants. In the generic case, $A\neq 0$, the solution has the form

$$\bar{u}(u)=\frac{p}{u+q}+m,\quad \bar{v}(v)=\frac{p}{v+q}+m, \tag{10}$$

where $p\neq 0$, q and m are arbitrary constants; the internal conformal factor F results from Eq. (7),

$$F(\bar{u},\bar{v})=\frac{p^2}{(\bar{u}-m)^2(\bar{v}-m)^2}.$$

Therefore, taking $\bar{u}=\bar{t}+\bar{r}$ and $\bar{v}=\bar{t}-\bar{r}$, we obtain the following proposition:

Proposition 3: The nonlinear coordinate transformations $\bar{t}=\bar{t}(t,r)$, $\bar{r}=\bar{r}(t,r)$, that maintain invariant the diagonal form of the Minkowski metric, except for an internal conformal factor F are given by

$$\bar{t}=\frac{-p(t+q)}{r^2-(t+q)^2}+m,\quad \bar{r}=\frac{pr}{r^2-(t+q)^2}$$

with $p\neq 0$, q and m as arbitrary constants. Then, this factor is

$$F(\bar{t},\bar{r})=\frac{p^2}{(\bar{r}^2-(\bar{t}-m)^2)^2}.$$

Note that, these *internal conformal transformations* in the Minkowski space–time, given by (10), also maintain invariant the form (4) of the RCKF, that is

$$\xi=(\bar{a}\bar{u}^2+\bar{b}\bar{u}+\bar{c})\frac{\partial}{\partial\bar{u}}+(\bar{a}\bar{v}^2+\bar{b}\bar{v}+\bar{c})\frac{\partial}{\partial\bar{v}},$$

where the new constants $\bar{a}$, $\bar{b}$, and $\bar{c}$ are obtained from the following matricial relation:

$$\begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} = \mathcal{M} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{with } \mathcal{M} \equiv \frac{1}{p} \begin{pmatrix} -q^2 & q & -1 \\ 2q(p+mq) & -(p+2mq) & 2m \\ -(p+mq)^2 & m(p+mq) & -m^2 \end{pmatrix}. \tag{11}$$

Moreover $b^2-4ac=\bar{b}^2-4\bar{a}\bar{c}$ and we have the following proposition:

Proposition 4: The form of a RCKF is invariant by the internal conformal transformations given in Proposition 3. Moreover, the quantity $\Delta=b^2-4ac$ is also invariant by these transformations.

Since $\det(\mathcal{M})=1$, the internal conformal transformations of the Minkowski metric are represented as orthogonal transformations on the algebra of the RCKF, considering its Killing form $\mathcal{K}$ as a metric. Hence, Δ is invariantly defined from the scalar product associated with $\mathcal{K}$, that is $\Delta=\mathcal{K}(\xi,\xi)$. The invariance of this quantity suggests us the possibility of a classification of the RCKF depending on the sign of Δ, which will be used to denote these classes. The classes $\Delta=0$, $\Delta>0$ and $\Delta<0$ can be represented by the fields ξ_1, ξ_2, and $\xi_3+\xi_1$, respectively.

Note that the fields ξ_1 and ξ_3 belong to the class $\Delta=0$ because the internal conformal transformation from Proposition 3 with $p=-1$, $q=0$ and m arbitrary lets us write the field ξ_3 as a timelike translation field in the new coordinates

$$\xi_3=(t^2+r^2)\frac{\partial}{\partial t}+2tr\frac{\partial}{\partial r}=\frac{\partial}{\partial \bar{t}}\equiv\bar{\xi}_1$$

as it follows from expression (11). The metric η in these coordinates has the form

$$\eta_{(\bar{t},\bar{r},\theta,\varphi)}=\frac{1}{[\bar{r}^2-(\bar{t}-m)^2]^2}\operatorname{diag}(-1,1,\bar{r}^2,\bar{r}^2\sin^2\theta). \tag{12}$$

Another interesting example is the equivalence between the fields ξ_2 and $\xi_3-\xi_1$ in the class $\Delta>0$. From Proposition 3 and Eq. (11), an internal conformal transformation with $p=-2m\neq 0$ and $q=1$ lets us write $\xi_3-\xi_1$ as the dilation field in the new coordinates $(\bar{t},\bar{r})$,

$$\xi_3-\xi_1=(t^2+r^2-1)\frac{\partial}{\partial t}+2tr\frac{\partial}{\partial r}=\bar{t}\frac{\partial}{\partial \bar{t}}+\bar{r}\frac{\partial}{\partial \bar{r}}\equiv\bar{\xi}_2$$

and now the metric η is written in the form,

$$\eta_{(\bar{t},\bar{r},\theta,\varphi)}=\frac{4m^2}{[\bar{r}^2-(\bar{t}-m)^2]^2}\operatorname{diag}(-1,1,\bar{r}^2,\bar{r}^2\sin^2\theta). \tag{13}$$

Then, as it is shown in last examples, we have the following result:

Proposition 5: The fields $\xi_3-k\xi_1$, with $k=0,+1,-1$, may be taken as representatives of the equivalence classes $\Delta=0$, $\Delta>0$, and $\Delta<0$, respectively.

Note that the representatives taken in the above proposition are obtained only from the fields ξ_1 and ξ_3, and then, they are not a basis of the radial conformal Killing algebra. But they generate by commutation the complete algebra, because ξ_2 is, up to a constant factor, the Lie bracket of any pair of these representatives.

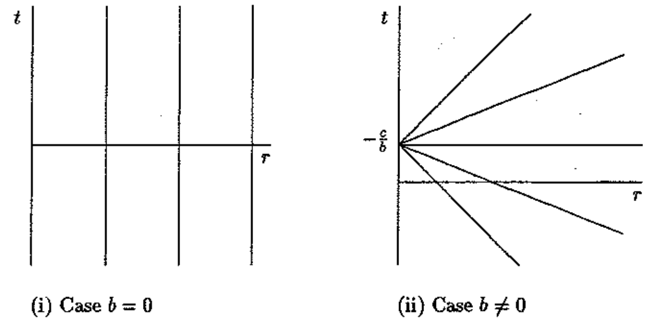

FIG. 2. Integral curves associated with a RCKF ξ given by expression (3) with $a=0$.

IV. INTEGRAL CURVES ASSOCIATED WITH A RADIAL CONFORMAL MOTION

The integral curves of a RCKF ξ given by (3) are the solution of the differential equation

$$\frac{dt}{a(t^2+r^2)+bt+c}=\frac{dr}{r(2at+b)}, \tag{14}$$

which has the following implicit form:

$$a(t^2-r^2)+bt-\omega r+c=0. \tag{15}$$

So, the integral curves are a one-parameter family of straight or hyperbolic lines, depending on the constants a, b, and c of the field and on the parameter ω. When $a=0$, Eq. (15) represents straight lines in the $\{t,r\}$-plane (Fig. 2). When $a\neq 0$, Eq. (15) can be written in the form

$$\left(t+\frac{b}{2a}\right)^2-\left(r+\frac{\omega}{2a}\right)^2=\frac{\Delta-\omega^2}{4a^2},$$

which represents a hyperbolic line for each value of the parameter ω except when $\omega^2=\Delta$ that corresponds to the light cone at the point $(t=-b/2a,\ r=-\omega/2a)$. The vertexes of each hyperbola are the points $[t=-(b/2a),\ r_{\pm}=(-\omega\pm\sqrt{\omega^2-\Delta})/2a]$ if $\Delta\leq 0$ or $\omega^2>\Delta>0$ and the points $[t_{\pm}=(-b\pm\sqrt{\Delta-\omega^2})/2a,\ r=-\omega/2a]$ if $\Delta>0$ and $\omega^2<\Delta$. We must consider only the part of the hyperbolic branches with $r>0$. Some of these integral curves are plotted in Fig. 3 for the different values of Δ. Note that in the case $\Delta>0$ there exist a double family of hyperbolic lines, Fig. 3(iii).

The vector field $\xi_2=t(\partial/\partial t)+r(\partial/\partial r)$ is called (the generator of) a dilation transformation since its integral curves are a radial congruence of straight lines [Figs. 1(i) and 2(ii)]. And the vector field $\xi_3=(t^2+r^2)(\partial/\partial t)+2tr(\partial/\partial r)$ is identified as (the generator of) an acceleration transformation along the t-axis, because each integral curve for $\omega\neq 0$ can be seen as a hyperbolic relativistic motion whose acceleration $\mathbf{a}$ has constant length, $|\mathbf{a}|=2|a/\omega|$ [Figs. 1(ii) and 3(ii)].

V. ORTHOGONAL SURFACES TO A RADIAL CONFORMAL KILLING FIELD

Let us consider the covector ξ_* associated by the metric to a RCKF ξ given by (3), that is

$$\xi_*=-(a(t^2+r^2)+bt+c)dt+r(2at+b)dr.$$

This 1-form is integrable, $\xi_*\wedge d\xi_*=0$, and admits as a potential the following function:

$$s(t,r)=\begin{cases} b(t^2-r^2)+2ct & \text{if } a=0\\ \dfrac{a(t^2-r^2)-c}{2at+b} & \text{if } a\neq 0\end{cases} \tag{16}$$

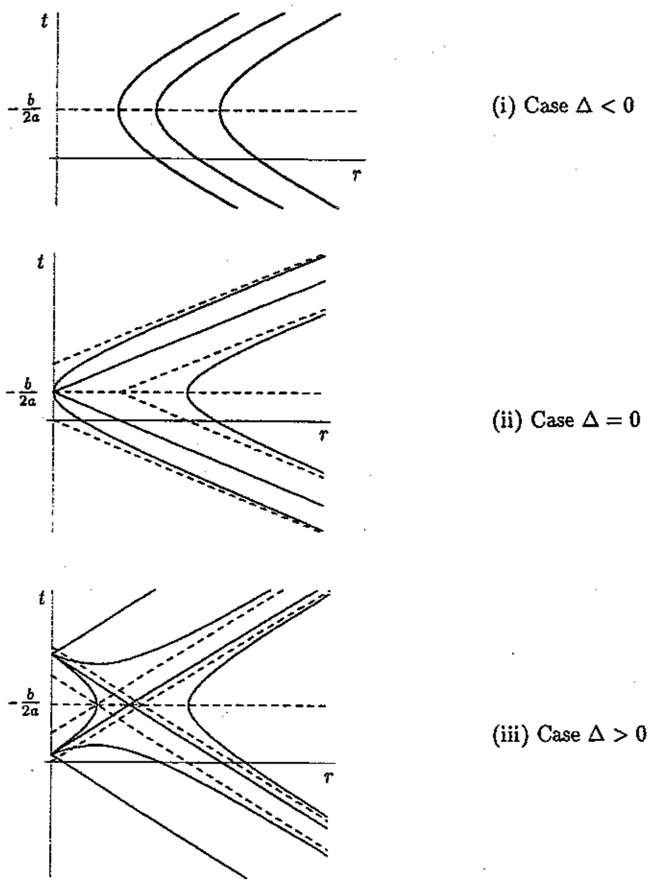

FIG. 3. Integral curves associated with a RCKF ξ given by expression (3) with $a\neq 0$, according to the sign of $\Delta = b^2 - 4ac$.

that is, $\xi_* \propto ds$ and the surfaces $\Sigma_s = \{(t,r,\theta,\varphi)/s = \text{constant}\}$ are orthogonal to the field ξ. Let us consider a domain where the field is not null on any point. The metric may be written using the coordinate s and another coordinate ω orthogonal to ξ, $\eta(d\omega,\xi)=0$. Such a coordinate has the expression

$$\omega(t,r) = \frac{a(t^2-r^2)+bt+c}{r} \tag{17}$$

for any value of a, according to Eq. (15). In order to express the flat metric in these coordinates, we need the inverse transformation of Eqs. (16) and (17), which is
(i) For $a=0$,

$$t(s,\omega) = \begin{cases} \dfrac{s}{2c} & \text{if } b=0 \\ -\dfrac{c}{b} + \dfrac{\omega}{b}\sqrt{\dfrac{bs+c^2}{\omega^2-b^2}} & \text{if } b\neq 0, \end{cases}$$

$$r(s,\omega) = \sqrt{\frac{bs+c^2}{\omega^2-b^2}}. \tag{18}$$

(ii) For $a\neq 0$,

$$t(s,\omega)=\frac{1}{\sigma^2-\omega^2}\left[(\Delta-\omega^2)s-\frac{b}{2a}(\sigma^2-\Delta)\pm\frac{\omega}{2a}\sqrt{(\sigma^2-\Delta)(\omega^2-\Delta)}\right],$$

$$r(s,\omega)=\frac{\sqrt{\sigma^2-\Delta}}{2a(\sigma^2-\omega^2)}[-\sigma\sqrt{\omega^2-\Delta}\pm\omega\sqrt{\sigma^2-\Delta}], \tag{19}$$

where $\sigma\equiv 2as+b$ and the $+$ $(-)$ sign in the t coordinate corresponds to the $+$ $(-)$ sign in the r coordinate.

Then η, written in the coordinates $(s,\omega,\theta,\varphi)$, has the following diagonal form:

$$\eta_{(s,\omega,\theta,\varphi)}=\begin{cases} r^2(s,\omega)\mathrm{diag}\left(\dfrac{b^2-\omega^2}{4(bs+c^2)^2},\dfrac{1}{\omega^2-b^2},1,\sin^2\theta\right) & \text{if } a=0\\ r^2(s,\omega)\mathrm{diag}\left(\dfrac{\Delta-\omega^2}{4(as^2+bs+c)^2},\dfrac{1}{\omega^2-\Delta},1,\sin^2\theta\right) & \text{if } a\neq 0.\end{cases} \tag{20}$$

Note that these coordinates $(s,\omega,\theta,\varphi)$ are not, in general, conformally flat coordinates. If we consider the coordinate transformations (18), (19) and Proposition 3, the resulting relation between (s,ω) and $(\bar{t},\bar{r})$ allows us to recover, from expression (20), the metric forms (12) and (13) presented in Sec. III.

From (20), the induced metric on the surfaces Σ_s by the Minkowski metric has the form

$$\gamma_{(\omega,\theta,\varphi)}=r^2(s,\omega)\mathrm{diag}\left(\frac{1}{\omega^2-\Delta},1,\sin^2\theta\right).$$

The Riemann double 2-form of curvature $\mathcal{R}$ of this induced metric can be expressed as $\mathcal{R}=[K(s)/2]\gamma\wedge\gamma$, where $\wedge$ denotes the exterior product of double 1-forms and $K(s)$ is given by

$$K(s)=\begin{cases}\dfrac{-b^2}{bs+c^2} & \text{for } a=0\\ \dfrac{-a}{as^2+bs+c} & \text{for } a\neq 0.\end{cases} \tag{21}$$

Therefore each surface Σ_s ($s=$constant) has constant sectional curvature, $K(s)$. If we take into account expressions (5) and (16), we obtain

$$P=\begin{cases} bs+c^2 & \text{if } a=0\\ \dfrac{as^2+bs+c}{a}(2at+b)^2 & \text{if } a\neq 0\end{cases} \tag{22}$$

and the sectional curvature of Σ_s can be written in the form

$$K(s)=-\frac{1}{P}(2at+b)^2$$

whose sign depends on the causal character of the field ξ, and the following result follows.

Proposition 6: In the Minkowski space–time, the surfaces orthogonal to a RCKF are three-dimensional spaces with constant curvature, which will be negative or positive if the field is timelike or spacelike, respectively; except for the field $\xi=(\partial/\partial t)$ (timelike everywhere), whose orthogonal 3-spaces are flat.

VI. TIMELIKE RADIAL CONFORMAL MOTIONS

The case of timelike RCKF is specially interesting because they are associated with particular, but in general noninertial, observers in Minkowski space–time. We are going to study their kinematical properties. The shear and the vorticity of (the unit vector **u** associated with) a timelike RCKF are zero. The expansion is

$$\theta=\frac{3(2at+b)}{\sqrt{P}}$$

with P given by (5), and the acceleration has the form

$$\mathbf{a}=\frac{2ar}{P}(-r(2at+b)dt+(a(t^2+r^2)+bt+c)dr).$$

Note that $\eta(\mathbf{a},\mathbf{a})=4a^2/(\omega^2-\Delta)$ is constant along each integral curve, as we can see from (15). This agrees with the fact that the integral curves associated with a timelike RCKF describe hyperbolic or inertial motions. Then we have the following proposition in the Minkowski space–time:

Proposition 7: In the Minkowski space–time, the acceleration of a timelike RCKF has constant length on each integral curve, that is

$$|\mathbf{a}|=\frac{2|a|}{\sqrt{\omega^2-\Delta}},$$

where ω is given by (15).

Note that, in inertial coordinates, a timelike RCKF (3) is geodesic iff $a=0$. And it will have null expansion (that is, it will be a Killing vector field) iff the constants a and b are equal to zero.

Let us consider the 4-velocity of a timelike RCKF,

$$\mathbf{u}=\frac{\xi}{\sqrt{P}}=\frac{1}{\sqrt{1-\mathrm{v}^2}}\left(\frac{\partial}{\partial t}+\mathrm{v}\frac{\partial}{\partial r}\right),$$

the velocity v relative to the inertial observer $\partial/\partial t$ is given by the quotient between the components ξ^1 and ξ^0 of the field,

$$\mathrm{v}=\frac{r(2at+b)}{a(t^2+r^2)+bt+c}. \tag{23}$$

From Fig. 1, we can clearly see that when we are approaching (inside the timelike regions) to the light cones where the field ξ is null, the relative velocity $\mathrm{v}\to 1$. In particular, when $a=c=0$ we have $\mathrm{v}=r/t$. This corresponds to the geodesic field ξ_2 and adapting coordinates to it, the Minkowski metric is written

$$\eta=-d\tau^2+\frac{\tau^2}{\left(1-\frac{\rho^2}{4}\right)^2}[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{24}$$

which has the form of a Robertson–Walker metric with expansion factor $R(\tau)=\tau$. This is Milne's expression of the flat metric used to give a kinematic interpretation of the Hubble law.[2] The timelike coordinate τ represents the proper time of the geodesic radial congruence associated with the field $\xi_2=\partial/\partial\tau$; and, according to Proposition 6, the surfaces $\tau=$constant are spaces of negative constant curvature.

$\xi_2 = \tau \frac{\partial}{\partial \tau}$

Expression (24) can be obtained from (20) redefining the (s,ω) coordinates in the following way:

$$s=\tau^2, \quad \omega=\frac{\rho}{4}+\frac{1}{\rho}.$$

In this sense, Milne's interpretation of the recession velocity of galaxies can be understood adapting coordinates to a RCKF in Minkowski space–time.

VII. DISCUSSION AND COMMENTS

We have analyzed the main properties of the RCKF in Minkowski space–time. In a conformally flat space–time, whose metric can be locally written as $g=e^{2\lambda}\eta$, the form of the RCKF will be given by the same expression (3) or (4) as for the Minkowski space–time. But the acceleration and expansion of these fields will depend on the function λ and its first derivatives. So, additional conditions imposed on these kinematic properties lead to a differential equation for this function λ that can be used to determine it. For instance, imposing that a conformally flat space–time admits a geodesic RCKF, the corresponding differential equation allows one to obtain the conformal factor of the Robertson–Walker metric found by Infeld and Schild[1] in their work on kinematic cosmology.

Therefore, it is natural to wonder whether the existence of a RCKF with certain kinematic properties can characterize the Robertson–Walker metrics and other generalized nonhomogeneous conformally flat cosmological models. In fact, Robertson–Walker universes are those conformally flat space–times which admit a timelike geodesic RCKF.[3] Other kinematic properties over these RCKF (nongeodesic with homogeneous expansion or admitting homogeneous orthogonal 3-spaces) could be used to characterize generalized conformally flat cosmologies. We shall soon develop this idea further.[4]

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