

Radial entire solutions of the linear equation

$$\Delta u + \lambda p(|x|)u = 0$$

Dedicated to Professor Tosihusa Kimura on his 60th birthday

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In this paper we are concerned with radial entire solutions of the linear elliptic differential equation

$$(1_\lambda) \quad \Delta u + \lambda p(|x|)u = 0, \quad x \in \mathbf{R}^N,$$

where Δ is the N -dimensional Laplacian, $|x|$ denotes the Euclidean length of $x \in \mathbf{R}^N$, and λ is a positive parameter. We always assume that $N \geq 3$ and p satisfies

$$(2) \quad p \in C[0, \infty), p(t) \geq 0 \text{ on } [0, \infty), \text{ and } p(t) \neq 0 \text{ on } [T, \infty) \text{ for every } T \geq 0.$$

The theorem below requires the further conditions

$$(3) \quad \int_0^\infty tp(t) dt < \infty$$

and

$$(4) \quad \int_0^\infty t^{N-1}p(t) dt < \infty.$$

The primary motivation for this paper comes from the observation that very little is known about the asymptotic property of radial entire solutions even for simple linear equations of the form (1_λ) , whereas there are many results concerning the existence and asymptotic property of positive entire solutions of the *nonlinear* equation

$$(5) \quad \Delta u + K(x)|u|^{\gamma-1}u = 0, \quad x \in \mathbf{R}^N, \quad \gamma \neq 1.$$

For some recent literature on equation (5) the reader is referred to the papers [1-3, 5-8] and the references cited therein.

Now let us consider the linear equation (1_λ) . Assume that (3) is satisfied, and put

$$(6) \quad P = \int_0^\infty tp(t) dt.$$

In [5, Remark 2.2], Kawano has shown that if the parameter $\lambda > 0$ is small enough so that $\lambda P < N - 2$, then equation (1_λ) has a radial entire solution $u(t)$, $t = |x|$, such that $u(t) > 0$ for $t \geq 0$ and $\lim u(t)$ as $t \rightarrow \infty$ exists and is positive. It is natural to consider the case of $\lambda P \geq N - 2$. However, as far as the author is aware, there is no result with respect to this case. The objective of this paper is to determine what happens to the numbers of zeros and the asymptotic behavior as $|x| \rightarrow +\infty$ of radial entire solutions of (1_λ) as the parameter λ grows to $+\infty$.

The next theorem can be shown.

THEOREM (I) *Let (3) be satisfied. There exist λ_0 and λ_1 with $0 < \lambda_0 \leq \lambda_1 < +\infty$ such that*

(i) *if $\lambda \in (0, \lambda_0)$, then every nontrivial radial entire solution $u(t)$ of (1_λ) has no zero in $[0, \infty)$ and has the asymptotic behavior that*

$$(7) \quad \lim_{t \rightarrow \infty} u(t) \text{ exists and is a non-zero finite value ;}$$

(ii) *if $\lambda \in [\lambda_0, \lambda_1]$, then every nontrivial radial entire solution $u(t)$ of (1_λ) has no zero in $[0, \infty)$ and has the asymptotic behavior that*

$$(8) \quad \lim_{t \rightarrow \infty} t^{N-2}u(t) \text{ exists and is a non-zero finite value ;}$$

(iii) *if $\lambda \in (\lambda_1, +\infty)$, then every nontrivial radial entire solution of (1_λ) has at least one zero in $[0, \infty)$.*

(II) *Let (4) be satisfied. Then, in addition to λ_1 in (I), there exist λ_k ($k = 2, 3, \dots$) with $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \lambda_{k+1} < \dots$ and $\lim_{k \rightarrow \infty} \lambda_k = +\infty$ such that if $\lambda \in (\lambda_k, \lambda_{k+1}]$ ($k = 1, 2, \dots$), then every nontrivial radial entire solution $u(t)$ of (1_λ) has exactly k zeros in $[0, \infty)$.*

A nontrivial radial entire function $u(t)$, where $t = |x|$, is a solution of (1_λ) if and only if $u(t)$ is a solution of the equation

$$(9_\lambda) \quad u'' + \frac{N-1}{t}u' + \lambda p(t)u = 0, \quad t > 0,$$

satisfying $u(0) = c$ and $u'(0) = 0$ for some real number $c \neq 0$. Then, since $\tilde{u}(t) = u(t)/u(0)$ is also a solution of (9_λ) satisfying $\tilde{u}(0) = 1$, $\tilde{u}'(0) = 0$, there is no loss of generality in assuming that u satisfies

$$(10) \quad u(0) = 1, \quad u'(0) = 0.$$

We denote by $u_\lambda(t)$ the solution of the initial value problem (9_λ) –(10). It can be easily verified that, for every $\lambda > 0$, $u_\lambda(t)$ is uniquely defined on $[0, \infty)$ and satisfies

$$(11) \quad u_\lambda(t) = 1 - \frac{\lambda}{N-2} \int_0^t \left[1 - \left(\frac{s}{t} \right)^{N-2} \right] sp(s) u_\lambda(s) ds, \quad t \geq 0.$$

In discussing the properties of $u_\lambda(t)$, the results for solutions of equations of the type

$$(12) \quad v'' + \frac{N-1}{t} v' + q(t)v = 0, \quad t \geq t_0,$$

will be effectively used.

LEMMA 1. Let $t_0 > 0$. Suppose that $q \in C[t_0, \infty)$ and

$$(13) \quad \int_{t_0}^\infty t |q(t)| dt < \infty.$$

Then equation (12) has a fundamental system of solutions $\{v_1(t), v_2(t)\}$ such that

$$\lim_{t \rightarrow \infty} v_1(t) = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} t^{N-2} v_2(t) = 1.$$

PROOF. Set $w(t) = tv([t/(N-2)]^{1/(N-2)})$. We find that $w(t)$ is a solution of the equation

$$(14) \quad w'' + Q(t)w = 0, \quad t \geq t_1,$$

where $Q(t) = t^{-4}[t/(N-2)]^{2(N-1)/(N-2)}q([t/(N-2)]^{1/(N-2)})$, $t_1 = (N-2)t_0^{N-2}$ and that condition (13) is rewritten as

$$(15) \quad \int_{t_1}^\infty t |Q(t)| dt < \infty.$$

It is well known that if $Q \in C[t_1, \infty)$ satisfies (15), then (14) has a fundamental system of solutions $\{w_1(t), w_2(t)\}$ such that

$$\lim_{t \rightarrow \infty} \frac{w_1(t)}{t} = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} w_2(t) = 1$$

(see, for example, Hartman [4, Corollary 9.1, p. 380]). Then $v_1(t) = (N-2)^{-1}t^{-N+2}w_1((N-2)t^{N-2})$ and $v_2(t) = t^{-N+2}w_2((N-2)t^{N-2})$ give the desired linearly independent solutions of (12). The proof is complete.

Lemma 1 implies in particular that, under condition (13), each nontrivial solution $v(t)$ of (12) has a finite number of zeros in $[t_0, \infty)$ and satisfies one of the next two asymptotic conditions:

$$\lim_{t \rightarrow \infty} v(t) \text{ exists and is a non-zero finite value,}$$

or

$\lim_{t \rightarrow \infty} t^{N-2}v(t)$ exists and is a non-zero finite value .

LEMMA 2. Let $t_0 > 0$. Suppose that $q \in C[t_0, \infty)$ and $q(t) \geq 0$ for $t \geq t_0$. If there is a solution $v(t)$ of (12) having no zero in $[t_0, \infty)$, then

$$(16) \quad (t^{N-2} - t_0^{N-2}) \int_t^\infty s^{-N+3} q(s) ds \leq N - 2, \quad t \geq t_0 .$$

PROOF. We may suppose that $v(t) > 0$ for $t \geq t_0$. An easy calculation shows that v satisfies

$$(17) \quad (t^{-N+3}(t^{N-2}v(t))')' + tq(t)v(t) = 0, \quad t \geq t_0 ,$$

and that

$$(18) \quad (t^{N-2}v(t))' \geq 0, \quad t \geq t_0 .$$

Then, integrating (17) once, we find that

$$(19) \quad t^{-N+3}(t^{N-2}v(t))' \geq \int_t^\infty sq(s)v(s) ds, \quad t \geq t_0 .$$

From (19) together with (18) it follows that, for $t \geq t_0$,

$$\begin{aligned} t^{N-2}v(t) &\geq t_0^{N-2}v(t_0) + \int_{t_0}^t s^{N-3} \left(\int_s^\infty \sigma q(\sigma)v(\sigma) d\sigma \right) ds \\ &\geq \frac{1}{N-2} (t^{N-2} - t_0^{N-2}) \int_t^\infty \sigma q(\sigma)v(\sigma) d\sigma \\ &\geq \frac{1}{N-2} (t^{N-2} - t_0^{N-2}) t^{N-2}v(t) \int_t^\infty \sigma^{-N+3} q(\sigma) d\sigma , \end{aligned}$$

yielding (16). The proof of Lemma 2 is complete.

We now return to studying the properties of $u_\lambda(t)$, the solution of the problem (9 _{λ})–(10). By Lemma 1 we can conclude that, if (3) is satisfied, then, for each $\lambda > 0$, $u_\lambda(t)$ has a finite number of zeros in $[0, \infty)$ and satisfies either

$$(20) \quad \lim_{t \rightarrow \infty} u_\lambda(t) \text{ exists and is a non-zero finite value ,}$$

or

$$(21) \quad \lim_{t \rightarrow \infty} t^{N-2}u_\lambda(t) \text{ exists and is a non-zero finite value .}$$

LEMMA 3. Let $0 < \mu < \lambda$. If $u_\lambda(t) > 0$ on $[0, t_\lambda)$, where $0 < t_\lambda \leq \infty$, then $u_\mu(t) \geq u_\lambda(t) (>0)$ on $[0, t_\lambda)$.

PROOF. First we claim that $u_\mu(t) > 0$ on $[0, t_\lambda)$. Assume to the contrary that $u_\mu(t)$ has a zero in $[0, t_\lambda)$. Then there is a $t_\mu \in (0, t_\lambda)$ such that $u_\mu(t_\mu) = 0$ and $u_\mu(t) > 0$ on $[0, t_\mu)$. Clearly, $u'_\mu(t_\mu) < 0$. Define the function $W(t)$ by

$$(22) \quad W(t) = t^{N-1}[u'_\mu(t)u_\lambda(t) - u_\mu(t)u'_\lambda(t)], \quad t \geq 0.$$

The derivative of W is given by

$$W'(t) = (\lambda - \mu)t^{N-1}p(t)u_\mu(t)u_\lambda(t), \quad t \geq 0.$$

By the assumptions, $W'(t) \geq 0$ on $[0, t_\mu]$; that is, $W(t)$ is nondecreasing on $[0, t_\mu]$. However, $W(0) = 0$ and $W(t_\mu) = t_\mu^{N-1}u'_\mu(t_\mu)u_\lambda(t_\mu) < 0$, contradicting the nondecreasing property of W . Thus $u_\mu(t) > 0$ on $[0, t_\lambda)$.

Reconsider the function $W(t)$ defined by (22). We have $W'(t) \geq 0$ on $[0, t_\lambda)$ and $W(0) = 0$, and so $W(t) \geq 0$ on $[0, t_\lambda)$. Then,

$$\left(\frac{u_\mu(t)}{u_\lambda(t)}\right)' = \frac{W(t)}{t^{N-1}[u_\lambda(t)]^2} \geq 0, \quad 0 \leq t < t_\lambda,$$

and consequently

$$\frac{u_\mu(t)}{u_\lambda(t)} \geq \frac{u_\mu(0)}{u_\lambda(0)} = 1, \quad 0 \leq t < t_\lambda.$$

This proves Lemma 3.

As a corollary of Lemma 3 we have the following lemma.

LEMMA 4. Let $0 < \mu < \lambda$. (I) If $u_\lambda(t) > 0$ for $t \geq 0$, then $u_\mu(t) \geq u_\lambda(t)$ for $t \geq 0$; in particular $u_\mu(t) > 0$ for $t \geq 0$.

(II) If $u_\mu(t)$ has a zero in $[0, \infty)$, then $u_\lambda(t)$ has a zero in $[0, \infty)$. Let t_λ^1 and t_μ^1 be the first zeros of $u_\lambda(t)$ and $u_\mu(t)$, respectively. Then $t_\lambda^1 \leq t_\mu^1$.

LEMMA 5. Suppose that (3) holds. Then, for each $\lambda > 0$,

$$(23) \quad |u_\lambda(t)| \leq \exp(\lambda P/(N - 2)), \quad t \geq 0,$$

where P is given by (6).

PROOF. Use of (11) gives

$$|u_\lambda(t)| \leq 1 + \frac{\lambda}{N - 2} \int_0^t sp(s)|u_\lambda(s)| ds, \quad t \geq 0,$$

and hence by Gronwall's inequality we have

$$|u_\lambda(t)| \leq \exp\left(\frac{\lambda}{N-2} \int_0^t sp(s) ds\right), \quad t \geq 0.$$

Then the assertion (23) is clear.

LEMMA 6. *Suppose that (3) holds. Then, for $\lambda > 0$ and $\mu > 0$,*

$$(24) \quad |u_\lambda(t) - u_\mu(t)| \leq |\lambda - \mu| \frac{P}{N-2} \exp\left(\frac{\lambda + \mu}{N-2} P\right), \quad t \geq 0,$$

where P is defined by (6).

PROOF. In view of (11) and (23) we can estimate as follows:

$$\begin{aligned} |u_\lambda(t) - u_\mu(t)| &\leq \frac{|\lambda - \mu|}{N-2} \int_0^t sp(s)|u_\lambda(s)| ds + \frac{\mu}{N-2} \int_0^t sp(s)|u_\lambda(s) - u_\mu(s)| ds \\ &\leq \frac{|\lambda - \mu|}{N-2} P \exp\left(\frac{\lambda P}{N-2}\right) + \frac{\mu}{N-2} \int_0^t sp(s)|u_\lambda(s) - u_\mu(s)| ds \end{aligned}$$

for $t \geq 0$. An application of Gronwall's inequality gives (24).

LEMMA 7. *Suppose that (3) holds. There is $\lambda' > 0$ such that if $\lambda \in (0, \lambda']$, then $u_\lambda(t) > 0$ for $t \geq 0$ and $u_\lambda(t)$ satisfies (20).*

This lemma is a consequence of Kawano's result [5, Remark 2.2].

PROOF OF THEOREM. (I) Define the subsets A_0 and A_1^+ of $\mathbf{R}_+ = (0, +\infty)$ by

$$\begin{aligned} A_0 &= \{\lambda \in \mathbf{R}_+ : u_\lambda(t) > 0 \text{ for } t \geq 0\}, \quad \text{and} \\ A_1^+ &= \{\lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ has at least one zero in } [0, \infty)\}. \end{aligned}$$

It is clear that $\mathbf{R}_+ = A_0 \cup A_1^+$, $A_0 \cap A_1^+ = \emptyset$. By Lemma 7, A_0 is non-empty. It can be shown that A_1^+ is also non-empty. To see this, assume the contrary. Then $\mathbf{R}_+ = A_0$, i.e., $u_\lambda(t) > 0$ on $[0, \infty)$ for every $\lambda > 0$. Applying Lemma 2 to the case of $q(t) = \lambda p(t)$, we find that

$$(25) \quad \lambda(t^{N-2} - t_0^{N-2}) \int_t^\infty s^{-N+3} p(s) ds \leq N - 2, \quad t \geq t_0,$$

for all $\lambda > 0$, where $t_0 > 0$ is an arbitrarily fixed number. In (25), fix t and let $\lambda \rightarrow +\infty$. Then we are led to a contradiction. Thus A_1^+ is non-empty. Besides, A_1^+ is an open subset of \mathbf{R}_+ because of the continuous dependence of

$u_\lambda(t)$ on λ . From (I) of Lemma 4 we see that if $0 < \mu < \lambda$ and $\lambda \in A_0$, then $\mu \in A_0$. Therefore we can conclude that A_0 and A_1^+ are of the forms

$$A_0 = (0, \lambda_1] \quad \text{and} \quad A_1^+ = (\lambda_1, +\infty)$$

for some $\lambda_1 > 0$.

Consider the subsets A_0^n and A_0^p of \mathbf{R}_+ defined by

$$A_0^n = \{ \lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ is positive on } [0, \infty) \text{ and satisfies (20)} \}, \quad \text{and}$$

$$A_0^p = \{ \lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ is positive on } [0, \infty) \text{ and satisfies (21)} \}.$$

We have $A_0 = A_0^n \cup A_0^p$ and $A_0^n \cap A_0^p = \phi$. Lemma 7 means that A_0^n is non-empty. By (I) of Lemma 4 we see that if $0 < \mu < \lambda$ and $\lambda \in A_0^n$, then $\mu \in A_0^n$. It follows from Lemma 6 that A_0^n is an open subset of \mathbf{R}_+ . Therefore A_0^n is of the form $A_0^n = (0, \lambda_0)$ for some λ_0 , $0 < \lambda_0 \leq \lambda_1$. Then $A_0^p = [\lambda_0, \lambda_1]$. The set A_0^p may consist of a single point. The proof of part (I) is complete.

(II) We have shown in (I) that, for every $\lambda > \lambda_1$, $u_\lambda(t)$ has a zero in $[0, \infty)$. Let t_λ^1 be the first zero of $u_\lambda(t)$. By (II) of Lemma 4, t_λ^1 is non-increasing for $\lambda > \lambda_1$. Since $u_{\lambda_1}(t)$ has no zero in $[0, \infty)$, the continuous dependence of u_λ on the parameter λ implies that $t_\lambda^1 \rightarrow +\infty$ as $\lambda \rightarrow \lambda_1 + 0$. Define the subsets A_1 and A_2^+ of $\mathbf{R}_+ = (0, +\infty)$ by

$$A_1 = \{ \lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ has exactly one zero in } [0, \infty) \}, \quad \text{and}$$

$$A_2^+ = \{ \lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ has at least two zeros in } [0, \infty) \}.$$

Obviously, $A_1^+ = A_1 \cup A_2^+$, $A_1 \cap A_2^+ = \phi$, where $A_1^+ = (\lambda_1, +\infty)$ consists of all $\lambda > 0$ such that $u_\lambda(t)$ has at least one zero in $[0, \infty)$. It will be shown that A_1 is non-empty. In fact, if this is not true, then, for every $\lambda > \lambda_1$, $u_\lambda(t)$ has at least two zeros in $[0, \infty)$. Let t_λ^2 be the second zero of $u_\lambda(t)$. It is clear that $0 < t_\lambda^1 < t_\lambda^2$ for $\lambda > \lambda_1$, and so $\lim t_\lambda^2 = +\infty$ as $\lambda \rightarrow \lambda_1 + 0$. Since t_λ^2 is the second zero of $u_\lambda(t)$, we have $u'_\lambda(t_\lambda^2) > 0$ ($\lambda > \lambda_1$). Noticing that $u'_\lambda(t)$ is given by

$$(26) \quad u'_\lambda(t) = -\frac{\lambda}{t^{N-1}} \int_0^t s^{N-1} p(s) u_\lambda(s) ds, \quad t \geq 0,$$

we see that

$$(27) \quad \int_0^{t_\lambda^2} s^{N-1} p(s) u_\lambda(s) ds < 0 \quad \text{for } \lambda > \lambda_1.$$

To take the limit as $\lambda \rightarrow \lambda_1 + 0$ in (27), first note that $u_\lambda(t) \rightarrow u_{\lambda_1}(t)$ as $\lambda \rightarrow \lambda_1 + 0$ at each point $t \in [0, \infty)$. Lemma 5 and condition (4) show that, for $\lambda \in (\lambda_1, \lambda_1 + 1]$, $t^{N-1} p(t) u_\lambda(t)$ is uniformly bounded on $[0, \infty)$ by an $L^1(0, \infty)$ -function. Then, by the dominated convergence theorem, it is seen that

$$\int_0^\infty s^{N-1}p(s)u_{\lambda_1}(s) ds = \lim_{\lambda \rightarrow \lambda_1+0} \int_0^{t_\lambda^2} s^{N-1}p(s)u_\lambda(s) ds \leq 0.$$

However this is a contradiction since $u_{\lambda_1}(t)$ is positive throughout $[0, \infty)$. Thus we conclude that A_1 is non-empty.

Assume next that A_2^+ is empty. Then $u_\lambda(t)$ has exactly one zero in $[0, \infty)$ for each $\lambda > \lambda_1$. (The zero of $u_\lambda(t)$ is denoted by t_λ^1 .) The nonincreasing property of t_λ^1 means that, for all $\lambda \geq \lambda_1 + 1$, $u_\lambda(t) < 0$ on $[t_{\lambda_1+1}^1 + 1, \infty)$. An application of Lemma 2 to the case of $q(t) = \lambda p(t)$ and $t_0 = t_{\lambda_1+1}^1 + 1$ shows that

$$\lambda [t^{N-2} - (t_{\lambda_1+1}^1 + 1)^{N-2}] \int_t^\infty s^{-N+3}p(s) ds \leq N - 2, \quad t \geq t_{\lambda_1+1}^1 + 1,$$

which leads to a contradiction in the limit $\lambda \rightarrow +\infty$. Thus A_2^+ is non-empty.

According to (II) of Lemma 4 and the well-known Sturm-type theorem (see, e.g., Swanson [9, Theorem 1.6]) we see that, if $\lambda_1 < \mu < \lambda$ and $\lambda \in A_1$, then $\mu \in A_1$. Furthermore it is clear that A_2^+ is an open subset of \mathbf{R}_+ . By these facts, it is seen that A_1 and A_2^+ are of the forms

$$A_1 = (\lambda_1, \lambda_2] \quad \text{and} \quad A_2^+ = (\lambda_2, +\infty)$$

for some $\lambda_2 (> \lambda_1)$.

We proceed with the same arguments. For every $\lambda > \lambda_2$, $u_\lambda(t)$ has at least two zeros in $[0, \infty)$. Denote the second zero of $u_\lambda(t)$ by t_λ^2 . It can be shown that $t_\lambda^2 \rightarrow +\infty$ as $\lambda \rightarrow \lambda_2 + 0$. Define the sets A_2 and A_3^+ by

$$A_2 = \{ \lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ has exactly two zeros in } [0, \infty) \},$$

$$A_3^+ = \{ \lambda \in \mathbf{R}_+ : u_\lambda(t) \text{ has at least three zeros in } [0, \infty) \}.$$

Clearly, $A_2^+ = A_2 \cup A_3^+$, $A_2 \cap A_3^+ = \emptyset$, and as in the above, A_3^+ is a non-empty open subset of \mathbf{R}_+ . We can show that A_2 is non-empty. To show this, assume the contrary. Then, for $\lambda > \lambda_2$, $u_\lambda(t)$ has at least three zeros in $[0, \infty)$. Arguing as in the above, we see that

$$(28) \quad \int_0^\infty s^{N-1}p(s)u_{\lambda_2}(s) ds = \lim_{\lambda \rightarrow \lambda_2+0} \int_0^{t_\lambda^3} s^{N-1}p(s)u_\lambda(s) ds \geq 0,$$

where t_λ^3 is the third zero of $u_\lambda(t)$. If $u_{\lambda_2}(t)$ has a non-zero finite limit as $t \rightarrow +\infty$, then it follows from Lemma 6 that there exists $\delta > 0$ and $T > 0$ such that $u_\lambda(t)$ has no zero in $[T, \infty)$ for $\lambda \in (\lambda_2, \lambda_2 + \delta]$. But this contradicts the fact that $u_\lambda(t) = 0$ at $t = t_\lambda^2$ and $\lim t_\lambda^2 = +\infty$ as $\lambda \rightarrow \lambda_2 + 0$. Hence the limit $\lim t^{N-2}u_{\lambda_2}(t)$ as $t \rightarrow +\infty$ exists and is negative. Since $\lim u_{\lambda_2}(t) = 0$ as $t \rightarrow +\infty$, $u_{\lambda_2}(t_{\lambda_2}^1) = 0$ and $u_{\lambda_2}(t) < 0$ on $(t_{\lambda_2}^1, \infty)$, it is possible to take a number $t^* \in (t_{\lambda_2}^1, \infty)$ such that $u'_{\lambda_2}(t^*) = 0$. Then equality (26) with $\lambda = \lambda_2$ implies

$$(29) \quad \int_0^{t^*} s^{N-1} p(s) u_{\lambda_2}(s) ds = 0.$$

From (28) and (29) it follows that

$$\int_{t^*}^{\infty} s^{N-1} p(s) u_{\lambda_2}(s) ds \geq 0,$$

which is a contradiction because $u_{\lambda_2}(t) < 0$ for $t \geq t^*$. Thus A_2 is non-empty. From (II) of Lemma 4 and the above-mentioned Sturm-type theorem we see that, if $\lambda_2 < \mu < \lambda$ and $\lambda \in A_2$, then $\mu \in A_2$. Therefore, A_2 and A_3^+ are of the forms

$$A_2 = (\lambda_2, \lambda_3] \quad \text{and} \quad A_3^+ = (\lambda_3, +\infty)$$

for some $\lambda_3 (> \lambda_2)$.

To complete the proof of part (II), continue the same arguments.

References

- [1] W.-Y. Ding and W.-M. Ni, On the elliptic equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ and related topics, *Duke Math. J.*, **52** (1985), 485–506.
- [2] A. L. Edelson, Entire solutions of singular elliptic equations, *J. Math. Anal. Appl.*, **139** (1989), 523–532.
- [3] N. Fukagai, Existence and uniqueness of entire solutions of second order sublinear elliptic equations, *Funkcial. Ekvac.*, **29** (1986), 151–165.
- [4] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.
- [5] N. Kawano, On bounded entire solutions of semilinear elliptic equations, *Hiroshima Math. J.*, **14** (1984), 125–158.
- [6] N. Kawano, J. Satsuma and S. Yotsutani, Existence of positive entire solutions of an Emden-type elliptic equation, *Funkcial. Ekvac.*, **31** (1988), 121–145.
- [7] T. Kusano and M. Naito, Oscillation theory of entire solutions of second order superlinear elliptic equations, *Funkcial. Ekvac.*, **30** (1987), 269–282.
- [8] E. S. Noussair and C. A. Swanson, Positive $L^q(\mathbb{R}^N)$ -solutions of subcritical Emden-Fowler problems, *Arch. Rat. Mech. Anal.*, **101** (1988), 85–93.
- [9] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

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