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Radial Growth and Hardy-Littlewood-Type Theorems on Hyperbolic Harmonic Functions

Shaolin Chen[a], **Zhenhua Su**[b]

In celebration of Matti Vuorinen's 65-th birthday

[a]*Department of Mathematics and Computational Science, Hengyang Normal University, Hengyang, Hunan 421008, People's Republic of China.*
[b]*Department of Mathematics, Huaihua University, Huaihua, Hunan 418008, People's Republic of China.*

Abstract. In this paper, we first show that a result of Girela et al. on analytic functions can be extended to hyperbolic-harmonic functions, and then we establish Hardy-Littlewood-type theorems on hyperbolic harmonic functions.

1. Introduction and main results

For $n \geq 2$, let $\mathbb{R}^n$ denote the usual real vector space of dimension n. Sometimes it is convenient to identify each point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with an $n \times 1$ column matrix so that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For $a = (a_1, \ldots, a_n)$ and $x \in \mathbb{R}^n$, we define the Euclidean inner product $\langle \cdot, \cdot \rangle$ by

$$\langle x, a \rangle = x_1 a_1 + \cdots + x_n a_n$$

so that the Euclidean length of x is defined by

$$|x| = \langle x, x \rangle^{1/2} = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}.$$

Denote a ball in $\mathbb{R}^n$ with center x_0 and radius r by

$$\mathbb{B}^n(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}.$$

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Email addresses: mathechen@126.com. (Shaolin Chen), szh820@163.com (Zhenhua Su)

In particular, $\mathbb{B}^n$ denotes the unit ball $\mathbb{B}^n(0, 1)$. Set $\mathbb{D} = \mathbb{B}^2$, the open unit disk in the complex plane $\mathbb{C}$.

Let Ω be a proper subdomain of $\mathbb{R}^n$. A function $f \in C^2(\Omega)$ is called *hyperbolic harmonic* (briefly, h-harmonic, in the following) function in Ω if it satisfies the *hyperbolic Laplace's equation*

$$\Delta_h u = (1 - |x|^2)^2 \Delta u + 2(n-2)(1-|x|^2)\langle \nabla u, x\rangle = 0,$$

where Δ denotes the ordinary *Laplacian operator* and ∇ denotes the gradient. Recall that hyperbolic harmonic functions are solutions of the *Laplace-Beltrami equation* with respect to the *Poincaré metric*

$$ds^2 = (1 - |x|^2)^2 \sum_{k=1}^{n} dx_k^2$$

in the unit ball $\mathbb{B}^n$.

Obviously, when $n = 2$, all h-harmonic functions are harmonic functions. We refer to [2, 3, 6, 14, 18, 28, 29] for more details of h-harmonic functions.

It turns out that if $\psi \in C(\partial\mathbb{B}^n)$, then the Dirichlet problem

$$\begin{cases} \Delta_h f = 0 & \text{in } \mathbb{B}^n \\ f = \psi & \text{on } \partial\mathbb{B}^n \end{cases}$$

has an unique solution in $C(\overline{\mathbb{B}}^n)$ and can be represented by

$$f(x) = \mathrm{P}_h[\psi](x) = \int_{\partial\mathbb{B}^n} \mathrm{P}_h(x, \zeta)\psi(\zeta) d\sigma(\zeta), \tag{1}$$

where $d\sigma$ is the unique normalized surface measure on $\partial\mathbb{B}^n$ and $\mathrm{P}_h(x, \zeta)$ is the *hyperbolic Poisson kernel* defined by

$$\mathrm{P}_h(x, \zeta) = \left(\frac{1 - |x|^2}{|x - \zeta|^2}\right)^{n-1} \quad (x \in \mathbb{B}^n,\ \zeta \in \partial\mathbb{B}^n).$$

Throughout this paper, we use C to denote the various positive constants, whose value may change from one occurrence to the next.

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset Ω of $\mathbb{R}^n$, a function $f : \Omega \to \mathbb{R}^m$ ($m \geq 1$) is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that

$$|f(x) - f(y)| \leq C\omega(|x - y|) \ \text{ for all } x,\ y \in \Omega. \tag{2}$$

For $\delta_0 > 0$, let

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq C \cdot \omega(\delta),\ 0 < \delta < \delta_0 \tag{3}$$

and

$$\delta \int_\delta^\infty \frac{\omega(t)}{t^2}\, dt \leq C \cdot \omega(\delta),\ 0 < \delta < \delta_0, \tag{4}$$

where ω is a majorant. A majorant ω is said to be *regular* if it satisfies the conditions (3) and (4) (see [12, 13, 26]).

Let Ω be a proper subdomain of $\mathbb{R}^n$. We use $d_\Omega(x)$ to denote the Euclidean distance from x to the boundary $\partial\Omega$ of Ω. In particular, we always use $d(x)$ to denote the Euclidean distance from x to the boundary of $\mathbb{B}^n$.

A proper subdomain G of $\mathbb{R}^n$ is said to be Λ_ω-*extension* if $\Lambda_\omega(G) = \text{loc}\Lambda_\omega(G)$, where $\text{loc}\Lambda_\omega(G)$ denotes the set of all functions $f: G \to \mathbb{R}^m$ ($m \geq 1$) satisfying (2) with a fixed positive constant C, whenever $x \in G$ and $y \in G$ such that $|x - y| < \frac{1}{2}d_G(x)$. Obviously, $\mathbb{B}^n$ is a Λ_ω-extension domain.

In [22], the author proved that G is a Λ_ω-extension domain if and only if each pair of points $x, y \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d_G(\tau))}{d_G(\tau)}\, ds(\tau) \leq C\omega(|x - y|) \tag{5}$$

with some fixed positive constant $C = C(G, \omega)$, where ds stands for the arc length measure on γ. Furthermore, the author also proved that Λ_ω-extension domains exist only for majorants ω satisfying (3). See [13, 15, 22] for more details on Λ_ω-extension domains.

For $p \in (0, \infty]$, the *Hardy class* $H^p(\mathbb{B}^n)$ consists of those functions $f: \mathbb{B}^n \to \mathbb{R}$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0<r<1} M_p(r, f), & \text{if } p \in (0, \infty), \\ \sup\limits_{z \in \mathbb{B}^n} |f(z)|, & \text{if } p = \infty \end{cases} \quad \text{and } M_p(r, f) = \left(\int_{\partial\mathbb{B}^n} |f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p}.$$

A classical result of Hardy and Littlewood asserts that if $p \in (0, \infty]$, $\alpha \in (1, \infty)$ and f is an analytic function in $\mathbb{D}$, then

$$M_p(r, f') = O\left(\left(\frac{1}{1 - r} \right)^\alpha \right) \text{ as } r \to 1$$

if and only if

$$M_p(r, f) = O\left(\left(\log \frac{1}{1 - r} \right)^{\alpha - 1} \right) \text{ as } r \to 1,$$

Indeed the above result of Hardy and Littlewood provides a close relationship between the integral means of analytic functions and those of their derivatives [11, 19, 20]). In [16, Theorem 1(a)], Girela and Peláez refined the above result for the case $\alpha = 1$ as follows.

Theorem 1.1. ([16, Theorem 1(a)]) *Let $p \in (2, \infty)$. For $r \in (0, 1)$, if f is an analytic function in $\mathbb{D}$ such that*

$$M_p(r, f') = O\left(\left(\frac{1}{1 - r} \right) \right) \text{ as } r \to 1,$$

then for all $\beta > 1/2$,

$$M_p(r, f) = O\left(\left(\log \frac{1}{1 - r} \right)^\beta \right) \text{ as } r \to 1. \tag{6}$$

In [16, P_{464}, Equation (26)], Girela and Peláez asked whether β in (6) can be substituted by 1/2. This problem was affirmatively settled by Girela, Pavlovic and Peláez in [17] (see [17, Theorem 1.1]). We show that Theorem 1.6 can be extended to h-harmonic functions in $\mathbb{B}^n$ with $\beta = 1/2$. On the related topics, see [5, 7, 8, 10, 27].

Theorem 1.2. *Let $p \in [2, \infty)$ and ω be a majorant. For $r \in (0, 1)$, if f is h-harmonic from $\mathbb{B}^n$ into $\mathbb{R}$ such that*

$$M_p(r, \nabla f) \leq C\omega\left(\frac{1}{1 - r} \right),$$

then

$$M_p(r, f) \leq \left[|f(0)|^2 + \frac{rp(p - 1)(1 + r)^{n-2}C^2\omega(1)}{n - 1} \int_0^1 \omega\left(\frac{1}{1 - rt} \right) dt \right]^{\frac{1}{2}}.$$

Especially, if $n = 2$ *and* $\omega(t) = t$*, then*

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \text{ as } r \to 1 \tag{7}$$

and the estimate of (7) *is sharp.*

Krantz [21] proved a Hardy-Littlewood-type theorem for harmonic functions in $\mathbb{B}^n$ with respect to the majorant $\omega(t) = \omega_\alpha(t) = t^\alpha$ $(0 < \alpha \leq 1)$ as follows. For the extended discussion on this topic, see [4, 9].

Theorem 1.3. ([21, Theorem 15.8]) *Let* u *be a harmonic function in* $\mathbb{B}^n$ *and* $0 < \alpha \leq 1$*. Then* u *satisfies*

$$|\nabla u(x)| \leq C\frac{\omega_\alpha\big(d(x)\big)}{d(x)} \text{ for any } x \in \mathbb{B}^n$$

if and only if

$$|u(x) - u(y)| \leq C\omega_\alpha(|x - y|) \text{ for any } x, y \in \mathbb{B}^n.$$

We generalize Theorem 1.3 to the following form.

Theorem 1.4. *Let* ω *be a majorant satisfying* (3)*,* Ω *be a* Λ_ω*-extension domain and* f *be a h-harmonic function from* Ω *into* $\mathbb{R}$*. Then* $f \in \Lambda_\omega(\Omega)$ *if and only if*

$$|\nabla f(x)| \leq C\frac{\omega\big(d_\Omega(x)\big)}{d_\Omega(x)} \text{ for any } x \in \Omega.$$

Let ω be a majorant and D be a bounded set of $\mathbb{R}^n$. We use $\Lambda^B_\omega(D)$ to denote all the bounded continuous functions f in D with the norm

$$\|f\|_{\omega,D} = \sup_{x,y\in D, x\neq y}\left\{\frac{|f(x) - f(y)|}{\omega(|x - y|)}\right\} < \infty.$$

Taking another majorant ω', we define the operator norm

$$\|\mathrm{P}_h\|_{\omega\to\omega'} = \sup_{f\in\Lambda^B_\omega(\partial\mathbb{B}^n),\ \|f\|_{\omega,\partial\mathbb{B}^n}\neq 0} \frac{\|\mathrm{P}_h[f]\|_{\omega',\mathbb{B}^n}}{\|f\|_{\omega,\partial\mathbb{B}^n}}.$$

For each $a \in \partial\mathbb{B}^n$, we define

$$\eta_{a,\omega}(\zeta) = \omega(|\zeta - a|) \text{ for } \zeta \in \partial\mathbb{B}^n.$$

We refer to [1] for the similar definitions of harmonic functions.

Proposition 1.5. *Let* ω *be a majorant. Then* $\eta_{a,\omega} \in \Lambda^B_\omega(\mathbb{B}^n)$*.*

The following result is the classical Hardy-Littlewood Theorem.

Theorem 1.6. ([11, Theorem 5.1]) *Let* f *be an analytic function in* $\mathbb{D}$ *and continuous in* $\overline{\mathbb{D}}$*. Then for some* $0 < \alpha \leq 1$*,*

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq C\omega_\alpha(|\theta_1 - \theta_2|) \text{ for any } 0 \leq \theta_1,\ \theta_2 < 2\pi$$

if and only if

$$|f'(z)| \leq C\frac{\omega_\alpha\big(d(z)\big)}{d(z)} \text{ for any } z \in \mathbb{D}.$$

The following result is a Hardy-Littlewood-type theorem for hyperbolic functions. For the extensive discussions on this topics, see [1, 13, 23–25].

Theorem 1.7. *Let* ω *be a majorant. Then for each* $a \in \partial\mathbb{B}^n$ *and any* $x \in \mathbb{B}^n$*, there is a constant* $C \geq 1$ *such that* $\mathrm{P}_h[\eta_{a,\omega}](x) \leq C\omega(|x - a|)$*, if and only if* $\|\mathrm{P}_h\|_{\omega\to\omega} < \infty$*.*

We will prove Theorems 1.2, 1.4 and 1.7 in section 2.

2. Proofs of the main results

Lemma 2.1. ([28, Lemma 3.2]) *If* $f \in C^2(\mathbb{B}^n)$, *then*
(*a*) $\frac{d}{dr}\int_{\partial\mathbb{B}^n} f(r\zeta)d\sigma(\zeta) = \frac{r^{1-n}(1-r^2)^{n-2}}{n}\int_{\mathbb{B}^n_{\mathbb{R}}(r)}\Delta_h f(x)d\tau(x)$ *and*
(*b*) $\int_{\partial\mathbb{B}^n} f(r\zeta)d\sigma(\zeta) = f(0) + \int_{\mathbb{B}^n_{\mathbb{R}}(r)} g(|x|, r)\Delta_h f(x)d\tau(x)$, *where*

$$g(|x|, r) = \frac{1}{n}\int_{|x|}^{r}\frac{(1-s^2)^{n-2}}{s^{n-1}}ds,$$

$d\tau = \frac{dV_N}{(1-|x|^2)^n}$ *and* dV_N *denotes the normalized Lebesgue volume measure on* $\mathbb{B}^n$.

Proof of Theorem 1.2

Let f be h-harmonic in $\mathbb{B}^n$. By elementary calculations, we see that

$$\Delta_h(|f|^p) = p(p-1)(1-|x|^2)^2|f|^{p-2}|\nabla f|^2. \tag{8}$$

For $r \in [0, 1)$, the Hölder's inequality yields

$$\int_{\partial\mathbb{B}^n} |f(r\zeta)|^{p-2}|\nabla f(r\zeta)|^2\, d\sigma(\zeta) \le M_p^2(r, \nabla f)\cdot M_p^{p-2}(r, f). \tag{9}$$

By (8), (9) and Lemma 2.1, we obtain

$$\begin{aligned} M_p^p(r, f) &= |f(0)|^p + \int_{\mathbb{B}^n(0,r)} g(|x|, r)\Delta_h(|f(x)|^p)d\tau(x) \\ &= |f(0)|^p + p(p-1)\int_{\mathbb{B}^n(0,r)} |f(x)|^{p-2}|\nabla f(x)|^2 g(|x|, r)(1-|x|^2)^2 d\tau(x) \\ &= |f(0)|^p + np(p-1)\int_0^r \frac{\rho^{n-1}g(\rho, r)}{(1-\rho^2)^{n-2}}\int_{\partial\mathbb{B}^n}|f(\rho\zeta)|^{p-2}|\nabla f(\rho\zeta)|^2 d\sigma(\zeta)d\rho \\ &\le |f(0)|^p + np(p-1)\int_0^r \frac{\rho^{n-1}g(\rho, r)}{(1-\rho^2)^{n-2}} M_p^2(\rho, \nabla f)M_p^{p-2}(\rho, f)d\rho. \end{aligned} \tag{10}$$

By computations, we obtain

$$\begin{aligned} g(\rho, r) &= \frac{1}{n}\int_\rho^r \frac{(1-s^2)^{n-2}}{s^{n-1}}ds \\ &\le \frac{1}{n\rho^{n-1}}\int_\rho^r (1-s^2)^{n-2}ds \\ &\le \frac{(1+r)^{n-2}}{n\rho^{n-1}}\int_\rho^r (1-s)^{n-2}ds \\ &\le \frac{(1+r)^{n-2}}{n(n-1)}\frac{(1-\rho)^{n-1}}{\rho^{n-1}}. \end{aligned} \tag{11}$$

Applying (10) and (11), we get

$$\begin{aligned} M_p^2(r,f) &\leq |f(0)|^2 + np(p-1)\int_0^r \frac{\rho^{n-1}g(\rho,r)}{(1-\rho^2)^{n-2}}M_p^2(\rho,\nabla f)d\rho \\ &\leq |f(0)|^2 + \frac{p(p-1)(1+r)^{n-2}}{n-1}\int_0^r (1-\rho)M_p^2(\rho,\nabla f)d\rho \\ &= |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}}{n-1}\int_0^1 (1-rt)M_p^2(tr,\nabla f)dt \\ &\leq |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2}{n-1}\int_0^1 (1-rt)\omega^2\left(\frac{1}{1-rt}\right)dt \\ &= |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2}{n-1}\int_0^1 \left[\omega\Big(\frac{1}{1-rt}\Big)(1-tr)\right]\omega\Big(\frac{1}{1-rt}\Big)dt \\ &\leq |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2\omega(1)}{n-1}\int_0^1 \omega\Big(\frac{1}{1-rt}\Big)dt, \end{aligned}$$

which gives

$$M_p(r,f) \leq \left[|f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2\omega(1)}{n-1}\int_0^1 \omega\Big(\frac{1}{1-rt}\Big)dt\right]^{\frac{1}{2}}.$$

In particular, if $n = 2$ and $\omega(t) = t$, then the estimate of (7) is sharp. The proof of the sharpness part follows from [16, Theorem 1(b)]. The proof of this theorem is complete. □

Lemma 2.2. *Let ω be a majorant. Then*
(1) *ω is subadditive, that is, if $t, s > 0$, then $\omega(s+t) \leq \omega(s) + \omega(t)$;*
(2) *for $t > 0$, if $\lambda \geq 1$, then $\omega(\lambda t) \leq \lambda\omega(t)$.*

Proof. We first prove (1). Since $\omega(t)/t$ is nonincreasing for $t > 0$, we see that for $s, t > 0$,

$$\begin{aligned} \omega(s) + \omega(t) - \omega(s+t) &= s\frac{\omega(s)}{s} + t\frac{\omega(t)}{t} - (s+t)\frac{\omega(t+s)}{t+s} \\ &= s\left(\frac{\omega(s)}{s} - \frac{\omega(s+t)}{s+t}\right) + t\left(\frac{\omega(t)}{t} - \frac{\omega(s+t)}{s+t}\right) \\ &\geq 0. \end{aligned}$$

(2) easily follows from the monotonicity of $\omega(t)/t$ for $t > 0$. The proof of this lemma is complete. □

Proof of Theorem 1.4

We first prove the sufficiency. Since Ω is a Λ_ω-extension domain, we see that for any $x, y \in \Omega$, by using (5), there is a rectifiable curve $\gamma \subset \Omega$ joining x to y such that

$$\begin{aligned} |f(x) - f(y)| &\leq \int_\gamma |\nabla f(\zeta)|ds(\zeta) \\ &\leq C\int_\gamma \frac{\omega\big(d_\Omega(\zeta)\big)}{d_\Omega(\zeta)}ds(\zeta) \\ &\leq C\omega(|x-y|). \end{aligned}$$

Now we come to prove the necessity. Let $x = (x_1, \cdots, x_n) \in \Omega$ and $r = d_\Omega(x)/2$. Then by Lemma 2.2, for all $y = (y_1, \cdots, y_n) \in \overline{\mathbb{B}^n(x,r)}$,

$$|f(x) - f(y)| \leq C\omega(|x-y|) \leq 2C\omega(d_\Omega(x)).$$

For all $y \in \mathbb{B}^n(x, r)$, using (1), we get

$$f(y) = \int_{\partial\mathbb{B}^n} \mathrm{P}_h(y, \zeta) f(r\zeta + x) d\sigma(\zeta),$$

where

$$\mathrm{P}_h(y, \zeta) = \left(\frac{r^2 - |y - x|^2}{|y - x - r\zeta|^2}\right)^{n-1} \text{ and } \zeta = (\zeta_1, \cdots, \zeta_n) \in \partial\mathbb{B}^n.$$

By elementary calculations, for each $k \in \{1, 2, \cdots, n\}$, we have

$$\frac{\partial \mathrm{P}_h(y, \zeta)}{\partial y_k} = -2(n-1)\Big(\mathrm{P}_h(y, \zeta)\Big)^{\frac{n-2}{n-1}} \frac{\Big[(y_k - x_k)|y - x - r\zeta|^2 + (r^2 - |y - x|^2)(y_k - r\zeta_k - x_k)\Big]}{|y - x - r\zeta|^4}.$$

Then for all $y \in \mathbb{B}^n(x, r/2)$,

$$\begin{aligned}
&\left|\frac{\partial \mathrm{P}_h(y, \zeta)}{\partial y_k}\right| \\
\leq\ & 2(n-1)\frac{\Big(r^2 - |y - x|^2\Big)^{n-2}\Big[|y_k - x_k||y - x - r\zeta|^2 + (r^2 - |y - x|^2)|y_k - r\zeta_k - x_k|\Big]}{|y - x - r\zeta|^{2n}} \\
\leq\ & 2(n-1)\frac{r^{2n-4}}{|y - x - r\zeta|^{2n}}\left(\frac{9r^3}{8} + \frac{3r^3}{2}\right) \\
\leq\ & \frac{21(n-1)}{4}\frac{r^{2n-1}}{\left(\frac{r}{2}\right)^{2n}} \\
=\ & \frac{2^{2(n-1)} \cdot 21(n-1)}{r},
\end{aligned}$$

which implies that

$$\begin{aligned}
|\nabla f(y)| &= \left[\sum_{k=1}^{n} f_{y_k}^2(y)\right]^{\frac{1}{2}} \\
&= \Big\{\sum_{k=1}^{n}\Big(\Big|\int_{\partial\mathbb{B}^n}\frac{\partial}{\partial y_k}\mathrm{P}_h(y, \zeta)(f(r\zeta + x) - f(x))d\sigma(\zeta)\Big|^2\Big\}^{\frac{1}{2}} \\
&\leq \sum_{k=1}^{n}\Big|\int_{\partial\mathbb{B}^n}\frac{\partial}{\partial y_k}\mathrm{P}_h(y, \zeta)(f(r\zeta + x) - f(x))d\sigma(\zeta)\Big| \\
&\leq \sum_{k=1}^{n}\int_{\partial\mathbb{B}^n}\left|\frac{\partial}{\partial y_k}\mathrm{P}_h(y, \zeta)\right|\Big|f(r\zeta + x) - f(x)\Big|d\sigma(\zeta) \\
&\leq \sqrt{n}\int_{\partial\mathbb{B}^n}\Big|\nabla\mathrm{P}_h(y, \zeta)\Big|\Big|f(r\zeta + x) - f(x)\Big|d\sigma(\zeta) \\
&\leq \frac{2^{2(n-1)} \cdot 21n(n-1)}{r}\int_{\partial\mathbb{B}^n}\Big|f(r\zeta + x) - f(x)\Big|d\sigma(\zeta) \\
&\leq \frac{2^{2(n-1)} \cdot 21n(n-1)C\omega(r)}{r} \\
&\leq 2^{2n-1} \cdot 21n(n-1)C\frac{\omega\Big(d_\Omega(x)\Big)}{d_\Omega(x)}.
\end{aligned}$$

The proof of this theorem is complete. □

Proof of Proposition 1.5

Let $\zeta_1, \zeta_2 \in \partial\mathbb{B}^n$. Without loss of generality, we may assume $|\zeta_1 - a| < |\zeta_2 - a|$. Then by Lemma 2.2, we have

$$\begin{aligned} |\eta_{a,\omega}(\zeta_2) - \eta_{a,\omega}(\zeta_1)| &= \omega(|\zeta_2 - a|) - \omega(|\zeta_1 - a|) \\ &= \omega(|\zeta_2 - a| - |\zeta_1 - a| + |\zeta_1 - a|) - \omega(|\zeta_1 - a|) \\ &\leq \omega(|\zeta_2 - a| - |\zeta_1 - a|) + \omega(|\zeta_1 - a|) - \omega(|\zeta_1 - a|) \\ &\leq \omega(|\zeta_2 - \zeta_1|), \end{aligned}$$

which gives

$$\frac{|\eta_{a,\omega}(\zeta_2) - \eta_{a,\omega}(\zeta_1)|}{\omega(|\zeta_2 - \zeta_1|)} \leq 1.$$

Hence $\|\eta_{a,\omega}\|_{\omega,\partial\mathbb{B}^n} \leq 1 < \infty$. On the other hand, for each $a \in \partial\mathbb{B}^n$,

$$\eta_{a,\omega}(\zeta) = \omega(|\zeta - a|) \leq \omega(2) < \infty \text{ for } \zeta \in \partial\mathbb{B}^n.$$

Then $\eta_{a,\omega}$ is bounded and $\eta_{a,\omega} \in \Lambda^B_\omega(\mathbb{B}^n)$. The proof of this Proposition is complete. □

Proof of Theorem 1.7

We first prove the necessity. Let $f \in \Lambda^B_\omega(\partial\mathbb{B}^n)$. We only need to prove that for any $x, y \in \mathbb{B}^n$, there is a positive constant C such that

$$\left|\mathrm{P}_h[f](x) - \mathrm{P}_h[f](x)\right| \leq C\|f\|_{w,\partial\mathbb{B}^n}\omega(|x - y|).$$

Without loss of generality, we assume that

$$0 < d(y) \leq d(x). \tag{12}$$

Let $x_0, y_0 \in \partial\mathbb{B}^n$ such that $d(x) = |x - x_0|$ and $d(y) = |y - y_0|$, respectively. For $\zeta \in \partial\mathbb{B}^n$, let $F(\zeta) = f(\zeta) - f(x_0)$.

We divide the proof into two cases.

Case 1. $|x - y| \leq d(x)/2$. For $z \in \mathbb{B}^n(x, d(x)/2)$, using Lemma 2.2, we see that

$$|z - x_0| \leq |z - x| + |x - x_0| \leq \frac{d(x)}{2} + d(x) = \frac{3d(x)}{2},$$

which yields that

$$\begin{aligned} |\mathrm{P}_h[F](z)| &\leq \|f\|_{w,\partial\mathbb{B}^n}|\mathrm{P}_h[\eta_{x_0,\omega}](z)| \\ &\leq C\|f\|_{w,\partial\mathbb{B}^n}\omega(|x - y|) \\ &\leq C\|f\|_{w,\partial\mathbb{B}^n}\omega\left(\frac{3d(x)}{2}\right) \\ &\leq \frac{3C}{2}\|f\|_{w,\partial\mathbb{B}^n}\omega(d(x)). \end{aligned}$$

Using arguments similar to those in the necessity's proof of Theorem 1.4, for $z \in \mathbb{B}^n(x, d(x)/2)$, there is a positive constant C such that

$$|\nabla\mathrm{P}_h[f](z)| = |\nabla\mathrm{P}_h[F](z)| \leq C\|f\|_{w,\partial\mathbb{B}^n}\frac{\omega(d(x))}{d(x)},$$

which gives that

$$\begin{aligned} \left|P_h[f](x) - P_h[f](y)\right| &\leq \int_{[x,y]} |\nabla P_h[f](z)||dz| \\ &\leq C\|f\|_{w,\partial\mathbb{B}^n} \frac{\omega(d(x))}{d(x)}|x-y| \\ &\leq C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(|x-y|), \end{aligned}$$

where $[x, y]$ denotes the segment from x to y.

Case 2. $|x - y| > d(x)/2$. We use the similar approach as in the proof of [1, Theorem 1.1] to prove this case. By (12), we know that $|x - y| > d(x)/2 \geq d(y)/2$. By elementary calculations, we see that there is positive constant C such that

$$\begin{aligned} \left|P_h[f](x) - f(x_0)\right| &= |P_h[F](x)| \leq C\|f\|_{w,\partial\mathbb{B}^n}|P_h[\eta_{x_0,\omega}](x)| \leq C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(d(x)) \\ &\leq C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(|x-y|). \end{aligned} \tag{13}$$

Similarly,

$$\left|P_h[f](y) - f(y_0)\right| \leq C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(|x-y|). \tag{14}$$

By using $|x_0 - y_0| \leq |x - y| + d(x) + d(y) < 5|x - y|$ and Lemma 2.2, we know that there is positive constant C such that

$$\left|f(x_0) - f(y_0)\right| \leq C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(|x_0-y_0|) \leq 5C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(|x-y|). \tag{15}$$

By (13), (14) and (15), we conclude that

$$\begin{aligned} \left|P_h[f](x) - P_h[f](y)\right| &\leq \left|P_h[f](x) - f(x_0) - \left(P_h[f](y) - f(y_0)\right) + f(x_0) - f(y_0)\right| \\ &\leq \left|P_h[f](x) - f(x_0)\right| + \left|P_h[f](y) - f(y_0)\right| + \left|f(x_0) - f(y_0)\right| \\ &\leq 7C\|f\|_{w,\partial\mathbb{B}^n}\,\omega(|x-y|). \end{aligned}$$

Now we come to prove the sufficiency. By Proposition 1.5, we have

$$\|P_h[\eta_{a,\omega}]\|_{\omega',\mathbb{B}^n} \leq \|P_h\|_{\omega\to\omega}\|\eta_{a,\omega}\|_{\omega,\partial\mathbb{B}^n} < \infty,$$

which implies that, for $x, y \in \mathbb{B}^n$, there is a positive constant C such that

$$\left|P_h[\eta_{a,\omega}](x) - P_h[\eta_{a,\omega}](y)\right| \leq C\omega(|x-y|).$$

Let $x_0 \in \partial\mathbb{B}^n$ such that $d(x) = |x_0 - x|$. By letting y tends to x_0 yields that

$$\left|P_h[\eta_{a,\omega}](x) - \eta_{a,\omega}(x_0)\right| \leq C\omega(|x_0-x|),$$

which gives

$$P_h[\eta_{a,\omega}](x) \leq \eta_{a,\omega}(x_0) + C\omega(|x_0-x|) \leq \omega(|x_0-a|) + C\omega(|x-a|). \tag{16}$$

Applying (16) and the inequality $|x_0 - a| \leq |x_0 - x| + |x - a| \leq 2|x - a|$, we conclude that

$$P_h[\eta_{a,\omega}](x) \leq C\omega(|x-a|) + \omega(|x_0-a|) \leq (C+2)\omega(|x-a|).$$

The proof of this theorem is complete. □

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