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Radial Symmetry of Positive Solutions of Nonlinear Elliptic Equations in Rn

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RADIAL SYMMETRY OF POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS IN $\mathbf{R}^n$†

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§1. Introduction.

In 1981 the method of "moving plane" (which goes back to A.D. Alexandroff; see [H]) was employed by [GNN] to study symmetry properties of positive solutions of the following problem

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \mathbf{R}^n, \ n \geq 2 \ , \\ \lim_{|x| \to \infty} u(x) = 0 \ . \end{cases} \tag{1.1}$$

Since then there have been lots of works published in this direction treating a variety of symmetry problems. (See e.g. [BN1,2,3], [CGS], [CL], [CN], [CS], [FL], [L], [LN1,2,3].) Generally speaking, in applying the "moving plane" device it is important to first obtain the asymptotic behavior of solutions near ∞ in order to get the device started near ∞. In case $f(u) \geq 0$ and $f(u) = o(u)$ near $u = 0$, the situation is quite involved since in this case solutions do have *different* asymptotic behavior at ∞. Thus symmetry conclusion usually can only be obtained for *certain* class of solutions with fast decay near ∞. (See e.g. [GNN], [LN1,2,3].) However, in the case where $f(0) = 0$ and $f'(0) < 0$, since solutions of (1.1) must decay exponentially at ∞, it follows that *all* positive solutions of (1.1) are radially symmetric as the following theorem shows.

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Theorem A. *([GNN]). Let u be a positive C^2 solution of (1.1) with $f(u) = -u + g(u)$ where g is continuous and $g(u) = O(u^\alpha)$ near $u = 0$ for some $\alpha > 1$. On the interval $0 \le s \le u_0 = \max u$, assume $g(s) = g_1(s) + g_2(s)$ with g_2 nondecreasing and $g_1 \in C^1$ satisfying, for some $C > 0, p > 1$*

$$|g_1(s) - g_1(t)| \le C|s - t|/|\log\min(s, t)|^p$$

for all $0 \le s, t \le u_0$. Then u is radially symmetric about some point x_0 in $\mathbf{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$. Furthermore,

$$\lim_{r \to \infty} r^{\frac{n-2}{2}} e^r u(r) = \mu > 0 .$$

This result covers the well-known scalar field equation

$$\Delta u - u + u^p = 0 \text{ in } \mathbf{R}^n, \; p > 1 ,$$

which has received much attention over the past several decades. Recently, Theorem A was generalized and improved by [L] with a different proof. (See Theorem C below.)

On the other hand it seems natural to consider (1.1) with $f(0) = f'(0) = 0$ and $f(s) \le 0$ for sufficiently small $s > 0$. In this case, (1.1) does have positive solutions with slow decay (say, power decay or even logarithmic decay; see Example 1 below) at ∞, and it seems that new ideas are needed to handle this case.

The purpose of this paper is to establish the following result which makes no assumption on the asymptotic behavior of solution and answers the question just posed above.

Theorem 1. *. Suppose that $f'(s) \le 0$ for sufficiently small $s > 0$. Then all positive solutions of (1.1) must be radially symmetric about the origin (up to translation) and $u_r < 0$ for $r = |x| > 0$.*

The novelty of Theorem 1 lies in the fact that no assumption is imposed on the decay rate of the solution u at ∞ . In general it seems very difficult to obtain a priori estimates for positive solutions of (1.1) with polynomial decay. The following example shows that one can even have solutions that go to zero slower than logarithmic decay at ∞.

Example 1. Let m be a positive integer and

$$v_m(x) = \log[\log[\cdots[\log(M + |x|^2)]\cdots]$$

where v_m is the m-th iterated log function, and M is a positive number such that $v_m(0) = 1$. Then $v_m(x)$ is an increasing function of $|x|$ with $\lim_{x\to\infty} v_m(x) = \infty$. Let $u_m(x) = 1/v_m(x)$ and $x \in \mathbf{R}^2$. Then u_m satisfies the following semilinear equation in $\mathbf{R}^2$

$$\Delta u_m(x) + f_m(u_m(x)) = 0 \ \text{ in } \ \mathbf{R}^2, \tag{1.2}$$

where

$$f_m(t) = -4t^2 exp\left(-\sum_{j=1}^{m} exp^{[j-1]}\left(\frac{1}{t}\right)\right)\left[\left(1 - Mexp\left(-exp^{[m-1]}\left(\frac{1}{t}\right)\right)\right)\cdot\right.$$

$$\cdot\left(2texp\left(-\sum_{j=1}^{m-1} exp^{[j-1]}\left(\frac{1}{t}\right)\right) + \sum_{i=1}^{m-1} exp\left(-\sum_{j=i}^{m-1} exp^{[j-1]}\left(\frac{1}{t}\right)\right)\right) -$$

$$\left. - Mexp\left(-exp^{[m-1]}\left(\frac{1}{t}\right)\right)\right] ,$$

and $exp^{[i]}$ is the i-th iterated exponential function; i.e. $exp^{[i]}(s) = exp(exp$ $(\cdots(exp(s))\cdots)$ with $exp^{[0]} = identity$. It can be easily verified that $f_m(0) = 0$ and there exists $r_m > 0$ such that $f'_m(t) < 0$ in $(0, r_m)$. Therefore (1.2) satisfies our hypotheses in Theorem 1. However, as m increases, the rate of decay of u_m gets slower and slower.

In a different direction, Franchi and Lanconelli [FL] generalized Theorem A to quasilinear equations.

Theorem B. *([FL]). Let u be a positive C^2 solution of*

$$div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + f(u) = 0 \ in \ \mathbf{R}^n, \ n \geq 2 \ ,$$

with $u(x) \to 0$ as $|x| \to \infty$. Assume that $f(u) = -u+g(u)$ near zero, where g is a $C^{1,\alpha}$-function such that $g(0) = g'(0) = 0$. Then u is radially symmetric about some point in $\mathbf{R}^n$.

More recently, C. Li [L] extended Theorem A further to fully nonlinear equations; that is,

$$\begin{cases} F[u] = F(x, u(x), Du(x), D^2u(x)) = 0 & \text{in } \mathbf{R}^n, n \geq 2 \ , \\ \lim_{|x|\to\infty} u(x) = 0 \ . \end{cases} \tag{1.3}$$

To state his result, we first introduce the following assumptions on F.

F1. F *is continuous in all of its variables, C^1 in p_{ij} and Lipschitz in s and p_i where p_{ij}'s are position variables for $\frac{\partial^2 u}{\partial x_i \partial x_j}$, p_i for $\frac{\partial u}{\partial x_i}$ and s for u.*

F2. $F_{p_{ij}}(x, s, p_i, p_{ij})\xi_i\xi_j \geq \overline{\lambda}(x, s, p_i, p_{ij})|\xi|^2, \xi \in \mathbf{R}^n$ *where $\overline{\lambda} > 0$ in $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2}$.*

F3. $F(x, s, p_i, p_{ij}) = F(|x|, s, p_i, p_{ij})$, *and F is nonincreasing in $|x|$.*

F4. $F(x, s, p_1, p_2, \ldots, p_{i_0-1}, -p_{i_0}, p_{i_0+1}, \ldots, p_n, p_{11}, \ldots, -p_{i_0 j_0}, \ldots, -p_{j_0 i_0}, \ldots, p_{nn}) = F(x, s, p_i, p_{ij})$ *for $1 \leq i_0 \leq n$, $1 \leq j_0 \leq n$, and $i_0 \neq j_0$.*

Theorem C. *([L]). Suppose that F satisfies (F1-4) and*

$$\overline{\lim_{\substack{|x|\to\infty \\ |s|+|p_i|+|p_{ij}|\to 0}}} F_s(x, s, p_i, p_{ij}) < -C < 0$$

for some positive constant C. Let u be a positive C^2 solution of (1.3) such that

$$\lim_{x\to\infty}(u(x) + |Du(x)| + |D^2u(x)|) = 0 \ .$$

Then u must be radially symmetric about some point x_0 in $\mathbf{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

Since our proof of Theorem 1 above rests on a simple observation that the usual maximum principle can be applied here to take care of the possible difficulties which might arise in getting the "moving plane" device started near ∞, our proof of Theorem 1 does generalize to cover the fully nonlinear equation (1.3) above.

Our main result of this paper is the following theorem which clearly contains Theorem 1 as a special case.

Theorem 2. *Suppose that F satisfies (F1-4) and*

$$F_s \leq 0 \text{ for } |x| \text{ large, and for } s \text{ small and positive.} \tag{1.4}$$

Let u be a positive C^2 solution of (1.3). Then u must be radially symmetric about some point x_0 in $\mathbf{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

Remark. We would like to point out if F is strictly decreasing in $|x|$, then it follows from the proof of Theorem 2 that a positive solution u of (1.3) must be radially symmetric about the origin. On the other hand if F depends on $|x|$ and is only nonincreasing in $|x|$ as in the assumption of Theorem 2, then the symmetric point needs not to be the origin as the following example shows.

Example 2. Let u_0 be the unique positive radial solution of the well-known scalar field equation

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \mathbf{R}^n, \ n > 2 \text{ and } 1 < p < \frac{n+2}{n-2} \ , \\ \lim_{|x| \to \infty} u(x) = 0 \ . \end{cases}$$

Define

$$f(|x|, u) = \begin{cases} -u + u^p & \text{if } u \leq u_0((|x| - 1)_+) , \\ -u + u_0{}^p((|x| - 1)_+) & \text{if } u > u_0((|x| - 1)_+) , \end{cases}$$

where $(|x| - 1)_+ = max(|x| - 1, 0)$. Then obviously $u(x) = u_0(|x - y|)$ is a solution of $F[u] \equiv \Delta u + f(|x|, u) = 0$ in $\mathbf{R}^n$ for every $|y| \leq 1$, and F satisfies $(F1 - 4)$ and (1.4).

§2. Proof of Theorem 2.

Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbf{R}^n$, we denote its reflection with respect to the hyperplane $T_\lambda = \{y \in \mathbf{R}^n | y_1 = \lambda\}$ by x^λ; i.e. $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$. Observe that if $\lambda > x_1$ and $\lambda > 0$

$$|x^\lambda| - |x| = \frac{4\lambda(\lambda - x_1)}{|x| + |x^\lambda|} > 0 . \tag{2.1}$$

Let u be a positive solution of (1.3). First, we define

$$\Lambda = \{\lambda \in \mathbf{R} | u(x) > u(x^\lambda) \text{ for all } x \in \mathbf{R}^n \text{ with } x_1 < \lambda \text{ and } \frac{\partial u}{\partial x_1} < 0 \text{ on } T_\lambda\} .$$

By condition (1.4), there exists a $r_0 > 0$ such that

$$F_s(x, s, p_i, p_{ij}) \leq 0 \text{ for } |x| \geq r_0{}^{-1} \text{ and } 0 \leq s \leq r_0 . \tag{2.2}$$

Since u goes to 0 at ∞, there exist $r_0{}^{-1} < R_0 < R_1$ such that

$$\begin{cases} \max\limits_{\mathbf{R}^n \backslash B_{R_0}(0)} u < r_0 \\ \text{and} \\ \max\limits_{\mathbf{R}^n \backslash B_{R_1}(0)} u < \min\limits_{\overline{B}_{R_0}(0)} u \equiv m_0 . \end{cases} \tag{2.3}$$

Step 1. $[R_1, \infty) \subset \Lambda$.

For each $\lambda \geq R_1$, let $w(x) = u(x) - u(x^\lambda)$ in $\Sigma_\lambda \equiv \{x \in \mathbf{R}^n | x_1 < \lambda\}$. Then

$$F(x, u(x), Du(x), D^2u(x)) - F(x, u(x^\lambda), Du(x^\lambda), D^2u(x^\lambda))$$

$$\leq F(x, u(x), Du(x), D^2u(x)) - F(x^\lambda, u(x^\lambda), Du(x^\lambda), D^2u(x^\lambda)) \qquad (2.4)$$

$$= F(x, u(x), Du(x), D^2u(x)) - F(x^\lambda, u(x^\lambda), (Du)(x^\lambda), (D^2u)(x^\lambda)) = 0$$

by (F3-4) and (2.1).

Therefore, it follows from the assumptions on F that
$$\begin{cases} Lw \leq 0 & \text{in } \Sigma_\lambda \text{ ,} \\ w = 0 & \text{on } T_\lambda \text{ ,} \end{cases}$$
where $L = a_{ij}D_{ij} + b_iD_i + c$ with

$$a_{ij}(x) = \int_0^1 F_{p_{ij}}(x, u(x), Du(x), D^2u(x^\lambda) + t(D^2u(x) - D^2u(x^\lambda)))dt \text{ ,}$$

$$b_i(x) = \int_0^1 F_{p_i}(x, u(x), Du(x^\lambda) + t(Du(x) - Du(x^\lambda)), D^2u(x^\lambda))dt \text{ ,}$$

$$\text{and } c(x) = \int_0^1 F_s(x, u(x^\lambda) + t(u(x) - u(x^\lambda)), Du(x^\lambda), D^2u(x^\lambda))dt \text{ .}$$

Since $\lambda \geq R_1$ and (2.3), $w > 0$ on $\overline{B}_{R_0}(0) \subset \Sigma_\lambda$. On the other hand, from (2.2), (2.3) and the definition of $c(x)$, we have

$$c(x) \leq 0 \text{ in } \Sigma_\lambda \setminus B_{R_0}(0) \text{ .} \qquad (2.5)$$

Therefore w satisfies
$$\begin{cases} a_{ij}D_{ij}w + b_iD_iw + cw \leq 0 \text{ in } \Sigma_\lambda \setminus \overline{B}_{R_0}(0) \text{ ,} \\ w \geq 0 \text{ on } \partial(\Sigma_\lambda \setminus \overline{B}_{R_0}(0)) \text{ and} \lim_{\substack{x \to \infty \\ x \in \Sigma_\lambda \setminus \overline{B}_{R_0}(0)}} w(x) = 0 \text{ .} \end{cases}$$

Since $w(x) \not\equiv 0$ and $c(x) \leq 0$ in $\Sigma_\lambda \setminus B_{R_0}(0)$, the strong maximum principle and the Hopf boundary lemma (see, e.g. [PW]) imply that
$$\begin{cases} w(x) > 0 \text{ in } \Sigma_\lambda \setminus \overline{B}_{R_0}(0) \text{ ,} \\ \frac{\partial w}{\partial x_1} < 0 \text{ on } T_\lambda \end{cases}$$
which, together with the fact that $w > 0$ on $\overline{B}_{R_0}(0)$ proves that $[R_1, \infty) \subset \Lambda$.

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Step 2. Λ is open in $(0, \infty)$.

Let $\lambda_0 \in \Lambda \cap (0, \infty)$. We claim that there exists an $\epsilon > 0$ such that $(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \subset \Lambda \cap (0, \infty)$. Without loss of generality, we may assume $0 < \lambda_0 \leq R_1$.

It follows from the assumption $\lambda_0 \in \Lambda \cap (0, \infty)$ that

$$\begin{cases} u(x) - u(x^{\lambda_0}) > 0 & \text{in } \Sigma_{\lambda_0} , \\ \frac{\partial u}{\partial x_1} < 0 & \text{on } T_{\lambda_0} . \end{cases} \qquad (2.6)$$

On T_{λ_0}, since $\frac{\partial u}{\partial x_1} < 0$, we can find an $\epsilon_1 \in (0, 1)$ such that

$$\frac{\partial u}{\partial x_1} < 0 \text{ in } \Pi_{\lambda_0, \epsilon_1} \equiv \{x = (x_1, x') \in \mathbf{R}^n | \lambda_0 - 4\epsilon_1 \leq x_1 \leq \lambda_0 + 4\epsilon_1, |x'| \leq R_1 + 1\}$$

$$(2.7)$$

where $x' = (x_2, \ldots, x_n)$. Therefore we conclude that

$$\begin{cases} u(x) - u(x^\lambda) > 0 & \text{in } \{x \in \overline{B}_{R_1+1}(0) | \lambda_0 - 2\epsilon_1 \leq x_1 < \lambda\} , \\ \frac{\partial u}{\partial x_1} < 0 & \text{on } T_\lambda \cap \overline{B}_{R_1+1}(0) \end{cases} \qquad (2.8)$$

for any $\lambda \in (\lambda_0 - \epsilon_1, \lambda_0 + \epsilon_1)$.

Let $M = 2 \max \left\{ \left| \frac{\partial u}{\partial x_1} \right| \; | \; |x_1| \leq 2(R_1 + 1), |x'| \leq R_1 + 1 \right\}$ and $\delta = \min\{u(x) - u(x^{\lambda_0})| \; |x'| \leq R_1 + 1, -(R_1 + 1) \leq x_1 \leq \lambda_0 - 2\epsilon_1\}$. Then (2.6) implies that $\delta > 0$, and hence

$$u(x) - u(x^\lambda) > 0 \text{ in } \{x \in \overline{B}_{R_1+1}(0) | -(R_1 + 1) \leq x_1 \leq \lambda_0 - 2\epsilon_1\} \quad (2.9)$$

for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ where $\epsilon = \min(\epsilon_1, \frac{\delta}{M}, \lambda_0)$. Combining (2.8) and (2.9), we obtain

$$\begin{cases} u(x) - u(x^\lambda) > 0 & \text{in } \overline{B}_{R_1+1}(0) \cap \Sigma_\lambda \text{ for } \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon) , \\ \frac{\partial u}{\partial x_1} < 0 & \text{on } \overline{B}_{R_1+1}(0) \cap T_\lambda . \end{cases} \qquad (2.10)$$

Now for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, let $w(x) = u(x) - u(x^\lambda)$. Then $w \not\equiv 0$ in $\Sigma_\lambda \setminus \overline{B}_{R_1+1}(0)$ and, similar to the proof of Step 1, w satisfies

$$\begin{cases} a_{ij} D_{ij} w + b_i D_i w + cw \leq 0 & \text{in } \Sigma_\lambda \setminus \overline{B}_{R_1+1}(0) , \\ w \geq 0 \text{ on } \partial(\Sigma_\lambda \setminus \overline{B}_{R_1+1}(0)) \text{ and } \lim_{x \to \infty} w(x) = 0 . \end{cases}$$

Once again, from the choice of R_1 (see (2.2) and (2.3)) and the definition for $c(x)$, we have

$$c(x) \leq 0 \text{ in } \Sigma_\lambda \setminus \overline{B}_{R_1+1}(0) \ .$$

Hence the strong maximum principle implies that for $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$

$$\begin{cases} w(x) > 0 & \text{in } \Sigma_\lambda \setminus \overline{B}_{R_1+1}(0) \ , \\ \dfrac{\partial w}{\partial x_1} < 0 & \text{on } T_\lambda \setminus \overline{B}_{R_1+1}(0) \ , \end{cases}$$

i.e.

$$\begin{cases} u(x) - u(x^\lambda) > 0 & \text{in } \Sigma_\lambda \setminus \overline{B}_{R_1+1}(0) \ , \\ \dfrac{\partial u}{\partial x_1} < 0 & \text{on } T_\lambda \setminus \overline{B}_{R_1+1}(0) \ , \end{cases} \tag{2.11}$$

for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. This, combined with (2.10), proves our assertion.

Step 3. $\Lambda \cap (0, \infty) = (0, \infty)$ or $u(x) \equiv u(x^{\lambda_1})$ for some $\lambda_1 \geq 0$.

Now we have shown that Λ is open and contains all large λ in $(0, \infty)$. Let (λ_1, ∞) be the component of $\Lambda \cap (0, \infty)$ containing (R_1, ∞) with $\lambda_1 \geq 0$.

From the continuity of u, we have

$$w(x) = u(x) - u(x^{\lambda_1}) \geq 0 \text{ in } \Sigma_{\lambda_1} \ ,$$

and it follows as in Step 1 that

$$\begin{cases} a_{ij} D_{ij} w + b_i D_i w + cw \leq 0 & \text{in } \Sigma_{\lambda_1} \ , \\ w \geq 0 \text{ in } \Sigma_{\lambda_1} \text{ and } w = 0 & \text{on } T_{\lambda_1} \ , \\ \displaystyle\lim_{x \to \infty} w(x) = 0 \ . \end{cases}$$

Hence, we have that either

$$w \equiv 0 \text{ in } \Sigma_{\lambda_1}, \text{ i.e. } u(x) \equiv u(x^{\lambda_1}) \text{ for } x_1 < \lambda_1 \ , \tag{2.12}$$

or

$$\begin{cases} w > 0 & \text{in } \Sigma_{\lambda_1}, \text{ i.e. } u(x) > u(x^{\lambda_1}) \text{ in } \Sigma_{\lambda_1} \ , \\ \frac{\partial w}{\partial x_1} < 0 & \text{on } T_{\lambda_1} \ . \end{cases} \tag{2.13}$$

(Note that here we do not need the sign condition of $c(x)$ in applying the maximum principle.) But if the latter happens, and if $\lambda_1 > 0$, it would imply that $\lambda_1 \in \Lambda \cap (0, \infty)$, which contradicts the fact that $\Lambda \cap (0, \infty)$ is open and (λ_1, ∞) is a component of it. Hence $\lambda_1 = 0$, and (2.13) becomes

$$u(x_1, x_2, x_3, \ldots, x_n) > u(-x_1, x_2, x_3, \ldots, x_n) \text{ for } x_1 < 0$$

which completes the proof of Step 3.

Finally if (2.12) occurs, then we have already shown that u is symmetric in the x_1 direction about the hyperplane T_{λ_1} and $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > \lambda_1$. On the other hand, if (2.13) occurs with $\lambda_1 = 0$, then we can repeat the previous Steps 1-3 for the negative x_1-direction for u to conclude that either

$$u(x) \equiv u(x^{\lambda_2}) \text{ and } \frac{\partial u}{\partial x_1} > 0 \text{ for } x_1 < \lambda_2 \text{ with } \lambda_2 \leq 0 , \qquad (2.12')$$

or

$$u(x_1, x_2, x_3, \ldots, x_n) < u(-x_1, x_2, x_3, \ldots, x_n) \text{ for } x_1 < 0 . \qquad (2.13')$$

But (2.13) and (2.13') can not occur simultaneously. Therefore u must be symmetric in x_1 direction about some hyperplane T_λ and strictly decreasing away from T_λ. Since the equation (1.3) is invariant under rotation, we may take any direction as the x_1-direction and conclude that u is symmetric in every direction about some hyperplane which is perpendicular to that direction and strictly decreasing away from that hyperplane. Hence u must be radially symmetric about some point x_0 in $\mathbf{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

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