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Radiative gravitational fields in general relativity II. Asymptotic behaviour at future null infinity

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We prove that Penrose's requirements for asymptotic simplicity are formally satisfied by the general metric, (1), which admits both post-Minkowskian and multipolar expansions, (2), which is stationary in the past and asymptotically Minkowskian in the past, (3), which admits harmonic coordinates, and (4), which is a solution of Einstein's vacuum equations outside a spatially bounded region. The proof is based on the setting up, by using the method of a previous work (L. Blanchet & T. Damour (*Phil. Trans. R. Soc. Lond.* A **320**, 379–430 (1986))), of an improved algorithm that generates a metric equivalent to the general harmonic metric of that work but written in radiative coordinates, i.e. admitting an expansion in powers of r^{-1} for $r \rightarrow \infty$ and $t-r$ fixed. The arbitrary parameters of the construction are the radiative multipole moments in the sense of K. S. Thorne (*Rev. mod. Phys.* **52**, 299 (1980)).

1. INTRODUCTION

Penrose (1963, 1965) has introduced the concept of an *asymptotically simple* space–time to geometrically formulate the asymptotic properties of radiative space–times that were investigated by Bondi *et al.* (1962) and Sachs (1962). Basically, an asymptotically simple space–time is a space–time sharing common local and global asymptotic properties with Minkowski space–time. However, this concept is only a *definition* of a class of space–times that we would like to associate with isolated systems and it has not been proven to be consistent with Einstein's equations: it is not known whether sufficiently general Einstein's space–times (notably, non-stationary space–times) satisfy the definition for asymptotic simplicity.

The purpose of this paper is to show that the definition is (formally) satisfied by the general metric constructed in a previous work (Blanchet & Damour 1986, hereafter referred to as paper I). This general metric is physically expected to be associated with an isolated system (stationary in the past) lying in a region $r \leqslant r_0$. The method of paper I is mainly an extension of the Bonnor–Thorne approach to gravitational radiation theory that combines the nonlinearity (or post-Minkowskian) expansions with multipolar expansions (Bonnor 1959; Bonnor &

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Rotenberg 1966; Couch *et al.* 1968; Hunter & Rotenberg 1969; Thorne 1977, 1980, 1983). The advantage of this method is that it is valid in all the 'weak-field' region outside an isolated system and thus, *in particular*, in all the 'asymptotic' region far from the system (e.g. the distant wave zone in Thorne's terminology). In contrast, the Bondi–Sachs–Penrose method is valid only in the asymptotic region.

Let us first recall the basic assumptions of paper I. We consider the class of metrics $\mathscr{g}^{\alpha\beta} = \sqrt{g}g^{\alpha\beta}$† satisfying the following properties.

1. $\mathscr{g}^{\alpha\beta}$ admits a multipolar post-Minkowskian expansion (MPM expansion), i.e. a formal expansion in powers of G (Newton's constant)

$$\mathscr{g}^{\alpha\beta} = f^{\alpha\beta} + \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}, \tag{1.1}$$

such that each $h_n^{\alpha\beta}$ admits a finite multipolar expansion

$$h_n^{\alpha\beta}(\boldsymbol{x}, t) = \sum_{l=0}^{l_{\max}} \hat{n}^L(\theta, \phi)\, h_{nL}^{\alpha\beta}(r, t), \tag{1.2}$$

where $l_{\max}$ is some maximum value of l (depending on n).

2. $\mathscr{g}^{\alpha\beta}$ is stationary in the past, i.e. there exists a time $-T$ such that

$$t \leqslant -T \Rightarrow (\partial/\partial t)\, \mathscr{g}^{\alpha\beta}(\boldsymbol{x}, t) = 0 \tag{1.3}$$

and $\mathscr{g}^{\alpha\beta}$ is asymptotically Minkowskian in the past in the sense that

$$t \leqslant -T \Rightarrow \lim_{r\rightarrow\infty} \mathscr{g}^{\alpha\beta}(\boldsymbol{x}, t) = f^{\alpha\beta}. \tag{1.4}$$

3. $\mathscr{g}^{\alpha\beta}$ satisfies Einstein's vacuum equations in a domain of the type $D = \{(\boldsymbol{x}, t); r > r_0\}$ for some $r_0 \geqslant 0$ which means that the h_ns satisfy in D the equations

$$\partial_\mu h_n^{\alpha\mu} = H_n^\alpha, \tag{1.5a}$$

$$\Box h_n^{\alpha\beta} = \partial H_n^{\alpha\beta} + N_n^{\alpha\beta}(h_m; m \leqslant n-1), \tag{1.5b}$$

where $\Box := f^{\mu\nu}\partial_{\mu\nu} = -\partial_t^2 + \Delta$ is the flat d'Alembertian operator, $N_n^{\alpha\beta}$ is a 'source term' depending on the h_ms (for $m \leqslant n-1$), and $\partial H_n^{\alpha\beta}$ is a 'gauge term' given by

$$\partial H_n^{\alpha\beta} = \partial^\alpha H_n^\beta + \partial^\beta H_n^\alpha - f^{\alpha\beta}\partial_\mu H_n^\mu, \tag{1.6a}$$

with $\partial^\alpha := f^{\alpha\mu}\partial_\mu$. Note that we have

$$\partial_\mu(\partial H_n^{\alpha\mu}) = \Box H_n^\alpha. \tag{1.6b}$$

In this paper we relax the last assumption of paper I, namely that $\partial_\beta \mathscr{g}^{\alpha\beta} = 0$ (harmonic coordinates), and thus we leave unspecified the H_n^αs in (1.5).

Let us employ, with Geroch & Horowitz (1978), a definition for asymptotic

† We use the notation of paper I: signature $-+++$; Greek indices $= 0, 1, 2, 3$; Latin indices $= 1, 2, 3$; $g := -\det(g_{\mu\nu})$; $f^{\alpha\beta} = f_{\alpha\beta} =$ flat metric $= \mathrm{diag}(-1, +1, +1, +1)$; $\mathbb{N}$, $\mathbb{R}$ are the usual sets of non-negative integers and real numbers; $C^p(U)$ is the set of p-times continuously differentiable functions in the open set U ($p \leqslant +\infty$); $r = (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$; $n^i = x^i/r$; $\partial_i = \partial/\partial x^i$; $n^L = n^{i_1}n^{i_2}\ldots n^{i_l}$ and $\partial_L = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$ where $L = i_1 i_2 \ldots i_l$ is a multi-index with l indices; $\hat{n}^L$ is the (symmetric) trace-free part (STF part) of n^L; we use $c = 1$ throughout the paper.

simplicity that slightly differs from Penrose's original definition by the following: we require that manifolds, Lorentz metrics and the conformal factor are C^∞ instead of C^4, C^3 and C^3 respectively, and that infinity is topologically $S^2 \times \mathbb{R}$ with complete '$\mathbb{R}$s' (so that the global asymptotic structure is the same as those of Minkowski space–time). However, we keep the terminology 'asymptotically simple' instead of 'asymptotically flat' to emphasize that the concept is only a definition. For reviews of properties satisfied by an asymptotically simple space–time see Geroch (1977), Schmidt (1979) and Ashtekar (1984).

Definition 1.1. A space–time $(M, g_{\alpha\beta})$, i.e. a C^∞ connected Hausdorff orientable manifold M with a C^∞ Lorentz metric $g_{\alpha\beta}$ on M, is said to be asymptotically simple at null infinity if there exists a C^∞ manifold $\tilde{M}$ with boundary $\mathscr{I}$ ($\subset \tilde{M}$) together with a C^∞ Lorentz metric $\tilde{g}_{\alpha\beta}$ on $\tilde{M}$ and a C^∞ scalar field Ω on $\tilde{M}$ such that:

(*a*) in the interior $\tilde{M}-\mathscr{I}$ we have $\Omega > 0$ and $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$;

(*b*) at the boundary $\mathscr{I}$ we have $\Omega = 0$, $\tilde{\nabla}_\alpha \Omega \neq 0$, $\tilde{\nabla}^\alpha \Omega \tilde{\nabla}_\alpha \Omega = 0$ and $\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Omega = 0$, where $\tilde{\nabla}^\alpha = \tilde{\mathscr{g}}^{\alpha\beta} \tilde{\nabla}_\beta$ is the covariant derivative operator associated with $\tilde{g}_{\alpha\beta}$;

(*c*) $\mathscr{I}$ consists of two parts, $\mathscr{I}^-$ and $\mathscr{I}^+$, each having topology $S^2 \times \mathbb{R}$, with the '$\mathbb{R}$s' being complete null generators.

In this paper we shall prove the following theorem.

THEOREM 1.1. *The general metric* $\mathscr{g}^{\alpha\beta} = f^{\alpha\beta} + \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}$, *which is a solution of* (1.1)–(1.5) *and which admits harmonic coordinates in the domain* D, *rewritten in covariant form* $g_{\alpha\beta} = f_{\alpha\beta} + \sum_{n=1}^{\infty} G^n g_{n\alpha\beta}$, *yields a sequence of space–times* $(M, g_{[p]\alpha\beta})$ *for* $p \geqslant 1$, *where* $g_{[p]\alpha\beta} = f_{\alpha\beta} + \sum_{n=1}^{p} G^n g_{n\alpha\beta}$, *which all satisfy definition* 1.1.

We require in theorem 1.1 that $\mathscr{g}$ admits, in the sense of power series in G, harmonic coordinates in D to connect $\mathscr{g}$ to the harmonic metrics of paper I. Theorem 1.1 gives us some confidence that an actual isolated system, stationary before a time $-T$ and non-stationary thereafter, generates a space–time solution of Einstein's equations that is asymptotically simple (definition 1.1). Note, however, that the assumption of stationarity before the time $-T$ is essential for theorem 1.1 to hold. Indeed, for a system that is always non-stationary (in particular, non-stationary in the remote past) asymptotic simplicity probably neither holds at $\mathscr{I}^-$ (see, for example, Bardeen & Press 1973; Walker & Will 1979) nor even at $\mathscr{I}^+$ (Damour 1986).

To see precisely what to do for proving theorem 1.1, let us consider the general MPM Einstein metric $\mathscr{g} = f + \sum_{n \geqslant 1} G^n h_n$ (stationary in the past and asymptotically Minkowskian in the past) in *harmonic coordinates* (i.e. such that $\partial_\beta h_n^{\alpha\beta} = 0$ for all n) that has been constructed and studied in paper I. This metric admits (paper I, theorem 7.2) the following asymptotic expansion at the infinity $r \to \infty$ with $t - r = \text{const.}$: $\forall N \geqslant N_0$ for N_0 large enough, we have

$$h_n(\boldsymbol{x}, t) = \sum_{l \geqslant 0} \hat{n}^L \left\{ \sum_{k=1}^{N} \sum_{p=0}^{n-1} (\lg r)^p r^{-k} F^L_{nkp}(t-r) + R^L_{nN}(r, t-r) \right\}, \tag{1.7}$$

where the functions $F^L_{nkp}(u)$ are $C^\infty(\mathbb{R})$ and constant when $u \leqslant -T$, and where the remainders $R^L_{nN}(r, u)$ are $O^\infty(r^{-N})$-functions (paper I, definition 7.1). Similarly, h_n admits an asymptotic expansion of the type $h_n \sim \sum \hat{n}^L (\lg r)^p r^a H^L_{nap}(t) + T_{nN}$ in the near zone $r \to 0$ with $t = \text{const.}$ (paper I, equation (5.4)).

In the near zone, the logarithms are probably caused by backscattering effects (Bonnor 1974; Thorne 1980) and thus they probably reflect physical properties of gravitational waves. On the contrary, the logarithms of (1.7) (in the 'outgoing' far zone), that have been known to appear since the work of Fock (1959), might be an artefact due to the use of harmonic coordinates (Isaacson & Winicour 1968; Madore 1970; Anderson 1979). These logarithms are troublesome because at large distances from the origin the nth post-Minkowskian approximation becomes larger than the $(n-1)$th one. (With a source at the origin, this occurs typically for $r \geqslant \lambda e^{\lambda/GM}$, where λ is the gravitational wavelength and M the mass of the source.) Furthermore, trying to apply definition 1.1 with the most natural conformal factor $\Omega = 1/r$, and by using coordinates $\tilde{x}^\alpha = (\Omega, u, \theta, \phi)$ with $u = t-r$, we are faced with powers of $\lg \Omega$ that prevent the conformal metric from being C^∞ in a neighbourhood of $\Omega = 0$. Therefore what is needed (and it is possible to do so) is to construct a new ('good') coordinate system (distinct from the harmonic coordinate system) such that the transformed h_ns admit expansions at infinity in powers of $1/r$ only (no $\lg r$ terms).

In fact, rather than exhibiting directly the coordinate transformation, we set up (in §2), by using the tools and the algorithmic method of paper I, an improved algorithm that constructs a particular metric (called $\mathscr{g}_{\text{rad}}$) expressed in 'good' coordinates. In §3 we prove that the space–time (M, g_{rad}) is asymptotically simple and we give some physical interpretations. Finally, in §4, we show that $\mathscr{g}_{\text{rad}}$ is 'equivalent' to the general metric of paper I. This will complete the proof of theorem 1.1.

2. Construction of a radiative field

We wish to construct a (stationary in the past and asymptotically Minkowskian in the past) MPM *radiative field* in the sense of Papapetrou (1969) and Madore (1970), i.e. an MPM field $\mathscr{g}_{\text{rad}} = f + \Sigma_{n \geqslant 1} G^n h_{\text{rad}\,n}$ such that each $h_{\text{rad}\,n}$ admits, when $r \to \infty$, an asymptotic expansion in powers of $1/r$ along a family of forward cones $t-r = \text{const}$. More precisely, by using the class of functions $\mathscr{L}^0$ defined in paper I (definition 7.2) we want, $\forall n \geqslant 1$, ${}_{\text{D}}h_{\text{rad}\,n}$ (the dynamic part, zero in the past, of $h_{\text{rad}\,n}$) to belong to $\mathscr{L}^0$, that is that there exist some functions $F^L_{nk}(u)$ that are both $C^\infty(\mathbb{R})$ and zero in the past ($u \leqslant -T$), and some functions $R^L_{nN}(r, u)$ that are $O^\infty(r^{-N})$ (paper I, definition 7.1) such that, $\forall N \geqslant 0$,

$$ {}_{\text{D}}h_{\text{rad}\,n}(\boldsymbol{x}, t) = \sum_{l=0}^{l_{\max}} \hat{n}^L \left\{ \sum_{k=1}^{N} r^{-k} F^L_{nk}(t-r) + R^L_{nN}(r, t-r) \right\}. \tag{2.1} $$

The stationary part of $h_{\text{rad}\,n}$, ${}_{\text{S}}h_{\text{rad}\,n}$ (constant in time), is simply a finite sum of terms $\hat{n}^L r^{-k} F^L_{nk}$ with $k \geqslant n$ (as shown in Appendix A).

2.1. *A linearized radiative field*

Our first problem is to find a convenient 'linearized radiative field', $h_{\text{rad}\,1}$, i.e. a solution of Einstein's linearized vacuum equations generating at further approximation steps a MPM radiative field. Consider the most general solution, satisfying the conditions (1.1)–(1.4), of the linearized vacuum equations in *harmonic* coordinates ($\Box h^{\alpha\beta} = \partial_\beta h^{\alpha\beta} = 0$), which has been dealt with by Sachs & Bergmann (1958); Sachs (1961); Pirani (1964); Thorne (1980) (for a careful derivation of this general solution see paper I, §2). This solution can be written, modulo an infinitesimal coordinate transformation, as the following 'canonical' multipolar expansion $h_{\text{can}\,1}[\mathscr{M}]$, depending on a finite set of $C^\infty(\mathbb{R})$ STF tensors (multipole moments) $\mathscr{M} = \{M_L(t), S_L(t)\}$, all constant in the past with M, M_i and S_i constant for all times. (We employ the notations of paper I, equations (2.32), except for the script letter $\mathscr{M}$ we use here to avoid confusion with the mass monopole M):

$$h^{00}_{\text{can}\,1}[\mathscr{M}] = -4 \sum_{l \geqslant 0} \frac{(-)^l}{l!} \partial_L [r^{-1} M_L(t-r)], \tag{2.2a}$$

$$h^{0i}_{\text{can}\,1}[\mathscr{M}] = +4 \sum_{l \geqslant 1} \frac{(-)^l}{l!} \partial_{L-1} [r^{-1}\, {}^{(1)}M_{iL-1}(t-r)] + 4 \sum_{l \geqslant 1} \frac{(-)^l\, l}{(l+1)!} \epsilon_{iab} \partial_{aL-1} [r^{-1} S_{bL-1}(t-r)], \tag{2.2b}$$

$$h^{ij}_{\text{can}\,1}[\mathscr{M}] = -4 \sum_{l \geqslant 2} \frac{(-)^l}{l!} \partial_{L-2} [r^{-1}\, {}^{(2)}M_{ijL-2}(t-r)] - 8 \sum_{l \geqslant 2} \frac{(-)^l\, l}{(l+1)!} \partial_{aL-2} [r^{-1} \epsilon_{ab(i}\, {}^{(1)}S_{j)\,bL-2}(t-r)]. \tag{2.2c}$$

This linearized field is the canonical linearized field used in Thorne (1980, equations (8.12)). Expanding the derivatives ∂_L, we see that $h_{\text{can}\,1}$ is a finite sum of terms $\hat{n}^L r^{-k} F^L_k(t-r)$ (with $k \geqslant 1$) and thus it is a finite expansion in powers of $1/r$ along the cones $t-r = \text{const.}$ (hence ${}_{\mathrm{D}}h_{\text{can}\,1} \in \mathscr{L}^0$). However, these cones are not null hypersurfaces for the metric $f^{\alpha\beta} + Gh^{\alpha\beta}_{\text{can}\,1}$ since the equations for these null hypersurfaces are easily seen to be (neglecting terms of order $O(G^2)$):

$$t - r - 2GM \lg r + G \sum_{l \geqslant 1} \sum_{k \geqslant 1} C^l_k\, n^L r^{-k}\, {}^{(l-k)}M_L(t-r) = \text{const.}, \tag{2.3}$$

where the C^l_ks are some constant coefficients, $n^L = n^{i_1} n^{i_2} \ldots n^{i_l}$ and ${}^{(p)}M(u) = (\mathrm{d}/\mathrm{d}u)^p M(u)$. Thus the cones $t-r = \text{const.}$ diverge (in harmonic coordinates) when $r \to \infty$ as $2GM \lg r$ from a family of null hypersurfaces of $f + Gh_{\text{can}\,1}$. Madore (1970) has proved that the cones $t-r = \text{const.}$ along which a radiative field admits a $1/r$-expansion must be, on the contrary, asymptotically tangent to a family of null hypersurfaces of the metric. He therefore concluded that a linearized harmonic metric, such as $h_{\text{can}\,1}$, cannot be the linearized approximation of a radiative field: $h_{\text{can}\,1}$ is not a convenient 'linearized radiative field'.

We can easily find a convenient $h_{\text{rad}\,1}$ such that the cones $t-r = \text{const.}$ are asymptotically tangent to null hypersurfaces of $f + Gh_{\text{rad}\,1}$. The simplest choice is

$$h^{\alpha\beta}_{\text{rad}\,1}[\mathscr{M}] = h^{\alpha\beta}_{\text{can}\,1}[\mathscr{M}] + \partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha - f^{\alpha\beta} \partial_\mu \xi^\mu, \tag{2.4a}$$

where the vector ξ^α is

$$\xi^\alpha = 2M f^{0\alpha} \lg r = (-2M \lg r, \mathbf{0}), \tag{2.4b}$$

M being the mass monopole associated with $\mathscr{M}$. $h_{\mathrm{rad}\,1}$ defined by (2.4) can be thought of as being the transformation of $h_{\mathrm{can}\,1}$ by the infinitesimal coordinate change $(t, \boldsymbol{x}) \rightarrow (t-2GM\lg r, \boldsymbol{x})$ so that it is clear from (2.3) that, after transformation, the new cones $t-r=$ const. are up to order $O(G^2)$ asymptotically tangent to null hypersurfaces of $f+Gh_{\mathrm{rad}\,1}$. The (new) coordinates are not harmonic coordinates for $h_{\mathrm{rad}\,1}$ because we have

$$\partial_\beta h^{\alpha\beta}_{\mathrm{rad}\,1} = \Delta \xi^\alpha = 2Mr^{-2} f^{0\alpha} \neq 0 \tag{2.5}$$

(with $\Delta = \delta_{ij}\partial_i\partial_j$). $h_{\mathrm{rad}\,1}$ is, like $h_{\mathrm{can}\,1}$, a finite sum of terms $\hat{n}^L r^{-k} F^L_k(t-r)$ with $k \geqslant 1$. In the following we will need to consider the dominant part of $h_{\mathrm{rad}\,1}$ for $r\rightarrow\infty$ with $t-r=$ const., namely $r^{-1}F_1(t-r, \boldsymbol{n}) = \Sigma_l \hat{n}^L r^{-1} F^L_1(t-r)$. From (2.2) and (2.4) we find

$$F^{\alpha\beta}_1(t-r, \boldsymbol{n}) = 2M(f^{0\alpha}k^\beta + f^{0\beta}k^\alpha) + z^{\alpha\beta}(t-r, \boldsymbol{n}), \tag{2.6}$$

where k^α is the Minkowski null vector $k^\alpha = (1, \boldsymbol{n})$ $(k_\alpha = f_{\alpha\beta}k^\beta = (-1, \boldsymbol{n}))$, and $z^{\alpha\beta}$ is given by

$$z^{00}(t-r, \boldsymbol{n}) = -4 \sum_{l\geqslant 2} \frac{1}{l!} n_L \, {}^{(l)}M_L(t-r), \tag{2.7a}$$

$$z^{0i}(t-r, \boldsymbol{n}) = -4 \sum_{l\geqslant 2} \frac{1}{l!} n_{L-1} \, {}^{(l)}M_{iL-1}(t-r) + 4 \sum_{l\geqslant 2} \frac{l}{(l+1)!} \epsilon_{iab} n_{aL-1} \, {}^{(l)}S_{bL-1}(t-r), \tag{2.7b}$$

$$z^{ij}(t-r, \boldsymbol{n}) = -4 \sum_{l\geqslant 2} \frac{1}{l!} n_{L-2} \, {}^{(l)}M_{ijL-2}(t-r) + 8 \sum_{l\geqslant 2} \frac{l}{(l+1)!} n_{aL-2} \epsilon_{ab(i} \, {}^{(l)}S_{j)bL-2}(t-r). \tag{2.7c}$$

Note that $z^{\alpha\beta}$ is zero for $t-r \leqslant -T$. Contractions with k^α are

$$k_\beta F^{\alpha\beta}_1 = 2Mk^\alpha \tag{2.8a}$$

and

$$k_\alpha k_\beta F^{\alpha\beta}_1 = 0. \tag{2.8b}$$

2.2. *A quadratic radiative field*

With $h_{\mathrm{rad}\,1}$ in hand, the next step is to consider $N_{\mathrm{rad}\,2} = N_2(h_{\mathrm{rad}\,1})$, namely the quadratic source of Einstein's equations (1.5) computed with $h_{\mathrm{rad}\,1}$. The exact expression of $N_2(h)$ is:

$$\begin{aligned} N^{\alpha\beta}_2(h) = &-\partial_{\mu\nu}(h^{\alpha\beta}h^{\mu\nu}) + 2h^{\mu(\alpha}\partial_{\mu\nu}h^{\beta)\nu} + \partial_\mu h^{\alpha\beta}\partial_\nu h^{\mu\nu} \\ &-\tfrac{1}{4}\partial^\alpha h^\mu_\mu \partial^\beta h^\nu_\nu + \tfrac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - 2\partial^{(\alpha} h_{\mu\nu}\partial^\mu h^{\beta)\nu} + \partial_\nu h^{\alpha\mu}\partial^\nu h^\beta_\mu \\ &+ \partial_\nu h^{\alpha\mu}\partial_\mu h^{\beta\nu} + f^{\alpha\beta}[\tfrac{1}{8}\partial_\mu h^\nu_\nu \partial^\mu h^\rho_\rho - \tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} + \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho}], \end{aligned} \tag{2.9}$$

with $h_{\mu\nu} = f_{\mu\rho}h^\rho_\nu = f_{\mu\rho}f_{\nu\sigma}h^{\rho\sigma}$ and $T^{(\alpha\beta)} = \frac{1}{2}(T^{\alpha\beta} + T^{\beta\alpha})$. Then we choose, following the method of paper I, a particular solution ${}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2}$ that is zero in the past of the d'Alembertian equation $\Box p^{\alpha\beta} = {}_{\mathrm{D}}N^{\alpha\beta}_{\mathrm{rad}\,2}$ (where ${}_{\mathrm{D}}N_{\mathrm{rad}\,2}$ is the dynamic part of $N_{\mathrm{rad}\,2}$) by using the operator $\mathrm{FP}\Box^{-1}_{\mathrm{R}}$ ('finite part' of the retarded integral), which has been defined in paper I (definition 3.3) by means of analytic continuation:

$$ {}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2} = \mathrm{FP}\Box^{-1}_{\mathrm{R}}\, {}_{\mathrm{D}}N^{\alpha\beta}_{\mathrm{rad}\,2}. \tag{2.10} $$

It is possible to do this because we check that ${}_{\mathrm{D}}N_{\mathrm{rad}\,2}(\boldsymbol{x}, t)$ belongs to the class of functions L^0 (paper I, definition 3.2) and thus, by paper I, theorem 3.1, we have $\Box\, {}_{\mathrm{D}}p_{\mathrm{rad}\,2} = {}_{\mathrm{D}}N_{\mathrm{rad}\,2}$ and ${}_{\mathrm{D}}p_{\mathrm{rad}\,2} \in L^1$. Now by exactly the same reasoning as used in proceeding from (4.10) to (4.13) in paper I, we can associate to ${}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2}$ a tensor ${}_{\mathrm{D}}q^{\alpha\beta}_{\mathrm{rad}\,2}$ that is zero in the past and satisfies $\Box\, {}_{\mathrm{D}}q^{\alpha\beta}_{\mathrm{rad}\,2} = 0$ (hence ${}_{\mathrm{D}}q_{\mathrm{rad}\,2}$ belongs to L^0 and $\mathscr{L}^0$), and furthermore is such that

$$ \partial_\beta\, {}_{\mathrm{D}}q^{\alpha\beta}_{\mathrm{rad}\,2} = -\partial_\beta\, {}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2}. \tag{2.11} $$

Then the sum ${}_{\mathrm{D}}p_{\mathrm{rad}\,2} + {}_{\mathrm{D}}q_{\mathrm{rad}\,2} + {}_{\mathrm{S}}h_{\mathrm{rad}\,2}$, where the stationary ${}_{\mathrm{S}}h_{\mathrm{rad}\,2}$ is given in Appendix A, is a particular divergence-free solution of Einstein's quadratic equations (1.5) with source $N^{\alpha\beta}_{\mathrm{rad}\,2}$. However, we will see just below that this particular quadratic solution, although coming from a 'good' radiative linearized $h_{\mathrm{rad}\,1}$, does not admit an asymptotic expansion in powers of $1/r$ only along the cones $t-r = \text{const.}$ (i.e. the dynamic part does not belong to $\mathscr{L}^0$).

Let us recall a result of paper I (lemma 7.4) according to which the (finite part of the) retarded integral of a source of the type $r^{-j}F(t-r, \boldsymbol{n})$, where F is zero in the past, belongs, for $j \geqslant 3$, to $\mathscr{L}^0$, and belongs, for $j = 2$, to $\mathscr{L}^1$ (i.e. admits an expansion that contains either zero or one power of $\lg r$; see definition 7.2 of paper I). The source $N_{\mathrm{rad}\,2}$, which is precisely a finite sum of terms $r^{-j}A_j(t-r, \boldsymbol{n})$ with $j \geqslant 2$, will therefore generate far-zone logarithms in ${}_{\mathrm{D}}p_{\mathrm{rad}\,2}$ if A_2 (which is zero in the past) is not always zero. It turns out that A_2, which can be computed by replacing the 'r^{-1}-part' of $h_{\mathrm{rad}\,1}$ in (2.9), is not zero (hence ${}_{\mathrm{D}}p_{\mathrm{rad}\,2} \in \mathscr{L}^1$) but has the particular form of the stress-energy tensor for a 'swarm of gravitons'. We find

$$ A_2^{\alpha\beta} = k^\alpha k^\beta \left(\tfrac{1}{2}\, {}^{(1)}z^{\mu\nu}\, {}^{(1)}z_{\mu\nu} - \tfrac{1}{4}\, {}^{(1)}z^{\mu}_{\mu}\, {}^{(1)}z^{\nu}_{\nu}\right), \tag{2.12} $$

where ${}^{(1)}z_{\mu\nu}$ is the time-derivative of $z_{\mu\nu} = f_{\mu\rho} z^\rho_\nu = f_{\mu\rho} f_{\nu\sigma} z^{\rho\sigma}$. It will be convenient to introduce the following integral $\mathscr{E}(t-r, \boldsymbol{n})$

$$ \mathscr{E}(t-r, \boldsymbol{n}) = \int_{-\infty}^{t-r} \mathrm{d}u \left(\tfrac{1}{8}\, {}^{(1)}z^{\mu\nu}\, {}^{(1)}z_{\mu\nu} - \tfrac{1}{16}\, {}^{(1)}z^{\mu}_{\mu}\, {}^{(1)}z^{\nu}_{\nu}\right)(u, \boldsymbol{n}), \tag{2.13a} $$

the interpretation of which will be given in §3. Note that $\mathscr{E}(t-r, \boldsymbol{n})$ is zero for $t-r \leqslant -T$. With this notation we have

$$ A_2^{\alpha\beta}(t-r, \boldsymbol{n}) = 4k^\alpha k^\beta\, {}^{(1)}\mathscr{E}(t-r, \boldsymbol{n}). \tag{2.13b} $$

Now the point is that, because of the particular form of $A_2^{\alpha\beta}$, we can find a vector λ^α such that the 'gauge term' $\partial\lambda^{\alpha\beta} = \partial^\alpha\lambda^\beta + \partial^\beta\lambda^\alpha - f^{\alpha\beta}\partial_\mu\lambda^\mu$, when added to the $\mathscr{L}^1$-expansion of the retarded integral of $r^{-2}A_2^{\alpha\beta}$, exactly cancels the logarithms in this expansion so that the result belongs to $\mathscr{L}^0$. We prove the following lemma in Appendix B.

Lemma 2.1. *Let λ^α be given by*

$$ \lambda^\alpha = \mathrm{FP}\Box^{-1}_{\mathrm{R}}[2r^{-2}k^\alpha\, \mathscr{E}(t-r, \boldsymbol{n})]. \tag{2.14a} $$

Then we have

$$\mathrm{FP}\Box_{\mathrm{R}}^{-1}[4r^{-2}k^\alpha k^{\beta(1)}\mathscr{E}(t-r,\boldsymbol{n})]+\partial\lambda^{\alpha\beta}\in\mathscr{L}^0. \tag{2.14b}$$

Therefore, because $r^{-2}A_2^{\alpha\beta}=4r^{-2}k^\alpha k^{\beta(1)}\mathscr{E}$ alone produces logarithms in ${}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2}$, we find

$${}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2}+\partial\lambda^{\alpha\beta}\in\mathscr{L}^0, \tag{2.15}$$

so the tensor $u^{\alpha\beta}_{\mathrm{rad}\,2}$ defined by

$$u^{\alpha\beta}_{\mathrm{rad}\,2}:={}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2}+{}_{\mathrm{D}}q^{\alpha\beta}_{\mathrm{rad}\,2}+{}_{\mathrm{S}}h^{\alpha\beta}_{\mathrm{rad}\,2}+\partial\lambda^{\alpha\beta} \tag{2.16}$$

is a solution of Einstein's quadratic equations (1.5*a*, *b*) (with $n=2$ and non-zero divergence*:* $H_2^\alpha=\Box\lambda^\alpha\neq 0$) that admits an asymptotic expansion in powers of $1/r$ ('beginning' at r^{-1}) along the cones $t-r=$ const. (indeed ${}_{\mathrm{D}}u_{\mathrm{rad}\,2}\in\mathscr{L}^0$ by (2.15) and ${}_{\mathrm{S}}h_{\mathrm{rad}\,2}$ has the structure (A 3)).

It would be possible to define the quadratic $h_{\mathrm{rad}\,2}$ to be simply $u_{\mathrm{rad}\,2}$, and then to continue the algorithm exactly in the same manner (cancelling at each step $n\geqslant 3$ the 'new' logarithms that appear by some $\partial\lambda_n^{\alpha\beta}$); see Blanchet (1986) for this way of proceeding. But it is convenient, and physically meaningful, to define a $h_{\mathrm{rad}\,2}$ the expansion of which at infinity 'begins' at r^{-2} instead of r^{-1} for $u_{\mathrm{rad}\,2}$. Then we will see that no new logarithms appear at the cubic step, and, continuing in this manner, at higher steps.

Let $-r^{-1}Z^{\alpha\beta}(t-r,\boldsymbol{n})$, where the minus sign is chosen for later convenience, be the (dominant) 'r^{-1}-term' at infinity in $u^{\alpha\beta}_{\mathrm{rad}\,2}$

$$u^{\alpha\beta}_{\mathrm{rad}\,2}=-r^{-1}Z^{\alpha\beta}(t-r,\boldsymbol{n})+O(r^{-2}). \tag{2.17}$$

Because the stationary ${}_{\mathrm{S}}h_{\mathrm{rad}\,2}$ is at least of order $O(r^{-2})$ (by (A 3)), this $-r^{-1}Z^{\alpha\beta}$ is equal to the r^{-1}-term in the $\mathscr{L}^0$-expansion of ${}_{\mathrm{D}}u^{\alpha\beta}_{\mathrm{rad}\,2}$. Thus $Z^{\alpha\beta}$ is zero in the past. Now consider the divergence $\partial_\beta u^{\alpha\beta}_{\mathrm{rad}\,2}$ of $u^{\alpha\beta}_{\mathrm{rad}\,2}$; on one hand we have, at infinity,

$$\partial_\beta u^{\alpha\beta}_{\mathrm{rad}\,2}=r^{-1}k_\beta(\partial/\partial t)Z^{\alpha\beta}(t-r,\boldsymbol{n})+O(r^{-2}) \tag{2.18a}$$

and, on the other hand (from (2.16)),

$$\partial_\beta u^{\alpha\beta}_{\mathrm{rad}\,2}=\Box\lambda^\alpha=2r^{-2}k^\alpha\mathscr{E}(t-r,\boldsymbol{n}). \tag{2.18b}$$

From (2.18*a*, *b*) we conclude that $k_\beta Z^{\alpha\beta}(t-r,\boldsymbol{n})$ must be a constant in time. But $Z^{\alpha\beta}(t-r,\boldsymbol{n})$ is zero in the past, therefore the constant is necessarily zero and we have

$$k_\beta Z^{\alpha\beta}(t-r,\boldsymbol{n})=0. \tag{2.19}$$

With this property we can prove (in Appendix B) the following lemma (adapted from §2 of paper I).

Lemma 2.2. *For* $Z^{\alpha\beta}$ *zero in the past and satisfying* (2.19), *there exist a set of multipole moments* $\mathscr{M}_1$ *and a d'Alembertian-free vector* Φ_1^α ($\Box\Phi_1^\alpha=0$), *with* $\mathscr{M}_1$ *and* Φ_1^α *both zero in the past, such that, for* $r\to\infty$ *and* $t-r=$ *const.*,

$$h^{\alpha\beta}_{\mathrm{rad}\,1}[\mathscr{M}_1]+\partial\Phi_1^{\alpha\beta}=r^{-1}Z^{\alpha\beta}+O(r^{-2}). \tag{2.20}$$

In other words, $r^{-1}Z^{\alpha\beta}$ can be regarded as the r^{-1}-term in $h^{\alpha\beta}_{\mathrm{rad}\,1}[\mathscr{M}_1]+\partial\Phi_1^{\alpha\beta}$. Note that because $\mathscr{M}_1$ is zero in the past we have $h_{\mathrm{rad}\,1}[\mathscr{M}_1]=h_{\mathrm{can}\,1}[\mathscr{M}_1]$. So, if we add to $u^{\alpha\beta}_{\mathrm{rad}\,2}$ the linearized (harmonic) $h^{\alpha\beta}_{\mathrm{rad}\,1}[\mathscr{M}_1]+\partial\Phi_1^{\alpha\beta}$, which is a solution of $\Box h^{\alpha\beta}=\partial_\beta h^{\alpha\beta}=0$, that is a solution of the 'homogeneous' problem, we obtain a solution of Einstein's quadratic equations that admits, like $u^{\alpha\beta}_{\text{rad}\,2}$, a radiative expansion at infinity in powers of $1/r$ and furthermore this expansion 'begins' at r^{-2}. This is exactly what we wanted so we pose

$$h^{\alpha\beta}_{\text{rad}\,2}[\mathscr{M}] := {}_{\text{D}}p^{\alpha\beta}_{\text{rad}\,2} + {}_{\text{D}}q^{\alpha\beta}_{\text{rad}\,2} + {}_{\text{S}}h^{\alpha\beta}_{\text{rad}\,2} + h^{\alpha\beta}_{\text{rad}\,1}[\mathscr{M}_1] + \partial\Phi^{\alpha\beta}_1 + \partial\lambda^{\alpha\beta}, \qquad (2.21)$$

and we are assured that ${}_{\text{D}}h_{\text{rad}\,2} \in \mathscr{L}^0$ and $h_{\text{rad}\,2} = O(r^{-2})$ (in the sense of the $\mathscr{L}^0$-expansions and of the known structure (A 3) of ${}_{\text{S}}h_{\text{rad}\,2}$).

2.3. *Post-quadratic radiative fields*

The following steps of the algorithm are straightforward. Given $n \geqslant 3$ we recursively assume that some $h_{\text{rad}\,m}$ for $2 \leqslant m \leqslant n-1$ have been constructed from $h_{\text{rad}\,1}$ to satisfy Einstein's equations and such that ${}_{\text{D}}h_{\text{rad}\,m}$ belongs both to L^{m-1} and to $\mathscr{L}^0$: ${}_{\text{D}}h_{\text{rad}\,m} \in L^{m-1} \cap \mathscr{L}^0$, with a radiative expansion at infinity beginning at r^{-2}: $h_{\text{rad}\,m} = O(r^{-2})$. Then we compute the source N_n with the $h_{\text{rad}\,m}$s and find, by lemmas 3.4 and 7.3 of paper I, that $N_{\text{rad}\,n} \in L^{n-2} \cap \mathscr{L}^0$. Furthermore, because $h_{\text{rad}\,m} = O(r^{-2})$ at infinity for $m \geqslant 2$ and $N_{\text{rad}\,n}$ (with $n \geqslant 3$) is at least a quadratic product of $h_{\text{rad}\,1}$ and $h_{\text{rad}\,m}$ ($m \geqslant 2$) or a cubic product of three $h_{\text{rad}\,1}$, we find that $N_{\text{rad}\,n} = O(r^{-3})$ at infinity: terms of order r^{-2}, which produce logarithms, are absent. Therefore, by theorem 7.1 and lemma 7.4 of paper I, the retarded integral of ${}_{\text{D}}N_{\text{rad}\,n}$ is a $\mathscr{L}^0$-expansion; we have

$${}_{\text{D}}p^{\alpha\beta}_{\text{rad}\,n} = \text{FP}\Box^{-1}_{\text{R}}\,{}_{\text{D}}N^{\alpha\beta}_{\text{rad}\,n} \in \mathscr{L}^0 \quad \text{for} \quad n \geqslant 3, \qquad (2.22)$$

(and also ${}_{\text{D}}p_{\text{rad}\,n} \in L^{n-1}$) in contrast with ${}_{\text{D}}p_{\text{rad}\,2} \in \mathscr{L}^1$. As announced, no new logarithms appear at the post-quadratic steps. Thus the desired $h_{\text{rad}\,n}$ is obtained by adding to ${}_{\text{D}}p_{\text{rad}\,n}$ its associated ${}_{\text{D}}q_{\text{rad}\,n}$ (so Einstein's equations are satisfied), the stationary ${}_{\text{S}}h_{\text{rad}\,n}$ given in Appendix A, and the convenient homogeneous solution $h_{\text{rad}\,1}[\mathscr{M}_{n-1}] + \partial\Phi_{n-1}$ that cancels, by lemma 2.2, the r^{-1}-part of ${}_{\text{D}}p_{\text{rad}\,n} + {}_{\text{D}}q_{\text{rad}\,n} + {}_{\text{S}}h_{\text{rad}\,n}$,

$$h^{\alpha\beta}_{\text{rad}\,n}[\mathscr{M}] := {}_{\text{D}}p^{\alpha\beta}_{\text{rad}\,n} + {}_{\text{D}}q^{\alpha\beta}_{\text{rad}\,n} + {}_{\text{S}}h^{\alpha\beta}_{\text{rad}\,n} + h^{\alpha\beta}_{\text{rad}\,1}[\mathscr{M}_{n-1}] + \partial\Phi^{\alpha\beta}_{n-1} \quad \text{for} \quad n \geqslant 3. \qquad (2.23)$$

This completes the construction of an MPM radiative field. We thus state the following.

PROPOSITION 2.1. *Given the linearized metric $h_{\text{rad}\,1}[\mathscr{M}]$ of (2.4), there exists an algorithm that constructs a formal metric solution of (1.1)–(1.4), $\mathscr{g}_{\text{rad}}[\mathscr{M}] = f + \Sigma_{n \geqslant 1} G^n h_{\text{rad}\,n}[\mathscr{M}]$ (the $h_{\text{rad}\,n}$s being given by (2.4), (2.21) and (2.23)), solving Einstein's vacuum equations (in D), and such that each $h_{\text{rad}\,n}[\mathscr{M}]$ admits the following asymptotic expansion ($\forall N \geqslant N_0$, where N_0 is chosen large enough so that R^L_{nN} is zero in the past):*

$$h_{\text{rad}\,n}(\boldsymbol{x}, t) = \sum_{l \geqslant 0} \hat{n}^L \left\{ \sum_{k=1}^{N} r^{-k} F^L_{nk}(t-r) + R^L_{nN}(r, t-r) \right\}, \qquad (2.24)$$

where the functions $F^L_{nk}(u)$ are $C^\infty(\mathbb{R})$ and constant when $u \leqslant -T$, and where the $R^L_{nN}(r, u)$ are $O^\infty(r^{-N})$.

This 'good' expansion at infinity is to be contrasted with the 'bad' logarithmic expansion of the h_ns in harmonic coordinates (paper I, equation (7.13), or (1.7) above).

3. Properties of the radiative field

3.1. *Asymptotic simplicity*

Let us prove, as a first step in the proof of theorem 1.1, that the MPM radiative field $\mathscr{g}^{\alpha\beta}_{\text{rad}} = f^{\alpha\beta} + \Sigma^{\infty}_{n=1} G^n h^{\alpha\beta}_{\text{rad}\,n}$, rewritten in covariant form $g_{\text{rad}\,\alpha\beta} = f_{\alpha\beta} + \Sigma^{\infty}_{n=1} G^n g_{\text{rad}\,n\alpha\beta}$, yields a formal space–time that is asymptotically simple in the following sense.

Proposition 3.1. *For any* $p \geqslant 1$, *the space–time* $(M, g_{\text{rad}\,[p]\alpha\beta})$, *where* $g_{\text{rad}\,[p]\alpha\beta} = f_{\alpha\beta} + \Sigma^{p}_{n=1} G^n g_{\text{rad}\,n\alpha\beta}$, *satisfies definition* 1.1 *for asymptotic simplicity.*

Proof. Let us deal first with $\mathscr{I}^+$. We introduce a conformal factor $\Omega = r^{-1}$, a retarded time $u = t - r$ and, for any $p \geqslant 1$, the conformal metric $\mathrm{d}\tilde{s}^2_{[p]} = \Omega^2 g_{\text{rad}\,[p]\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$. In the coordinates $\tilde{x}^\mu = (\Omega, u, \theta, \phi)$ we have $\mathrm{d}\tilde{s}^2_{[p]} = \tilde{g}_{\text{rad}\,[p]\mu\nu}\,\mathrm{d}\tilde{x}^\mu\,\mathrm{d}\tilde{x}^\nu$. Then it can be checked, thanks to the asymptotic expansions (2.24) of the $h_{\text{rad}\,n}$s (and by using the properties of the $\mathscr{L}^0$ class) that each $\tilde{g}_{\text{rad}\,[p]\mu\nu}$ admits the following expansion ($\forall N \geqslant N_0$)

$$\tilde{g}_{\text{rad}\,[p]}(\Omega, u, \theta, \phi) = \sum_{l \geqslant 0} \hat{n}^L(\theta, \phi) \left\{ \sum_{k=0}^{N+1} \Omega^k G^L_{pk}(u) + S^L_{pN+1}(\Omega, u) \right\}, \tag{3.1}$$

where the functions $G^L_{pk}(u)$ are $C^\infty(\mathbb{R})$ and constant when $u \leqslant -T$, and the $S^L_{pN+1}(\Omega, u)$ are of the type (skipping the letters L and p)

$$S_{N+1}(\Omega, u) = R_{N+1}(\Omega^{-1}, u), \tag{3.2}$$

where $R_{N+1}(r, u)$ is a $O^\infty(r^{-N-1})$-function. The only problem we find in proving (3.1) is caused by the coefficient of $\mathrm{d}\Omega^2 : \tilde{g}_{\text{rad}\,[p]\Omega\Omega} = \Sigma^{p}_{n=1} \Omega^{-2} k^\alpha k^\beta G^n g_{\text{rad}\,n\alpha\beta}$, which is not apparently of 'order $O(\Omega^0)$'. But, in fact, it is because of (2.8*b*) and our choice $h_{\text{rad}\,n} = O(r^{-2})$ for $n \geqslant 2$. (Note, however, that in the other radiative algorithm we mentioned in §2, in which $h_{\text{rad}\,2} = u_{\text{rad}\,2}$ given by (2.16) and where we subtract at each step $n \geqslant 3$ the 'new' logarithms, we would have as well find $\tilde{g}_{\text{rad}\,[p]\Omega\Omega} = O(\Omega^0)$, so the proof of the proposition 3.1 would be exactly the same.)

The first and main task is to prove that the functions $(\Omega, u, \theta, \phi) \rightarrow \tilde{g}_{\text{rad}\,[p]}(\Omega, u, \theta, \phi)$, which are initially defined in $]0, +\infty[\times \mathbb{R}^3$, can be extended for $\Omega = 0$ to functions belonging to $C^\infty([0, +\infty[\times \mathbb{R}^3)$, i.e. belonging to $C^N([0, +\infty[\times \mathbb{R}^3)$ for any $N \in \mathbb{N}$, where $[0, +\infty[\times \mathbb{R}^3$ is endowed with the induced topology of $\mathbb{R}^4$. For doing this we will prove the equivalent requirement (see, for example, Schwartz 1966, p. 313) that, $\forall N \in \mathbb{N}$ (or rather $\forall N \geqslant N_0$), we can find extensions (which will depend on N) of these functions for all values of $\Omega \in \mathbb{R}$ belonging to $C^N(\mathbb{R}^4)$. Since each function $(\Omega, u, \theta, \phi) \rightarrow \hat{n}^L(\theta, \phi)\, \Omega^k G^L_{pk}(u)$ in (3.1), considered as a function on $\mathbb{R}^4$, belongs to $C^\infty(\mathbb{R}^4)$ it is sufficient to prove that, $\forall N \geqslant N_0$, we can extend $S_{N+1}(\Omega, u)$ to a function on $\mathbb{R}^2$ belonging to $C^N(\mathbb{R}^2)$. We pose, for $\Omega \neq 0$,

$$\mathscr{S}_{N+1}(\Omega, u) := S_{N+1}(|\Omega|, u) = R_{N+1}(|\Omega|^{-1}, u) \tag{3.3}$$

(say). First we have $\mathscr{S}_{N+1}(\Omega, u) \in C^\infty(\mathbb{R}^* \times \mathbb{R})$ where $\mathbb{R}^* = \mathbb{R} - \{0\}$. Indeed we know that the function $(\boldsymbol{x}, t) \rightarrow \hat{n}^L R_{N+1}(r, u)$ belongs to L^{p-1} (because each ${}_{\mathrm{D}}h_{\text{rad}\,n} \in L^{n-1}$) therefore we have $\hat{n}^L R_{N+1}(r, u) \in C^\infty\,[(\mathbb{R}^3 - \{\mathbf{0}\}) \times \mathbb{R}]$ (as a function of the variables $\boldsymbol{x}$ and t), which implies $\mathscr{S}_{N+1}(\Omega, u) \in C^\infty(\mathbb{R}^* \times \mathbb{R})$. Second, to prove that in fact

$\mathscr{S}_{N+1}(\Omega, u) \in C^N(\mathbb{R}^2)$, we use lemma E1 of paper I (Appendix E), which requires that the derivatives with respect to Ω of $\mathscr{S}_{N+1}$ be uniformly (in u) bounded by adequate powers of $|\Omega|$. Let $m \leqslant N$ and $\Omega \neq 0$, then we have (from the properties of the $O^\infty(r^{-N})$-functions, lemma 7.1 of paper I), $\forall k \geqslant 0$,

$$(\mathrm{d}/\mathrm{d}\Omega)^m (\mathrm{d}/\mathrm{d}u)^k \mathscr{S}_{N+1}(\Omega, u) = R_{N+1-m}(|\Omega|^{-1}, u), \tag{3.4}$$

where R_{N+1-m} is a $O^\infty(r^{-N-1+m})$-function. Thus, by definition of the $O^\infty(r^{-N})$-functions we find $\forall (u_0, u_1) \in \mathbb{R}^2$, there exist $M > 0$ and $A > 0$ such that $[u_0 \leqslant u \leqslant u_1$ and $0 < |\Omega| < A^{-1}]$ imply

$$|(\mathrm{d}/\mathrm{d}\Omega)^m (\mathrm{d}/\mathrm{d}u)^k \mathscr{S}_{N+1}(\Omega, u)| \leqslant M|\Omega|^{N+1-m}. \tag{3.5}$$

So the hypotheses of lemma E1 are satisfied (with $K = N$) and therefore $S_{N+1}(\Omega, u)$ can be extended to $\mathscr{S}_{N+1}(\Omega, u) \in C^N(\mathbb{R}^2)$.

The other requirements for asymptotic simplicity at $\mathscr{I}^+$ are easy to prove. Indeed we have $\tilde{\nabla}_\alpha \Omega = (1, 0, 0, 0) \neq 0$, $\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Omega = 0$ and $[\tilde{\nabla}^\alpha \Omega \tilde{\nabla}_\alpha \Omega]_{|\Omega=0}$ equals $[\Omega^2(1+\Sigma_{n=1}^p G^n n_i n_j g^{ij}_{\mathrm{rad}\,n})]_{|\Omega=0} = 0$. And, at $\Omega = 0$, $\mathrm{d}\tilde{s}^2_{[p]}$ is equal to $0 \cdot \mathrm{d}u^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ so that $\mathscr{I}^+$ is topologically $S^2 \times \mathbb{R}$. The completeness is a consequence of our assumption according to which the construction is valid in the whole of the 'exterior' region $D = \{(\boldsymbol{x}, t), r > r_0 \geqslant 0\}$.

Finally, the same reasoning applies to $\mathscr{I}^-$ with the simplification that, because the metric is stationary in the past (and has the structure (A 3)), it is an easy matter to prove that the conformal metric in coordinates $(\Omega, t+r, \theta, \phi)$ is C^∞. ■

3.2. *Interpretation of $\mathscr{M}$ and $\mathscr{E}$*

We found it convenient to define $h_{\mathrm{rad}\,n}$ for $n \geqslant 2$ in such a way that its expansion at infinity 'begins' at the order r^{-2}. Thus the dominant, or radiative, part at infinity (the r^{-1}-part) of the full non-linear $g_{\mathrm{rad}} = f + \Sigma_{n \geqslant 1} G^n h_{\mathrm{rad}\,n}$ comes exclusively from the linearized $h_{\mathrm{rad}\,1}$. In other words, only the linearized $h_{\mathrm{rad}\,1}$ 'radiates' at infinity and the nonlinear $h_{\mathrm{rad}\,n}$s for $n \geqslant 2$ 'do not radiate'. This means that the functions $\mathscr{M} = \{M_L(t), S_L(t)\}$ that serve as unknown parameters in g_{rad} are in fact exactly the *radiative*, or far-zone, multipole moments in the sense of Thorne (1980, p. 331),

$$M_L(t) = M_L^{\mathrm{rad}}(t);\ S_L(t) = S_L^{\mathrm{rad}}(t). \tag{3.6}$$

We expect these moments to be measured on a detector far from the system. In other algorithms, such as the harmonic algorithms in paper I or the second radiative algorithm mentioned in §2 (in which $h_{\mathrm{rad}\,2} = u_{\mathrm{rad}\,2}$), the functions $M_L(t)$ and $S_L(t)$ do not have this interpretation.

Finally, let us compute the energy–momentum vector $P^\alpha(u)$, at $\mathscr{I}^+$, associated with g_{rad} by the following method (see, for example, Streubel 1978; Porrill & Stewart 1979): we integrate by using Gauss's theorem the usual Landau–Lifschitz (1971) energy–momentum pseudo-tensor over a sphere $r = $ const., $t = $ const. and then take the limit $r \to \infty$ with $t - r = $ const. The result is

$$P^\alpha(u) = \begin{pmatrix} GM \\ \mathbf{0} \end{pmatrix} - G^2 \int \frac{\mathrm{d}\Omega}{4\pi}\, k^\alpha \mathscr{E}(u, \boldsymbol{n}), \tag{3.7}$$

(with $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$). Thus we can interpret $G\mathscr{E}(u, \boldsymbol{n})$ as being 4π times the energy per steradian carried off by the radiation field before the time u and in the direction $\boldsymbol{n}$. Note that $\mathscr{E}$ is positive as easily deduced from the algebraic properties of $z^{\alpha\beta}$ (equations (2.7)),

$$\mathscr{E}(u, \boldsymbol{n}) = \int_{-\infty}^{u} du' \, [\tfrac{1}{16}(p_i p_j \, {}^{(1)}z_{ij} - q_i q_j \, {}^{(1)}z_{ij})^2 + \tfrac{1}{4}(p_i q_j \, {}^{(1)}z_{ij})^2], \tag{3.8}$$

where $p_i = \partial n_i/\partial\theta$ and $q_i \sin\theta = \partial n_i/\partial\phi$. It can be recovered by integration of $k^\alpha \mathscr{E}$ over angles equations (4.16′) and (4.20′) of Thorne (1980) giving the energy and linear momentum carried off at infinity by the waves as infinite formal multipolar series (the dominant energy contribution agreeing with the usual Einstein quadrupole formula).

4. Coordinate transformation between the canonical and the radiative metrics

Up to now we have proven asymptotic simplicity only for a particular MPM metric, namely the radiative metric $g_{\text{rad}}[\mathscr{M}]$ constructed in §2. To prove theorem 1.1, we need to show that the general metric g_{gen} satisfying our assumptions ((1.1)–(1.5) and existence of harmonic coordinates) can be connected to some $g_{\text{rad}}[\mathscr{M}]$ by a coordinate transformation.

In paper I (theorem 4.5) this has been shown for a particular 'canonical' MPM metric $g_{\text{can}}[\mathscr{M}]$ in harmonic coordinates whose linearized approximation is $h_{\text{can}\,1}[\mathscr{M}]$ ((2.2)); given any g_{gen}, there exist $\mathscr{M}$ and a coordinate transformation T such that

$$T g_{\text{gen}} = g_{\text{can}}[\mathscr{M}]. \tag{4.1}$$

We will prove in this section that we can also connect g_{can} and g_{rad}; given $\mathscr{M}$, there exist $\mathscr{M}'$ and a coordinate transformation T' such that

$$T' g_{\text{can}}[\mathscr{M}] = g_{\text{rad}}[\mathscr{M}']. \tag{4.2}$$

Equation (4.2), combined with (4.1), is the needed result that completes the proof of theorem 1.1. Note that in (4.2) the coordinate transformation must be done conjointly with a change $\mathscr{M} \to \mathscr{M}'$ of multipole moments (T. Damour, personal communication, 1984).

So let $\mathscr{M}$ be given. We first consider the coordinate transformation

$$x'^\alpha = x^\alpha + G\xi^\alpha, \tag{4.3}$$

where ξ^α is the vector defined by (2.4*b*): $\xi^\alpha = 2Mf^{0\alpha} \lg r$ in which M is the mass monopole associated with $\mathscr{M}$. This coordinate transformation transforms the harmonic $g_{\text{can}}[\mathscr{M}]$ into a non-harmonic $g'_{\text{can}}[\mathscr{M}]$ whose linearized approximation is equal to $h_{\text{rad}\,1}[\mathscr{M}]$ (because $h_{\text{rad}\,1}$ given by (2.4*a*) differs from $h_{\text{can}\,1}$ by the 'gauge' $\partial\xi^{\alpha\beta}$):

$$g'_{\text{can}}[\mathscr{M}] = f + G h_{\text{rad}\,1}[\mathscr{M}] + O(G^2). \tag{4.4}$$

Consider now the quadratic approximation of g'_{can}, namely $h'_{\text{can}\,2}$. This $h'_{\text{can}\,2}$, whose divergence is $h^{\mu\nu}_{\text{can}\,1}\partial_{\mu\nu}\xi^\alpha$, satisfies Einstein's quadratic equations (1.5) with source $N_{\text{rad}\,2} = N_2(h_{\text{rad}\,1})$:

$$\Box h'^{\alpha\beta}_{\text{can}\,2} = N^{\alpha\beta}_{\text{rad}\,2} + \partial H^{\alpha\beta}_{\text{can}\,2}, \tag{4.5a}$$

$$\partial_\beta h'^{\alpha\beta}_{\text{can}\,2} = : H^{\alpha}_{\text{can}\,2} = h^{\mu\nu}_{\text{can}\,1}\partial_{\mu\nu}\xi^\alpha, \tag{4.5b}$$

(with $\partial H^{\alpha\beta} = \partial^\alpha H^\beta + \partial^\beta H^\alpha - f^{\alpha\beta}\partial_\mu H^\mu$). To relate the unknown $h'_{\mathrm{can}\,2}$ to $h_{\mathrm{rad}\,2}$, we define, by means of the operator $\mathrm{FP}\Box_{\mathrm{R}}^{-1}$, the vector

$$\omega^\alpha := \mathrm{FP}\Box_{\mathrm{R}}^{-1}(h^{\mu\nu}_{\mathrm{can}\,1}\,\partial_{\mu\nu}\xi^\alpha). \tag{4.6}$$

This is possible because $h^{\mu\nu}_{\mathrm{can}\,1}\,\partial_{\mu\nu}\xi^\alpha$ is zero in the past (because ${}_{\mathrm{S}}h^{ij}_{\mathrm{can}\,1} = 0$) and belongs to L^0. Thus $\omega^\alpha \in L^1$. Then we easily check that the quantity ${}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2} + {}_{\mathrm{D}}q^{\alpha\beta}_{\mathrm{rad}\,2} + {}_{\mathrm{S}}h^{\alpha\beta}_{\mathrm{rad}\,2} + \partial\omega^{\alpha\beta}$ where ${}_{\mathrm{D}}p_{\mathrm{rad}\,2}$, ${}_{\mathrm{D}}q_{\mathrm{rad}\,2}$ and ${}_{\mathrm{S}}h_{\mathrm{rad}\,2}$ are pieces of the construction of $h_{\mathrm{rad}\,2}$ (2.21) is a solution of the same equations as those that are satisfied by $h'_{\mathrm{can}\,2}$ (4.5). Therefore the difference between this quantity and $h'_{\mathrm{can}\,2}$ is equal to a solution of the (harmonic) homogeneous problem to which we know the general solution, i.e. this difference is equal to a linearized harmonic metric $h^{\alpha\beta}_{\mathrm{can}\,1}[\mathscr{M}'_1] + \partial\Psi^{\alpha\beta}_1$, where $\mathscr{M}'_1$ is some set of multipole moments and Ψ^α_1 a d'Alembertian-free vector. Moreover, because the stationary ${}_{\mathrm{S}}h_{\mathrm{rad}\,2}$ and ${}_{\mathrm{S}}h'_{\mathrm{can}\,2}$ are both of order $O(r^{-2})$ when $r \to \infty$ (Appendix A), we find that the mass monopole M'_1 associated with $\mathscr{M}'_1$ is zero, so that we can write $h_{\mathrm{can}\,1}[\mathscr{M}'_1] = h_{\mathrm{rad}\,1}[\mathscr{M}'_1]$. Now we have

$${}_{\mathrm{D}}p^{\alpha\beta}_{\mathrm{rad}\,2} + {}_{\mathrm{D}}q^{\alpha\beta}_{\mathrm{rad}\,2} + {}_{\mathrm{S}}h^{\alpha\beta}_{\mathrm{rad}\,2} + \partial\omega^{\alpha\beta} - h'^{\alpha\beta}_{\mathrm{can}\,2} = h^{\alpha\beta}_{\mathrm{rad}\,1}[\mathscr{M}'_1] + \partial\Psi^{\alpha\beta}_1, \tag{4.7}$$

and thus, by using definition (2.21) of $h_{\mathrm{rad}\,2}$,

$$h'^{\alpha\beta}_{\mathrm{can}\,2} = h^{\alpha\beta}_{\mathrm{rad}\,2}[\mathscr{M}] - h^{\alpha\beta}_{\mathrm{rad}\,1}[\mathscr{M}_1 + \mathscr{M}'_1] - \partial V^{\alpha\beta}, \tag{4.8}$$

where V^α is the vector

$$V^\alpha = \lambda^\alpha - \omega^\alpha + \Phi^\alpha_1 + \Psi^\alpha_1. \tag{4.9}$$

Now we perform the coordinate transformation

$$x''^\alpha = x'^\alpha + G^2 V^\alpha. \tag{4.10}$$

Then $\mathscr{g}'_{\mathrm{can}}$ is transformed into $\mathscr{g}''_{\mathrm{can}}$ whose linearized and quadratic approximations are precisely $h_{\mathrm{rad}\,1}$ and $h_{\mathrm{rad}\,2}$ being computed not with the original $\mathscr{M}$ but with a 'corrected' $\mathscr{M} - G(\mathscr{M}_1 + \mathscr{M}'_1)$,

$$\mathscr{g}''_{\mathrm{can}}[\mathscr{M}] = f + Gh_{\mathrm{rad}\,1}[\mathscr{M} - G(\mathscr{M}_1 + \mathscr{M}'_1)] + G^2 h_{\mathrm{rad}\,2}[\mathscr{M} - G(\mathscr{M}_1 + \mathscr{M}'_1)] + O(G^3). \tag{4.11}$$

The same reasoning can easily be extended to any order G^n.† The final multipole moment $\mathscr{M}'$ is a complicated functional of $\mathscr{M}$: $\mathscr{M}' = \mathscr{M} - G(\mathscr{M}_1 + \mathscr{M}'_1) + \ldots$. ∎

Appendix A. The stationary metric

We wish to construct a stationary MPM metric $f + \Sigma_{n \geqslant 1} G^n\, {}_{\mathrm{S}}h_{\mathrm{rad}\,n}$ such that ${}_{\mathrm{S}}h_{\mathrm{rad}\,1}$ is the stationary part of $h_{\mathrm{rad}\,1}$ (2.4*a*)

$${}_{\mathrm{S}}h^{\alpha\beta}_{\mathrm{rad}\,1} = {}_{\mathrm{S}}h^{\alpha\beta}_{\mathrm{can}\,1} + \partial\xi^{\alpha\beta}. \tag{A 1}$$

A possible choice is simply to define this metric as the transformed of the stationary canonical metric of paper I by the coordinate transformation $x'^\alpha = x^\alpha + G\xi^\alpha$ (T. Damour, personal communication 1985). However, we prefer for practical

† For $n \geqslant 3$, it is necessary to study separately the stationary part of ω^α (4.6) in the manner of Appendix C, §C2, in paper I.

reasons to define the stationary ${}_{\rm S}h_{{\rm rad}\,n}$ for $n \geqslant 2$ 'similarly' to ${}_{\rm D}h_{{\rm rad}\,n}$ i.e. by using the operator FPΔ^{-1} (paper I, Appendix C) on the stationary source ${}_{\rm S}N_{{\rm rad}\,n}$,

$$ {}_{\rm S}h^{\alpha\beta}_{{\rm rad}\,n} := {\rm FP}\Delta^{-1}\,{}_{\rm S}N^{\alpha\beta}_{{\rm rad}\,n}. \tag{A 2}$$

By the reasoning of paper I (Appendix C), ${}_{\rm S}h_{{\rm rad}\,n}$ solves Einstein's stationary equations with zero divergence ($\Delta_{\rm S}h^{\alpha\beta}_{{\rm rad}\,n} = {}_{\rm S}N^{\alpha\beta}_{{\rm rad}\,n}$ and $\partial_\beta\,{}_{\rm S}h^{\alpha\beta}_{{\rm rad}\,n} = 0$) and has the structure

$$ {}_{\rm S}h^{\alpha\beta}_{{\rm rad}\,n} = \sum_{l\geqslant 0} \hat{n}^L \sum_{k \geqslant n} r^{-k}\,F^L_{nk}, \tag{A 3}$$

(where the F^L_{nk} are constant) provided that 'critical terms' in the quadratic ${}_{\rm S}N^{0i}_{{\rm rad}\,2}$, ${}_{\rm S}N^{ij}_{{\rm rad}\,2}$ and the cubic ${}_{\rm S}N^{ij}_{{\rm rad}\,3}$, which produce logarithms, are absent (as it is the case for ${}_{\rm S}N^{0i}_{{\rm can}\,2}$, ${}_{\rm S}N^{ij}_{{\rm can}\,2}$ and ${}_{\rm S}N^{ij}_{{\rm can}\,3}$ in paper I). We now prove that these 'critical terms' are absent. Indeed, under the coordinate transformation $x'^\alpha = x^\alpha + G\xi^\alpha$, ${}_{\rm S}h_{{\rm can}\,1}$ is transformed into ${}_{\rm S}h'_{{\rm can}\,1} = {}_{\rm S}h_{{\rm rad}\,1}$ (by (A 1)), ${}_{\rm S}h_{{\rm can}\,2}$ is transformed into

$$ {}_{\rm S}h'^{00}_{{\rm can}\,2} = {}_{\rm S}h^{00}_{{\rm can}\,2} + 4M^2r^{-2}, \tag{A 4a}$$

$$ {}_{\rm S}h'^{0i}_{{\rm can}\,2} = {}_{\rm S}h^{0i}_{{\rm can}\,2}, \tag{A 4b}$$

$$ {}_{\rm S}h'^{ij}_{{\rm can}\,2} = {}_{\rm S}h^{ij}_{{\rm can}\,2}, \tag{A 4c}$$

and ${}_{\rm S}h_{{\rm can}\,3}$ is transformed into

$$ {}_{\rm S}h'^{00}_{{\rm can}\,3} = {}_{\rm S}h^{00}_{{\rm can}\,3} + 2\,{}_{\rm S}h^{0i}_{{\rm can}\,2}\,\partial_i\,\xi^0, \tag{A 5a}$$

$$ {}_{\rm S}h'^{0i}_{{\rm can}\,3} = {}_{\rm S}h^{0i}_{{\rm can}\,3} + {}_{\rm S}h^{ij}_{{\rm can}\,2}\,\partial_j\,\xi^0, \tag{A 5b}$$

$$ {}_{\rm S}h'^{ij}_{{\rm can}\,3} = {}_{\rm S}h^{ij}_{{\rm can}\,3}. \tag{A 5c}$$

Now ${}_{\rm S}h'_{{\rm can}\,2}$ must satisfy Einstein's equations with source ${}_{\rm S}N_{{\rm rad}\,2}$ (because ${}_{\rm S}h'_{{\rm can}\,1} = {}_{\rm S}h_{{\rm rad}\,1}$) i.e. $\Delta_{\rm S}h'_{{\rm can}\,2} = {}_{\rm S}N_{{\rm rad}\,2}$. By using the values (A 4) for ${}_{\rm S}h'_{{\rm can}\,2}$ we find

$$ {}_{\rm S}N^{00}_{{\rm rad}\,2} = {}_{\rm S}N^{00}_{{\rm can}\,2} + \Delta(4M^2r^{-2}), \tag{A 6a}$$

$$ {}_{\rm S}N^{0i}_{{\rm rad}\,2} = {}_{\rm S}N^{0i}_{{\rm can}\,2}, \tag{A 6b}$$

$$ {}_{\rm S}N^{ij}_{{\rm rad}\,2} = {}_{\rm S}N^{ij}_{{\rm can}\,2}, \tag{A 6c}$$

so no 'critical terms' occur in ${}_{\rm S}N^{0i}_{{\rm rad}\,2}$ and ${}_{\rm S}N^{ij}_{{\rm rad}\,2}$ because they do not occur in ${}_{\rm S}N^{0i}_{{\rm can}\,2}$ and ${}_{\rm S}N^{ij}_{{\rm can}\,2}$. We thus pose ${}_{\rm S}h_{{\rm rad}\,2} = {\rm FP}\Delta^{-1}\,{}_{\rm S}N_{{\rm rad}\,2}$, and find ${}_{\rm S}h_{{\rm rad}\,2} = {}_{\rm S}h'_{{\rm can}\,2}$ (A 4). Now ${}_{\rm S}h'_{{\rm can}\,3}$ must satisfy Einstein's equations with source ${}_{\rm S}N_{{\rm rad}\,3}$ (because ${}_{\rm S}h'_{{\rm can}\,1} = {}_{\rm S}h_{{\rm rad}\,1}$ and ${}_{\rm S}h'_{{\rm can}\,2} = {}_{\rm S}h_{{\rm rad}\,2}$), i.e.

$$ \Delta_{\rm S}h'^{\alpha\beta}_{{\rm can}\,3} = {}_{\rm S}N^{\alpha\beta}_{{\rm rad}\,3} + \partial({}_{\rm S}h^{ij}_{{\rm can}\,2}\,\partial_{ij}\,\xi)^{\alpha\beta}. \tag{A 7}$$

From (A 5) we then find

$$ {}_{\rm S}N^{00}_{{\rm rad}\,3} = {}_{\rm S}N^{00}_{{\rm can}\,3} + \Delta(2\,{}_{\rm S}h^{0i}_{{\rm can}\,2}\,\partial_i\,\xi^0), \tag{A 8a}$$

$$ {}_{\rm S}N^{0i}_{{\rm rad}\,3} = {}_{\rm S}N^{0i}_{{\rm can}\,3} + \Delta({}_{\rm S}h^{ij}_{{\rm can}\,2}\,\partial_j\,\xi^0) - \partial^i({}_{\rm S}h^{jk}_{{\rm can}\,2}\,\partial_{jk}\,\xi^0), \tag{A 8b}$$

$$ {}_{\rm S}N^{ij}_{{\rm rad}\,3} = {}_{\rm S}N^{ij}_{{\rm can}\,3}. \tag{A 8c}$$

Therefore no 'critical' terms appear in ${}_{\rm S}N^{ij}_{{\rm rad}\,3}$. ■

Appendix B. Proofs of lemmas 2.1 and 2.2

B 1. *Proof of lemma* 2.1

Let us first recall the following result of paper I (equation (7.11)): for any function F that is zero in the past, we have

$$\mathrm{FP}\Box_{\mathrm{R}}^{-1}[k^{\alpha_1}k^{\alpha_2}\dots k^{\alpha_l}r^{-2}F(t-r)]+\tfrac{1}{2}(-)^l\,\lg r\,\partial^{\alpha_1}\partial^{\alpha_2}\dots\partial^{\alpha_l}[r^{-1\,(-l-1)}F(t-r)]\in\mathscr{L}^0, \tag{B 1}$$

where $k^\alpha=(1,n^i)$, $\partial^\alpha=(-\partial_0,\partial_i)$ and ${}^{(-l-1)}F$ is the $(l+1)$th anti-derivative of F that is zero, together with all its derivatives, in the past. Now consider the unique decomposition of $\mathscr{E}(t-r,\boldsymbol{n})$ into symmetric trace-free (STF) tensors $\mathscr{E}_L(t-r)$:

$$\mathscr{E}(t-r,\boldsymbol{n})=\sum_{l\geqslant 0}n_L\,\mathscr{E}_L(t-r). \tag{B 2}$$

Then, applying equation (B 1), we readily find for λ^α given by (2.14a)

$$\partial\lambda^{\alpha\beta}-2\,\lg r\sum_{l\geqslant 0}(-)^l\,\partial^\alpha\partial^\beta\partial_L[r^{-1\,(-l-2)}\mathscr{E}_L(t-r)]\in\mathscr{L}^0, \tag{B 3}$$

and also

$$\mathrm{FP}\Box_{\mathrm{R}}^{-1}[4r^{-2}k^\alpha k^{\beta\,(1)}\mathscr{E}(t-r,\boldsymbol{n})]+2\,\lg r\sum_{l\geqslant 0}(-)^l\,\partial^\alpha\partial^\beta\partial_L[r^{-1\,(-l-2)}\mathscr{E}_L(t-r)]\in\mathscr{L}^0. \tag{B 4}$$

The sum of (B 3) and (B 4) is exactly the equation to be proved (2.14b). ■

Note that the choice we have made for λ^α (2.14a) is not unique. For instance, another choice as valid as λ^α but less elegant would have been

$$\lambda'^\alpha=\lg r\sum_{l\geqslant 0}(-)^l\,\partial^\alpha\partial_L[r^{-1\,(-l-2)}\mathscr{E}_L(t-r)]. \tag{B 5}$$

B 2. *Proof of lemma* 2.2

Consider the unique decomposition of $Z^{\alpha\beta}(t-r,\boldsymbol{n})$ into ten STF tensors $A_L(t-r)$, ..., $J_L(t-r)$, all zero for $t\leqslant -T$:

$$Z^{00}=\sum_{l\geqslant 0}n_L A_L, \tag{B 6a}$$

$$Z^{0i}=\sum_{l\geqslant 0}n_{iL}B_L+\sum_{l\geqslant 1}\{n_{L-1}C_{iL-1}+\epsilon_{iab}\,n_{aL-1}D_{bL-1}\}, \tag{B 6b}$$

$$Z^{ij}=\sum_{l\geqslant 0}\{n_{ijL}E_L+\delta_{ij}\,n_L F_L\}+\sum_{l\geqslant 1}\{n_{L-1\,(i}G_{j)\,L-1}+\epsilon_{ab\,(i}n_{j)\,aL-1}H_{bL-1}\}$$
$$+\sum_{l\geqslant 2}\{n_{L-2}I_{ijL-2}+n_{aL-2}\,\epsilon_{ab\,(i}J_{j)\,bL-2}\}. \tag{B 6c}$$

By the four constraints $k_\mu Z^{\alpha\mu}=0$ we find

$$A=B=E+F, \tag{B 7a}$$

$$A_i-2B_i+E_i+F_i=2D_i-H_i=0, \tag{B 7b}$$

$$C_L=A_L-B_L\quad\text{for}\quad l\geqslant 1, \tag{B 7c}$$

$$G_L=2(B_L-E_L-F_L)\quad\text{for}\quad l\geqslant 1, \tag{B 7d}$$

$$I_L=A_L-2B_L+E_L+F_L\quad\text{for}\quad l\geqslant 2, \tag{B 7e}$$

$$J_L=2D_L-H_L\quad\text{for}\quad l\geqslant 2. \tag{B 7f}$$

From (B 7c)–(B 7f) we see that $Z^{\alpha\beta}$ is completely characterized by the set $\{A_L, B_L, D_L, E_L, F_L, H_L\}$ with the constraints (B 7a, b). We now define a new set $\{M_L, S_L, W_L, X_L, Y_L, Z_L\}$ by the equations

$$M_L := -\tfrac{1}{4}l!\,[{}^{(-l)}A_L - 2{}^{(-l)}B_L + {}^{(-l)}E_L + {}^{(-l)}F_L] \quad \text{for} \quad l \geqslant 0, \tag{B 8a}$$

$$S_L := \tfrac{1}{4}[(l+1)!/l]\,[{}^{(-l)}D_L - \tfrac{1}{2}{}^{(-l)}H_L] \quad \text{for} \quad l \geqslant 1, \tag{B 8b}$$

$$W_L := (-)^l\,[-{}^{(-l-1)}B_L + \tfrac{1}{2}{}^{(-l-1)}E_L] \quad \text{for} \quad l \geqslant 0, \tag{B 8c}$$

$$X_L := \tfrac{1}{2}(-)^{l\,(-l-2)}E_L \quad \text{for} \quad l \geqslant 0, \tag{B 8d}$$

$$Y_L := (-)^l\,[{}^{(-l)}B_L - {}^{(-l)}E_L - {}^{(-l)}F_L] \quad \text{for} \quad l \geqslant 1, \tag{B 8e}$$

$$Z_L := \tfrac{1}{2}(-)^{l+1\,(-l-1)}H_L \quad \text{for} \quad l \geqslant 1. \tag{B 8f}$$

(Apart from simple substitutions, these equations are the same as (2.26) of paper I.) Then, the constraints (B 7a, b) become

$$M = M_i = S_i = 0, \tag{B 9}$$

and, by a straightforward calculation, we find that $r^{-1}Z^{\alpha\beta}$ (B 6) is exactly the r^{-1}-term in $h^{\alpha\beta}_{\mathrm{rad}\,1}\,[\mathcal{M}_1] + \partial\Phi^{\alpha\beta}_1$ (i.e. (2.20) holds), where $\mathcal{M}_1$ is the set of multipole moments $\mathcal{M}_1 = (M_L(u), S_L(u))$ given by equations (B 8a, b), where $h_{\mathrm{rad}\,1}\,[\;]$ is the functional given by (2.4), i.e. since $M = 0$, also by (2.2), and where the vector Φ^{α}_1 is

$$\Phi^0_1 = \sum_{l \geqslant 0} \partial_L\,[r^{-1}W_L(t-r)], \tag{B 10a}$$

$$\Phi^i_1 = \sum_{l \geqslant 0} \partial_{iL}\,[r^{-1}X_L(t-r)] + \sum_{l \geqslant 1} \{\partial_{L-1}\,[r^{-1}Y_{iL-1}(t-r)] + \epsilon_{iab}\,\partial_{aL-1}\,[r^{-1}Z_{bL-1}(t-r)]\}. \tag{B 10b}$$

This completes the proof of lemma 2.2. ■

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