Czechoslovak Mathematical Journal

Ján Jakubík
Radical subgroups of lattice ordered groups

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 2, 285–297

Persistent URL: http://dml.cz/dmlcz/102092

Terms of use:

© Institute of Mathematics AS CR, 1986

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

RADICAL SUBGROUPS OF LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received December 19, 1984)

The lattice $c(G)$ of all convex l-subgroups of a lattice ordered group G was studied in [3]. A lattice ordered group $H \in c(G)$ will be said to be a *radical sugroup of* G (shortly: r-subgroup of G) if, whenever $G_1 \in c(G)$ and $H_1 \in c(H)$ such that G_1 is isomorphic to H_1, then $G_1 \subseteq H$. The system $R(G)$ of all r-subgroups of G is partially ordered by inclusion.

Radical classes of lattice ordered groups were investigated in [2], [6], [7], [8] and [9]. The collection of all radical classes of lattice ordered groups will be denoted by $\mathscr{R}$; this collection is partially ordered by inclusion. Let $\mathscr{G}$ be the class of all lattice ordered groups. For $G \in \mathscr{G}$ and $A \in \mathscr{R}$ we denote by $A(G)$ the largest convex l-subgroup of G belonging to A.

It turns out that the partially ordered set $R(G)$ is a closed sublattice of the lattice $c(G)$ and that for each $H \in c(G)$ the following conditions are equivalent: (i) H is an r-subgroup of G; (ii) there exists $A \in \mathscr{R}$ such that $H = A(G)$.

If G has no nontrivial r-subgroup (i.e., if card $R(G) \leqq 2$), then G is said to be *r-homogeneous*. G will be said to be *totally r-inhomogeneous* if, whenever $\{0\} \neq H \in R(G)$, then there exists $H_1 \in R(G)$ such that $\{0\} \subset H_1 \subset H$ (i.e., the lattice $R(G)$ has no atom).

The main results of this paper concern the lattice $R(G)$ for the case when G is a complete lattice ordered group. Let us mention the following existence results:

For each cardinal $\alpha > 0$ there is a proper class A_α of mutually nonisomorphic complete lattice ordered groups such that for each $G \in A_\alpha$, $R(G)$ is isomorphic to the Boolean algebra 2^α. (Hence, in particular, there exists a proper class of mutually nonisomorphic r-homogeneous complete lattice ordered groups.) For each ordinal δ there is a complete lattice ordered group G such that $R(G)$ is a chain isomorphic to δ. For each complete lattice ordered group G there exists a complete lattice ordered group G_1 such that $G \in R(G_1)$ and G is covered by G_1 in the lattice $R(G_1)$. There exists a proper class of mutually nonisomorphic totally r-inhomogeneous lattice ordered groups. The question whether there exists a complete totally r-inhomogeneous lattice ordered group $G \neq \{0\}$ remains open. Some results on the lattice $\mathscr{R}_c$ of all radical classes of complete lattice ordered groups will be also established; e.g., it will be shown that $\mathscr{R}_c$ is a Stone lattice.

1. PRELIMINARIES

The standard notations for lattice ordered groups will be applied (cf. [1] and [4]). The group operation will be written additively.

When considering a subclass Y of $\mathscr{G}$ we always assume that Y is closed with respect to isomorphisms and that the zero group $\{0\}$ belongs to Y.

Let $G \in \mathscr{G}$. The system $c(G)$ is a complete lattice (the operation $\wedge$ in $c(G)$ coincides with the set theoretical intersection; the join in $c(G)$ will be denoted by $\vee^c$).

A subclass X of $\mathscr{G}$ is said to be a *radical class* if it is closed with respect to

a) convex l-subgroups, and

b) joins of convex l-subgroups.

Hence $\mathscr{G}$ is the largest element in $\mathscr{R}$. The class containing one-element lattice ordered groups only is the least in $\mathscr{R}$; this class will be denoted by 0^-.

For $X \subseteq \mathscr{G}$ we denote by

Sub X – the class of all convex l-subgroups of lattice ordered groups belonging to X;

$\mathrm{Join}_c X$ – the class of all lattice ordered groups G having a system $\{G_i\}_{i \in I} \subseteq c(G)$ with $G_i \in X$ for each $i \in I$ such that $\bigvee^c_{i \in I} G_i = G$.

The following three propositions were proved in [6].

1.1. Proposition. *$\mathscr{R}$ is a complete lattice in which the meet coincides with the intersection of classes. Let I be a nonempty class and for each $i \in I$ let $X_i \in \mathscr{R}$. Then $\bigvee_{i \in I} X_i = \mathrm{Join}_c (\bigcup_{i \in I} X_i)$.*

For $X \subseteq \mathscr{G}$ we denote by $T(X)$ the intersection of all $Y \in \mathscr{R}$ with $X \subseteq Y$. In view of 1.1, $T(X)$ belongs to $\mathscr{R}$; it is said to be the *radical class generated by* X.

1.2. Proposition. *Let $X \subseteq \mathscr{G}$. Then $T(X) = \mathrm{Join}_c \,\mathrm{Sub}\, X$.*

1.3. Proposition. *The lattice $\mathscr{R}$ satisfies the infinite distributive law*

$$X \wedge (\bigvee_{i \in I} Y_i) = \bigvee_{i \in I} (X \wedge Y_i) . \tag{1}$$

If $X_1, X_2 \in \mathscr{R}$ and $X_1 \leqq X_2$, then $[X_1, X_2]$ denotes the collection of all $Y \in \mathscr{R}$ with $X_1 \leqq Y \leqq Y_2$.

2. BASIC PROPERTIES OF THE LATTICE $R(G)$

Let $G \in \mathscr{G}$ and $A \in \mathscr{R}$. Let $\{H_i\}_{i \in I}$ be the set of all elements of $c(G)$ which belong to A. According to the definition of the notion of a radical class (cf. the condition b) in Section 1) the lattice ordered group $A(G) = \bigvee^c_{i \in I} H_i$ belongs to A. We obviously have $A(G) \in R(G)$.

If $G_1 \in \mathscr{G}$ and if X is the class of all lattice ordered groups G_2 such that either G_2 is a zero group or G_2 is isomorphic to G_1, then we denote $T(X) = T(G_1)$. The radical class $T(G_1)$ is said to be *principal* (and *generated by* G_1).

Now let $H \in R(G)$. Put $A = T(H)$. Clearly $H \in A$, hence $H \subseteq A(G)$. Because

$A(G) \in A$, in view of 1.2 there are elements H_i $(i \in I)$ of $c(H)$ such that $A(G) = \bigvee^c_{i\in I} H_i$. Thus $A(G) \subseteq H$ and therefore $A(G) = H$. We obtain:

2.1. Proposition. *Let $G \in \mathscr{G}$ and $H \in c(G)$. Then the following conditions are equivalent:*

(i) *H belongs to $R(G)$.*

(ii) *There exists $A \in \mathscr{R}$ such that $H = A(G)$.*

2.2. Proposition. *Let $G \in \mathscr{G}$. Then $R(G)$ is a closed sublattice of $c(G)$.*

Proof. Let $I \neq \emptyset$ be a set and for each $i \in I$ let $H_i \in R(G)$. In view of the definition of $R(G)$ we have $\bigcap_{i\in I} H_i \in R(G)$.

Put $\bigvee^c_{i\in I} H_i = H$. We have to verify that H belongs to $R(G)$. Let $H_1 \in c(H)$, $G_1 \in c(G)$ and suppose that φ is an isomorphism of H_1 onto G_1. It is well-known (cf., e.g., [3]) that for any $G_0 \in c(G)$ and $\{G_j\}_{j\in J} \subseteq c(G)$ the following infinite distributive law is valid:

(1a) $$G_0 \wedge (\bigvee^c_{j\in J} G_j) = \bigvee^c_{j\in J} (G_0 \wedge G_j)\,.$$

Hence
$$H_1 = H_1 \wedge H = H_1 \wedge (\bigvee^c_{i\in I} H_i) = \bigvee^c_{i\in I} (H_1 \wedge H_i)\,.$$

Put $G_i = \varphi(H_1 \wedge H_i)$. From $H_i \in R(G)$ we infer that $G_i \subseteq H_i$; moreover, $G_1 = \bigvee^c_{i\in I} G_i$. Thus $G_1 \subseteq H$ and therefore $H \in R(G)$.

From 2.2 and 1.3 we obtain:

2.2.1. Corollary. *Let $G \in \mathscr{G}$. The lattice $R(G)$ satisfies the infinite distributive law* (1).

From 2.1 and [6], Corollary 2 of Proposition 4.2 we infer:

2.3. Proposition. *Let $G \in \mathscr{G}$. Then $R(G)$ is isomorphic to the interval $[0^-, T(G)]$ of the lattice $\mathscr{R}$.*

Let us remark that Corollary 2.2.1 can be obtained also as a consequence of 2.3 and 1.3.

The following example shows that the lattice $R(G)$ need not satisfy the infinite distributive law dual to (1a).

2.4. Example. Let R_0 be the additive group of all reals with the natural linear order. Let P be the set of all positive primes and for each $p \in P$ let G_p be the l-subgroup of R consisting of all elements of R_0 which can be written in the form mp^{-n}, where m and n are integers, $n > 0$. Let G be the (complete) direct product
$$G = \prod_{p\in P} G_p\,.$$

We denote by H the discrete direct product (= direct sum) of the system $\{G_p\}_{p\in P}$. For each $p \in P$ let $I(p) = \{q \in P: q > p\}$ and
$$H_p = \prod_{i\in I(p)} G_i\,.$$

Then $H \in R(G)$ and $H_p \in R(G)$. We have
$$\bigwedge_{p\in P} H_p = \{0\}$$

and

$$H \vee^c H_p = G \quad \text{for each} \quad p \in P .$$

Therefore

$$H \vee^c (\bigwedge_{p \in P} H_p) = H ,$$

$$\bigwedge_{p \in P} (H \vee^c H_p) = G .$$

Since $G \neq H$, the infinite distributive law dual to (1a) does not hold in the lattice $R(G)$.

Let $G \in \mathscr{G}$ and $M \subseteq G$. The set

$$M^{\perp} = \{g \in G : |g| \wedge |m| = 0 \text{ for each } m \in M\}$$

is a *polar* of G; $M^{\perp}$ and $M^{\perp\perp}$ are *complementary polars* of G.

Let $X \subseteq \mathscr{G}$. We denote by X^{δ} the class of all lattice ordered groups G such that, whenever $H \in c(G) \cap X$, then $H = \{0\}$.

From 1.2 we infer that for each $Y \in \mathscr{R}$ the relation

$$T(X) \wedge Y = 0^{-} \Leftrightarrow Y \leqq X^{\delta}$$

is valid. Hence $X^{\delta\delta\delta} = X^{\delta}$ for each $X \subseteq \mathscr{G}$.

2.5. Lemma. (Cf. [6], Lemma 2.1.) *Let $X \subseteq \mathscr{G}$. Then $X^{\delta} \in \mathscr{R}$.*

For each $g \in G$ we denote by $[g]$ the convex l-subgroup of G generated by g. If $g > 0$, then g is a strong unit in $[g]$; in particular, for each $0 < g_1 \in [g]$ we have $0 < g_1 \wedge g$.

2.6. Lemma. *Let $X \subseteq \mathscr{G}$ and $G \in \mathscr{G}$. Then $X^{\delta}(G)$ and $X^{\delta\delta}(G)$ are complementary polars of G.*

Proof. We obviously have $X^{\delta} \wedge X^{\delta\delta} = 0^{-}$, whence

$$X^{\delta}(G) \wedge X^{\delta\delta}(G) = (X^{\delta} \wedge X^{\delta\delta})(G) = 0^{-}(G) = 0^{-} .$$

Thus $X^{\delta\delta}(G) \subseteq (X^{\delta}(G))^{\perp}$ and $X^{\delta}(G) \subseteq (X^{\delta\delta}(G))^{\perp}$.

We shall show that

(2) $$(X^{\delta}(G))^{\perp} \subseteq X^{\delta\delta}(G)$$

is valid. Let $0 < y \in (X^{\delta}(G))^{\perp}$. For proving that y belongs to $X^{\delta\delta}(G)$ it suffices to verify that $[y]$ belongs to the class $X^{\delta\delta}$. By way of contradiction, assume that $[y]$ does not belong to $X^{\delta\delta}$. Hence there exists $H \in c([y]) \cap X^{\delta}$ such that $H \neq \{0\}$. Choose $0 < y_1 \in H$. Then $y_1 \in [y]$, hence $y_1 \wedge y > 0$. On the other hand, we have $H \in X^{\delta}$, whence $H \subseteq X^{\delta}(G)$, thus $y_1 \in X^{\delta}(G)$ and therefore $y_1 \wedge y = 0$, which is a contradiction. Thus (2) is valid and hence

(3) $$(X^{\delta}(G))^{\perp} = X^{\delta\delta}(G)$$

holds. By putting X^{δ} instead of X in (3) we obtain

$$(X^{\delta\delta}(G))^{\perp} = X^{\delta\delta\delta}(G) = X^{\delta}(G) ,$$

completing the proof.

3.1. Lemma. (Cf. [6], Corollary 2 to Proposition 4.1.) *Let $G \in \mathcal{G}$ and $Y \in \mathcal{R}$. Assume that $Y \leqq T(G)$. Then $Y = T(G_1)$, where $G_1 = Y(G)$.*

3.2. Lemma. *Let $G \in \mathcal{G}$. For each $G_1 \in R(G)$ we put $\varphi(G_1) = T(G_1)$. Then φ is an isomorphism of the lattice $R(G)$ onto the interval $[0^-, T(G)]$ of $\mathcal{R}$.*

Proof. In view of 3.1 and 2.1, the mapping φ is an epimorphism. If $G_1, G_1' \in R(G)$ such that $\varphi(G_1) = \varphi(G_1')$, then $G_1' \in T(G_1)$, whence (in view of 2.1) $G_1' \subseteq G_1$; similarly we have $G_1 \subseteq G_1'$. Thus φ is a monomorphism.

Let $G_1, G_2 \in R(G)$ be such that $G_1 \subseteq G_2$. According to 1.2 we have $\varphi(G_1) \leqq \varphi(G_2)$. Now let $Y_1, Y_2 \in [0^-, T(G)]$ be such that $Y_1 \leqq Y_2$. Put $G_1 = Y_1(G)$, $G_2 = Y_2(G)$. Hence $G_1 = \varphi^{-1}(Y_1)$ and $G_2 = \varphi^{-1}(Y_2)$. Because of $G_1 \in Y_2$ we infer that $G_1 \subseteq G_2$ (by applying 1.2 again). Thus φ is an isomorphism.

3.3. Corollary. *A lattice ordered group $G \neq \{0\}$ is r-homogeneous if and only if $T(G)$ is an atom of the lattice $\mathcal{R}$.*

If $M \subseteq \mathcal{R}$ ($\mathcal{G}_1 \subseteq \mathcal{G}$) and if there exists an injective mapping of the class of all cardinals into M (or $\mathcal{G}_1$, respectively), then M is said to be a *proper collection of radical classes* (*a proper class of lattice ordered groups*).

3.4. Proposition. *There exists a proper collection $\mathcal{A} \subseteq \mathcal{R}$ such that* (i) *for each $X \in \mathcal{A}$ there is a linearly ordered group G xuch that $X = T(G)$;* (ii) *each $X \in \mathcal{A}$ is an atom in $\mathcal{R}$.*

From 3.3 and 3.4 we infer:

3.5. Theorem. *There exists a proper class $\mathcal{G}_1$ of linearly ordered groups such that*

(i) *if G_1 and G_2 are distinct elements of $\mathcal{G}_1$, then G_1 is not isomorphic to G_2;*

(ii) *if $G \in \mathcal{G}_1$, then G is r-homogeneous.*

The class of all nonisomorphic types of complete linearly ordered groups fails to be a proper class, hence Theorem 3.5 cannot be sharpened by assuming that all linearly ordered groups of the class $\mathcal{G}_1$ are complete. Thus if we search for a large collection of nonisomorphic complete lattice ordered groups, then we must cancel the assumption of linear ordering.

Let B be a Boolean algebra. Let us recall the notion of Carathéodory functions corresponding to B (cf. [5], or [10], p. 97).

Let $E(B)$ be the system consisting of all forms

$$f = a_1 b_1 + \ldots + a_n b_n \tag{4}$$

(where $a_i \neq 0$ are reals and $b_i \in B$, $b_i > 0$, $b_{i_1} \wedge b_{i_2} = 0$ for any $i_1, i_2 \in \{1, 2, \ldots, n\}$, $i_1 \neq i_2$) and of the empty form; if g is another such form,

$$g = a_1' b_1' + \ldots + a_m' b_m',$$

then f and g are considered equal if $\bigvee_{i=1}^{n} b_i = \bigvee_{j=1}^{m} b'_j$ and $a_i = a'_j$ whenever $b_i \wedge b'_j \neq 0$. For any $b, b' \in B$ let $b - b'$ be the relative complement of $b \wedge b'$ in the interval $[0, b]$. The operation $+$ in $E(B)$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a'_j)(b_i \wedge b'_j) + \sum_{i=1}^{n} a_i(b_i - \bigvee_{j=1}^{m} b'_j) + \\ + \sum_{j=1}^{m} a'_j(b'_j - \bigvee_{i=1}^{n} b_i),$$

where in the summations only those terms are taken into account in which $a_j + a'_j \neq 0$ and the elements $b_i \wedge b'_j$, $b_i - \bigvee_{j=1}^{m} b'_j$ or $b'_j - \bigvee_{i=1}^{n} b_i$ are non-zero. The multiplication by a real $a \neq 0$ is defined by $af = (aa_1) b_1 + \ldots + (aa_n) b_n$; $0f$ is the empty form. The form (4) is positive if $a_i > 0$ for $i = 1, 2, \ldots, n$. Then $E(B)$ is a vector lattice; in particular, $E(B)$ is a lattice ordered group. Elements of $E(B)$ are said to be the *elementary Carathéodory functions.*

Let us denote by $G_c(B)$ the subset of $E(B)$ consisting of the empty form and of all forms (4) such that a_i are integers ($i = 1, 2, \ldots, n$). Then $G_c(B)$ is an l-subgroup of the l-group $E(B)$. The empty form is the zero element of $G_c(B)$. If $0 \neq b \in B$, then the form $1b$ will be identified with b.

It is easy to verify that if B is a complete Boolean algebra, then $G_c(B)$ is a complete lattice ordered group.

From the definition of $G_c(B)$ we immediately obtain:

3.6. Lemma. *Let* $0 < b \in B$. *Then* $[b] = G_c([0, b])$.

A Boolean algebra B is said to be *homogeneous* if for each $0 < b \in B$, the Boolean algebra $[0, b]$ is isomorphic to B.

The following proposition is a consequence of [11] (Corollaries 3.12 and 3.14).

3.7. Proposition. *For each cardinal* α *there exists a homogeneous Boolean algebra* B *such that* (i) B *is complete, and* (ii) card $B \geqq \alpha$.

3.8. Lemma. *Let* B *be a homogeneous Boolean algebra. Then the lattice ordered group* $G_c(B)$ *is* r-*homogeneous.*

Proof. Let $G_1 \in R(G_c(B))$, $G_1 \neq \{0\}$. Choose $0 < g_1 \in G_1$. There exists $0 < b \in B$ such that $b \leqq g_1$, hence $[b] \subseteq G_1$. Let $0 < g \in G_c(B)$. There are nonzero elements $b_1, b_2, \ldots, b_n$ in B and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $g = \alpha_1 b_1 + \alpha_2 b_2 + \ldots \ldots + \alpha_n b_n$. In view of 3.6, each lattice ordered group $[b_i]$ ($i = 1, 2, \ldots, n$) is isomorphic to $[b]$, hence $[b_i] \subseteq G_1$. Thus $g \in G_1$. We infer that $G_1 = G_c(B)$; hence $G_c(B)$ is r-homogeneous.

From 3.7 and 3.8 we obtain:

3.9. Theorem. *There exists a proper class* $\mathscr{G}_2$ *of nonzero complete lattice ordered groups such that* (i) *if* G_1 *and* G_2 *are distinct elements of* $\mathscr{G}_2$, $0 < g_1 \in G_1$, $0 < g_2 \in G_2$, *then* $[g_1]$ *is not isomorphic to* $[g_2]$; (ii) *if* $G \in \mathscr{G}_2$, *then* G *is* r-*homogeneous.*

In this section we will construct complete lattice ordered groups whose lattice of radical subgroups is isomorphic to the Boolean algebra 2^α, where α is a given cardinal. Further it will be shown that the lattice $R(G)$ corresponding to a nonzero lattice ordered group G is an atomic Boolean algebra if and only if G is a direct sum of r-homogeneous lattice ordered groups belonging to $R(G)$.

4.1. Lemma. *Let $K \neq \{0\}$ be an r-homogeneous lattice ordered group, $G \in \mathscr{G}$, $K \in R(G)$. Then K is an atom in $R(G)$.*

This is an immediate consequence of the definition of r-homogeneity.

The direct sum G of lattice ordered groups G_i $(i \in I)$ is denoted by $\sum_{i\in I} G_i$. For $g \in G$ we denote by $g(i)$ the i-th component of g; we put $I(g) = \{i \in I: g(i) \neq 0\}$. For $H \subseteq G$ we set $I(H) = \bigcup_{h\in H} I(h)$.

4.2. Lemma. *Let $\{0\} \neq G_i$ $(i \in I \neq \emptyset)$ be r-homogeneous lattice ordered groups and let $G = \sum_{i\in I} G_i$. Assume that $G_i \in R(G)$ for each $i \in I$. Let $H \in R(G)$, $H \neq \{0\}$. Then $H = \sum_{i\in I(H)} G_i$.*

Proof. If $i \in I(H)$, then $H \cap G_i \neq \{0\}$, hence in view of 4.1 we have $H \supseteq G_i$. If $i \in I \setminus I(H)$, then $H \cap G_i = \{0\}$. Therefore $H = \sum_{i\in I(H)} G_i$.

4.3. Lemma. *Let $G_i (i \in I)$ be as in* 4.2. *Let $\emptyset \neq I_1 \subseteq I$, $H_1 = \sum_{i\in I_1} G_i$. Then $H_1 \in R(G)$.*

Proof. Let $\{0\} \neq K \in c(H_1)$ and $K' \in c(G)$. Assume that φ is an isomorphism of K onto K'. Clearly $G_i \in R(H_1)$ for each $i \in I_1$. Moreover, in view of 4.2 we have

$$K = \sum_{i\in I(K)} G_i\,, \quad K' = \sum_{i\in I(K')} G_i\,.$$

Let $j \in I(K')$. Then $\{0\} \neq \varphi^{-1}(G_j) \in c(K) \subseteq c(H_1)$. Because $G_j \in R(G)$ we have $\varphi^{-1}(G_j) \subseteq G_j$, thus $G_j \cap H_1 \neq \{0\}$; therefore $j \in I_1$. Hence $K' \subseteq H_1$ and so $H_1 \in R(G)$.

From 4.2 and 4.3 we obtain:

4.4. Lemma. *Let G and G_i $(i \in I)$ be as in* 4.2. *Then the lattice $R(G)$ is isomorphic to the Boolean algebra 2^α, where $\alpha = \operatorname{card} I$.*

4.5. Lemma. *Let G_i $(i \in I)$ be nonzero r-homogeneous lattice ordered groups and let $G = \sum_{i\in I} G_i$. Then the following conditions are equivalent*: (i) *all G_i belong to $R(G)$*; (ii) *if $i_1, i_2 \in I$, $i_1 \neq i_2$, $0 < g_1 \in G_{i_1}$, $0 < g_2 \in G_{i_2}$, then $[g_1]$ is not isomorphic to $[g_2]$.*

Proof. Let (i) be valid. Let $i_1, i_2 \in I$, $i_1 \neq i_2$, $0 < g_1 \in G_{i_1}$, $0 < g_2 \in G_{i_2}$. Assume that $[g_1]$ is isomorphic to $[g_2]$. Because $G_{i_1} \in R(G)$ we infer that $[0, g_2] \subseteq G_{i_1}$, hence $G_{i_1} \cap G_{i_2} \neq \{0\}$, which is a contradiction; thus (ii) holds. Conversely, assume that (ii) is fulfilled. Let $i \in I$. By way of contradiction, assume that G_i does not belong to $R(G)$. Hence there are $H_1 \in c(G_i)$ and $H \in c(G)$ such that H_1 is isomorphic to H but H is not a subset of G_i. Hence there is $0 < h \in H \setminus G_i$. If $h(G_j) = 0$ for each

$j \in I \setminus \{i\}$, then we should have $h \in G_i$; thus there is $j \in I \setminus \{i\}$ such that $h(G_j) > 0$. Because H_1 and H are isomorphic there is $0 < h_i \in G_i$ such that $[h(G_j)]$ is isomorphic to $[h_i]$, which contradicts (ii). Thus (i) must be valid.

From 4.4, 4.5 and 3.9 we infer:

4.6. Theorem. *Let α be a cardinal. There exists a proper class $\mathcal{G}_\alpha$ of complete lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathcal{G}_\alpha$, then G_1 is not isomorphic to G_2;* (ii) *if $G \in \mathcal{G}_\alpha$, then the lattice $R(G)$ is isomorphic to 2^α.*

4.7. Lemma. *Let $G \neq \{0\}$ be a lattice ordered group such that $R(G)$ is an atomic Boolean algebra. Let $\{G_i\}_{i \in I}$ be the set of all atoms of $R(G)$. Then all G_i are r-homogeneous and $G = \sum_{i \in I} G_i$.*

Proof. Since G_i is an atom in $R(G)$, it is r-homogeneous. If i, j are distinct elements in I, then $G_i \cap G_j = \{0\}$; hence whenever $g_i \in G_i$ and $g_j \in G_j$, then $g_i + g_j = g_j + g_i$. Because $R(G)$ is atomic, we have $G = \bigvee_{i \in I} G_i$. Therefore for each nonzero element $g \in G$ there are distinct indices $i_1, i_2, \ldots, i_n \in I$ and elements $g_1 \in G_{i_1}, \ldots, g_n \in G_{i_n}$ such that $g = g_1 + \ldots + g_n$. Hence $G = \sum_{i \in I} G_i$.

From 4.2 and 4.7 we obtain:

4.8. Proposition. *Let G be a nonzero lattice ordered group. The following conditions are equivalent:* (i) *$R(G)$ is an atomic Boolean algebra.* (ii) *G is a direct sum of r-homogeneous lattice ordered groups belonging to $R(G)$.*

5. AN EXAMPLE

The direct product of lattice ordered groups G_i $(i \in I)$ will be denoted by $\prod_{i \in I} G_i$. Let α be an infinite cardinal. By the α-direct product of the given system $\{G_i\}_{i \in I}$ we shall mean the l-subgroup of $\prod_{i \in I} G_i = G^0$ consisting of all elements $g \in G^0$ such that card $\{i \in I: g(i) \neq 0\} < \alpha$.

By means of α-products we shall construct complete lattice ordered groups whose lattice of radical subgroups is a well-ordered chain having a given cardinality β.

Let $G \in \mathcal{G}$. An element of G will be said to be an *s-element of* G (Sptize in the terminology of [12]) if $g > 0$ and the interval $[0, g]$ is a chain. A system $\{g_j\}_{j \in J}$ of elements of G is said to be *disjoint* if $g_j > 0$ for each $j \in J$ and $g_{j_1} \wedge g_{j_2} = 0$ whenever j_1 and j_2 are distinct elements of J.

Let G_0 be the additive group of all integers with the natural linear order. Let I be an infinite set of indices, card $I = \gamma$, and for each $i \in I$ let G_i be a lattice ordered group isomorphic to G_0. Put $G^0 = \prod_{i \in I} G_i$. Let G be the l-subgroup of G^0 consisting of all bounded elements of G^0 (i.e., an element g of G^0 belongs to G iff there is a positive integer n such that $g(i) \leqq n$ for each $i \in I$). For any $g \in G$ let $I(g)$ be as in Section 4.

Let α be an infinite cardinal, $\alpha \leqq \gamma$. We denote by G^α the set of all $g \in G$ such that card $I(g) < \alpha$ (i.e., G^α is the set of all bounded elements of G^0 which belong to the

α-product of the system $\{G_i\}_{i\in I}$). Then $G^\alpha \in c(G)$. The following lemma is obvious (under the notations as above.).

5.1. Lemma. *Let $0 < g \in G$. Then the following conditions are equivalent:*

(i) *g belongs to G^α.*

(ii) *If $\{g_j\}_{j\in J}$ is a disjoint system of s-elements of the lattice ordered group $[g]$, then* card $J < \alpha$.

5.2. Lemma. *Let α be an infinite cardinal, $\alpha \leqq \gamma$. Then $G^\alpha \in R(G)$.*

Proof. Let $H_1 \in c(G^\alpha)$, $H \in c(G)$ and let φ be an isomorphism of H_1 onto H. Let $0 < h \in H$, $g = \varphi^{-1}(h)$. In view of 5.1, the condition (ii) from 5.1 is valid; thus the analogous condition holds for the element h. Therefore $h \in G^\alpha$. This implies that $H \subseteq G^\alpha$ and thus $G^\alpha \in R(G)$.

5.3. Lemma. *Let $G' \in R(G)$, $\{0\} \neq G' \neq G$. Then there is an infinite cardinal α with $\alpha \leqq \gamma$ such that $G' = G^\alpha$.*

Proof. There exists $0 < g \in G'$. Let H be the set of all $h \in G$ such that $I(h) \subseteq I(g)$. There is a positive integer n with $|h| \leqq ng$; hence $H \subseteq G'$. Let $I_2 \subseteq I$, card $I_2 =$ $=$ card $I(g)$. Next, let H' be the l-subgroup of G consisting of all $h' \in G$ with $I(h') \subseteq$ $\subseteq I_2$. Then $H' \in c(G)$ and H' is isomorphic to $H \in c(G')$. Thus $H' \subseteq G$. Hence $G^\beta \subseteq G'$, where $\beta =$ card $I(g)$.

If for each β with $\beta \leqq \gamma$ there exists $0 < g \in G'$ with card $I(g) = \beta$, then we should have $G' = G$, which is a contradiction. Hence there exists a least cardinal $\alpha \leqq \gamma$ with $g_1 \notin G'$ for some g_1 such that card $I(g_1) = \alpha$. Then $G' = G^\alpha$. It is easy to verify that the cardinal α must be infinite.

Let us denote by C_γ the set of all infinite cardinals $\alpha \leqq \gamma$ (with the natural linear order). From 5.2 and 5.3 we obtain:

5.4. Lemma. *Let S be the set of all radical subgroups of G which are distinct from $\{0\}$ and G; S is partially ordered by inclusion. Then S is isomorphic to C_γ.*

Since the infinite cardinal γ considered above was chosen arbitrarily, from 5.4 we infer:

5.5. Theorem. *Let δ be an ordinal. There exists a complete lattice ordered group G such that the lattice $R(G)$ is a chain isomorphic to δ.*

Also, if we consider γ as running over the class of all infinite cardinals, then we obtain:

5.6. Theorem. *There exists a proper class $\mathscr{G}_4$ of complete lattice ordered groups such that the following conditions are valid:* (i) *If G_1 and G_2 are distinct elements of $\mathscr{G}_4$, then G_1 is not isomorphic to G_2; moreover, either G_1 is isomorphic to some radical subgroup of G_2, or G_2 is isomorphic to some radical subgroup of G_1.* (ii) *For each $G \in \mathscr{G}_4$, $R(G)$ is a well-ordered chain.*

The following question remains open: to what extent do the results of this section remain valid if G is an arbitrary nonzero r-homogeneous complete lattice ordered group?

6. THE COVERING RELATION

Let $G \in \mathscr{G}$. If H is a dual atom of the lattice $R(G)$, then H will be said to be *covered by* G. If G is a nonzero lattice ordered group, then the following questions can be proposed:

(Q_1) Does there exist a lattice ordered group H_1 such that H_1 is covered by G?

(Q_2) Does there exist a lattice ordered group H_2 such that G is covered by H_2?

Both (Q_1) and (Q_2) can be modified in such a way that G, H_1 and H_2 are assumed to be complete.

From 5.5 we obtain as a corollary:

6.1. Proposition. *There exists a proper class $\mathscr{G}_5$ of complete lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathscr{G}_5$, then G_1 is not isomorphic to G_2*; (ii) *if $G \in \mathscr{G}_5$, then no lattice ordered group is covered by G.*

6.2. Lemma. *Let $G \in \mathscr{G}$. There exists a proper class $\mathscr{G}_6(G)$ of nonzero complete r-homogeneous lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathscr{G}_6(G)$ and $0 < g_1 \in G_1$, $0 < g_2 \in G_2$, then $[g_1]$ is not isomorphic to $[g_2]$, and* (ii) *if $G_1 \in \mathscr{G}_6(G)$ and $0 < g_1 \in G$, then no convex l-subgroup of G is isomorphic to $[g_1]$.*

This is an immediate consequence of 3.9.

6.3. Lemma. *Let G and $\mathscr{G}_6(G)$ be as in 6.2. Let $G_1 \in \mathscr{G}_6(G)$. Put $H = G \times G_1$. Then G is covered by H.*

Proof. From 6.2 we infer that both G and G_1 belong to $R(H)$ and that $G \cap G_1 = \{0\}$ is valid. Moreover, $G \vee G_1 = H$ holds. As G_1 is r-homogeneous, $\{0\}$ is covered by G_1. In view of the distributivity of $R(H)$, G is covered by H.

6.4. Lemma. *Let G and $\mathscr{G}_6(G)$ be as in 6.2. Let $G_1, G_2 \in \mathscr{G}_6(G)$, $G_1 \neq G_2$. Then $G \times G_1$ is not isomorphic to $G \times G_2$.*

Proof. By way of contradiction, assume that φ is an isomorphism of $G \times G_1$ onto $G \times G_2$. Then there are $P \in c(G)$ and $Q \in c(G_2)$ such that $\varphi(G_1) = P \times Q$. In view of 6.2 (i) we must have $Q = \{0\}$. Similarly, according to 6.2 (ii) the relation $P = \{0\}$ must be valid. Hence $G_1 = \{0\}$, which is a contradiction.

Let us remark that if G, G_1 and H are as in 6.3 and if G is complete, then H is complete as well. Thus 6.2, 6.3 and 6.4 yield:

6.5. Theorem. *Let $G \in \mathscr{G}$. There exists a proper class $\mathscr{G}_7(G)$ of lattice ordered groups such that* (i) *the elements of $\mathscr{G}_7(G)$ are mutually nonisomorphic*; (ii) *if*

$H \in \mathscr{G}_7(G)$, *then* G *is covered by* H; (iii) *if* G *is complete, then all elements of* $\mathscr{G}_7(G)$ *are complete.*

Next, we may ask whether there exists a lattice ordered group $G \neq \{0\}$ is covered by no element of $R(G)$; i.e., $R(G)$ has no atoms. Such a lattice ordered group G will called *totally r-inhomogeneous.*

From 2.3 and from the construction established in [6], Section 5 (cf. Proposition 5.4) we obtain:

6.6. Proposition. *There exists a proper class* $\mathscr{G}_8$ *of linearly ordered groups such that* (i) *the elements of* $\mathscr{G}_8$ *are mutually nonisomorphic;* (ii) *if* $G \in \mathscr{G}_8$, *then* G *is totally r-inhomogeneous.*

The question whether there exists a complete totally r-inhomogeneous lattice ordered group remains open.

7. THE LATTICE $\mathscr{R}_c$

We denote by $\mathscr{R}_c$ the collection of all radical classes $A \in \mathscr{R}$ such that each lattice ordered group belonging to A is complete. Similarly as $\mathscr{R}$, the collection $\mathscr{R}_c$ is partially ordered by inclusion.

Let $\mathscr{G}_c$ be the class of all complete lattice ordered groups; then $\mathscr{G}_c$ is a radical class (cf. [6]). Hence $\mathscr{R}_c$ is the interval $[0^-, \mathscr{G}_c]$ of the lattice $\mathscr{R}$.

(For $\mathscr{R}$ and $\mathscr{R}_c$ we apply the usual lattice theoretic notations, though $\mathscr{R}$ and $\mathscr{R}_c$ fail to be sets.) Hence we have:

7.1. Lemma. $\mathscr{R}_c$ *is a closed sublattice of* $\mathscr{R}$; *thus the infinite distributive law* (1) *is valid in* $\mathscr{R}_c$.

In [6] it was shown that no element of R distinct from 0^- and $\mathscr{G}$ has a complement in the lattice $\mathscr{R}$. Thus $\mathscr{R}$ is pseudocomplemented, but it fails to be a Stone lattice.

7.2. Proposition. $\mathscr{R}_c$ *is a Stone lattice.*

Proof. Let $A \in \mathscr{R}_c$. Put $A^{\delta o} = A^\delta \cap \mathscr{G}_c$. Then obviously, $A^{\delta o}$ is a pseudocomplement of A in the lattice $\mathscr{R}_c$. We have to verify that $A^{\delta o} \vee A^{\delta o \delta o} = \mathscr{G}_c$ is valid for each $A \in \mathscr{R}_c$.

We have $A^{\delta o \delta o} = A^{\delta\delta} \cap \mathscr{G}_c$, hence

$$A^{\delta o} \vee A^{\delta o \delta o} = (A^\delta \wedge \mathscr{G}_c) \vee (A^{\delta\delta} \wedge \mathscr{G}_c) = (A^\delta \vee A^{\delta\delta}) \wedge \mathscr{G}_c .$$

Let $G \in \mathscr{G}_c$. Then

$$(A^{\delta o} \vee A^{\delta o \delta o})(G) = ((A^\delta \vee A^{\delta\delta}) \wedge \mathscr{G}_c)(G) = (A^\delta \vee A^{\delta\delta})(G) \cap \mathscr{G}_c(G) =$$
$$= ((A^\delta \vee A^{\delta\delta})(G)) \cap G = (A^\delta \vee A^{\delta\delta})(G) = A^\delta(G) \vee^c A^{\delta\delta}(G) .$$

In view of 2.6, $A^\delta(G)$ and $A^{\delta\delta}(G)$ are complementary polars of G. Since G is complete, $A^\delta(G)$ and $A^{\delta\delta}(G)$ are complementary direct factors of G. Hence $A^\delta(G) \vee^c A^{\delta\delta}(G) = G$. Therefore G belongs to $A^{\delta o} \vee A^{\delta o \delta o}$ and thus $A^{\delta o} \vee A^{\delta o \delta o} = \mathscr{G}_c$.

Since for each nonzero r-homogeneous complete lattice G the radical class $T(G)$ is an atom of $\mathscr{R}_c$, 3.9 implies:

7.3. Proposition. *There exists a proper collection of atoms in* $\mathscr{R}_c$.

7.4. Lemma. *Let* $G \in \mathscr{G}$, $\{G_i\}_{i\in I} \subseteq \mathscr{G}$, $H = \prod_{i\in I} G_i$, $0 < h \in H$, $\operatorname{card} I(h) > \operatorname{card} G$, $A = T(G)$. *Then* h *does not belong to* $A(H)$.

Proof. By way of contradiction, assume that $h \in A(H)$. Hence in view of 1.2 there exist $\{H_j\}_{j\in J} \subseteq c(H)$ and $\{G'_j\}_{j\in J} \subseteq c(G)$ such that for each $j \in J$, H_j is isomorphic to G'_j and $[h] = \bigvee_{j\in J} H_j$. Thus there exists a finite subset J_1 of J such that for some $0 < h_j \in H_j$ $(j \in J_1)$ we have $h = \sum_{j\in J_1} h_j$. For each element $0 \leqq h' \leqq h$ there are $h'_j \in [0, h_j]$ $(j \in J_1)$ with $h' = \sum_{j\in J_1} h'_j$. We obviously have $\operatorname{card} I(h) \leqq \operatorname{card} [0, h]$, whence $\operatorname{card} I(h)$ is equal or less than the product of the cardinals $\operatorname{card} [0, h_j]$ (where j runs over the set J_1). Because $\operatorname{card} [0, h_j] \leqq \operatorname{card} G$ for each $j \in J_1$, we obtain $\operatorname{card} I(h) \leqq \operatorname{card} G$, which is a contradiction.

Next, $\mathscr{R}$ has no dual atom. (This a consequence of Corollary 1 of Propos. 3.4, [6].) Similarly we have:

7.5. Proposition. *The lattice* $\mathscr{R}_c$ *has no dual atom.*

Proof. By way of contradiction, assume that A is a dual atom of $\mathscr{R}_c$. Hence there exists $G \in \mathscr{G}_c$ such that G does not belong to A. Put $B = T(G)$. Let I be a system of indices, $\operatorname{card} I > G$. Denote $H = \prod_{i\in I} G_i$, where each G_i is equal to G. Then H belongs neither to A nor to B. (In fact, the relation $H \in A$ would imply $G \in A$, which is a contradiction; in view of 7.4, H does not belong to $T(G)$.) We have $A \vee B = \mathscr{G}_c$, hence

$$H = \mathscr{G}_c(H) = (A \vee B)(H) = A(H) \vee^c B(H)\,.$$

If $0 < h_1 \in H$ is such that $h_1(i) > 0$ for each $i \in I$, then h does not belong to B (cf. 7.4). There exists $0 < g_0 \in G$ with $g_0 \notin A(G)$. Let $h \in H$ be such that $h(i) = g_0$ for each $i \in I$. We have $h \in H = A(H) \vee^c B(H) = A(H) + B(H)$, hence there are $u \in A(H)$ and $v \in B(H)$ with $h = u + v$. There exists $i \in I$ such that $v(i) = 0$. Hence $h(i) = u(i)$. Because $0 < u(i) \leqq u \in A(H)$, we obtain $g_0 \in A(H)$. Next, from

$$A(G_i) = G_i \cap A(H)$$

we infer that $g_0 \in A(G_i)$, which is a contradiction.

References

[1] *P. Conrad:* Lattice ordered groups, Tulane University 1970.

[2] *P. Conrad:* K-radical classes of lattice ordered groups. Algebra, Proc. Conf. Carbondale (1980), Lecture Notes Math. 848, 1981, 186–207.

[3] *P. Conrad:* The lattice of all convex l-subgroups of a lattice ordered group. Czech. M. J. 15, 1965, 101–123.

[4] *L. Fuchs:* Partially ordered algebraic systems, Oxford 1963.

[5] *C. Gofman:* Remarks on lattice ordered groups and vector lattices I. Carathéodory functions. Trans. Amer. Math. Soc. 88, 1958, 107—120.
[6] *J. Jakubík:* Radical mappings and radical classes of lattice ordered groups. Symposia Math., 31 Academic Press, New York—London 1977, 451—477.
[7] *J. Jakubík:* Products of radical classes of lattice ordered groups. Acta Math. Univ. Comen. 39, 1980, 31—42.
[8] *J. Jakubík:* On K-radical classes of lattice ordered groups. Czech. Math. J. 33, 1983, 149 to 163.
[9] *J. Jakubík:* Projectable kernel of a lattice ordered group. Universal algebra and applications. Banach Center Publ. Vol. 9, 1982, 105—112.
[10] *J. Jakubík:* Cardinal properties of lattice ordered groups. Fund. Math. 74, 1972, 85—98.
[11] *R. S. Pierce:* Complete Boolean algebras. Proc. Symp. Pure Math., Vol. 2, Amer. Math. Soc., 1961, 129—140.
[12] *F. Šik:* Über die Beziehungen zwischen eigenen Sptizen und minimalen Komponenten einer l-Gruppe. Acta math. Acad. Sci. Hung. 13, 1962, 83—93.

Author's address: 040 01 Košice, Karpatská 5, Czechoslovakia (Matematický ústav SAV).