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Research Article

Radius and Differential Subordination Results for Starlikeness Associated with Limaçon Class

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In this study we investigate the sharp radius of starlikeness of subclasses of Ma and Minda class for the ratio of analytic functions which are related to limaçon functions. This survey is connected also to the first-order differential subordinations. In this context, we get the condition on β for which certain differential subordinations associated with limaçon functions imply Ma and Minda starlike functions. Simple corollaries are provided for certain examples of our results. Finally, we present several geometries related to our study.

1. Introduction and Preliminaries

Let A be the class of normalized analytic functions $f(z)$ having the seres form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n z \in U \coloneqq \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

Let $\mathscr{P}(\alpha)$ denotes the class of all functions $p(z)$ such that Re $p(z) > \alpha, z \in U, 0 \le \alpha < 1$. The case $\mathscr{P}(0) = \mathscr{P}$ is the usual class of Carathéodory functions. Let S denotes the subclasses of A consisting of functions which are univalent in U. We say $f(z) \in A$ is subordinate to $g(z) \in A$ (written as $f \prec g$ or $f(z) \prec g(z)$) if there exists a Schwarz function $w(z)$ such that $f(z) = g(w(z))$ for all $z \in U$. The class S is one of the most vital categories of geometric function theory due to its wide applications in sciences and engineering. For example, univalent functions are extensively used in ODEs and PDEs and operators' theory. Also, they are important in image processing techniques. Among the earlier subclasses of S that had recieved tremendous attention are the classes C and S^* of convex and starlike functions, respectively.

In 1992, Ma and Minda [1] gave a unified characterization of the subclasses of S, which consist of functions that map U onto starlike domains. For this purpose, they considered analytic functions $\mathscr{V}(z)$ with Re $\mathscr{V}(z) > 0$ in U and normalized by $\mathscr{V}(0) = 1$ and $\mathscr{V}'(z) > 0$. Thus, the Ma and Minda class of starlike functions denoted by $S^*(\mathscr{V})$ was defined by the subordination

$$\frac{zf'(z)}{f(z)} \prec \mathscr{V}(z), f \in A, z \in U. \tag{2}$$

In particular, for $\mathscr{V}(z) = (1 + Az)/(1 + Bz), -1 \le B < A \le 1$, the class $S^*(\mathscr{V})$ reduces to the class $S^*(A, B)$ of Janowski starlike functions [2]. The class $S^*(1 - 2\alpha, -1) = S^*(\alpha) = \{f \in A : \text{Re } (zf'(z))/(f(z)) > \alpha, z \in U\}$ is a well-known class of starlike functions of order α. For $\alpha = 0$, $S^*(0) \equiv S^*$ of starlike functions. More recently, many researchers have been investigating subclasses of S^* having nice geometries in the right-half plane. In this direction, let $\mathcal{V} : U \longrightarrow \mathbb{C}$ be defined by

$$\begin{aligned}
&\mathcal{V}_1(z) = \cos(z),\\
&\mathcal{V}_2(z) = 1 + \sin(z),\\
&\mathcal{V}_3(z) = 1 + \sin h^{-1}(z),\\
&\mathcal{V}_4(z) = e^z,\\
&\mathcal{V}_5(z) = z + \sqrt{1+z^2},\\
&\mathcal{V}_6(z) = 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\right]^2,\\
&\mathcal{V}_7(z) = 1 + \left(\frac{4}{3}\right)z + \left(\frac{2}{3}\right)z^2,\\
&\mathcal{V}_8(z) = \sqrt{1+z}.
\end{aligned} \tag{3}$$

Then, for $\mathcal{V}(z) = \mathcal{V}_k(z), k = 1, 2, \cdots 8$, the class $S^*(\mathcal{V})$ reduces to $S^*_{\cos}$,$S^*_{\sin}$, $S^*_{\sin h^{-1}}$,S^*_e, S^*_{cr}, S^*_p, S^*_c, and S^*_l. The geometric properties of these classes have been demonstrated in [3–10] and the references therein.

In 2020, Masih and Kanas [11] explored another novel subclass of $S^*(\mathcal{V})$ with $\mathcal{V}(z) = (1 + sz)^2, 0 < s \le 1/\sqrt{2}$. This class was denoted by $S^*_{\mathcal{L}}$ and functions in it map U onto a region bounded by limaçon. Saliu et al. [12] furthered the investigation of this class and obtained the bounds of the Hankel determinants, sharp radius, and differential implications associated with it. Also, Kanaga and Ravichandran [13] examined $\mathcal{L}_s(z) = (1 + sz)^2, 0 < s \le 1/\sqrt{2}$ and found the smallest disc $\mathcal{D}_a(R_a)$ and the largest disc $\mathcal{D}_a(r_a)$ centered at $(a, 0)$ such that

$$\mathcal{D}_a(r_a) \subset \mathcal{L}(U) \subset \mathcal{D}_a(R_a). \tag{4}$$

This concept was then applied to find the radius of limaçon starlikeness for $S^*(\alpha)$. For more findings associated with $\mathcal{L}_s(z)$, we refer to [14–16].

Let $\mathcal{P}$ and $\mathcal{Q}$ be two subclasses of A defined on U. The $\mathcal{Q}$ radius of $\mathcal{P}$ denoted by $R_{\mathcal{Q}}(\mathcal{P}) \le 1$ is the largest radius such that $f \in \mathcal{P}$ implies the function f_r, defined by $f_r(z) = r^{-1}f(rz) \in \mathcal{Q}$, for all $0 < r \le R_{\mathcal{Q}}(\mathcal{P})$. Radius result with ratio of analytic functions $f, g \in A$ satisfying

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \text{ or } \left|\frac{f(z)}{g(z)} - 1\right| < 1, z \in U \tag{5}$$

was studied by MacGregor [17], and the ones satisfying

$$\left|\frac{f'(z)}{g'(z)} - 1\right| < 1, z \in U \tag{6}$$

was examined by Ratti [18]. Presently, the radius of $S^*(\mathcal{V})$ of various choices of $\mathcal{V}$ for ratio of analytic functions has attracted the interest of researchers. To this end, Ali et al. [19] considered the functions $f \in A$ whose ratio $(f(z))/(g(z))$, $(g(z))/(zp(z))$, and $p(z)$ are each subordinate to $\sqrt{1+z}$ or e^z, for some analytic functions $g(z)$ and $p(z)$. They obtained various radii of starlikeness for these classes. In clases where these are subordinate to $z + \sqrt{1+z^2}$ or e^z, Yadav et al. [20] also obtained various radii of starlikeness. Zhang et al. [21] found the radius of starlikeness connected with subclasses of $S^*(\mathcal{V})$ for the ratio of analytic functions $f(z)$ and $g(z)$ satisfying

$$\begin{gathered}
\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0,\\
\operatorname{Re}\left(\frac{(1+z)^{2n}}{z}\right)g(z) > 0,\\
z \in U, n \in \mathbb{N}.
\end{gathered} \tag{7}$$

The theory of first-order differential subordinations arose from the work of Goluzin and Robertson in 1935 and 1947, respectively. Later, Miller and Mocanu [22, 23] developed and generalized this idea. Using this theory, many results related with the Ma and Minda class have emerged in different directions and perspectives in the literature. For more information in this direction, we refer to the recent work of Cho et al. [24] and Kumar and Gangania [25] with the references therein.

Let $P(\mathcal{L}_s)$ be the class of analytic functions $p(z)$ satisfying the subordination:

$$p(z) \prec (1 + sz)^2, z \in U, 0 < s \le \frac{1}{\sqrt{2}}. \tag{8}$$

Motivated with these aforecited works, we initiated the following classes of analytic functions:

$$\begin{aligned}
\mathcal{H}_1 &= \left\{ f \in A : \frac{f}{g} \in P(\mathcal{L}_s), \text{ for some } g \in A \text{ with } \frac{g}{zp} \in P(\mathcal{L}_s) \text{ and } p \in P(\mathcal{L}_s) \right\},\\
\mathcal{H}_2 &= \left\{ f \in A : \left|\frac{f(z)}{g(z)} - 1\right| < 1, \text{ for some } g \in A \text{ with } \frac{g}{zp} \in P(\mathcal{L}_s) \text{ and } p \in P(\mathcal{L}_s) \right\},\\
\mathcal{H}_3 &= \left\{ f \in A : \frac{f}{zp} \in P(\mathcal{L}_s) \text{ for some } p \in P(\mathcal{L}_s) \right\}.
\end{aligned} \tag{9}$$

Then, we investigate the sharp radius of starlikeness for various subclasses of Ma and Minda class. Moreover, the first-order differential subordination implications are also studied. Some special cases of our findings are given as a simple corollaries. Finally, we illustrate the geometries of some of our findings.

The following lemmas are required for our investigations.

Lemma 1. *Let $p \in P(\mathscr{L}_s)$. Then,*

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2sr}{1-sr}, |z| \le r. \tag{10}$$

This bound is sharp for $p(z) = (1+sz)^2$ with $z = -r$.

Proof. From the property of subordination, we have $p(z) = (1+sw(z))^2$, where $w(z)$ is a Schwarz function. Therefore, a simple computation and Schwarz lemma ([26], p. 166) give

$$\left|\frac{zp'(z)}{p(z)}\right| = \frac{2rs|w'(z)|}{|1+sw(z)|} \le \frac{2sr\left(1-|w(z)|^2\right)}{(1-s|w(z)|)(1-r^2)}. \tag{11}$$

Let $x = |w(z)|$ with $0 \le x \le r$. Then,

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2sr}{1-r^2}\phi(x), \tag{12}$$

where $\phi(x) = (1-x^2)/(1-sx)$. It is easy to see that $\phi(x)$ is continous on $[0, r]$. Then, by the elementary theorem of Real analysis, we have that

$$\left(\inf_{x\in[0,r]} \phi(x), \sup_{x\in[0,r]} \phi(x)\right) \subset R(\phi), \tag{13}$$

where $R(\phi)$ denotes the range of $\phi(x)$. Therefore,

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2sr}{1-r^2} \sup_{x\in[0,r]} \phi(x) = \frac{2sr}{1-sr}. \tag{14}$$

□

Lemma 2 (see [23], Theorem 3.4h, p. 132). *Let $t(z)$ be univalent in U and let ψ and φ be analytic in a domain D containing $t(U)$ with $\psi(w) \ne 0$, where $w \in t(U)$. Set $Q(z) = zt'(z) \cdot \psi(t(z))$, $h(z) = \varphi(t(z)) + Q(z)$, and suppose that either*

(i) $h(z)$ is convex or $Q(z)$ is starlike univalent in U

(ii) $\mathrm{Re}\,(zh'(z))/(Q(z)) = \mathrm{Re}\,((\varphi'(t(z)))/(\psi(t(z))) + (zQ'(z))/(Q(z))) > 0$

If $p(z)$ is analytic in U with $p(0) = t(0)$, $p(U) \subset D$ and

$$\varphi(p(z)) + zp'(z)\psi(p(z)) \prec \varphi(t(z)) + zt'(z)\psi(t(z)) = h(z), \tag{15}$$

then $p(z) \prec t(z)$, and $t(z)$ is the best dominant in the sense that $p \prec q \Rightarrow t \prec q$ for all q.

2. The Class $\mathscr{H}_1$

Define the function $f_1, g_1 : U \longrightarrow \mathbb{C}$ by

$$f_1(z) = z(1+sz)^6\, g_1(z) = z(1+sz)^4, 0 < s \le \frac{1}{\sqrt{2}}. \tag{16}$$

It is easy to see that $f_1 \in \mathscr{H}_1$ and, thus, $\mathscr{H}_1 \ne \varnothing$. Also, the class contains nonunivalent functions, since

$$f'(z) = (1+7s)(1+sz)^5 \tag{17}$$

varnishes at $z = -1/7s$. Hence, the radius of univalence for $\mathscr{H}_1$ is $1/7s$ for $1/7 < s \le 1/\sqrt{2}$. It is observed that for $\alpha = 0$ in Corollary 4 (i), this radius coincides with the radius of starlikeness for the class.

Let $f \in \mathscr{H}_1$; then, there exists $g \in A$ such that

$$\begin{aligned} \frac{f(z)}{g(z)} &= p_1(z), \\ \frac{zp(z)}{g(z)} &= p_2(z), \end{aligned} \tag{18}$$

for $p, p_1, p_2 \in P(\mathscr{L}_s)$. Therefore, a computation yields

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \left|\frac{zp_1'(z)}{p_1(z)}\right| + \left|\frac{zp_2'(z)}{p_2(z)}\right| + \left|\frac{zp'(z)}{p(z)}\right|. \tag{19}$$

Then, from Lemma 1, it follows that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{6sr}{1-sr}, |z| \le r. \tag{20}$$

3. The Class $\mathscr{H}_2$

Let the function $f_2 : U \longrightarrow \mathbb{C}$ be defined by

$$f_2(z) = z(1+z)(1+sz)^4, 0 < s \le \frac{1}{\sqrt{2}}. \tag{21}$$

Then, the functions $f_2(z)$ and $g_1(z)$ satisfy

$$\left|\frac{f_2(z)}{g_1(z)} - 1\right| < 1\,\frac{g_1}{zp} \in P(\mathscr{L}_s), \tag{22}$$

for some $p = (1+sz)^2$. Thus, $f_2 \in \mathscr{H}_2$ and $\mathscr{H}_2 \ne \varnothing$. Since

$$\begin{aligned} f_2'(z) &= \left(6sz^2 + (5s+2)z + 1\right)(1+sz) \\ &= 0 \text{ for } z = \frac{-2}{5s+2+\sqrt{25s^2-4s+4}}, \end{aligned} \tag{23}$$

the class $\mathscr{H}_2$ contains some nonunivalent functions. Hence, the radius of univalence for this class is $2/(5s+2+\sqrt{25s^2-4s+4})$.

Let $f \in \mathscr{H}_2$. Then, there exists an analytic function $g(z)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1, \quad \frac{g}{zp} \in P(\mathscr{L}_s), \tag{24}$$

for some $p \in P(\mathscr{L}_s)$. Since the disc $|w-1|<1$ implies Re $1/(w)>1/2$, then Re $g(z)/f(z)>1/2$. Let $p_1, p_2 : U \longrightarrow \mathbb{C}$ be defined such that

$$\frac{g(z)}{f(z)} = p_1(z), \quad \frac{g(z)}{zp(z)} = p_2(z), \quad p_1, p_2 \in P(\mathscr{L}_s). \tag{25}$$

Then,

$$f(z) = \frac{zp(z)p_2(z)}{p_1(z)}, \tag{26}$$

and a computation gives

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{zp_2'(z)}{p_2(z)}\right| + \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zp_1'(z)}{p_1(z)}\right|. \tag{27}$$

It is known from Lemma 2 of [27] that for $p \in P(\alpha)$, we have

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)} |z| \le r. \tag{28}$$

Using this fact for the case $\alpha = 1/2$ and Lemma 1 in (27), we arrive at

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{(1+4s)r-5sr^2}{(1-r)(1-sr)}. \tag{29}$$

It also follows from (29) that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{6sr^2-(5s+2)r+1}{(1-r)(1-sr)} > 0, \tag{30}$$

provided $r \le R_{S^*}(\mathscr{H}_2) = 2/(5s+2+\sqrt{25s^2-4s+4})$. For $f_2(z)$, we have

$$\frac{zf_2'(z)}{f_2(z)} = \frac{6sz^2+(5s+2)z+1}{(1+z)(1+sz)} = 0, \tag{31}$$

at $z = -R_{S^*}(\mathscr{H}_2)$. This shows that the radius is sharp. Hence, the radius of starlikeness for the class $\mathscr{H}_2$ is the same as the radius of univalence for the class.

4. The Class $\mathscr{H}_3$

Let $f_3(z) = g_1(z)$. Then, $f_3 \in \mathscr{H}_3$. It is obvious that $\mathscr{H}_3$ contains nonunivalent functions, since

$$f_3'(z) = (1+5sz)(1+sz)^3 \tag{32}$$

varnishes at $z = -1/5s$ for $1/5 < s \le 1/\sqrt{2}$. Hence, the radius of univalence for $\mathscr{H}_3$ is $1/5s$ for $1/5 < s \le 1/\sqrt{2}$. We notice that this radius coincides with the radius of starlikess of the class for the choice of $\alpha = 0$ in Corollary 4 (iii).

Let $p_1, p : U \longrightarrow \mathbb{C}$ be defined such that for $f \in \mathscr{H}_3$,

$$\frac{zf(z)}{p(z)} = p_1(z), p_1, p \in P(\mathscr{L}_s). \tag{33}$$

Then, an obvious calculation yields

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zp_1'(z)}{p_1(z)}\right|. \tag{34}$$

In view of Lemma 1, we arrive at

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{4sr}{1-sr}. \tag{35}$$

5. Radius of Starlikeness

In this section, we obtain the radii of starlikeness of the classes $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ for different Ma and Minda starlike classes of functions.

Silverman ([28], pp. 50-51) showed that

$$\{w \in \mathbb{C} : |w-c| < d\} \subset \{w \in \mathbb{C} : |w-a| < b\} \tag{36}$$

holds if and only if $|a-c| \le |b-d|$. Using this fact, we obtain the radii of Janowski starlikeness for $\mathscr{H}_1$ and $\mathscr{H}_3$.

Theorem 3. *The following sharp results hold for Janowski starlike class $S^*(A,B)$:*

(i) $R_{S^*(A,B)}(\mathscr{H}_1) = (A-B)/(A-7B+6)s$, $(A-B)/(A-7B+6) < s \le 1/\sqrt{2}$

(ii) $R_{S^*(A,B)}(\mathscr{H}_3) = (A-B)/(A-5B+4)s$, $(A-B)/(A-5B+4) < s \le 1/\sqrt{2}$

Proof.

(i) Let $f \in \mathscr{H}_1$. Then, f satisfies (20). To prove our result, it suffices to show that

$$\left\{w \in \mathbb{C} : |w-1| < \frac{6sr}{1-sr}\right\} \subset \left\{w \in \mathbb{C} : \left|w - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}. \tag{37}$$

Then, from (36), it follows that

$$\frac{1-AB}{1-B^2} - 1 \le \frac{A-B}{1-B^2} - \frac{6sr}{1-sr}, \tag{38}$$

where we have that

$$r \le R_{S^*(A,B)} = \frac{A-B}{(A-7B+6)s}, s > \frac{A-B}{(A-7B+6)}. \tag{39}$$

For $f_1(z)$ in (16), we have

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1+7sz}{1+sz}. \tag{40}$$

At $z = -R_{S^*(A,B)}$,

$$\left|\frac{zf_1'(z)}{f_1(z)} - \frac{1-AB}{1-B^2}\right| = \left|\frac{1-A}{1-B} - \frac{1-AB}{1-B^2}\right| = \frac{A-B}{1-B^2} \tag{41}$$

(ii) Let $f \in \mathscr{H}_3$. Then, f satisfies (35). We need to show that

$$\left\{w : |w-1| < \frac{4sr}{1-sr}\right\} \subset \left\{w : \left|w - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}. \tag{42}$$

Then, from (36), it follows that

$$\frac{1-AB}{1-B^2} - 1 \le \frac{A-B}{1-B^2} - \frac{4sr}{1-sr}, \tag{43}$$

which gives

$$r \le R_{S^*(A,B)} = \frac{A-B}{(A-5B+4)s}, s > \frac{A-B}{(A-5B+4)}. \tag{44}$$

For $f_3(z)$ in Section 4 at $z = -R_{S^*(A,B)}$, we have

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1+5sz}{1+sz} = \frac{1-A}{1-B}, \tag{45}$$

which implies

$$\left|\frac{zf_3'(z)}{f_3(z)} - \frac{1-AB}{1-B^2}\right| = \left|\frac{1-A}{1-B} - \frac{1-AB}{1-B^2}\right| = \frac{A-B}{1-B^2} \tag{46}$$

□

Corollary 4. *The following sharp results hold for $S^*(\alpha)$:*

(i) $R_{S^*(\alpha)}(\mathscr{H}_1) = (1-\alpha)/(7-\alpha)s$, $(1-\alpha)/(7-\alpha) < s \le 1/\sqrt{2}$

(ii) $R_{S^*(\alpha)}(\mathscr{H}_2) = (2(1-\alpha))/((5-\alpha)s + 2 - \alpha + \sqrt{(5-\alpha)^2 s^2 + 4(1-s) + \alpha(14s+\alpha) - 2\alpha(1-\alpha)s})$

(iii) $R_{S^*(\alpha)}(\mathscr{H}_2) = (1-\alpha)/(5-\alpha)s$, $(1-\alpha)/(5-\alpha) < s \le 1/\sqrt{2}$

Proof. The proof of (i) and (iii) are direct from Theorem 3 for $A = 1 - 2\alpha$.

$(0 < \alpha \le 1)$ and $B = -1$. For (ii), $f \in S^*(\alpha)$ provided (30) satisfies

$$\frac{6sr^2 - (5s+2)r + 1}{(1-r)(1-sr)} > \alpha. \tag{47}$$

That is, $T(r) := (6-\alpha)sr^2 - [(5-\alpha)s + 2 - \alpha]r + (1-\alpha) \ge 0$. Then, $T(0) = 1 - \alpha > 0$ and $T(1) = s - 2 < 0$. Therefore, there exists $R_{S^*(\alpha)}(\mathscr{H}_2) \in (0,1)$ such that $T(R_{S^*(\alpha)}(\mathscr{H}_2)) = 0$, where we obtain $R_{S^*(\alpha)}(\mathscr{H}_2)$ given in the theorem. For sharpness, consider the function $f_2(z)$ in (21). Then, for $z_0 = -R_{S^*(\alpha)}(\mathscr{H}_2)$, we have that

$$\frac{zf_2'(z_0)}{f_2(z_0)} = \frac{6sz_0^2 + (5s+2)z_0 + 1}{(1+sz_0)(1+z_0)} = \alpha. \tag{48}$$

□

The class S_p^* is the class of parabolic starlike functions introduced by Rønning [8]. This class consists of functions $f \in A$ satisfying the inequality

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in U. \tag{49}$$

For $1/2 < a < 3/2$, Shanmugam and Ravichandran [29] proved the following inclusion relation for the class S_p^*.

$$\{w \in \mathbb{C} : |w-a| < a - 1/2\} \subset \{w \in \mathbb{C} : \operatorname{Re}\, w > |w-a|\} := \Delta_p. \tag{50}$$

Theorem 5. *The following sharp results hold for S_p^* :*

(i) $R_{S_p^*}(\mathscr{H}_1) = 1/(13s)$, $1/13 < s \le 1/\sqrt{2}$

(ii) $R_{S_p^*}(\mathscr{H}_2) = 2/(3(1+2s)+\sqrt{9+10s+81s^2})$

(iii) $R_{S_p^*}(\mathscr{H}_3) = 1/(9s)$, $1/9 < s \le 1/\sqrt{2}$

Proof.

(i) For $a=1$, relation (50) implies that disc (20) lies in Δ_p provided

$$\frac{6rs}{1-sr} \le \frac{1}{2}, \tag{51}$$

or if $r \le R_{S_p^*}(\mathscr{H}_1) = 1/(13s)$, $1/13 < s \le 1/\sqrt{2}$. The function $f_1(z)$ shows that the radius of parabolic starlike functions is sharp. For this function, we have

$$\frac{zf_1'(z)}{f_1(z)} = 1 + \frac{6sz}{1+sz}. \tag{52}$$

At $z = -R_{S_p^*}(\mathscr{H}_1)$, a calculation shows that

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)} = \frac{1}{2} = \left|\frac{zf_1'(z)}{f_1(z)} - 1\right| \tag{53}$$

(ii) Since the center of disc (29) is 1, then it stays inside Δ_p if

$$\frac{(1+4s)r - 5sr^2}{(1-r)(1-sr)} \le \frac{1}{2}, \tag{54}$$

or equivalently $r \le R_{S_p^*}(\mathscr{H}_2) = 2/(3(1+2s)+\sqrt{9+10s+81s^2})$. For $f_2(z)$ at $z = -R_{S_p^*}(\mathscr{H}_2)$, a computation gives

$$\frac{zf_2'(z)}{f_2(z)} = \frac{6sz^2 + (2+5s)z + 1}{(1+z)(1+sz)},$$
$$\left|\frac{zf_2'(z)}{f_2(z)} - 1\right| = \frac{1}{2} = \operatorname{Re}\frac{zf_2'(z)}{f_2(z)} \tag{55}$$

(iii) Since the center of disc (35) is $a=1$, then inclusion (50) holds provided

$$\frac{4sr}{1-sr} < \frac{1}{2}, \tag{56}$$

which implies $r \le R_{S_p^*}(\mathscr{H}_3) = 1/7s$, $1/7 < s \le 1/\sqrt{2}$. For the function $f_3(z)$ at $z = -R_{S_p^*}(\mathscr{H}_3)$, we have

$$\operatorname{Re}\frac{zf_3'(z)}{f_3(z)} = \frac{1}{2} = \left|\frac{zf_3'(z)}{f_3(z)} - 1\right| \tag{57}$$

□

The class S_e^* of the exponential starlike functions $f(z)$ satisfies the inequality

$$\left|\log\left(\frac{zf'(z)}{f(z)}\right)\right| < 1. \tag{58}$$

This class was initiated by Mendiratta et al. [6] in 2015. They also proved that for $e^{-1} < a \le (e+e^{-1})/2$,

$$\{w \in \mathbb{C} : |w-a| < a - 1/e\} \subset \{w \in \mathbb{C} : |\log(w)| < 1\} := \Delta_e. \tag{59}$$

Theorem 6. *The following sharp results hold for S_e^* :*

(i) $R_{S_e^*}(\mathscr{H}_1) = (e-1)/((7e-1)s)$, $(e-1)/(7e-1) < s \le 1/\sqrt{2}$

(ii) $R_{S_e^*}(\mathscr{H}_2) = (2(e-1))/((5e-1)s + 2e - 1 + \sqrt{(5e-1)^2 s^2 - (4e^2 - 14e + 2)s + (2e-1)^2})$

(iii) $R_{S_e^*}(\mathscr{H}_3) = (e-1)/(5e-1)s$, $(e-1)/(5e-1) < s \le 1/\sqrt{2}$

Proof.

(i) For $a=1$, relation (59) implies that disc (20) lies inside Δ_e whenever

$$\frac{6rs}{1-sr} \le 1 - \frac{1}{e}. \tag{60}$$

Equivalently, if $r \le R_{S_e^*}(\mathscr{H}_2) = (e-1)/(7e-1)s$, $(e-1)/(7e-1) < s \le 1/\sqrt{2}$. The function $f_1(z)$ shows that this radius is best possible since

$$\frac{zf_1'(z)}{f_1(z)} = 1 + \frac{6rs}{1-sr}, \tag{61}$$

and at $z = -R_{S_e^*}(\mathscr{H}_2)$, we have

$$\left|\log\left(\frac{zf_1'(z)}{f_1(z)}\right)\right| = \left|\log\left(e^{-1}\right)\right| = 1 \tag{62}$$

(ii) Since $e^{-1} < 1 = a < (e^{-1} + e)/2$, inclusion (59) implies that disc (29) is in Δ_e provided

$$\frac{(1+4s)r - 5sr^2}{(1-r)(1-sr)} \le 1 - \frac{1}{e}, \tag{63}$$

which at the same time means

$$r \le R_{S_e^*}(\mathscr{H}_2) = \frac{2(e-1)}{(5e-1)s + 2e - 1 + \sqrt{(5e-1)^2 s^2 - (4e^2 - 14e + 2)s + (2e-1)^2}}. \tag{64}$$

At $z = -R_{S_e^*}(\mathscr{H}_2)$ for $f_2(z)$, we arrive at

$$\left|\log\left(\frac{zf_2'(z)}{f_2(z)}\right)\right| = \left|\log\left(e^{-1}\right)\right| = 1 \tag{65}$$

(iii) For $a = 1$, disc (35) lies completely in Δ_e if

$$\frac{4rs}{1-sr} \le 1 - \frac{1}{e}, \tag{66}$$

or $(5e-1)sr + 1 - e \le 0$. Thus, the S_e^* of the function $f \in \mathscr{H}_3$ is $R_{S_e^*}(\mathscr{H}_3) = (e-1)/(5e-1)s$, $(e-1)/(5e-1) < s \le 1/\sqrt{2}$. The sharpness is seen from the function $f_3(z)$ at $z = -R_{S_e^*}(\mathscr{H}_3)$, i.e.,

$$\left|\log\left(\frac{zf_3'(z)}{f_3(z)}\right)\right| = \left|\log\left(e^{-1}\right)\right| = 1 \tag{67}$$

□

In 2016, Sharma et al. [9] introduced the class S_c^* of functions that map the open unit disc onto a Cardiod domain. A function $f \in S_c^*$ if it satisfies the subordination:

$$\frac{zf'(z)}{f(z)} \prec 1 + \left(\frac{4}{3}\right)z + \left(\frac{2}{3}\right)z^2. \tag{68}$$

The following inclusion relation was also established in [9]:

$$\{w \in \mathbb{C} : |w - a| < (3a-1)/3\} \subset \Delta_c, \tag{69}$$

provided $1/3 < a < 5/3$, where

$$\Delta_c = \left\{x + iy : \left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0\right\}. \tag{70}$$

Theorem 7. *The following sharp results hold for the class S_c^* :*

(i) $R_{S_c^*}(\mathscr{H}_1) = 1/10s$, $1/10 < s \le 1/\sqrt{2}$

(ii) $R_{S_c^*}(\mathscr{H}_2) = 4/(5 + 14s + \sqrt{196s^2 + 4s + 25})$

(iii) $R_{S_c^*}(\mathscr{H}_3) = 1/7s$, $1/7 < s \le 1/\sqrt{2}$

Proof.

(i) Since the center of (20) is 1, then by relation (69), we have that $f \in \mathscr{H}_1$ belongs to S_c^* provided

$$\frac{6sr}{1-sr} \le 1 - \frac{1}{3}, \tag{71}$$

or $r \le R_{S_c^*}(\mathscr{H}_1) = 1/10s$, $1/10 < s \le 1/\sqrt{2}$. This radius cannot be improved since the function $f_1(z)$ assumes the extremum, i.e., at $z = -R_{S_c^*}(\mathscr{H}_1)$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1}{3} = \mathscr{V}_7(-1) \in \partial\mathscr{V}_7(U) \tag{72}$$

(ii) From (69), it follows that disc (29) stays inside Δ_c whenever

$$\frac{(1+4s)r - 5sr^2}{(1-r)(1-sr)} \le \frac{2}{3}. \tag{73}$$

This gives $r \le R_{S_c^*}(\mathscr{H}_2) = 4/(5 + 14s + \sqrt{196s^2 + 4s + 25})$. To prove that this radius is sharp, we consider the function $f_2(z)$ such that at $z=-R_{S_c^*}(\mathscr{H}_2)$, we arrive at

$$\frac{zf_2'(z)}{f_2(z)}=\frac{1}{3}=\mathscr{V}_7(-1)\in\partial\mathscr{V}_7(U) \quad (74)$$

(iii) Let $f\in\mathscr{H}_3$. Then, we see that $f\in S_c^*$ if

$$\frac{4sr}{1-sr}\le\frac{2}{3}, \quad (75)$$

which gives $r\le R_{S_c^*}(\mathscr{H}_3)=1/7s, 1/7<s\le 1/\sqrt{2}$. For the function $f_3(z)$, we see that

$$\frac{zf_3'(z)}{f_3(z)}=\frac{1}{3}=\mathscr{V}_7(-1)\in\partial\mathscr{V}_7(U), \quad (76)$$

which shows that the radius is sharp

□

In 2019, Cho et al. [4] considered and investigated the class $S_{\sin}^*=\{f\in A: zf'(z)/f(z)\prec 1+\sin(z)\}$. They also proved the following:

$$\{w\in\mathbb{C}:|w-a|<\sin(1)-|a-1|\}\subset\Delta_s, \quad (77)$$

if $1-\sin(1)<a<1+\sin(1)$, where Δ_s is the image of U under the function $\mathscr{V}_2(z)$.

Theorem 8. *The following sharp results hold for* $S_{\sin}^*$:

(i) $R_{S_{\sin}^*}(\mathscr{H}_1)=\sin(1)/(6+\sin(1))s, \sin(1)/(6+\sin(1))<s\le 1/\sqrt{2}$

(ii) $R_{S_{\sin}^*}(\mathscr{H}_2)=2\sin(1)/((4+\sin(1))s+1+\sin(1)+\sqrt{(4+\sin(1))^2s^2+(8-10\sin(1)-2\sin^2(1))s+(1+\sin(1))^2})$

(iii) $R_{S_{\sin}^*}(\mathscr{H}_3)=\sin(1)/(4+\sin(1))s, \sin(1)/(4+\sin(1))<s\le 1/\sqrt{2}$

Proof.

(i) It is easy to see from (77) that disc (20) stays inside the region $\Delta_{\sin}$ provided

$$\frac{6sr}{1-sr}\le\sin(1), \quad (78)$$

which implies that $r\le R_{S_{\sin}^*}(\mathscr{H}_1)=\sin(1)/(6+\sin(1))s, \sin(1)/(6+\sin(1))<s\le 1/\sqrt{2}$. To show that this radius is best possible, we consider the function $f_1(z)$ so that

$$\frac{zf_1'(z)}{f_1(z)}=1+\frac{6sz}{1+sz}. \quad (79)$$

At $z=-R_{S_{\sin}^*}(\mathscr{H}_1)$, we arrive at

$$\frac{zf_1'(z)}{f_1(z)}=1-\sin(1)=\mathscr{V}_2(-1)\in\partial\mathscr{V}_2(U) \quad (80)$$

(ii) Since $1-\sin(1)<1=a<1+\sin(1)$, then disc (29) satisfies relation (77) provided

$$\frac{(1+4s)r-5sr^2}{(1-r)(1-sr)}\le\sin(1). \quad (81)$$

This gives $r\le R_{S_{\sin}^*}(\mathscr{H}_2)=2\sin(1)/((4+\sin(1))s+1+\sin(1)+\sqrt{(4+\sin(1))^2s^2+(8-10\sin(1)-2\sin^2(1))s+(1+\sin(1))^2})$.

For the function $f_2(z)$, the result is sharp. Indeed, at $z=-R_{S_{\sin}^*}(\mathscr{H}_2)$, we have

$$\frac{zf_2'(z)}{f_2(z)}=1-\sin(1)=\mathscr{V}_2(-1)\in\partial\mathscr{V}_2(U) \quad (82)$$

(iii) Proceeding as in the above cases and using inclusion (77), we find that $f\in S_{\sin}^*$ if

$$\frac{4sr}{1-sr}\le\sin(1), \quad (83)$$

which holds for $r\le R_{S_{\sin}^*}(\mathscr{H}_3)=\sin(1)/(4+\sin(1))s, \sin(1)/(4+\sin(1))<s\le 1/\sqrt{2}$. The sharpness of the radius is assumed by the function $f_3(z)$

□

Kumar and Arora [5] investigated the subclass $S_{\sin h^{-1}}^*$ of Ma and Minda class of functions that map U onto a petal domain. They also proved the inclusion relation

$$\{w\in\mathbb{C}:|w-a|<a-(1-\sin h^{-1}(1))\}\subset\Delta_{\sin h^{-1}}, \quad (84)$$

where $\Delta_{\sin h^{-1}}$ is the image of U under the function $\mathscr{V}_3(z)$. In view of the procedure of Theorem 8, (20), (29), and (35), we have the following theorem.

Theorem 9. *The following sharp results hold for* $S_{\sin h^{-1}}^*$:

(i) $R_{S_{\sin h^{-1}}^*}(\mathscr{H}_1)=\sin h^{-1}(1)/(6+\sin h^{-1}(1))s, \sin h^{-1}(1)/(6+\sin h^{-1}(1))<s\le 1/\sqrt{2}$

(ii) $R_{S^*_{\sin h^{-1}}}(\mathscr{H}_2) = 2\sin h^{-1}(1)/((4+\sin h^{-1}(1))s + 1 + \sin h^{-1}(1) + \sqrt{(4+\sin h^{-1}(1))^2 s^2 + (8 - 10\sin h^{-1}(1) - 2(\sin h^{-1}(1))^2)s + (1+\sin h^{-1}(1))^2})$

(iii) $R_{S^*_{\sin h^{-1}}}(\mathscr{H}_3) = \sin h^{-1}(1)/(4+\sin h^{-1}(1))s$, $\sin h^{-1}(1)/(4+\sin h^{-1}(1)) < s \le 1/\sqrt{2}$

In 2021, Bano and Mohsan [3] introduced the subclass $S^*_{\cos}$ of analytic functions characterized by the subordination

$$\frac{zf'(z)}{f(z)} \prec \cos(z), z \in U. \tag{85}$$

They also proved that

$$\{w \in \mathbb{C} : |w-a| < a - \cos(1)\} \subset \Delta_{\cos}, \tag{86}$$

where $\Delta_{\cos}$ is the image domain of U under the mapping $\mathscr{V}_1(z) = \cos(z)$.

Theorem 10. *The following sharp results hold for* $S^*_{\cos}$:

(i) $R_{S^*_{\cos}}(\mathscr{H}_1) = (1-\cos(1))/(7-\cos(1))s$, $(1-\cos(1))/(7-\cos(1)) < s \le 1/\sqrt{2}$

(ii) $R_{S^*_{\cos}}(\mathscr{H}_2) = 2(1-\cos(1))/((5-\cos(1))s + (2-\cos(1)) + \sqrt{(5-\cos(1))^2 s^2 - (2\cos^2(1) - 14\cos(1) + 4)s + (2-\cos(1))^2})$

(iii) $R_{S^*_{\cos}}(\mathscr{H}_3) = (1-\cos(1))/(5-\cos(1))s$, $(1-\cos(1))/(5-\cos(1)) < s \le 1/\sqrt{2}$

Proof.

(i) Since disc (20) has a center 1, then relation (86) implies that this disc lies inside $\Delta_{\cos}$ whenever

$$\frac{6rs}{1-sr} \le 1 - \cos(1), \tag{87}$$

or $r \le R_{S^*_{\cos}}(\mathscr{H}_1) = (1-\cos(1))/(7-\cos(1))s$, $(1-\cos(1))/(7-\cos(1)) < s \le 1/\sqrt{2}$. The sharpness of this radius is achieved by the function $f_1(z)$ at $z = -R_{S^*_{\cos}}(\mathscr{H}_1)$

(ii) For $a = 1$, disc (29) is inside the domain $\Delta_{\cos}$ provided

$$\frac{(1+4s)r - 5sr^2}{(1-r)(1-sr)} \le 1 - \cos(1), \tag{88}$$

which means that $r \le R^*_{S^*_{\cos}}(\mathscr{H}_2) = 2(1-\cos(1))/((5-\cos(1))s + (2-\cos(1)) + \sqrt{(5-\cos(1))^2 s^2 - (2\cos^2(1) - 14\cos(1) + 4)s + (2-\cos(1))^2})$.

The function $f_2(z)$ shows that the radius cannot be improved since at $z = -R^*_{S^*_{\cos}}(\mathscr{H}_2)$, we have

$$\frac{zf_2'(z)}{f_2(z)} = \cos(1) = \mathscr{V}_1(-1) \in \partial\mathscr{V}_1(U) \tag{89}$$

(iii) It is observed that disc (35) is in $\Delta_{\cos}$ if

$$\frac{4sr}{1-sr} \le 1 - \cos(1), \tag{90}$$

which equivalently implies $r \le R_{S^*_{\cos}}(\mathscr{H}_3) = (1-\cos(1))/(5-\cos(1))s$, $(1-\cos(1))/(5-\cos(1)) < s \le 1/\sqrt{2}$. For the function $f_3(z)$, we see that at $z = -R_{S^*_{\cos}}(\mathscr{H}_3)$,

$$\frac{zf_3'(z)}{f_3(z)} = \cos(1) = \mathscr{V}_1(-1) \in \partial\mathscr{V}_1(U) \tag{91}$$

□

6. Differential Subordination Implications

In this section, we present some first-order differential subordinations associated with $\mathscr{L}_s(z)$.

Theorem 11. *Let $p(z)$ be analytic in $\mathscr{U}$ with $p(0) = 1$, and suppose*

$$1 + \beta z p'(z) \prec \mathscr{L}_s(z), z \in \mathscr{U}. \tag{92}$$

Then, we have the following subordination results:

(i) $p(z) \prec (1+Az)/(1+Bz)$, *for* $\beta \ge \max\{\beta_1, \beta_2\}$ *and* $-1 < B < A \le 1$ *with* $\beta_1 = (s(4-s)(1-B))/(2(A-B))$, $\beta_2 = s(4+s)(1+B)/2(A-B)$ *and*

$$\max\{\beta_1, \beta_2\} = \begin{cases} \beta_2, & \text{if } B > -\dfrac{s}{4}, \\ \beta_1, & \text{if } B < -\dfrac{s}{4} \end{cases} \tag{93}$$

(ii) $p(z) \prec \mathscr{V}_8(z)$ *for* $\beta \ge (s(4+s))/(2(\sqrt{2}-1))$

(iii) $p(z) \prec \mathscr{V}_4(z)$ *for* $\beta \ge es(4-s)/2(e-1)$

(iv) $p(z) \prec \mathscr{V}_7(z)$ *for* $\beta \ge 3s(4-s)/4$

(v) $p(z) \prec \mathscr{V}_2(z)$ *for* $\beta \ge s(4+s)/2\sin(1)$

(vi) $p(z) \prec \mathscr{V}_3(z)$ *for* $\beta \ge s(4+s)/2\sin h^{-1}(1)$

(vii) $p(z) \prec \mathscr{V}_5(z)$ *for* $\beta \ge s(4-s)/2(2-\sqrt{2})$

These bounds are sharp.

Proof. Consider the analytic function $t_\beta : \bar{U} \longrightarrow \mathbb{C}$ such that

$$t_\beta(z) = 1 + \frac{sz(sz+4)}{2\beta} \tag{94}$$

is a solution of the differential equation

$$1 + \beta z t'_\beta(z) = (1+sz)^2. \tag{95}$$

Let $\varphi(w) = 1$ and $\psi(w) = \beta$. Then, the function $Q : \bar{U} \longrightarrow \mathbb{C}$ is defined by $Q(z) = z t'_\beta(z)\psi(t_\beta(z)) = sz(sz+2)$ such that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = 2\left[1 - \operatorname{Re}\left(\frac{1}{2+sz}\right)\right] \geq \frac{2(1-s)}{2-s} > 0. \tag{96}$$

This means that $Q(z)$ is a starlike univalent function in U. It is easy to see that the function $h(z) = \varphi(t_\beta(z)) + Q(z) = (1+sz)^2$ satisfies

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0. \tag{97}$$

Therefore, by Lemma 2, we have that

$$1 + \beta z p'(z) \prec 1 + \beta z t'_\beta(z) \Rightarrow p(z) \prec t_\beta(z). \tag{98}$$

Each subordinations of Theorem 11 is equivalent to

$$p(z) \prec \mathbb{P}(z), \tag{99}$$

for each superordinate function $\mathbb{P}(z)$ in the theorem, which holds if $t_\beta(z) \prec \mathbb{P}(z), z \in U$. Then,

$$\mathbb{P}(-1) < t_\beta(-1) < t_\beta(1) < \mathbb{P}(1). \tag{100}$$

This yields the necessary condition for which $p(z) \prec \mathbb{P}(z), z \in U$. Looking at the geometry of each of the functions $\mathbb{P}(z)$, it is observed that this condition is also sufficient.

(i) Let $\mathbb{P}(z) = (1+Az)/(1+Bz)$. Then,

$$\begin{aligned} t_\beta(-1) &\geq \frac{1-A}{1-B}, \text{whenever } \beta \geq \frac{s(4-s)(1-B)}{2(A-B)} \coloneqq \beta_1, \\ t_\beta(1) &\leq \frac{1+A}{1+B}, \text{whenever } \beta > \frac{s(4+s)(1+B)}{2(A-B)} \coloneqq \beta_2. \end{aligned} \tag{101}$$

We notice that

$$\max\{\beta_1, \beta_2\} = \beta = \begin{cases} \beta_2, & \text{if } B > -\dfrac{s}{4}, \\ \beta_1, & \text{if } B < -\dfrac{s}{4}. \end{cases} \tag{102}$$

Thus,

$$t_\beta(z) \prec \frac{1+Az}{1+B} \text{ if } \beta \geq \max\{\beta_1, \beta_2\} \tag{103}$$

(ii) $\mathbb{P}(z) = \sqrt{1+z}$. Then, the inequalities $t_\beta(-1) \geq 0$ and $t_\beta(1) \leq \sqrt{2}$ reduce to $\beta \geq s(4-s)/2 = \beta_1$ and $\beta \geq s(4+s)/2(\sqrt{2}-1) = \beta_2$. We note that $\beta_1 - \beta_2 < 0$. Thus,

$$t_\beta(z) \prec \sqrt{1+z} \text{ if } \beta \geq \max\{\beta_1, \beta_2\} = \beta_2 \tag{104}$$

(iii) Let $\mathbb{P}(z) = e^z$. Then, from (100), we have

$$\begin{aligned} t_\beta(-1) &\geq e^{-1}, \text{whenever } \beta \geq \frac{es(4-s)}{2(e-1)} \coloneqq \beta_1, \\ t_\beta(1) &\leq e, \text{whenever } \beta > \frac{es(4+s)}{2(e-1)} \coloneqq \beta_2. \end{aligned} \tag{105}$$

We notice that $\beta_1 - \beta_2 > 0$. Therefore, the subordination $t_\beta(z) \prec e^z$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$

(iv) Let $\mathbb{P}(z) = 1 + (4/3)z + (2/3)z^2$. Then, the inequalities $t_\beta(-1) \geq 1/3$ and $t_\beta(1) \leq 3$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = 3s(4-s)/4$ and $\beta_1 = s(4+s)/4$. A calculation shows that $\beta_1 - \beta_2 = s(2-s) > 0$. Therefore, the subordination $t_\beta(z) \prec 1 + (4/3)z + (2/3)z^2$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$

(v) Let $\mathbb{P}(z) = 1 + \sin(z)$. Then, the inequalities $t_\beta(-1) \geq 1 - \sin(1)$ and $t_\beta(1) \leq 1 + \sin(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = s(4-s)/2 \sin(1)$ and $\beta_2 = s(4+s)/2 \sin(1)$. A calculation shows that $\beta_1 - \beta_2 = s(2-s) > 0$. Therefore, the subordination $t_\beta(z) \prec 1 + \sin(z)$ holds provided $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$

(vi) Let $\mathbb{P}(z) = 1 + \sin h^{-1}(z)$. Then, following the same line of proof of (v), we show that the subordination $t_\beta(z) \prec 1 + \sin h^{-1}(z)$ holds provided $\beta \geq s(4+s)/2 (\sin h^{-1}(1))$

(vii) Let $\mathbb{P}(z) = z + \sqrt{1+z^2}$. Then, the inequalities $t_\beta(-1) \geq \sqrt{2} - 1$ and $t_\beta(1) \leq 1 + \sqrt{2}$ yield $\beta \geq \beta_1 = s(4-s)/2(2-\sqrt{2})$ and $\beta \geq \beta_2 = s(4+s)/2\sqrt{2}$. A calculation shows that $\beta_1 - \beta_2 > 0$. Therefore, the subordination $t_\beta(z) \prec z + \sqrt{1+z^2}$ holds provided $\beta \geq s(4-s)/2(2-\sqrt{2})$

□

As a direct consequence of Theorem 11, we have the following results.

Corollary 12. *Let $f \in A$ and suppose it satisfies the following subordination*

$$\frac{zf'(z)}{f(z)}\left(\frac{\left(zf'(z)\right)'}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{\sqrt{2}z + 1/2z^2}{\beta}, \quad (106)$$

then

(i) $f \in S^*(A,B)$ *provided* $\beta \geq \max\{\beta_1, \beta_2\}$ *with*

$$\max\{\beta_1,\beta_2\} = \beta = \begin{cases} \beta_2, & \text{if } B > -\dfrac{\sqrt{2}}{8}, \\ \beta_1, & \text{if } B < -\dfrac{\sqrt{2}}{8}, \end{cases} \quad (107)$$

where

$$\beta_1 = \frac{\left(4\sqrt{2}-1\right)(1-B)}{(A-B)}, \quad \beta_2 = \frac{\left(4\sqrt{2}-1\right)(1+B)}{4(A-B)}, \quad (108)$$

$$-1 < B < A \leq 1$$

(ii) $f \in S_l^*$ *provided* $\beta \geq (5\sqrt{2}+9)/4 \approx 4.0178$

(iii) $f \in S_e^*$ *provided* $\beta \geq e\sqrt{2}(8-\sqrt{2})/8(e-1) \approx 1.8418$

(iv) $f \in S_c^*$ *provided* $\beta \geq (12\sqrt{2}-3)/8 \approx 1.7463$

(v) $f \in S_{\sin}^*$ *provided* $\beta \geq \sqrt{2}(8+\sqrt{2})/(8\sin(1)) \approx 1.9778$

(vi) $f \in S_{\sin h^{-1}}^*$ *provided* $\beta \geq \sqrt{2}(8+\sqrt{2})/8(\log_e(1+\sqrt{2}) \approx 1.882$

(vii) $f \in S_{cr}^*$ *provided* $\beta \geq (7\sqrt{2}+6)/8 \approx 1.9874$

Theorem 13. *Let $p(z)$ be analytic in U with $p(0)=1$, and suppose*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \mathscr{L}_s(z), z \in U. \quad (109)$$

Then, the following subordination results are true:

(i) $p(z) \prec (1+Az)/(1+Bz)$, *for* $\beta \max\{\beta_1,\beta_2\}$ *and* $-1 < B < A \leq 1$ *with*

$$\beta_1 = \frac{s(4-s)}{\log_e(1-B)^2 - \log_e(1-A)^2}, \quad \beta_2 = \frac{s(4+s)}{\log_e(1+A)^2 - \log_e(1+B)^2} \quad (110)$$

(ii) $p(z) \prec \mathscr{V}_8(z)$ *for* $\beta \geq s(4+s)/\log_e(2)$

(iii) $p(z) \prec \mathscr{V}_4(z)$ *for* $\beta \geq s(4+s)/2$

(iv) $p(z) \prec \mathscr{V}_7(z)$ *for* $\beta \geq s(4+s)/\log_e(9)$

(v) $p(z) \prec \mathscr{V}_2(z)$ *for* $\beta \geq s(4+s)/2\log_e(\sin(1))$

(vi) $p(z) \prec \mathscr{V}_3(z)$ *for* $\beta \geq s(4+s)/2\log_e(1+\sin h^{-1}(1))$

(vii) $p(z) \prec \mathscr{V}_5(z)$ *for* $\beta \geq s(4+s)/2\log_e(\sqrt{2}+1)$

These bounds on β cannot be improved.

Proof. Let the analytic function $q_\beta : \bar{U} \longrightarrow \mathbb{C}$ be defined by

$$q_\beta(z) = \exp\left(\frac{4sz + s^2z^2}{2\beta}\right). \quad (111)$$

Then, $q_\beta(z)$ is a solution of the differential equation

$$1 + \beta\frac{zq_\beta'(z)}{q_\beta(z)} = (1+sz)^2. \quad (112)$$

Let $\varphi(w)=1$ and $\psi(w)=\beta/w$. Then, the function $Q : \bar{U} \longrightarrow \mathbb{C}$ is defined by $Q(z) = zq_\beta'(z)\psi(q_\beta(z)) = sz(sz+2)$. As it was demonstrated in Theorem 11, we note that $Q(z)$ is starlike in U and that $h(z) = \varphi(q_\beta(z)) + Q(z) = (1+sz)^2$ satisfies

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0. \quad (113)$$

Therefore, by Lemma 2, we have that

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq_\beta'(z)}{q_\beta(z)} \Rightarrow p(z) \prec q_\beta(z). \quad (114)$$

Each subordinations of Theorem 11 is equivalent to

$$p(z) \prec \mathbb{P}(z), \quad (115)$$

for each superordinate function $\mathbb{P}(z)$ in the theorem. From here, we obtain the required results by following the same line of proof as in Theorem 11. □

The following results are direct consequences of Theorem 13

Corollary 14. *Let $f \in A$ and suppose it satisfies the following subordination*

$$\left(\frac{\left(zf'(z)\right)'}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{\sqrt{2}z + 1/2z^2}{\beta}, \qquad (116)$$

then

(i) $f \in S^*(A,B)$ *provided* $\beta \geq \max\{\beta_1, \beta_2\}$, *where for* $-1 < B < A \leq 1$,

$$\beta_1 = \frac{\sqrt{2}\left(8 - \sqrt{2}\right)}{4\left[\log_e(1-B)^2 - \log_e(1-A)^2\right]}, \quad \beta_2 = \frac{\sqrt{2}\left(8 + \sqrt{2}\right)}{4\left[\log_e(1+A)^2 - \log_e(1+B)^2\right]} \qquad (117)$$

(ii) $f \in S_l^*$ *provided* $\beta \geq \sqrt{2}(\sqrt{2}+8)/4\log_e(2) \approx 4.8020$

(iii) $f \in S_e^*$ *provided* $\beta \geq e\sqrt{2}(8+\sqrt{2})/8 \approx 1.6642$

(iv) $f \in S_c^*$ *provided* $\beta \geq \sqrt{2}(\sqrt{2}+8)/8\log_e(3) \approx 1.5149$

(v) $f \in S_{\sin}^*$ *provided* $\beta \geq \sqrt{2}(8+\sqrt{2})/8\log_e(1+\sin(1)) \approx 2.7256$

(vi) $f \in S_{\sin h^{-1}}^*$ *provided* $\beta \geq \sqrt{2}(8+\sqrt{2})/\log_e(1+\log_e(1+\sqrt{2})) \approx 2.6332$

(vii) $f \in S_{cr}^*$ *provided* $\beta \geq \sqrt{2}(8+\sqrt{2})/8\log_e(1+\sqrt{2}) \approx 1.8882$

Theorem 15. *Let $p(z)$ be analytic in U with $p(0) = 1$, and suppose*

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec \mathscr{L}_s(z), z \in U. \qquad (118)$$

Then, the following subordination results hold:

(i) $p(z) \prec (1+Az)/(1+Bz)$, *for* $\beta \geq \max\{\beta_1, \beta_2\}$ *and* $-1 \leq B < A < 1$ *with* $\beta_1 = s(4-s)(1-A)/2(A-B)$, $\beta_2 = s(4+s)(1+A)/2(A-B)$ *and*

$$\max\{\beta_1, \beta_2\} = \begin{cases} \beta_1, & \text{if } A < -\dfrac{s}{4}, \\ \beta_2, & \text{if } A > -\dfrac{s}{4} \end{cases} \qquad (119)$$

(ii) $p(z) \prec \mathscr{V}_8(z)$ *for* $\beta \geq s(4+s)(\sqrt{2}+2)/2$

(iii) $p(z) \prec \mathscr{V}_4(z)$ *for* $\beta \geq es(4+s)/2(e-1)$

(iv) $p(z) \prec \mathscr{V}_7(z)$ *for* $\beta \geq 3s(4+s)/4$

(v) $p(z) \prec \mathscr{V}_2(z)$ *for* $\beta \geq s(4+s)(1+\sin(1))/2\sin(1)$

(vi) $p(z) \prec \mathscr{V}_3(z)$ *for* $\beta \geq s(4+s)(1+\log_e(1+\sqrt{2}))/2\log_e(1+\sqrt{2})$

(vii) $p(z) \prec \mathscr{V}_5(z)$ *for* $\beta \geq s(4+s)(2+\sqrt{2})/4$

These bounds on β are sharp.

Proof. Consider the analytic function $r_\beta : \bar{U} \longrightarrow \mathbb{C}$ such that

$$r_\beta(z) = \left(1 - \frac{sz(sz+4)}{2\beta}\right)^{-1}. \qquad (120)$$

Then, r_β is a solution of the differential equation

$$1 + \frac{\beta z r_\beta'(z)}{r_\beta^2(z)} = (1+sz)^2, z \in U. \qquad (121)$$

Let $\varphi(w) = 1$ and $\psi(w) = \beta/w^2$. Then, the function $Q : \bar{U} \longrightarrow \mathbb{C}$ is defined by $Q(z) = zr_\beta'(z)\psi(r_\beta(z)) = sz(sz+2)$. A computation shows that $Q(z)$ is starlike in U. Also, we have that $h(z) = \varphi(r_\beta(z)) + Q(z) = (1+sz)^2$ satisfies

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0. \qquad (122)$$

Therefore, by Lemma 2, we have that

$$1 + \frac{\beta zp'(z)}{p^2(z)} \prec 1 + \frac{\beta z r_\beta'(z)}{r_\beta^2(z)} \Rightarrow p(z) \prec r_\beta(z). \qquad (123)$$

Hence, we obtain the desired results by following the same line of proof of Theorem 11 □

From Theorem 15, we have the following consequences.

Corollary 16. *Let $f \in A$ and suppose it satisfies the following subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(\frac{\left(zf'(z)\right)'}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{\sqrt{2}z + 1/2z^2}{\beta}, \qquad (124)$$

then

(i) $f \in S^*(A,B)$ *provided* $\beta \geq \max\{\beta_1, \beta_2\}$ *with*

$$\max\{\beta_1, \beta_2\} = \beta = \begin{cases} \beta_1, & \text{if } A < -\dfrac{\sqrt{2}}{8}, \\ \beta_2, & \text{if } A > -\dfrac{\sqrt{2}}{8}, \end{cases} \qquad (125)$$

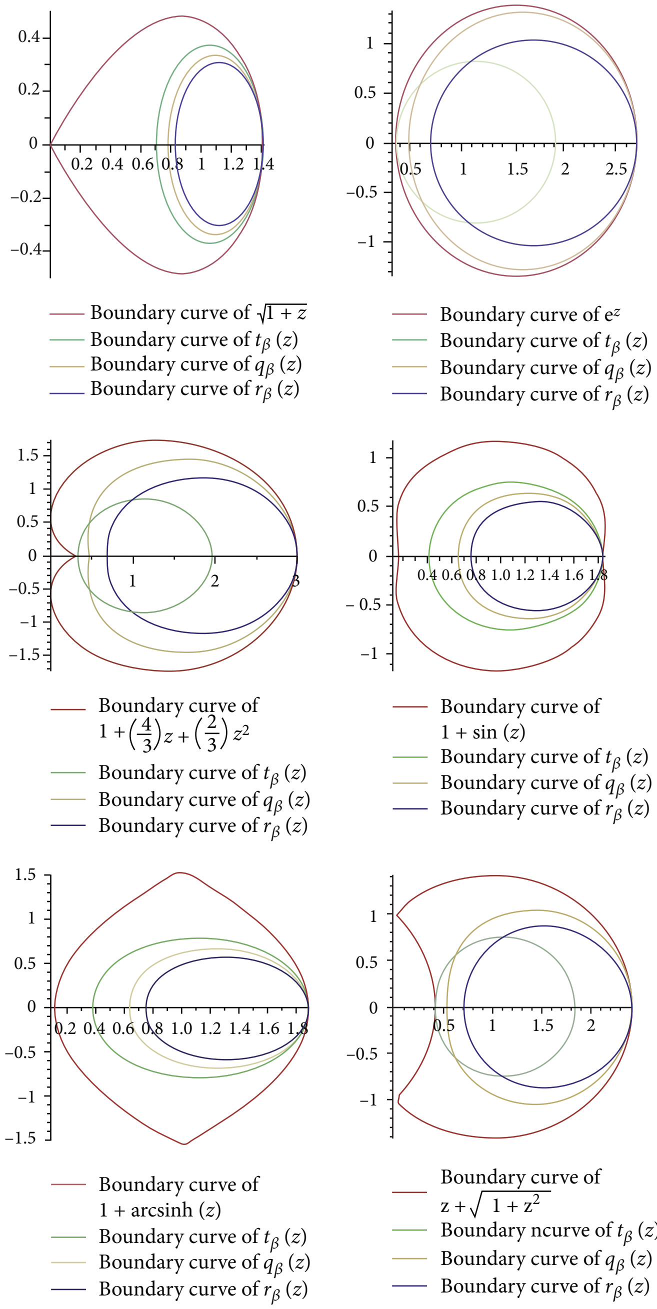

Figure 1

where

$$\beta_1 = \frac{\left(4\sqrt{2}-1\right)(1-A)}{4(A-B)},$$
$$\beta_2 = \frac{\left(4\sqrt{2}-1\right)(1+A)}{4(A-B)}, \tag{126}$$
$$-1 \le B < A < 1$$

(ii) $f \in S_l^*$ *provided* $\beta \ge (9\sqrt{2}+10)/4 \approx 5.6820$

(iii) $f \in S_e^*$ *provided* $\beta \ge e\sqrt{2}(8+\sqrt{2})e/8(e-1) \approx 2.6328$

(iv) $f \in S_c^*$ *provided* $\beta \ge (12\sqrt{2}+3)/8 \approx 2.4963$

(v) $f \in S_{\sin}^*$ *provided* $\beta \ge \sqrt{2}(8+\sqrt{2})(1+\sin(1))/(8\sin(1)) \approx 3.6421$

(vi) $f \in S_{\sin h^{-1}}^*$ *provided* $\beta \ge \sqrt{2}(8+\sqrt{2})(1+\log(1+\sqrt{2}))/(\log_e(1+\sqrt{2}) \approx 3.5526$

(vii) $f \in S_{cr}^*$ *provided* $\beta \ge (9\sqrt{2}+10)/8 \approx 2.8410$

The following geometries illustrate the sharpness of the Corollaries 12, 14 and 16 from (ii) to (vii) as we can see in Figure 1.

7. Conclusion

We introduced the classes $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ and then obtained the sharp radii for which normalized analytic functions $f(z)$ in these classes belonged to Ma and Minda starlike class. Moreover, applying the well-known theory of differential subordination developed by Miller and Mocanu, we found the restriction on β for which the differential subordination

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec \mathscr{L}_s(z), k=0,1,2, z\in U \tag{127}$$

implies certain Ma and Minda subclasses. We gave some special cases of our findings. Lastly, geometries of parts of our investigations were also illustrated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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