RADON TRANSFORM INVERSION VIA WIENER FILTERING OVER THE EUCLIDEAN MOTION GROUP

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ABSTRACT

In this paper we formulate the Radon transform as a convolution integral over the Euclidean motion group ($SE(2)$) and provide a minimum mean square error (MMSE) stochastic deconvolution method for the Radon transform inversion. Proposed approach provides a fundamentally new formulation that can model nonstationary signal and noise fields. Key components of our development are the Fourier transform over $SE(2)$, stochastic processes indexed by groups and fast implementation of the $SE(2)$ Fourier transform. Numerical studies presented here demonstrate that the method yields image quality that is comparable or better than the filtered backprojection algorithm. Apart from X-ray tomographic image reconstruction, the proposed deconvolution method is directly applicable to inverse radiotherapy, and broad range of science and engineering problems in computer vision, pattern recognition, robotics as well as protein science.

1. INTRODUCTION

The Radon transform and its generalizations arise in diverse engineering applications, including medical imaging, synthetic aperture radar (SAR), radio astronomy and pattern recognition [1]. The Radon transform plays an important role in image reconstruction problems because it constitutes a good model of tomographic acquisition process for X-ray CT, SPECT, PET and SAR. The problem of image reconstruction is equivalent to computing the inverse Radon transform.

In X-ray computed tomography, an X-ray beam with known energy is sent trough the object and the attenuated X-ray is collected by an array of collimated detectors. The attenuation in the final X-ray beam provides a means of determining the integral of the mass density of the object along the path of the X-ray. In 2D, the relationship between the mass density along the path and the attenuation at angle θ, and radius r, is given by the Radon transform:

$$\mathcal{R}f(r,\theta) = \int_{\mathbf{R}^2} f(x_1,x_2)\delta(r-\cos\theta x_1 - \sin\theta x_2)dx_1dx_2, \quad (1)$$

where δ is the Dirac delta function. Similarly, in PET and SPECT, the line projections and the attenuation coefficients are related by the Radon transform. The objective in X-ray tomography is to recover the function f from the measurements $\mathcal{R}f(r,\theta)$.

Here, we propose a new forward model for Radon transform inversion by modeling projections as a convolution integral over the Euclidean motion group and provide a minimum mean square error solution within a generalized Wiener filtering framework [2]. The proposed group theoretical approach offers the following advantages:

- Fundamentally a new statistical formulation for the inversion of the Radon transform that can operate in nonstationary noise and signal fields.
- Can be utilized for radiation treatment planning and for inverse source problems.
- Potential applications in non-rigid body, such as cardiac imaging and local tomography.
- Ability to model finite beam width as opposed to infinite width X-ray beam assumption used for filter back projection algorithms.
- Directly applicable to a broad range of problems in computer vision pattern recognition, robotics, as well as protein science where Euclidean group convolution is extensively utilized [3]-[5].
- Furthermore, group theoretic formulation of the problem offers the possibility to develop optimal sampling and fast algorithm development for Radon transform inversion. Currently, most data acquisition systems sample data at uniformly spaced intervals. It is important to analyze and understand the ramifications of

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this sampling scheme in the reconstructed data for X-ray tomographic image reconstruction. Such a study may lead to an understanding of the reconstruction artifacts in standard image reconstruction algorithms and may provide methods to lessen them.

The paper is organized as follows: In Section 2, group representations of $SE(2)$ followed by convolution integral representations of the Radon transform are provided. Fourier transform over $SE(2)$ and its operational properties are given in Section 3. Wiener filtering over $SE(2)$ is given in Section 4. Algorithmic and numerical details are discussed in Section 5. Numerical results are presented in Section 6. Lastly, Section 7 is the conclusion part.

2. RADON TRANSFORM AS A CONVOLUTION OVER SE(2)

The Euclidean motion group in two dimension is the semidirect product of the rotation group $SO(2)$ and the additive group $\mathbb{R}^2$. It is a locally compact topological group with elements $g = g(\mathbf{r}, \theta)$ consisting of a translation vector $r \in \mathbb{R}^2$ and a counter clockwise rotation angle $\theta \in [0, 2\pi)$. Intuitively speaking, the group operation is a rotation followed by a translation operation. To be precise, $SE(2)$ is a subgroup of 3×3 real matrices, $GL_2(\mathbb{R})$, where each element g is represented with a translation vector $r \in \mathbb{R}^2$ and a counter clockwise rotation angle $\theta \in [0, 2\pi)$, i.e.,

$$g(\mathbf{r}, \theta) = \begin{pmatrix} R(\theta) & \mathbf{r} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The group operation is the usual matrix multiplication. Let f_1 and f_2 be two real valued integrable functions defined over $SE(2)$. The convolution integral is given by

$$f_1 *_{SE(2)} f_2(g) = \int_{SE(2)} f_1(h) f_2(h^{-1}g) dg \tag{3}$$

$$= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \int_0^{2\pi} f_1(h) f_2(h^{-1}g) dr_1 dr_2 d\theta \tag{4}$$

$$= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \int_0^{2\pi} f_1(h) f_2(h^{-1}g) r dr d\phi d\theta \tag{5}$$

Radon transform can be formulated as a convolution integral over the Euclidean motion group,

$$f_3 = f_1 *_{SE(2)} f_2^* \tag{6}$$

where $f_1(\mathbf{r}, \theta) = \delta(r_1)$, $f_2(\mathbf{r}, \theta) = f(-\mathbf{r})$ and $f_3(\mathbf{r}, \theta) = \mathcal{R}f(-\theta, -r_1)$ with $r \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$ and $f^*(g) = \overline{f(g^{-1})}$. Observe that the above convolution formulation of Radon transform is equivalent to

$$f_3^* = f_2 *_{SE(2)} f_1^*. \tag{7}$$

With in the context of this formulation f_1 is called as the blurring filter. In the following Sections, we shall introduce a Fourier domain solution for the group deconvolution problem.

3. FOURIER TRANSFORM OVER SE(2)

In this section we shall provide the definition and the properties of the Fourier transform over $SE(2)$. Detailed discussion of the topic can be found in [6]. The unitary irreducible representations of $SE(2)$ on the square integrable function of the unit circle are given by

$$(U_g^p f)(\mathbf{x}) = e^{-ip(\mathbf{r}\cdot\mathbf{x})} f(R(\theta)^{-1})\mathbf{x}) \tag{8}$$

with a nonnegative $p \in \mathbb{R}$ and $g \in SE(2)$. The matrix elements of the unitary irreducible representations can be expressed as follows:

$$u_{m,n}(g,p) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} e^{-ip(r_1\cos\psi + r_2\sin\psi)} \times \times e^{in(\psi-\theta)} d\psi, \; m, n \in \mathbb{Z} \tag{9}$$

Properties of unitary representations can be summarized by the following equation

$$u_{m,n}(g^{-1}, p) = \overline{u_{m,n}(g,p)} = (-1)^{m-n} u_{-m,-n}(g,p). \tag{10}$$

Using the unitary representations, Fourier transform of any complex valued function over $SE(2)$ is given by

$$\mathcal{F}\{f\} = \hat{f}(p) = \int_{SE(2)} f(g) U_{g^{-1}}^p dg, \tag{11}$$

where is a nonnegative real number. The corresponding inverse Fourier transform is given by

$$\mathcal{F}\{\hat{f}\} = \int_0^\infty trace(f(g) U_g^p) p \, dp \tag{12}$$

Alternatively, the Fourier transform and its inversion can be expressed in terms of the matrix elements of the unitary irreducible representations as follows:

$$\hat{f}_{m,n}(p) = \int_{SE(2)} f(g) u_{m,n}(g^{-1}, p) dg \tag{13}$$

$$f_{m,n}(g) = \sum_{m,n\in\mathbb{Z}} \int_0^\infty \hat{f}(p) u_{m,n}(g,p) p \, dp. \tag{14}$$

Fourier transform over $SE(2)$ has properties similar to the ordinary Fourier transform. Let f, f_1 and f_2 be square integrable functions over $SE(2)$. Then,

1. $\int_{SE(2)} |f(g)|^2 dg = \int_0^\infty ||\hat{f}_{m,n}(p)||_2^2 p \, dp,$

2. $\mathcal{F}\{f_1 *_{SE(2)} f_2\} = \mathcal{F}\{f_2\}\mathcal{F}\{f_1\}$,

3. $\hat{f}^*(p) = (\hat{f}(p))^\dagger$,

where † denotes the adjoint operator. Note that using the matrix representation of the Fourier transform and the second property above, the Radon transform can be expressed in terms of matrix products for each p.

4. WIENER FILTER OVER SE(2) AND INVERSION OF RADON TRANSFORM

A second order stationary process over $SE(2)$ is defined as a random process with a covariance function, $\gamma(s,t) = E[x(s), x(t)]$, $s, t \in SE(2)$, independent of transitive action of $SE(2)$

$$\gamma(gs, gt) = \gamma(s,t),\ g \in SE(2). \tag{15}$$

Thus γ is defined on $SE(2)$ instead of $SE(2) \times SE(2)$, i.e.,

$$\gamma(s,t) = \gamma(t^{-1}s, e) = \gamma(g, e) = \Gamma(g), \tag{16}$$

where e is the identity element of $SE(2)$. In other words, a second order stationary process on $SE(2)$ can be viewed as rotation and translation invariant process. In [2], a spectral density function for group stationary processes is defined via group Fourier transform. Namely,

$$S(p) = \mathcal{F}\{\Gamma\}. \tag{17}$$

For detailed information on stochastic processes over groups the reader is referred to [7]-[8]. Let x and n be two zero mean, statistically independent stationary processes over $SE(2)$, referred to as signal and noise, respectively. Let the measurements $y(g)$ be the blurred and noisy signal given as,

$$y = x *_{SE(2)} f + n, \tag{18}$$

then the minimum mean square error filter in Fourier transform domain is given by [2]

$$\hat{W}_{opt}(p) = S_x(p)\hat{f}^\dagger(p)\left[\hat{f}(p)S_x(p)\hat{f}^\dagger(p) + S_n(p)\right]^{-1}, \tag{19}$$

where S_x and S_n are signal and noise power spectral density functions. Note that MMSE estimate of the signal in Fourier domain is given by $\hat{W}_{\text{opt}}\hat{y} = \hat{\tilde{x}}$.

Observe that if there is no noise then the optimal filter is nothing but the inverse of the Fourier transform of f. If there is no a priori information available on the unknown signal x and that the noise is "white" in the $SE(2)$ sense, then the optimal Wiener filter for the Radon transform inversion can be written by

$$\hat{W}_{\text{opt}}(p) = \hat{f}_1(p)\left[\hat{f}_1^{\dagger}(p)\hat{f}_1(p) + \sigma^2 I\right]^{-1}, \tag{20}$$

where σ^2 is the noise variance. This filter is also the linear least square filter with zeroth order Tikhonov regularization. Therefore MMSE estimate of $f_2(\mathbf{r}, \theta) = f(-\mathbf{r})$ is given by

$$\tilde{f}_2 = \mathcal{F}^{-1}\{\hat{W}_{opt}\hat{f}_3^{\dagger}\}. \tag{21}$$

In the following Section we shall address the numerical issues involved in the implementation of the Fourier transform of $SE(2)$ and the Wiener filter.

5. IMPLEMENTATION

Fourier Transform over SE(2): A fast algorithm for the Fourier Transform over $SE(2)$ based on fast Fourier transforms (FFT) was given in [4]. This algorithm is based on step wise computation of the integral of $\hat{F}_{mn}$ using the inner product expression for matrix elements of unitary representations, given in Equation 9,

$$\hat{F}_{mn}(p) = \int_{\mathbf{r}\in\mathbf{R}^2}\int_{\theta=0}^{2\pi}\int_{\psi=0}^{2\pi} F(r,\theta)e^{in\psi} \times \tag{22}$$
$$\times e^{i(p\cos\psi, p\sin\psi\cdot(r_1,r_2))} \times e^{-m(\psi-\theta)}\, d^2r\, d\theta\, d\psi.$$

The integral is computed in 4 steps:

1. $F_1(\mathbf{p},\theta) = \int_{\mathbf{R}^2} F(\mathbf{r},\theta)e^{i(\mathbf{p}\cdot\mathbf{r})}d^2\mathbf{r}$,
2. Interpolate $F_1(p,\psi,\theta) = F_1(\mathbf{p},\theta)$ from Cartesian coordinates to polar coordinates for each θ.
3. $F_2^{(m)}(p,\psi) = \int_{SO(2)} F_1(p,\psi,\theta)e^{im\theta}d\theta$,
4. $\hat{F}_{mn}(p) = \int_o^{2\pi}\left[F_2^{(m)}(p,\psi)e^{-im\psi}\right]e^{in\psi}d\psi$,

The 1^{st}, 3^{rd} and 4^{th} steps can be computed using the ordinary Fast Fourier Transforms (FFT). If the sampling rates are given as in Table 1 then Fourier transform over $SE(2)$ requires $O(S^3\log(S^3))$ total number of computations.

N_R	Number of samples on $SO(2)$	$O(S)$
$N_\mathbf{r}$	Number of samples on $\mathbb{R}^2$	$O(S^2)$
N	Total number of samples on $SE(2)$	$O(S^3)$
N_p	Number of samples on p interval	$O(S)$
N_ψ	Number of samples on $[0, 2\pi)$	$O(S)$
N_{mn}	Total number of harmonics	$O(S^2)$

Table 1. Sampling in $SE(2)$ as an order of S, $O(S)$. ($N = N_R N_\mathbf{r}$, $N_{mn} = N_p N_\psi$)

Bandlimitedness over SE(2): Note that the Fourier transform of any function on $SE(2)$ does not necessarily have a finite rank. Therefore, for numerical implementation, any function must be approximated by a finite number of harmonics based on the Fourier algorithm discussed above.

$\mathbf{FFT_{SE(2)}}$ of functions over $\mathbb{R}^2$: Let $f(r_1, r_2)$ be a function defined over $\mathbb{R}^2$. f can be treated as a θ invariant function over $SE(2)$, $f(g) = f(r_1, r_2)$. The matrix elements of the Fourier transform of a θ invariant function, $\hat{f}_{m,n}$ is independent of m, i.e. $\hat{f}_{m,n} = \hat{f}_{q,n}$. Therefore it is enough to compute one row of the matrix.

Blurring filter: Note that the blurring filter is given by, $f_1(\mathbf{r}, \theta) = \delta(r_1)$. Assuming that the unknown object lies within the unit ball centered at origin, i.e. $f_2(\mathbf{r}, \theta) = f_2(\mathbf{r}, \theta) w(r_1) w(r_2) = f_2(\mathbf{r}, \theta) w(r_2)$, where $w(r) = 1$ only if $|r| \leq 1$,then $f_1(\mathbf{r}, \theta)$ can be constructed as a periodic extension of $f_1(\mathbf{r}, \theta) = f_1(\mathbf{r}, \theta) w(r_2)$ with period 2. As a result, the Fourier coefficients of f_1, $\hat{f}_1$, becomes diagonal and non-negative, i.e. $\hat{f}_{1_{mn}} \geq 0$ when $m = n$ and 0 otherwise.

6. RESULTS AND DISCUSSION

All numerical experiments were performed using the Shepp-Logan phantom of size 129×129. All functions were zero padded in r_1 and r_2 directions to prevent aliasing. During the 2^{nd} step of the Fourier transform, linear interpolation was used. Regularized linear least squares version of the Wiener filter for different σ^2 values was implemented. For $\sigma^2 = 10^-4, 10^-6$, the reconstructed images are shown in Figure 1. Visual comparison indicates that the proposed algorithm produce images that are comparable or better than that of standard filtered backprojection (FBP) algorithm. The reconstructed images have sharper and more consistent edges than the images reconstructed by the FBP algorithm.

7. CONCLUSION

In this paper, we present a new formulation of the Radon transform as a convolution integral over the Euclidean motion group and a deconvolution method based on the regularized linear least squares version of the Wiener filtering method, developed in abstract harmonic analysis. The formulation and the proposed solution are applicable to a wide range of problems involving Radon transform and convolution integrals over the Euclidean motion group. Furthermore, the deconvolution method allows nonstationary signal and noise modeling. Numerical studies involving nonstationary noise and a priori knowledge are on going and will be reported in our future work.

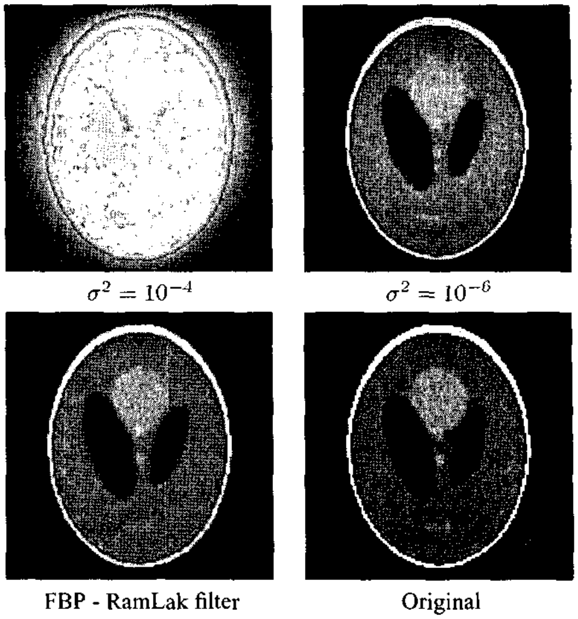

Fig. 1. Wiener filter design for the noiseless case using various σ^2 values

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