## Mathematic Bohemica

Gary Chartrand; Garry L. Johns; Kathleen A. McKeon; Ping Zhang Rainbow connection in graphs

Mathematic Bohemica, Vol. 133 (2008), No. 1, 85-98

Persistent URL: http: //dml.cz/dmlcz/133947

## Terms of use:

(C) Institute of Mathematics AS CR, 2008

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.


This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://dml.cz

# RAINBOW CONNECTION IN GRAPHS 

Gary Chartrand, Kalamazoo, Garry L. Johns, Saginaw Valley, Kathleen A. McKeon, New London, Ping Zhang, Kalamazoo

(Received July 31, 2006)

Abstract. Let $G$ be a nontrivial connected graph on which is defined a coloring $c: E(G) \rightarrow$ $\{1,2, \ldots, k\}, k \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A path $P$ in $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is rainbow-connected if $G$ contains a rainbow $u-v$ path for every two vertices $u$ and $v$ of $G$. The minimum $k$ for which there exists such a $k$-edge coloring is the rainbow connection number $\operatorname{rc}(G)$ of $G$. If for every pair $u, v$ of distinct vertices, $G$ contains a rainbow $u-v$ geodesic, then $G$ is strongly rainbow-connected. The minimum $k$ for which there exists a $k$-edge coloring of $G$ that results in a strongly rainbow-connected graph is called the strong rainbow connection number $\operatorname{src}(G)$ of $G$. Thus $\operatorname{rc}(G) \leqslant \operatorname{src}(G)$ for every nontrivial connected graph $G$. Both $\operatorname{rc}(G)$ and $\operatorname{src}(G)$ are determined for all complete multipartite graphs $G$ as well as other classes of graphs. For every pair $a, b$ of integers with $a \geqslant 3$ and $b \geqslant(5 a-6) / 3$, it is shown that there exists a connected graph $G$ such that $\operatorname{rc}(G)=a$ and $\operatorname{src}(G)=b$.

Keywords: edge coloring, rainbow coloring, strong rainbow coloring
MSC 2000: 05C15, 05C38, 05C40

## 1. Introduction

Let $G$ be a nontrivial connected graph on which is defined a coloring $c: E(G) \rightarrow$ $\{1,2, \ldots, k\}, k \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A $u-v$ path $P$ in $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is rainbow-connected (with respect to $c$ ) if $G$ contains a rainbow $u-v$ path for every two vertices $u$ and $v$ of $G$. In this case, the coloring $c$ is called a rainbow coloring of $G$. If $k$ colors are used, then $c$ is a rainbow $k$-coloring. The minimum $k$ for which there exists a rainbow $k$-coloring of the edges of $G$ is the rainbow connection number $\operatorname{rc}(G)$ of $G$. A rainbow coloring of $G$ using $\operatorname{rc}(G)$ colors is called a minimum rainbow coloring of $G$.

Let $c$ be a rainbow coloring of a connected graph $G$. For two vertices $u$ and $v$ of $G$, a rainbow $u-v$ geodesic in $G$ is a rainbow $u-v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between $u$ and $v$ (the length of a shortest $u-v$ path in $G$ ). The graph $G$ is strongly rainbow-connected if $G$ contains a rainbow $u-v$ geodesic for every two vertices $u$ and $v$ of $G$. In this case, the coloring $c$ is called a strong rainbow coloring of $G$. The minimum $k$ for which there exists a coloring $c: E(G) \rightarrow\{1,2, \ldots, k\}$ of the edges of $G$ such that $G$ is strongly rainbow-connected is the strong rainbow connection number $\operatorname{src}(G)$ of $G$. A strong rainbow coloring of $G$ using $\operatorname{src}(G)$ colors is called a minimum strong rainbow coloring of $G$. Thus $\operatorname{rc}(G) \leqslant \operatorname{src}(G)$ for every connected graph $G$.

Since every coloring that assigns distinct colors to the edges of a connected graph is both a rainbow coloring and a strong rainbow coloring, every connected graph is rainbow-connected and strongly rainbow-connected with respect to some coloring of the edges of $G$. Thus the rainbow connection numbers $\operatorname{rc}(G)$ and $\operatorname{src}(G)$ are defined for every connected graph $G$. Furthermore, if $G$ is a nontrivial connected graph of size $m$ whose diameter (the largest distance between two vertices of $G$ ) is $\operatorname{diam}(G)$, then

$$
\begin{equation*}
\operatorname{diam}(G) \leqslant \operatorname{rc}(G) \leqslant \operatorname{src}(G) \leqslant m \tag{1}
\end{equation*}
$$

To illustrate these concepts, consider the Petersen graph $P$ of Figure 1, where a rainbow 3 -coloring of $P$ is also shown. Thus $\operatorname{rc}(P) \leqslant 3$. On the other hand, if $u$ and $v$ are two nonadjacent vertices of $P$, then $d(u, v)=2$ and so the length of a $u-v$ path is at least 2. Thus any rainbow coloring of $P$ uses at least two colors and so $\operatorname{rc}(P) \geqslant 2$. If $P$ has a rainbow 2-coloring $c$, then there exist two adjacent edges of $G$ that are colored the same by $c$, say $e=u v$ and $f=v w$ are colored the same. Since there is exactly one $u-w$ path of length 2 in $P$, there is no rainbow $u-w$ path in $P$, which is a contradiction. Therefore, $\operatorname{rc}(P)=3$.


Figure 1. A rainbow 3 -coloring and a strong rainbow 4 -coloring of the Petersen graph

Since $\operatorname{rc}(P)=3$, it follows that $\operatorname{src}(P) \geqslant 3$. Furthermore, since the edge chromatic number of the Petersen graph is known to be 4, any 3 -coloring $c$ of the edges of $P$ results in two adjacent edges $u v$ and $v w$ being assigned the same color. Since $u, v, w$ is the only $u-w$ geodesic in $P$, the coloring $c$ is not a strong rainbow coloring. Because the 4 -coloring of the edges of $P$ shown in Figure 2 is a strong rainbow coloring, $\operatorname{src}(P)=4$.

As another example, consider the graph $G$ of Figure 2(a), where a rainbow 4coloring $c$ of $G$ is also shown. In fact, $c$ is a minimum rainbow coloring of $G$ and so $\operatorname{rc}(G)=4$, as we now verify.

(a)

(b)

Figure 2. A graph $G$ with $\operatorname{rc}(G)=\operatorname{src}(G)=4$
Since $\operatorname{diam}(G) \geqslant 3$, it follows that $\operatorname{rc}(G) \geqslant 3$. Assume, to the contrary, $\operatorname{rc}(G)=3$. Then there exists a rainbow 3-coloring $c^{\prime}$ of $G$. Since every $u-v$ path in $G$ has length 3 , at least one of the three $u-v$ paths in $G$ is a rainbow $u-v$ path, say $u, u_{1}, v_{1}, v$ is a rainbow $u-v$ path. We may assume that $c^{\prime}\left(u u_{1}\right)=1, c^{\prime}\left(u_{1} v_{1}\right)=2$, and $c^{\prime}\left(v_{1} v\right)=3$. (See Figure 2(b).)

If $x$ and $y$ are two vertices in $G$ such that $d(x, y)=2$, then $G$ contains exactly one $x-y$ path of length 2 , while all other $x-y$ paths have length 4 or more. This implies that no two adjacent edges can be colored the same. Thus we may assume, without loss of generality, that $c^{\prime}\left(u u_{2}\right)=2$ and $c^{\prime}\left(u u_{3}\right)=3$. (See Figure 2(b).) Thus $\left\{c^{\prime}\left(v v_{2}\right), c^{\prime}\left(v v_{3}\right)\right\}=\{1,2\}$. If $c^{\prime}\left(v v_{2}\right)=1$ and $c^{\prime}\left(v v_{3}\right)=2$, then $c^{\prime}\left(u_{2} v_{2}\right)=3$ and $c^{\prime}\left(u_{3} v_{3}\right)=1$. In this case, there is no rainbow $u_{1}-v_{3}$ path in $G$. On the other hand, if $c^{\prime}\left(v v_{2}\right)=2$ and $c^{\prime}\left(v v_{3}\right)=1$, then $c^{\prime}\left(u_{2} v_{2}\right) \in\{1,3\}$ and $c^{\prime}\left(u_{3} v_{3}\right)=2$. If $c^{\prime}\left(u_{2} v_{2}\right)=1$, then there is no rainbow $u_{2}-v_{3}$ path in $G$; while if $c^{\prime}\left(u_{2} v_{2}\right)=3$, there is no rainbow $u_{2}-v_{1}$ path in $G$, a contradiction. Therefore, as claimed, $\operatorname{rc}(G)=4$.

Since $4=\operatorname{rc}(G) \leqslant \operatorname{src}(G)$ for the graph $G$ of Figure 2 and the rainbow 4-coloring of $G$ in Figure 2(a) is also a strong rainbow 4-coloring, $\operatorname{src}(G)=4$ as well.

If $G$ is a nontrivial connected graph of size $m$, then we saw in (1) that diam $(G) \leqslant$ $\operatorname{rc}(G) \leqslant \operatorname{src}(G) \leqslant m$. In the following result, it is determined which connected graphs $G$ attain the extreme values 1,2 or $m$.

Proposition 1.1. Let $G$ be a nontrivial connected graph of size $m$. Then
(a) $\operatorname{src}(G)=1$ if and only if $G$ is a complete graph,
(b) $\operatorname{rc}(G)=2$ if and only if $\operatorname{src}(G)=2$,
(c) $\operatorname{rc}(G)=m$ if and only if $G$ is a tree.

Proof. We first verify (a). If $G$ is a complete graph, then the coloring that assigns 1 to every edge of $G$ is a strong rainbow 1-coloring of $G$ and so $\operatorname{src}(G)=1$. On the other hand, if $G$ is not complete, then $G$ contains two nonadjacent vertices $u$ and $v$. Thus each $u-v$ geodesic in $G$ has length at least 2 and so $\operatorname{src}(G) \geqslant 2$.

To verify (b), first assume that $\operatorname{rc}(G)=2$ and $\operatorname{so} \operatorname{src}(G) \geqslant 2$ by (1). Since $\operatorname{rc}(G)=2$, it follows that $G$ has a rainbow 2-coloring, which implies that every two nonadjacent vertices are connected by a rainbow path of length 2 . Because such a path is a geodesic, $\operatorname{src}(G)=2$. On the other hand, if $\operatorname{src}(G)=2$, then $\operatorname{rc}(G) \leqslant 2$ by (1) again. Furthermore, $\operatorname{since} \operatorname{src}(G)=2$, it follows by (a) that $G$ is not complete and so $\operatorname{rc}(G) \geqslant 2$. Thus $\operatorname{rc}(G)=2$.

We now verify (c). Suppose first that $G$ is not a tree. Then $G$ contains a cycle $C: v_{1}, v_{2}, \ldots, v_{k}, v_{1}$, where $k \geqslant 3$. Then the ( $m-1$ )-coloring of the edges of $G$ that assigns 1 to the edges $v_{1} v_{2}$ and $v_{2} v_{3}$ and assigns the $m-2$ distinct colors from $\{2,3, \ldots, m-1\}$ to the remaining $m-2$ edges of $G$ is a rainbow coloring. Thus $\operatorname{rc}(G) \leqslant m-1$. Next, let $G$ be a tree of size $m$. Assume, to the contrary, that $\operatorname{rc}(G) \leqslant m-1$. Let $c$ be a minimum rainbow coloring of $G$. Then there exist edges $e$ and $f$ such that $c(e)=c(f)$. Assume, without loss generality, that $e=u v$ and $f=x y$ and $G$ contains a $u-y$ path $u, v, \ldots, x, y$. Then there is no rainbow $u-y$ path in $G$, which is a contradiction.

Proposition 1.1 also implies that the only connected graphs $G$ for which $\operatorname{rc}(G)=1$ are the complete graphs and that the only connected graphs $G$ of size $m$ for which $\operatorname{src}(G)=m$ are trees.

## 2. Some rainbow connection numbers of graphs

In this section, we determine the rainbow connection numbers of some well-known graphs. We refer to the book [1] for graph-theoretical notation and terminology not described in this paper. We begin with cycles of order $n$. Since diam $\left(C_{n}\right)=\lfloor n / 2\rfloor$, it follows by (1) that $\operatorname{src}\left(C_{n}\right) \geqslant \operatorname{rc}\left(C_{n}\right) \geqslant\lfloor n / 2\rfloor$. This lower bound for $\operatorname{rc}\left(C_{n}\right)$ and $\operatorname{src}\left(C_{n}\right)$ is nearly the exact value of these numbers.

Proposition 2.1. For each integer $n \geqslant 4, \operatorname{rc}\left(C_{n}\right)=\operatorname{src}\left(C_{n}\right)=\lceil n / 2\rceil$.
Proof. Let $C_{n}: v_{1}, v_{2}, \ldots, v_{n}, v_{n+1}=v_{1}$ and for each $i$ with $1 \leqslant i \leqslant n$, let $e_{i}=v_{i} v_{i+1}$. We consider two cases, according to whether $n$ is even or $n$ is odd.

Case 1. $n$ is even. Let $n=2 k$ for some integer $k \geqslant 2$. Thus $\operatorname{src}\left(C_{n}\right) \geqslant \operatorname{rc}\left(C_{n}\right) \geqslant$ $\operatorname{diam}\left(C_{n}\right)=k$. Since the edge coloring $c_{0}$ of $C_{n}$ defined by $c_{0}\left(e_{i}\right)=i$ for $1 \leqslant i \leqslant k$ and $c_{0}\left(e_{i}\right)=i-k$ if $k+1 \leqslant i \leqslant n$ is a strong rainbow $k$-coloring, it follows that $\operatorname{rc}\left(C_{n}\right) \leqslant \operatorname{src}\left(C_{n}\right) \leqslant k$ and so $\operatorname{rc}\left(C_{n}\right)=\operatorname{src}\left(C_{n}\right)=k$.

Case 2. $n$ is odd. Then $n=2 k+1$ for some integer $k \geqslant 2$. First define an edge coloring $c_{1}$ of $C_{n}$ by $c_{1}\left(e_{i}\right)=i$ for $1 \leqslant i \leqslant k+1$ and $c_{1}\left(e_{i}\right)=i-k-1$ if $k+2 \leqslant i \leqslant n$. Since $c_{1}$ is a strong rainbow $(k+1)$-coloring of $C_{n}$, it follows that $\operatorname{rc}\left(C_{n}\right) \leqslant \operatorname{src}\left(C_{n}\right) \leqslant k+1$.

Since $\operatorname{rc}\left(C_{n}\right) \geqslant \operatorname{diam}\left(C_{n}\right)=k$, it follows that $\operatorname{rc}\left(C_{n}\right)=k$ or $\operatorname{rc}\left(C_{n}\right)=k+1$. We claim that $\operatorname{rc}\left(C_{n}\right)=k+1$. Assume, to the contrary, that $\operatorname{rc}\left(C_{n}\right)=k$. Let $c^{\prime}$ be a rainbow $k$-coloring of $C_{n}$ and let $u$ and $v$ be two antipodal vertices of $C_{n}$. Then the $u-v$ geodesic in $C_{n}$ is a rainbow path and the other $u-v$ path in $C_{n}$ is not a rainbow path since it has length $k+1$. Suppose, without loss of generality, that $c^{\prime}\left(v_{k+1} v_{k+2}\right)=k$.

Consider the vertices $v_{1}, v_{k+1}$, and $v_{k+2}$. Since the $v_{1}-v_{k+1}$ geodesic $P: v_{1}$, $v_{2}, \ldots, v_{k+1}$ is a rainbow path and the $v_{1}-v_{k+2}$ geodesic $Q: v_{1}, v_{n}, v_{n-1}, \ldots, v_{k+2}$ is a rainbow path, some edge on $P$ is colored $k$ as is some edge on $Q$. Since the $v_{2}-v_{k+2}$ geodesic $v_{2}, v_{3}, \ldots, v_{k+2}$ is a rainbow path, it follows that $c^{\prime}\left(v_{1} v_{2}\right)=k$. Similarly, the $v_{n}-v_{k+1}$ geodesic $v_{n}, v_{n-1}, v_{n-2}, \ldots, v_{k+1}$ is a rainbow path and so $c^{\prime}\left(v_{n} v_{1}\right)=k$. Thus $c^{\prime}\left(v_{1} v_{2}\right)=c^{\prime}\left(v_{n} v_{1}\right)=k$. This implies that there is no rainbow $v_{2}-v_{n}$ path in $G$, producing a contradiction. Thus $\operatorname{rc}\left(C_{n}\right)=\operatorname{src}\left(C_{n}\right)=k+1$.

A well-known class of graphs constructed from cycles are the wheels. For $n \geqslant 3$, the wheel $W_{n}$ is defined as $C_{n}+K_{1}$, the join of $C_{n}$ and $K_{1}$, constructed by joining a new vertex to every vertex of $C_{n}$. Thus $W_{3}=K_{4}$. Next, we determine rainbow connection numbers of wheels.

Proposition 2.2. For $n \geqslant 3$, the rainbow connection number of the wheel $W_{n}$ is

$$
\operatorname{rc}\left(W_{n}\right)= \begin{cases}1 & \text { if } n=3 \\ 2 & \text { if } 4 \leqslant n \leqslant 6 \\ 3 & \text { if } n \geqslant 7\end{cases}
$$

Proof. Suppose that $W_{n}$ consists of an $n$-cycle $C_{n}: v_{1}, v_{2}, \ldots, v_{n}, v_{n+1}=v_{1}$ and another vertex $v$ joined to every vertex of $C_{n}$. Since $W_{3}=K_{4}$, it follows by Proposition 1.1 that $\operatorname{rc}\left(W_{3}\right)=1$. For $4 \leqslant n \leqslant 6$, the wheel $W_{n}$ is not complete and
so $\mathrm{rc}\left(W_{n}\right) \geqslant 2$. Since the 2-coloring $c: E\left(W_{n}\right) \rightarrow\{1,2\}$ defined by $c\left(v_{i} v\right)=1$ if $i$ is odd, $c\left(v_{i} v\right)=2$ if $i$ is even, and $c\left(v_{i} v_{i+1}\right)=1$ if $i$ is odd, and $c\left(v_{i} v_{i+1}\right)=2$ if $i$ is even is a rainbow coloring, it follows that $\operatorname{rc}\left(W_{n}\right)=2$ for $4 \leqslant n \leqslant 6$.

Finally, suppose that $n \geqslant 7$. Since the 3 -coloring $c: E\left(W_{n}\right) \rightarrow\{1,2,3\}$ defined by $c\left(v_{i} v\right)=1$ if $i$ is odd, $c\left(v_{i} v\right)=2$ if $i$ is even, and $c(e)=3$ for each $e \in E\left(C_{n}\right)$ is a rainbow coloring, it follows that $\operatorname{rc}\left(W_{n}\right) \leqslant 3$. It remains to show that $\operatorname{rc}\left(W_{n}\right) \geqslant 3$. Since $W_{n}$ is not complete, $\operatorname{rc}\left(W_{n}\right) \geqslant 2$. Assume, to the contrary, that $\operatorname{rc}\left(W_{n}\right)=$ 2. Let $c^{\prime}$ be a rainbow 2 -coloring of $W_{n}$. Without loss of generality, assume that $c^{\prime}\left(v_{1} v\right)=1$. For each $i$ with $4 \leqslant i \leqslant n-2, v_{1}, v, v_{i}$ is the only $v_{1}-v_{i}$ path of length 2 in $W_{n}$ and so $c^{\prime}\left(v_{i} v\right)=2$ for $4 \leqslant i \leqslant n-2$. Since $c\left(v_{4} v\right)=2$, it follows that $c\left(v_{n} v\right)=1$. This forces $c\left(v_{3} v\right)=2$, which in turn forces $c\left(v_{n-1} v\right)=1$. Similarly, $c\left(v_{n-1} v\right)=1$ forces $c\left(v_{2} v\right)=2$. Since $c\left(v_{2} v\right)=2$ and $c\left(v_{5} v\right)=2$, there is no rainbow $v_{2}-v_{5}$ path in $W_{n}$, which is a contradiction. Therefore, $\operatorname{rc}\left(W_{n}\right)=3$ for $n \geqslant 7$.

Proposition 2.3. For $n \geqslant 3$, the strong rainbow connection number of the wheel $W_{n}$ is

$$
\operatorname{src}\left(W_{n}\right)=\lceil n / 3\rceil .
$$

Proof. Suppose that $W_{n}$ consists of an $n$-cycle $C_{n}: v_{1}, v_{2}, \ldots, v_{n}, v_{n+1}=v_{1}$ and another vertex $v$ joined to every vertex of $C_{n}$. Since $W_{3}=K_{4}$, it follows by Proposition 1.1 that $\operatorname{src}\left(W_{3}\right)=1$. If $4 \leqslant n \leqslant 6$, then $\operatorname{rc}\left(W_{n}\right)=2$ by Proposition 2.2 and $\operatorname{so} \operatorname{src}\left(W_{n}\right)=2$ by Proposition 1.1. Therefore, $\operatorname{src}\left(W_{n}\right)=\lceil n / 3\rceil$ for $4 \leqslant n \leqslant 6$.

Thus we may assume $n \geqslant 7$. Then there is an integer $k$ such that $3 k-2 \leqslant n \leqslant 3 k$. We first show that $\operatorname{src}\left(W_{n}\right) \geqslant k$. Assume, to the contrary, that $\operatorname{src}\left(W_{n}\right) \leqslant k-1$. Let $c$ be a strong rainbow $(k-1)$-coloring of $W_{n}$. Since $\operatorname{deg} v=n>3(k-1)$, there exists $S \subseteq V\left(C_{n}\right)$ such that $|S|=4$ and all edges in $\{u v: u \in S\}$ are colored the same. Thus there exist at least two vertices $u^{\prime}, u^{\prime \prime} \in S$ such that $d_{C_{n}}\left(u^{\prime}, u^{\prime \prime}\right) \geqslant 3$ and $d_{W_{n}}\left(u^{\prime}, u^{\prime \prime}\right)=2$. Since $u^{\prime}, v, u^{\prime \prime}$ is the only $u^{\prime}-u^{\prime \prime}$ geodesic in $W_{n}$, it follows that there is no rainbow $u^{\prime}-u^{\prime \prime}$ geodesic in $W_{n}$, which is a contradiction. Thus $\operatorname{src}\left(W_{n}\right) \geqslant k$.

To show that $\operatorname{src}\left(W_{n}\right) \leqslant k$, we provide a strong rainbow $k$-coloring $c: E\left(W_{n}\right) \rightarrow$ $\{1,2, \ldots, k\}$ of $W_{n}$ defined by

$$
c(e)= \begin{cases}1 & \text { if } e=v_{i} v_{i+1} \text { and } i \text { is odd } \\ 2 & \text { if } e=v_{i} v_{i+1} \text { and } i \text { is even } \\ j+1 & \text { if } e=v_{i} v \text { if } i \in\{3 j+1,3 j+2,3 j+3\} \text { for } 0 \leqslant j \leqslant k-1\end{cases}
$$

Therefore, $\operatorname{src}\left(W_{n}\right)=k=\lceil n / 3\rceil$ for $n \geqslant 7$ as well.

We now determine the rainbow connection numbers of all complete multipartite graphs, beginning with the strong connection number of the complete bipartite graph $K_{s, t}$ with $1 \leqslant s \leqslant t$.

Theorem 2.4. For integers $s$ and $t$ with $1 \leqslant s \leqslant t$,

$$
\operatorname{src}\left(K_{s, t}\right)=\lceil\sqrt[s]{t}\rceil
$$

Proof. Since $\operatorname{src}\left(K_{1, t}\right)=t$, the result follows for $s=1$. So we may assume that $s \geqslant 2$. Let $\lceil\sqrt[s]{t}\rceil=k$. Hence

$$
1 \leqslant k-1<\sqrt[s]{t} \leqslant k
$$

Therefore, $(k-1)^{s}<t \leqslant k^{s}$ and so $(k-1)^{s}+1 \leqslant t \leqslant k^{s}$.
First, we show that $\operatorname{src}\left(K_{s, t}\right) \geqslant k$. Assume, to the contrary, that $\operatorname{src}\left(K_{s, t}\right) \leqslant k-1$. Then there exists a strong rainbow $(k-1)$-coloring of $K_{s, t}$. Let $U$ and $W$ be the partite sets of $K_{s, t}$, where $|U|=s$ and $|W|=t$. Suppose that $U=\left\{u_{1}, u_{2}, \ldots, u_{s}\right\}$. Let there be given a strong rainbow $(k-1)$-coloring $c$ of $K_{s, t}$. For each vertex $w \in W$, we can associate an ordered $s$-tuple $\operatorname{code}(w)=\left(a_{1}, a_{2}, \ldots, a_{s}\right)$ called the color code of $w$, where $a_{i}=c\left(u_{i} w\right)$ for $1 \leqslant i \leqslant s$. Since $1 \leqslant a_{i} \leqslant k-1$ for each $i(1 \leqslant i \leqslant s)$, the number of distinct color codes of the vertices of $W$ is at most $(k-1)^{s}$. However, since $t>(k-1)^{s}$, there exists at least two distinct vertices $w^{\prime}$ and $w^{\prime \prime}$ of $W$ such that code $\left(w^{\prime}\right)=\operatorname{code}\left(w^{\prime \prime}\right)$. Since $c\left(u_{i} w^{\prime}\right)=c\left(u_{i} w^{\prime \prime}\right)$ for all $i(1 \leqslant i \leqslant s)$, it follows that $K_{s, t}$ contains no rainbow $w^{\prime}-w^{\prime \prime}$ geodesic in $K_{s, t}$, contradicting our assumption that $c$ is a strong rainbow $(k-1)$-coloring of $K_{s, t}$. Thus, as claimed, $\operatorname{src}\left(K_{s, t}\right) \geqslant k$.

Next, we show that $\operatorname{src}\left(K_{s, t}\right) \leqslant k$, which we establish by providing a strong rainbow $k$-coloring of $K_{s, t}$. Let $A=\{1,2, \ldots, k\}$ and $B=\{1,2, \ldots, k-1\}$. The sets $A^{s}$ and $B^{s}$ are Cartesian products of the $s$ sets $A$ and $s$ sets $B$, respectively. Thus $\left|A^{s}\right|=k^{s}$ and $\left|B^{s}\right|=(k-1)^{s}$. Hence $\left|B^{s}\right|<t \leqslant\left|A^{s}\right|$. Let $W=\left\{w_{1}, w_{2}, \ldots, w_{t}\right\}$, where the vertices of $W$ are labeled with $t$ elements of $A^{s}$ and such that the vertices $w_{1}, w_{2}, \ldots, w_{(k-1)^{s}}$ are labeled by the $(k-1)^{s}$ elements of $B^{s}$. For each $i$ with $1 \leqslant i \leqslant t$, denote the label of $w_{i}$ by

$$
\begin{equation*}
\mathbf{w}_{i}=\left(w_{i, 1}, w_{i, 2}, \ldots, w_{i, s}\right) \tag{2}
\end{equation*}
$$

For each $i$ with $1 \leqslant i \leqslant(k-1)^{s}$, we have $1 \leqslant w_{i, j} \leqslant k-1$ for $1 \leqslant j \leqslant s$. We now define a coloring $c: E\left(K_{s, t}\right) \rightarrow\{1,2, \ldots, k\}$ of the edges of $K_{s, t}$ by

$$
c\left(w_{i} u_{j}\right)=w_{i, j} \quad \text { where } 1 \leqslant i \leqslant t \text { and } 1 \leqslant j \leqslant s
$$

Thus for $1 \leqslant i \leqslant t$, the color code code $\left(w_{i}\right)$ of $w_{i}$ provided by the coloring $c$ is in fact $\mathbf{w}_{i}$, as described in (2). Hence distinct vertices in $W$ have distinct color codes.

We show that $c$ is a strong rainbow $k$-coloring of $K_{s, t}$. Certainly, for $w_{i} \in W$ and $u_{j} \in U$, the $w_{i}-u_{j}$ path $w_{i}, u_{j}$ is a rainbow geodesic. Let $w_{a}$ and $w_{b}$ be two vertices of $W$. Since these vertices have distinct color codes, there exists some $l$ with $1 \leqslant l \leqslant s$ such that code $\left(w_{a}\right)$ and code $\left(w_{b}\right)$ have different $l$-th coordinates. Thus $c\left(w_{a} u_{l}\right) \neq c\left(w_{b} u_{l}\right)$ and $w_{a}, u_{l}, w_{b}$ is a rainbow $w_{a}-w_{b}$ geodesic in $K_{s, t}$. We now consider two vertices $u_{p}$ and $u_{q}$ in $U$, where $1 \leqslant p<q \leqslant s$. Since there exists a vertex $w_{i} \in W$ with $1 \leqslant i \leqslant(k-1)^{s}$ such that $w_{i, p} \neq w_{i, q}$, it follows that $u_{p}, w_{i}, u_{q}$ is a rainbow $u_{p}-u_{q}$ geodesic in $K_{s, t}$. Thus, as claimed, $c$ is a strong rainbow $k$-coloring of $K_{s, t}$ and $\operatorname{so~} \operatorname{src}\left(K_{s, t}\right) \leqslant k$.

With the aid of Theorem 2.4, we are now able to determine the strong rainbow connection numbers of all complete multipartite graphs.

Theorem 2.5. Let $G=K_{n_{1}, n_{2}, \ldots, n_{k}}$ be a complete $k$-partite graph, where $k \geqslant 3$ and $n_{1} \leqslant n_{2} \leqslant \ldots \leqslant n_{k}$ such that $s=\sum_{i=1}^{k-1} n_{i}$ and $t=n_{k}$. Then

$$
\operatorname{src}(G)= \begin{cases}1 & \text { if } n_{k}=1 \\ 2 & \text { if } n_{k} \geqslant 2 \text { and } s>t \\ \lceil\sqrt[s]{t}\rceil & \text { if } s \leqslant t\end{cases}
$$

Proof. Let $n=\sum_{i=1}^{k} n_{i}$. If $n_{k}=1$, then $G=K_{n}$ and by Proposition 1.1, $\operatorname{src}(G)=1$. Suppose next that $n_{k} \geqslant 2$ and $s>t$. Since $n_{k} \geqslant 2$, it follows that $G \neq K_{n}$ and $\operatorname{so} \operatorname{src}(G) \geqslant 2$ by Proposition 1.1. It remains to show that $\operatorname{src}(G) \leqslant 2$ in this case.

Partition the multiset $S=\left\{n_{1}, n_{2}, \ldots, n_{k}\right\}$ into two submultisets

$$
A=\left\{a_{1}, a_{2}, \ldots, a_{p}\right\} \text { and } B=\left\{b_{1}, b_{2}, \ldots, b_{q}\right\}
$$

where then $p+q=k$, such that

$$
a=\sum_{i=1}^{p} a_{i} \leqslant \sum_{j=1}^{q} b_{j}=b
$$

and $b-a$ is the minimum nonnegative integer among all such partitions of $S$. Hence $K_{a, b}$ is a spanning subgraph of $G$. Since $\operatorname{diam}\left(K_{a, b}\right)=2$, for every two nonadjacent
vertices $u$ and $v$ of $K_{a, b}$, a path $P$ is a $u-v$ geodesic in $K_{a, b}$ if and only if $P$ is a $u-v$ geodesic in $G$. Thus, from Theorem 2.4,

$$
\operatorname{src}(G) \leqslant \operatorname{src}\left(K_{a, b}\right)=\lceil\sqrt[a]{b}\rceil
$$

We claim that $b \leqslant 2^{a}$. Assume, to the contrary, that $b>2^{a}$. Since $s>t$, it follows that $q \geqslant 2$. We consider two cases, according to $a \leqslant 3$ or $a \geqslant 4$. If $G$ is a complete $k$ partite graph with $a \leqslant 3$, then the only ordered pairs $(a, b)$ for $K_{a, b}$ are: $(2,3),(2,4)$, $(3,3),(3,4),(3,5),(3,6)$. In all cases, $\operatorname{src}(G) \leqslant \operatorname{src}\left(K_{a, b}\right)=\lceil\sqrt[a]{b}\rceil=2$. Hence we may assume that $a \geqslant 4$. Let $b_{1}$ be the smallest element of $B$. Hence $a+b_{1}>b-b_{1}$. Because $a \geqslant 4$, it follows that

$$
b_{1}>\frac{b-a}{2}>\frac{2^{a}-a}{2}>\frac{3 a-a}{2}=a .
$$

Let $A^{\prime}=\left\{b_{1}\right\}$ and let the multiset $B^{\prime}=S-\left\{b_{1}\right\}$. Since $b_{2} \in B^{\prime}, b_{1} \leqslant b_{2}$, and $a<b_{1}$, this contradicts the defining properties of the sets $A$ and $B$. Hence, as claimed, $b \leqslant 2^{a}$. Thus

$$
\operatorname{src}(G) \leqslant\lceil\sqrt[a]{b}\rceil \leqslant\left\lceil\sqrt[a]{2^{a}}\right\rceil=2
$$

giving us the desired result.
Next, suppose that $s \leqslant t$. Let $W$ be the unique independent set of $n_{k}=t$ vertices of $G$. Since $K_{s, t}$ is a connected spanning subgraph of $G$, it follows again, since $\operatorname{diam}(G)=2$, that

$$
\operatorname{src}(G) \leqslant \operatorname{src}\left(K_{s, t}\right)=\lceil\sqrt[s]{t}\rceil
$$

We claim that $\operatorname{src}(G)=\lceil\sqrt[s]{t}\rceil$. Assume, to the contrary, that $\operatorname{src}(G)=l<\lceil\sqrt[s]{t}\rceil$. Then $t>l^{s}$. This implies that there exists a strong rainbow $l$-coloring $c$ of $G$. Since every vertex of $G$ belonging to $W$ has degree $s$ in $G$, the coloring $c$ produces a color code $\operatorname{code}(w)$ for each vertex $w$ of $W$ consisting of an ordered $s$-tuple, each entry of which is an element of $\{1,2, \ldots, l\}$. Since the number of distinct color codes for the vertices of $W$ is at most $l^{s}$ and $|W|=t>l^{s}$, there exist two vertices $w^{\prime}$ and $w^{\prime \prime}$ in $W$ having the same color code. This, however, implies that the two edges in each $w^{\prime}-w^{\prime \prime}$ geodesic in $G$ have the same color, contradicting the assumption that $c$ is a strong rainbow $l$-coloring of $G$.

According to Theorems 2.4 and 2.5, the strong rainbow connection number of a complete multipartite graph can be arbitrarily large. This is not the case for the rainbow connection number of a complete multipartite graph however, as we show next. We begin with complete bipartite graphs.

Theorem 2.6. For integers $s$ and $t$ with $2 \leqslant s \leqslant t$,

$$
\operatorname{rc}\left(K_{s, t}\right)=\min \{\lceil\sqrt[s]{t}\rceil, 4\}
$$

Proof. First, observe that for $2 \leqslant s \leqslant t,\lceil\sqrt[s]{t}\rceil \geqslant 2$. Let $U$ and $W$ be the partite sets of $K_{s, t}$, where $|U|=s$ and $|W|=t$. Suppose that $U=\left\{u_{1}, u_{2}, \ldots, u_{s}\right\}$. We consider three cases.

Case 1. $\lceil\sqrt[s]{t}\rceil=2$. Then $s \leqslant t \leqslant 2^{s}$. Since

$$
2 \leqslant \operatorname{rc}\left(K_{s, t}\right) \leqslant \operatorname{src}\left(K_{s, t}\right)=\lceil\sqrt[s]{t}\rceil=2
$$

it follows that $\operatorname{rc}\left(K_{s, t}\right)=2$.
Case $2 .\lceil\sqrt[s]{t}\rceil=3$. Then $2^{s}+1 \leqslant t \leqslant 3^{s}$. Since

$$
2 \leqslant \operatorname{rc}\left(K_{s, t}\right) \leqslant \operatorname{src}\left(K_{s, t}\right)=\lceil\sqrt[s]{t}\rceil=3,
$$

it follows that $\operatorname{rc}\left(K_{s, t}\right)=2$ or $\operatorname{rc}\left(K_{s, t}\right)=3$. We claim that $\operatorname{rc}\left(K_{s, t}\right)=3$. Assume, to the contrary, that there exists a rainbow 2 -coloring of $K_{s, t}$. Corresponding to this rainbow 2-coloring of $K_{s, t}$, there is a color code code $(w)$ assigned to each vertex $w \in W$, consisting of an ordered $s$-tuple $\left(a_{1}, a_{2}, \ldots, a_{s}\right)$, where $a_{i}=c\left(u_{i} w\right) \in\{1,2\}$ for $1 \leqslant i \leqslant s$. Since $t>2^{s}$, there exist two distinct vertices $w^{\prime}$ and $w^{\prime \prime}$ of $W$ such that code $\left(w^{\prime}\right)=\operatorname{code}\left(w^{\prime \prime}\right)$. Since the edges of every $w^{\prime}-w^{\prime \prime}$ path of length 2 are colored the same, there is no rainbow $w^{\prime}-w^{\prime \prime}$ path in $K_{s, t}$, a contradiction. Thus, as claimed, $\operatorname{rc}\left(K_{s, t}\right)=3$.

Case $3 .\lceil\sqrt[s]{t}\rceil \geqslant 4$. Then $t \geqslant 3^{s}+1$. We claim that $\operatorname{rc}\left(K_{s, t}\right)=4$. First, we show that $\operatorname{rc}\left(K_{s, t}\right) \geqslant 4$. Assume, to the contrary, that there exists a rainbow 3-coloring of $K_{s, t}$. In this case, corresponding to this rainbow 3-coloring of $K_{s, t}$, there is a color code, code $(w)$, assigned to each vertex $w \in W$, consisting of an ordered $s$-tuple $\left(a_{1}, a_{2}, \ldots, a_{s}\right)$, where $a_{i}=c\left(u_{i} w\right) \in\{1,2,3\}$ for $1 \leqslant i \leqslant s$. Since $t>3^{s}$, there exist two distinct vertices $w^{\prime}$ and $w^{\prime \prime}$ of $W$ such that $\operatorname{code}\left(w^{\prime}\right)=\operatorname{code}\left(w^{\prime \prime}\right)$. Since every $w^{\prime}-w^{\prime \prime}$ path in $K_{s, t}$ has even length, the only possible rainbow $w^{\prime}-w^{\prime \prime}$ path must have length 2 . However, since $\operatorname{code}\left(w^{\prime}\right)=\operatorname{code}\left(w^{\prime \prime}\right)$, the colors of the edges of every $w^{\prime}-w^{\prime \prime}$ path of length 2 are the same. Hence there is no rainbow $w^{\prime}-w^{\prime \prime}$ path in $K_{s, t}$, a contradiction. Thus, as claimed, $\mathrm{rc}\left(K_{s, t}\right) \geqslant 4$.

To verify that $\operatorname{rc}\left(K_{s, t}\right) \leqslant 4$, we show that there exists a rainbow 4-coloring of $K_{s, t}$. Let $A=\{1,2,3\}, W=\left\{w_{1}, w_{2}, \ldots, w_{t}\right\}, W^{\prime}=\left\{w_{1}, w_{2}, \ldots, w_{3 s}\right\}$, and $W^{\prime \prime}=$ $W-W^{\prime}$. Assign to the vertices in $W^{\prime}$ the $3^{s}$ distinct elements of $A^{s}$ and assign to the vertices in $W^{\prime \prime}$ the identical code whose first coordinate is 4 and all whose remaining coordinates are 3 . Corresponding to this assignment of codes is a coloring
of the edges of $K_{s, t}$, where $c\left(w_{i} u_{j}\right)=k$ if the $j$ th coordinate of $\operatorname{code}\left(w_{i}\right)$ is $k$. We claim that this coloring is, in fact, a rainbow 4-coloring of $K_{s, t}$. Let $x$ and $y$ be two nonadjacent vertices of $K_{s, t}$. Suppose first that $x, y \in W$. We consider three cases.

Case i. $x, y \in W^{\prime}$. Since code $(x) \neq \operatorname{code}(y)$, there exists $i$ with $1 \leqslant i \leqslant s$ such that code $(x)$ and code $(y)$ have different $i$ th coordinates. Then the path $x, u_{i}, y$ is a rainbow $x-y$ path of length 2 in $K_{s, t}$.

C ase ii. $x \in W^{\prime}$ and $y \in W^{\prime \prime}$. Suppose that the first coordinate of $\operatorname{code}(x)$ is $a$, where $1 \leqslant a \leqslant 3$. Then $x, u_{1}, y$ is a rainbow $x-y$ path of length 2 in $K_{s, t}$ whose edges are colored $a$ and 4 .

Case iii. $x, y \in W^{\prime \prime}$. Let $z \in W^{\prime}$ such that the first coordinate of code $(z)$ is 1 and the second coordinate of $\operatorname{code}(z)$ is 2 . Then $x, u_{1}, z, u_{2}, y$ is a rainbow $x-y$ path of length 4 in $K_{s, t}$ whose edges are colored $4,1,2,3$, respectively.

Finally, suppose that $x, y \in U$. Then $x=u_{i}$ and $y=u_{j}$, where $1 \leqslant i<j \leqslant s$. Then there exists a vertex $w \in W^{\prime}$ whose $i$ th and $j$ th coordinates are distinct. Then $x, w, y$ is a rainbow $x-y$ path in $K_{s, t}$.

Thus this coloring is a rainbow 4-coloring of $K_{s, t}$ and so $\operatorname{rc}\left(K_{s, t}\right)=4$ in this case.

Next, we determine rainbow connection numbers of all complete multipartite graphs.

Theorem 2.7. Let $G=K_{n_{1}, n_{2}, \ldots, n_{k}}$ be a complete $k$-partite graph, where $k \geqslant 3$ and $n_{1} \leqslant n_{2} \leqslant \ldots \leqslant n_{k}$ such that $s=\sum_{i=1}^{k-1} n_{i}$ and $t=n_{k}$. Then

$$
\operatorname{rc}(G)= \begin{cases}1 & \text { if } n_{k}=1 \\ 2 & \text { if } n_{k} \geqslant 2 \text { and } s>t \\ \min \{\lceil\sqrt[s]{t}\rceil, 3\} & \text { if } s \leqslant t\end{cases}
$$

Proof. Let $n=s+t=\sum_{i=1}^{k} n_{i}$. If $n_{k}=1$, then $G=K_{n}$ and by Proposition 1.1, $\operatorname{rc}(G)=1$. Suppose next that $n_{k} \geqslant 2$ and $s>t$. By Theorem $2.5, \operatorname{src}(G)=2$ and so $\operatorname{rc}(G)=2$ by Proposition 1.1.

Next, suppose that $s \leqslant t$. Since $n_{k} \geqslant 2$, it follows that $G \neq K_{n}$ and so $\operatorname{rc}(G) \geqslant 2$. By Theorem 2.5, $\operatorname{src}(G)=\lceil\sqrt[s]{t}\rceil$ and so $\operatorname{rc}(G) \leqslant\lceil\sqrt[s]{t}\rceil$. To show that $\operatorname{rc}(G) \leqslant 3$ as well, we provide a rainbow 3 -coloring of $G$. Let $V_{1}, V_{2}, \ldots, V_{k}$ be the partite sets of $G$ with

$$
V_{i}=\left\{v_{i, 1}, v_{i, 2}, \ldots, v_{i, n_{i}}\right\}
$$

for $1 \leqslant i \leqslant k$. Furthermore, let

$$
U=V_{1} \cup V_{2} \cup \ldots \cup V_{k-1}=\left\{u_{1}, u_{2}, \ldots, u_{s}\right\}
$$

such that $u_{i}=v_{k-1, i}$ for $1 \leqslant i \leqslant n_{k-1}$. Thus $|U|=s$. Define a coloring $c^{*}$ of the edges of $G$ by

$$
c^{*}(e)= \begin{cases}1 & \text { if } e=v_{i, j} v_{i+1, j} \text { for } 1 \leqslant i \leqslant k-2 \text { and } 1 \leqslant j \leqslant n_{i} \text { or } \\ & \text { if } e=u_{l} v_{k, l} \text { for } 1 \leqslant l \leqslant s, \\ 2 & \text { if } e=v_{1, j} v_{k, l} \text { for } 1 \leqslant j \leqslant n_{1} \text { and } s+1 \leqslant l \leqslant t \\ 3 & \text { otherwise }\end{cases}
$$

Let $x$ and $y$ be two nonadjacent vertices of $G$. Then $x, y \in V_{i}$ for some $i$ with $1 \leqslant i \leqslant k$. Let $x=v_{i, p}$ and $y=v_{i, q}$, where $1 \leqslant p<q \leqslant n_{i}$. If $1 \leqslant i \leqslant k-1$, then $x, v_{i+1, p}, y$ is a rainbow $x-y$ path in $G$ whose edges are colored 1 and 3 . Thus we may assume that $i=k$. If $1 \leqslant p<q \leqslant s$, then $x, u_{p}, y$ is a rainbow $x-y$ path in $G$ whose edges are colored 1 and 3 . If $s+1 \leqslant p<q \leqslant t$, then $x, v_{1,1}, v_{2,1}, y$ is a rainbow $x-y$ path in $G$ whose edges are colored 2,1 and 3 , respectively. If $1 \leqslant p \leqslant s$ and $s+1 \leqslant q \leqslant t$, then $x, v_{1,1}, y$ is a rainbow $x-y$ path whose edges are colored 3 and 2. Thus $\operatorname{rc}(G) \leqslant 3$. Therefore, as claimed, $\operatorname{rc}(G) \leqslant \min \{\lceil\sqrt[s]{t}\rceil, 3\}$.

Assume, to the contrary, that $\operatorname{rc}(G)<\min \{\lceil\sqrt[s]{t}\rceil, 3\} \leqslant 3$. Since $\operatorname{rc}(G) \geqslant 2$, it follows that $\operatorname{rc}(G)=2$. Let $c^{\prime}$ be a rainbow 2 -coloring of $G$. Thus, we can associate a color code $\operatorname{code}(w)=\left(a_{1}, a_{2}, \ldots, a_{s}\right)$ to each vertex $w \in W$, where $a_{i}=c\left(u_{i} w\right) \in\{1,2\}$ for $1 \leqslant i \leqslant s$. Since $\sqrt[s]{t}>2$, it follows that $t>2^{s}$ and so there exist two distinct vertices $w^{\prime}$ and $w^{\prime \prime}$ of $W$ such that $\operatorname{code}\left(w^{\prime}\right)=\operatorname{code}\left(w^{\prime \prime}\right)$. Hence the two edges of each $w^{\prime}-w^{\prime \prime}$ path of length 2 are colored the same and so there is no rainbow $w^{\prime}-w^{\prime \prime}$ path in $K_{s, t}$, producing a contradiction. Thus, as claimed, $\operatorname{rc}\left(K_{s, t}\right)=3=\min \{\lceil\sqrt[s]{t}\rceil, 3\}$ in this case.

## 3. On Rainbow connection numbers with prescribed values

We have seen that $\operatorname{rc}(G) \leqslant \operatorname{src}(G)$ for every nontrivial connected graph $G$. By Proposition 1.1, it follows that for every positive integer $a$ and for every tree $T$ of size $a, \operatorname{rc}(T)=\operatorname{src}(T)=a$. Furthermore, for $a \in\{1,2\}, \operatorname{rc}(G)=a$ if and only if $\operatorname{src}(G)=a$. If $a=3$ and $b \geqslant 4$, then by Propositions 2.2 and $2.3, \operatorname{rc}\left(W_{3 b}\right)=3$ and $\operatorname{src}\left(W_{3 b}\right)=b$. For $a \geqslant 4$, we have the following.

Theorem 3.1. Let $a$ and $b$ be integers with $a \geqslant 4$ and $b \geqslant(5 a-6) / 3$. Then there exists a connected graph $G$ such that $\operatorname{rc}(G)=a$ and $\operatorname{src}(G)=b$.

Proof. Let $n=3 b-3 a+6$ and let $W_{n}$ be the wheel consisting of an $n$-cycle $C_{n}: v_{1}, v_{2}, \ldots, v_{n}, v_{1}$ and another vertex $v$ joined to every vertex of $C_{n}$. Let $G$ be the graph constructed from $W_{n}$ and the path $P_{a-1}: u_{1}, u_{2}, \ldots, u_{a-1}$ of order $a-1$ by identifying $v$ and $u_{a-1}$.

First, we show that $\operatorname{rc}(G)=a$. Since $b \geqslant(5 a-6) / 3$ and $a \geqslant 4$, it follows that $b>a$ and so $n=3 b-3 a+6 \geqslant 7$. By Proposition 2.2, we then have $\operatorname{rc}\left(W_{n}\right)=3$. Define a coloring $c$ of the graph $G$ by

$$
c(e)= \begin{cases}i & \text { if } e=u_{i} u_{i+1} \text { for } 1 \leqslant i \leqslant a-2 \\ a & \text { if } e=v_{i} v \text { and } i \text { is odd } \\ a-1 & \text { if } e=v_{i} v \text { and } i \text { is even } \\ 1 & \text { otherwise }\end{cases}
$$

Since $c$ is a rainbow $a$-coloring of the edges of $G$, it follows that $\operatorname{rc}(G) \leqslant a$.
It remains to show that $\operatorname{rc}(G) \geqslant a$. Assume, to the contrary, that $\operatorname{rc}(G) \leqslant a-1$. Let $c^{\prime}$ be a rainbow ( $a-1$ )-coloring of $G$. Since the path $u_{1}, u_{2}, \ldots, u_{a-1}$ is the only $u_{1}-u_{a-1}$ path in $G$, the edges of this path must be colored differently by $c^{\prime}$. We may assume, without loss of generality, that $c^{\prime}\left(u_{i} u_{i+1}\right)=i$ for $1 \leqslant i \leqslant a-2$. For each $j$ with $1 \leqslant j \leqslant 3 b-3 a+6$, there is a unique $u_{1}-v_{j}$ path of length $a-1$ in $G$ and so $c^{\prime}\left(v v_{j}\right)=a-1$ for $1 \leqslant j \leqslant 3 b-3 a+6$. Consider the vertices $v_{1}$ and $v_{a+1}$. Since $b \geqslant(5 a-6) / 3$, any $v_{1}-v_{a+1}$ path of length $a-1$ or less must contain $v$ and thus two edges colored $a-1$, contradicting our assumption that $c^{\prime}$ is a rainbow $(a-1)$-coloring of $G$. This implies that $\operatorname{rc}(G) \geqslant a$ and so $\operatorname{rc}(G)=a$.

Next, we show that $\operatorname{src}(G)=b$. Since $n=3 b-3 a+6=3(b-a+2) \geqslant 7$, it follows by Proposition 2.3 that $\operatorname{src}\left(W_{n}\right)=b-a+2$. Let $c_{1}$ be a strong rainbow $(b-a+2)$-coloring of $W_{n}$. Define a coloring $c$ of the graph $G$ by

$$
c(e)= \begin{cases}c_{1}(e) & \text { if } e \in E\left(W_{n}\right) \\ b-a+2+i & \text { if } e=u_{i} u_{i+1} \text { for } 1 \leqslant i \leqslant a-2\end{cases}
$$

Since $c$ is a strong rainbow $b$-coloring of $G$, it follows that $\operatorname{src}(G) \leqslant b$.
It remains to show that $\operatorname{src}(G) \geqslant b$. Assume, to the contrary, that $\operatorname{src}(G) \leqslant b-1$. Let $c^{*}$ be a strong rainbow $(b-1)$-coloring of $G$. We may assume, without loss of generality, that $c^{*}\left(u_{i} u_{i+1}\right)=i$ for $1 \leqslant i \leqslant a-2$. For each $j$ with $1 \leqslant j \leqslant 3 b-3 a+6$, there is a unique $u_{1}-v_{j}$ geodesic in $G$, implying $c^{*}\left(v v_{j}\right) \in C=\{a-1, a, \ldots, b-1\}$. Let $S=\left\{v v_{j}: 1 \leqslant j \leqslant 3 b-3 a+6\right\}$. Then $|S|=3 b-3 a+6$ and $|C|=b-a+1$. Since at most three edges in $S$ can be colored the same, the $b-a+1$ colors in $C$ can
color at most $3(b-a+1)=3 b-3 a+3$ edges, producing a contradiction. Therefore, $\operatorname{src}(G) \geqslant b$ and $\operatorname{so~} \operatorname{src}(G)=b$.

Combining Propositions 1.1, 2.2, 2.3 and Theorem 3.1, we have the following.

Corollary 3.2. Let $a$ and $b$ be positive integers. If $a=b$ or $3 \leqslant a<b$ and $b \geqslant(5 a-6) / 3$, then there exists a connected graph $G$ such that $\operatorname{rc}(G)=a$ and $\operatorname{src}(G)=b$.

We conclude with two conjectures and a result.

Conjecture 3.3. Let $a$ and $b$ be positive integers. Then there exists a connected graph $G$ such that $\operatorname{rc}(G)=a$ and $\operatorname{src}(G)=b$ if and only if $a=b \in\{1,2\}$ or $3 \leqslant a \leqslant b$.

It is easy to see that if $H$ is a connected spanning subgraph of a nontrivial (connected) graph $G$, then $\operatorname{rc}(G) \leqslant \operatorname{rc}(H)$. We have already noted that if, in addition, $\operatorname{diam}(H)=2$, then $\operatorname{src}(G) \leqslant \operatorname{src}(H)$. However, the question arises as to whether this is true when $\operatorname{diam}(H) \geqslant 3$.

Conjecture 3.4. If $H$ is a connected spanning subgraph of a nontrivial (connected) graph $G$, then $\operatorname{src}(G) \leqslant \operatorname{src}(H)$.

If Conjecture 3.4 is true, then for every nontrivial connected graph $G$ of order $n$,

$$
\operatorname{diam}(G) \leqslant \operatorname{rc}(G) \leqslant \operatorname{src}(G) \leqslant n-1
$$

The following can be proved immediately.

Proposition 3.5. For each triple $d, k, n$ of integers with $2 \leqslant d \leqslant k \leqslant n-1$, there exists a connected graph $G$ of order $n$ with $\operatorname{diam}(G)=d$ such that $\operatorname{rc}(G)=$ $\operatorname{src}(G)=k$.

## References

[1] G. Chartrand, P. Zhang: Introduction to Graph Theory. McGraw-Hill, Boston, 2005.

Authors' addresses: Kathy McKeon, Connecticut College, Department of Mathematics, 270 Mohegan Avenue, Box 5561, New London, CT 06320, USA; Garry L. Johns, Saginaw Valley State University, University Center, Department of Mathematical Sciences, MI 48710-0001, USA; Ping Zhang, Gary Chartrand, Western Michigan University, Department of Mathematics, Kalamazoo, MI 49008, USA, e-mail: ping.zhang@wmich.edu.

