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## Rainbow Generalizations of Ramsey Theory - A Dynamic Survey

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## Abstract

In this work, we collect Ramsey-type results concerning rainbow edge colorings of graphs.

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If you have corrections, updates or new results which fit the scope of this work, please contact Colton Magnant at coltonmagnant@yahoo.com.

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# 1 Introduction

We will almost entirely focus on coloring edges so “coloring” will mean edge coloring. In most cases,  $k$  will be used to denote the number of colors used on the edges. Also define the color degree  $d^c(v)$  to be the number of colors on edges incident to  $v$ . A colored graph is called *rainbow* if each edge receives a distinct color. For all other notation, we refer the reader to [47].

The original publication of this work was [78]. There are some other surveys of edge coloring that we should mention. The first is the dynamic survey [158] by Radziszowski which contains a wonderful list of known (monochromatic) Ramsey numbers. There is a brief survey of anti-Ramsey results in [165]. Also there is a survey by Kano and Li [122] which discusses some rainbow coloring. There is also a forthcoming survey by Fujita, Liu and Magnant [74] related to this survey but focusing more on large monochromatic structures.

It should be noted that in [173], Voloshin demonstrates very interesting relationships between rainbow / monochromatic subgraphs and mixed hypergraph colorings. In fact, many of the notions of generalized Ramsey colorings are very closely related to upper and lower chromatic numbers of the derived mixed hypergraph.

## 2 Anti-Ramsey Theory

The anti-Ramsey problem is stated as follows.

**Definition 1** *Given graphs  $G$  and  $H$ , the anti-Ramsey number  $ar(G, H)$  is defined to be the maximum number of colors  $k$  such that there exists a coloring of the edges of  $G$  with exactly  $k$  colors in which every copy of  $H$  in  $G$  has at least two edges with the same color ( $H$  is not rainbow colored).*

Classically, the graph  $G$  is a large complete graph and the graph  $H$  comes from some class.

This is equivalent to the *rainbow number*  $rb(G, H)$  which is defined to be the minimum number of colors  $k$  such that any coloring, using  $k$  colors, of the edges of  $G$  contains a rainbow  $H$ . Thus, the relationship is  $rb(G, H) = ar(G, H) + 1$ . In order to be consistent with the majority of the results, we state all results in terms of anti-Ramsey numbers.

The study of anti-Ramsey theory began with a paper by Erdős, Simonovits, and Sós [62] in 1975 (note that related ideas were studied even earlier in [61]). Since then, the field has blossomed in a wide variety of papers. See [165] for a brief survey.

### 2.1 Cycles

In the original work by Erdős, Simonovits and Sós, the authors stated the following conjecture.

**Conjecture 1 ([62])** *For all  $n \geq k \geq 3$ ,*

$$ar(K_n, C_k) = \left( \frac{k-2}{2} + \frac{1}{k-1} \right) n + O(1).$$

The authors provided the following lower bound construction (as presented in [113]). For  $n = (k - 1)q + r$ , partition  $V(G)$  into sets  $V_1, \dots, V_q$  of size  $k - 1$  and one set  $V_{q+1}$  of size  $r$ . The edges with endpoints in the same set receive  $q \binom{k-1}{2} + \binom{r}{2}$  different colors. On the remaining edges, we use  $q$  more colors  $c_1, \dots, c_q$  with  $c_{\min\{i,j\}}$  on the edges between the sets  $V_i$  and  $V_j$  when  $i \neq j$ . Each set is too small to contain the desired cycle and there can be no cycle between sets so this example provides the stated lower bound.

Erdős, Simonovits and Sós proved the conjecture in the case when  $k = 3$  by showing that  $ar(K_n, C_3) = n - 1$ . Alon [3] proved the conjecture for  $k = 4$  by showing that  $ar(K_n, C_4) = \lfloor \frac{4n}{3} \rfloor - 1$ . He also provided a general upper bound of  $ar(K_n, C_k) \leq (k - 2)n - \binom{k-1}{2}$ . In 2000, Montellano-Ballesteros and Neumann-Lara [150] provided another upper bound. Jiang and West [113] later improved the upper bound to  $ar(K_n, C_k) \leq (\frac{k+1}{2} - \frac{2}{k-1})n - (k - 2)$  with a slight improvement when  $k$  is even. In 2004, Jiang, Schiermeyer and West [111] (see also [163]) proved the conjecture for  $k \leq 7$  but finally, in 2005, Montellano-Ballesteros and Neumann-Lara [153] completely proved Conjecture 1 with a simplified proof by Choi in [51].

**Theorem 1 ([153])** *For all  $n \geq k \geq 3$ ,*

$$ar(K_n, C_k) = \left( \frac{k - 2}{2} + \frac{1}{k - 1} \right) n + O(1).$$

In a related work, Axenovich, Jiang and Kündgen [18], proved the following result for finding even cycles in complete bipartite graphs.

**Theorem 2 ([18])** *For all positive integers  $m, n, k$  with  $m \leq n$  and  $k \geq 2$ ,*

$$ar(K_{m,n}, C_{2k}) = \begin{cases} (k - 1)(m + n) - 2(k - 1)^2 + 1 & \text{for } m \geq 2k - 1, \\ (k - 1)n + m - (k - 1) & \text{for } k - 1 \leq m \leq 2k - 1, \\ mn & \text{for } m \leq k - 1. \end{cases}$$

For the general class of all rainbow cycles, the following was shown.

**Theorem 3 ([116])** *For positive integers  $m$  and  $n$ , the maximum number of colors that can appear in an edge coloring of  $K_{m,n}$  with no rainbow cycles is  $m + n - 1$ .*

It was also shown in [116] that the colorings that achieve the bound in Theorem 3 can be encoded by special vertex labelings of full binary trees with  $m + n$  leaves.

Looking within hypercubes, the authors of [36] consider cycles and provide some bounds on  $ar(Q_n, C_k)$  and the exact results when  $n \leq 4$ .

Let  $\Omega_k$  be the set of graphs containing  $k$  vertex disjoint disjoint cycles. In [115], the following result was proven along with some general bounds for  $ar(K_n, \Omega_k)$ .

**Theorem 4 ([115])** *For  $n \geq 7$ ,*

$$ar(K_n, \Omega_2) = 2n - 2.$$

*Also  $ar(K_6, \Omega_2) = 11$ .*

Gorgol [84] considered using a split graph as the underlying host graph.

**Theorem 5 ([84])** *Let  $H$  be a graph with  $\delta(G) \geq 2$ . Then*

$$ar(K_n + \overline{K_s}, H) \geq ar(K_n, H) + s$$

for  $n, s \geq 1$ .

**Theorem 6 ([84])** *If  $|V(H)| \leq n$  and  $H$  is a subgraph of  $K_{n,s}$ , then*

$$ar(K_n + \overline{K_s}, H) \geq ar(K_n, H) + ar(K_{n,s}, H).$$

**Theorem 7 ([84])** *Let  $n \geq 2$  and  $s \geq 1$ . Then  $ar(K_n + \overline{K_s}, C_3) = n + s - 1$ .*

**Proposition 1 ([84])** *Let  $n \in \{2, 3\}$  and  $n + s \geq 4$ . Then*

$$ar(K_n + \overline{K_s}, C_4) = ar(K_{n,s}, C_4) + 1.$$

**Theorem 8 ([84])** *Let  $n \geq 4$  and  $s \geq n$ . Then*

$$ar(K_n + \overline{K_s}, C_4) \leq ar(K_n, C_4) + ar(K_{n,s}, C_4) - 1.$$

Letting  $C_3^+$  denote a triangle with a pendant edge, the following was obtained.

**Theorem 9 ([84])** *Let  $n \geq 3$  and  $s \geq 1$ . Then*

$$ar(K_n + \overline{K_s}, C_3^+) \leq n + s - 1.$$

Let  $B$  be the bull, the triangle with two disjoint pendant edges.

**Theorem 10 ([84])** *Let  $n, s \geq 1$  and  $n + s \geq 5$ . Then*

$$ar(K_n + \overline{K_s}, B) \geq n + s,$$

and this bound is sharp for  $n = 2, 3$ .

Let  $K_{1,4}^+$  denote the triangle with two pendant edges incident to a single vertex of the triangle.

**Theorem 11 ([84])** *Let  $s \geq 3$ . Then*

$$\begin{aligned} ar(K_2 + \overline{K_s}, K_{1,4}^+) &\leq s + 1, \\ ar(K_3 + \overline{K_s}, K_{1,4}^+) &\leq \max\{7, s + 3\}. \end{aligned}$$

**Theorem 12 ([84])** *Let  $n \geq 4$  and  $s \leq n$ . Then*

$$ar(K_n + \overline{K_s}, K_{1,4}^+) \leq n + s + 1.$$

## 2.2 Cliques

Let  $\text{ex}(n, \mathcal{H})$  be the maximum number of edges in a graph  $G$  on  $n$  vertices containing no subgraph isomorphic to  $H \in \mathcal{H}$ . This function has been called the Turán function since it was first studied in [171] where the set  $\mathcal{H}$  consists of a single clique of order  $k + 1$ . In this work, Turán proved the following theorem.

**Theorem 13** ([171])

$$\text{ex}(n, K_{k+1}) = \binom{k}{2}t^2 + i(k-1)t + \binom{i}{2}$$

where the sharpness is given by the complete  $k$ -partite graph with partite sets  $V_1, \dots, V_k$  where  $|V_j| = t + 1$  for  $1 \leq j \leq i$  and  $|V_j| = t$  for  $i + 1 \leq j \leq k$  where  $n = tk + i$  (i.e. an almost balanced complete multipartite graph).

On the surface, Theorem 13 may seem to have little in common with anti-Ramsey theory but in [62], Erdős, Simonovits, and Sós proved the following relationship.

**Theorem 14** ([62]) *Given an integer  $k$ , there exists an integer  $n(k)$  such that*

$$\text{ar}(K_n, K_k) = \text{ex}(n, K_{k-1}) + 1 \tag{1}$$

for all  $n \geq n(k)$ .

The authors also showed, in [62], that Equation (1) holds for  $k = 3$  for all  $n \geq 4$ .

Independently, Montellano-Ballesteros and Neumann-Lara [151] and Schiermeyer [164] proved that Equation (1) holds for all  $n > k \geq 3$ .

The lower bound, as observed in [62], uses a different color on each edge of Turán's construction and then a single new color on all other edges to complete the coloring. This coloring certainly has no rainbow  $K_k$  but it uses  $\text{ex}(n, K_{k-1}) + 1$  colors. Both proofs of the upper bound are by induction on  $n$  but each uses a different counting strategy within the induction.

The idea of anti-Ramsey numbers for cliques was extended in [34] to coloring in rounds. For positive integers  $k \leq n$  and  $t$ , let  $\chi^t(k, n)$  denote the minimum number  $\chi$  of colors such that there exists a sequence of length  $t$  of  $\chi$ -colorings  $\psi_1, \psi_2, \dots, \psi_t$  of the edges of  $K_n$  such that all  $\binom{k}{2}$  edges of each  $K_k \subseteq K_n$  get different colors in at least one coloring  $\psi_i$ . Conversely, let  $t(k, n)$  denote the minimum length of such a sequence of colorings each using  $\binom{k}{2}$  colors such that each  $K_k$  is rainbow in at least one coloring. The main result of [34] is the following concerning rainbow triangles.

**Theorem 15** ([34]) *For all  $n \geq 3$  and  $t$ ,*

$$(n - 1)^{1/t} \leq \chi^t(3, n) \leq 4n^{1/t} - 1.$$

This result generalizes an earlier result of Körner and Simonyi [125] which is stated as follows.

**Theorem 16** ([125]) *For all  $n \geq 3$ ,*

$$\lceil \log(n-1)/\log 3 \rceil \leq t(3, n) \leq \lceil \log n \rceil.$$

The authors of [34], then go on to explore 2-round colorings, providing bounds on  $\chi^2(i, n)$  for  $i = 4, 5, 6, 7$ . Further, they studied  $t(k, n)$  for  $k = n - 1, n - 2$  and  $\frac{n}{2}$ .

### 2.3 Trees

For general trees, Jiang and West [112] provide exact numbers for some families of trees and bounds for some individual trees. Let  $\mathcal{T}_k$  be the family of all trees on  $k$  edges and let  $\ell(n, k)$  denote the maximum size of an  $n$ -vertex graph in which every two components together have at most  $k$  vertices.

For the sake of notation, for any set of graphs  $\mathcal{H}$ , let  $ar(K_n, \mathcal{H})$  be the maximum number of colors  $k$  such that there exists a coloring of  $K_n$  with exactly  $k$  colors in which, for all  $H \in \mathcal{H}$ , no copy of  $H$  in the colored  $K_n$  is rainbow.

**Theorem 17** ([112]) *If  $n > k$ , then:*

$$ar(K_n, \mathcal{T}_k) - 1 = \ell(n, k) = \begin{cases} \binom{k-1}{2} & \text{if } k < n \leq 2k - 1, \\ \binom{\lceil k/2 \rceil}{2} + r \binom{\lfloor k/2 \rfloor}{2} + \binom{s}{2} & \text{if } n \geq 2k. \end{cases}$$

where  $r = \lfloor (n - \lceil k/2 \rceil) / \lfloor k/2 \rfloor \rfloor$  and  $s = n - \lceil k/2 \rceil - r \lfloor k/2 \rfloor$ .

This result is proven by finding  $\ell(k, n)$  and then showing the relationship to the anti-Ramsey number. The bipartite version of this problem is considered in [114]. Also in [112], the authors prove the following for an individual tree  $T$ .

**Theorem 18** ([112]) *Let  $T$  be a tree with  $k$  edges and  $n \geq 2k$ . Then*

$$\frac{n}{2} \left\lfloor \frac{k-2}{2} \right\rfloor + c_k \leq ar(K_n, T) \leq n(k-1)$$

where  $c_k$  does not depend on  $n$ .

The upper bound in Theorem 18 comes from the known bound of  $ex(n, T) \leq n(k-1)$ . Regarding this quantity, Erdős and Sós conjectured the following.

**Conjecture 2**

$$ex(n, T) \leq \frac{n(k-1)}{2}.$$

If this conjecture is true, then the upper bound of Theorem 18 can also be reduced to  $\frac{n(k-1)}{2}$ .

More specifically, Jiang and West also proved the following result for brooms. Let  $B_{s,t}$  be the broom consisting of  $s + t$  edges obtained by identifying the center of  $K_{1,s}$  with an end-vertex of  $P_{t+1}$ .

**Theorem 19 ([112])** For  $n$  sufficiently large,

$$\frac{1}{2}nr_1 + c_k \leq ar(K_n, B_{s,t}) \leq \frac{1}{2}nr_2 + 1$$

where  $r_1 = \max\{s - 1, 2\lfloor (t - 1)/2 \rfloor\}$ ,  $r_2 = \max\{s - 1, t\}$  and  $c_k$  does not depend on  $n$ .

Jiang [107] and Montellano-Ballesteros [147] independently found the anti-Ramsey number for stars, improving upon bounds in [142].

**Theorem 20 ([107, 147])**

$$ar(K_n, K_{1,k}) = \left\lfloor \frac{n(k-2)}{2} \right\rfloor + \left\lfloor \frac{n}{n-k+2} \right\rfloor$$

or possibly this value plus one if certain conditions hold.

In [142], Manoussakis, Spyratos, Tuza and Voigt found the number for spanning rainbow stars.

**Theorem 21 ([142])**

$$ar(K_n, K_{1,n-1}) = \frac{n(n-3)}{2} + \left\lfloor \frac{n}{3} \right\rfloor + 1.$$

Also in [147], the author found the anti-Ramsey numbers for stars  $K_{1,k}$  in host graphs such as the hypercube  $Q_n$ , the grid  $C_m \times C_n$  and a general graph  $G$  with  $\delta(G) \geq k + 4$ . Similarly in [152], the authors consider rainbow stars within chosen subsets of vertices in colored multigraphs.

The anti-Ramsey numbers for paths were considered by Simonovits and Sós [168].

**Theorem 22 ([168])** There exists a constant  $c$  such that if  $t \geq 5$  and  $n > ct^2$ , then for  $\epsilon = 0, 1$ , we have

$$ar(K_n, P_{2t+3+\epsilon}) = tn - \binom{t+1}{2} + 1 + \epsilon.$$

Simonovits and Sós also defined the  $H_0$  spectra of colorings as follows. Given a particular graph  $H \subseteq K_n$  and a coloring  $\phi_r$  of  $K_n$  using  $r$  colors, let  $c(H; \phi_r)$  denote the number of colors on  $H$ . For a given graph  $H_0$ , define the spectrum to be

$$S(H_0; n, \phi_r) = \{i : H \sim H_0, c(H; \phi_r) = i\}.$$

Let  $\mathcal{T}_n$  be the set of all trees on  $n$  vertices and let  $\mathcal{T}_n^*$  be the set of all graphs obtained from graphs in  $\mathcal{T}_n$  by the removal of a single edge. Then the following was proven by Bialostocki and Voxman [32].

**Theorem 23 ([32])**

$$ar(K_n, \mathcal{T}_n) - ex(n, \mathcal{T}_n^*) = 1.$$

For a given set  $S \subset \{1, \dots, r\}$ , a general question is whether or not there exists a coloring  $\phi_r$  of  $K_n$  such that  $S(H_0; n, \phi_r) = S$ . Some cases of this problem are considered in [168] and it is noted that this is a generalization of work presented in [53].



## 2.4 Matchings

In 2004, Schiermeyer observed the following easy proposition. Unfortunately, it became a rather difficult problem to pin down the exact anti-Ramsey numbers for matchings.

**Proposition 2 ([164])**

$$ex(n, (k-1)K_2) + 1 \leq ar(K_n, kK_2) \leq ex(n, kK_2).$$

The extremal number for a matching is known from [57] to be as follows.

**Theorem 24 ([57])**

$$ex(n, kK_2) = \max \left\{ \binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1) \right\}.$$

Also in [164], Schiermeyer used a counting technique to show that the lower bound is, in fact, the correct number for all  $k \geq 2$  and  $n \geq 3k + 3$ . This was later improved by Fujita, Kaneko, Schiermeyer and Suzuki [73] for all  $n \geq 2k + 1$ . For  $k = 2, 3, 4$  the same result was proven by Kaneko, Saito, Schiermeyer and Suzuki [121]. Finally, Chen, Li and Tu [50] used the Gallai-Edmonds Structure Theorem for matchings to prove the following, which shows that the lower bound of Proposition 2 is almost always the correct number.

**Theorem 25 ([50])**

$$ar(K_n, kK_2) = \begin{cases} 4, & n = 4 \text{ and } k = 2, \\ ex(n, (k-1)K_2) + 2, & n = 2k \text{ and } k \geq 7, \\ ex(n, (k-1)K_2) + 1, & \text{otherwise.} \end{cases}$$

Haas and Young [97] verified a conjecture from [73] in the following result.

**Theorem 26 ([97])** For  $k \geq 3$ , if  $M_k$  is a matching on  $k$  edges,

$$ar(K_{2k}, M_k) = \max \left\{ \binom{2k-3}{2} + 3, \binom{k-2}{2} + k^2 - 2 \right\}.$$

Others have studied rainbow matchings in bipartite graphs. Li, Tu and Jin [134] determined the anti-Ramsey number for matchings in complete bipartite graphs as follows.

**Theorem 27 ([134])** For all  $m \geq n \geq k \geq 3$ ,

$$ar(K_{m,n}, kK_2) = m(k-2) + 1.$$

In looking at more sparse graphs, Li and Xu [133] determined the anti-Ramsey number for matchings in  $m$ -regular bipartite graphs of order  $2n$ , denoted  $B_{n,m}$ .

**Theorem 28 ([133])** For all  $k \geq 2$  and  $m \geq 3$ , if  $n > (3k-1)$ , then

$$ar(B_{n,m}, kK_2) = m(k-2) + 1.$$

Gilboa and Roditty [82] proved reduction results of the form ‘if  $\text{ar}(K_n, L \cup t_1P_s) \leq f(n, t_1, L)$  then  $\text{ar}(K_n, L \cup tP_s) \leq f(n, t, L)$ ’ where  $s = 2$  or  $3$ . These results lead to the following.

**Corollary 29 ([82])** *For sufficiently large  $n$ ,*

- $\text{ar}(K_n, P_3 \cup tP_2) = (t - 1)(n - t/2) + 2$  for  $t \geq 2$ ,
- $\text{ar}(K_n, P_4 \cup tP_2) = t(n - (t + 1)/2) + 2$  for  $t \geq 1$ ,
- $\text{ar}(K_n, C_3 \cup tP_2) = t(n - (t + 1)/2) + 2$  for  $t \geq 1$ ,
- $\text{ar}(K_n, tP_3) = (t - 1)(n - t/2) + 2$  for  $t \geq 1$ ,
- $\text{ar}(K_n, P_{k+1} \cup tP_3) = (t + \lfloor k/2 \rfloor - 1)(n - \frac{t + \lfloor k/2 \rfloor}{2}) + 2 + k \pmod 2$  for  $k \geq 3$  and  $t \geq 0$ ,
- $\text{ar}(K_n, P_2 \cup tP_3) = (t - 1)(n - t/2) + 3$  for  $t \geq 1$ ,
- $\text{ar}(K_n, kP_2 \cup tP_3) = (t + k - 2)(n - (t + k - 1)/2) + 2$  for  $k \geq 2$  and  $t \geq 2$ .

## 2.5 Other Graphs

In full generality, Erdős, Simonovits and Sós [62] proved the following proposition which cements the relationship between the anti-Ramsey numbers and the extremal numbers of Turán. For this statement, given a set of graphs  $\mathcal{H}$ , let  $\text{ex}(G, \mathcal{H})$  be the maximum number of edges in a subgraph of  $G$  containing no copy of  $H$  for any  $H \in \mathcal{H}$ .

**Proposition 3 ([62])** *Given graphs  $G$  and  $H$ , we have*

$$\text{ex}(G, \mathcal{H}) + 1 \leq \text{ar}(G, H) \leq \text{ex}(G, H),$$

where  $\mathcal{H} = \{H - e : e \in E(H)\}$ .

For graphs containing at least one vertex of degree 2, Jiang [106] proved the following theorem.

**Theorem 30 ([106])** *Given a graph  $H$ , let  $\mathcal{H} = \{H - v : v \in V(H), d_H(v) = 2\}$  and suppose  $H$  has  $p$  vertices and  $q$  edges. For all positive integers  $n$ , we have*

$$\text{ar}(K_n, H) \leq \text{ex}(n, \mathcal{H}) + bn,$$

where  $b = \max\{2p - 2, q - 2\}$ .

This eventually led to the following result for subdivided graphs.

**Theorem 31 ([106])** *If  $H$  is a graph containing at least two cycles in which each edge is incident to a vertex of degree two, then*

$$\text{ar}(K_n, H) = \text{ex}(n, \mathcal{H})(1 + o(1)),$$

where  $\mathcal{H} = \{H - e : e \in E(H)\}$ .

Theorem 30 also implies the following result from [17].

**Theorem 32 ([17])**

$$ar(K_n, K_{2,t}) = ex(K_n, K_{2,t-1}) + O(n).$$

A related result was shown in [128].

**Theorem 33 ([128])** For all  $s \leq t$ , there exists  $c = c(s, t)$  such that

$$ar(K_n, K_{s,t}) - ex(K_n, K_{s,t-1}) < cn.$$

In [17], the authors also provided the following general results for finding  $K_{2,t}$  in complete bipartite graphs.

**Theorem 34 ([17])**

$$ar(K_{m,n}, K_{2,t}) = ex(K_{m,n}, K_{2,t-1}) + O(m + n).$$

**Theorem 35 ([17])**

$$ar(K_{n,n}, K_{2,t}) = \sqrt{t-2}n^{3/2} + O(n^{4/3}).$$

Theorem 35 follows immediately from Theorem 34 and the following result of Füredi.

**Theorem 36 ([79])**

$$ex(K_{n,n}, K_{2,t}) = \sqrt{t-1}n^{3/2} + O(n^{4/3}).$$

For a  $C_4$  with a single chord, which we will denote  $D$  for diamond, the following result was proven in [148, 149].

**Theorem 37 ([148, 149])** For  $n \geq 4$ ,

$$ex(K_n, \{C_3, C_4\}) + 1 \leq ar(K_n, D) \leq ex(K_n, \{C_3, C_4\}) + n.$$

For a cycle with a pendant edge, denoted by  $C_k^+$ , Gorgol showed the following interesting result.

**Theorem 38 ([83])** For  $n \geq k + 1$ ,

$$ar(K_n, C_k^+) = ar(n, C_k).$$

If you add one additional pendant to the cycle, creating a graph denoted by  $C_k^{++}$ , Gorgol showed the following.

**Theorem 39 ([83])** For  $n \geq k + 2$ ,

$$ar(K_n, C_k^{++}) > ar(n, C_k).$$

The proof of this result involves a slight modification of the coloring providing the lower bound of Conjecture 1.

A related result for a triangle with two pendant edges off a single vertex is the following.

**Theorem 40 ([87])** For  $n \geq 5$ ,

$$ar(K_n, K_{1,4} + e) = n + 1.$$

Let  $Q_n$  be the hypercube of dimension  $n$ , i.e. the graph of order  $2^n$  in which the vertices are binary  $n$ -tuples and two vertices are adjacent if and only if the corresponding tuples differ by one term. Regarding the hypercube, Axenovich, Harborth, Kemnitz, Möller and Schiermeyer [14] provided a collection of results for finding one hypercube in another.

**Theorem 41 ([14])**

$$n2^{n-1} - \left\lfloor \frac{n}{k}(2^{n-1} - k + 1) \right\rfloor \leq ar(Q_n, Q_k) \leq n2^{n-1} \left( 1 - \frac{n-k}{(n-1)k2^{k-2}} \right).$$

More specifically, the authors also proved the following.

**Theorem 42 ([14])**

$$ar(Q_n, Q_{n-1}) = \begin{cases} n2^{n-1} - 4 & \text{for } n = 3, 4, 5, \\ n2^{n-1} - 3 & \text{for } n \geq 6. \end{cases}$$

and  $ar(Q_4, Q_2) = 18$ .

Bode et al. [35] provide exact results for  $ar(Q_5, Q_2)$  and  $ar(Q_5, Q_3)$ .

In other work, Gorgol and Łazuka computed the following anti-Ramsey numbers for stars with an added edge.

**Theorem 43 ([86])** For all  $n \geq 4$ ,

$$ar(K_n, K_{1,3} + e) = n - 1,$$

and for all  $n \geq 5$ ,

$$ar(K_n, K_{1,4} + e) = n + 1.$$

We say that a graph  $H$  is doubly edge-critical if  $\chi(H \setminus e) \geq p + 1$  for any edge  $e \in E(H)$  and there exists a pair of edges  $e, f$  for which  $\chi(H \setminus \{e, f\}) = p$ . Jiang and Pikhurko [110] obtained exact values of  $ar(K_n, H)$  for doubly edge-critical graphs  $H$  and classified all sharpness examples. This result generalizes Theorem 14 since  $K_{p+2}$  is doubly edge-critical.

The *cyclomatic number* of a connected graph  $G$ , denoted  $v(G)$ , is the minimum number of edges that must be removed from  $G$  to make the resulting graph acyclic, that is,  $v(G) = |E(G)| - |V(G)| + 1$ .

**Theorem 44 ([166])** Let  $H$  be a connected graph of order  $p \geq 4$  and cyclomatic number  $v(H) \geq 2$ . Then  $ar(K_n, H)$  cannot be bounded from above by a function which is linear in  $n$ .

**Theorem 45 ([166])** *Let  $H$  be a graph of order  $p \geq 5$  and cyclomatic number  $v(H) = 1$ . If  $H$  contains a cycle with  $k$  vertices for some  $k$  with  $3 \leq k \leq p - 2$ , then*

$$\left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r}{2} + \left\lceil \frac{n}{k-1} \right\rceil - 1 \leq ar(K_n, H) \leq (p-2)n - p \cdot \frac{p-3}{2} - 1$$

where  $n \geq p$  and  $r$  is the residue of  $n \pmod{k-1}$ .

Let  $B$  be the *bull* graph, the unique graph on 5 vertices with degree sequence  $(1, 1, 2, 3, 3)$ .

**Theorem 46 ([166])**  $ar(K_5, B) = 5$  and  $ar(K_n, B) = n + 1$  for  $n \geq 6$ .

Gorgol and Görlich considered anti-Ramsey numbers for disjoint copies of a graph  $G$ , denoted by  $pG$ .

**Theorem 47 ([85])** *For any graph  $G$  on  $n \geq 3$  vertices and for any  $p$ , we have*

$$ar(m, pG) \geq \max \left\{ \binom{pn-2}{2} + 1, ar(m-p+1, G) + (p-1)m - \binom{p}{2} \right\}.$$

The same authors also offered the following conjecture.

**Conjecture 3 ([85])** *For any graph  $G$  on  $n \geq 3$  vertices and for any  $p$ , if  $m \geq p|V(G)|$ , then*

$$ar(m, pG) = ar(m-p+1, G) + (p-1)m - \binom{p}{2}$$

if and only if  $G$  is a tree.

For specific graphs, the following result was shown.

**Theorem 48 ([85])** *For any integer  $m \geq 6$ , we have*

$$ar(m, 2P_3) = \begin{cases} 7 & \text{if } m = 6, \\ m & \text{if } m \geq 7. \end{cases}$$

For any integer  $m > 12$ , we have

$$ar(m, 3P_3) = 2m - 2.$$

## 2.6 Generalizations of Anti-Ramsey Theory

For given integers  $p$  and  $q$ , a  $(p, q)$ -coloring of  $K_n$  is a coloring in which the edges of every  $K_p$  subgraph uses at least  $q$  colors. Let  $f(n, p, q)$  be the minimum number of colors in a  $(p, q)$ -coloring of  $K_n$ . When  $q = 2$ , this reduces down to the classical anti-Ramsey problem

This problem was first considered by Elekes, Erdős and Füredi in [55] and later revisited by Erdős and Gyárfás in [58]. They used the Local Lemma to provide the following general upper bound on  $f(n, p, q)$ .

**Theorem 49 ([58])** For some  $c$  depending only on  $p$  and  $q$ ,

$$f(n, p, q) \leq cn^{(p-2)/\binom{p}{2}-q+1}.$$

In general, they also proved the following.

**Theorem 50 ([58])** For any  $p$ , let  $q = \binom{p}{2} - p + 3$ . Then  $f(n, p, q)$  is linear in  $n$  while  $f(n, p, q - 1)$  is sublinear (less or equal to  $cn^{1-1/(p-1)}$ ).

The authors also considered  $f(n, p, q)$  for many small fixed values of  $p$  and  $q$ . In particular, they struggled with finding upper and lower bounds on  $f(n, 5, 9)$  and  $f(n, 4, 3)$ . Since then, Mubayi [155], Axenovich [10] and Krop and Krop [127] have improved the bounds on these small cases.

In similar work, Axenovich, Füredi and Mubayi [13] studied the function  $r(G, H, q)$  which is defined to be the minimum number of colors in a coloring of  $G$  in which every copy of  $H \subseteq G$  together receive at least  $q$  colors. The paper includes a variety of results concerning the case when  $G$  and  $H$  are complete bipartite graphs. Mubayi and West also considered bipartite graphs in [156]. Improvements were made by Ling in [135].

As another variation, Axenovich and Kündgen [21] defined the function  $R(n, p, q_1, q_2)$  to be the maximum number of colors in a coloring of  $K_n$  such that the number of colors used on a subgraph  $A$  is between  $q_1$  and  $q_2$  (inclusive) for every subgraph  $A$  with  $|A| = p$ . The case when  $q_1 = 1$  reduces to the problem studied in [62]. In [21], the authors mention general results from [1] for a variety of values for  $p, q_1, q_2$  and bounds are proven for  $p = 4, q_2 = 4$  or  $5$ . A similar problem, when  $q_1 = 2, q_2 = |E(H)| - 1$  and we restrict only the copies of  $H$  in a general graph  $G$ , was considered in [15] under the title of mixed anti-Ramsey numbers.

The problem of finding the minimum number of colors  $f(n, e, L)$  necessary to color a graph on  $n$  vertices and  $e$  edges such that every copy of  $L$  has all edges of different colors was studied in [7, 42, 162]. In [42], the following question was asked.

**Question 1** Let  $L$  be a connected bipartite graph that is not a star. Is it true that

$$\lim_{n \rightarrow \infty} \frac{f(n, \alpha n^2, L)}{n} = \infty?$$

This question is answered in the affirmative in [162] for the case when  $L$  is a connected, bipartite graph that is not complete bipartite. The function  $f(n, e, L)$  was also studied in [43] where some bounds were provided for some classes of graphs  $L$ . These are roughly described as follows:

- A lower bound when  $L$  is bipartite with  $\Delta(L) \geq 2$  and having at least two strongly independent edges (meaning that the end vertices of the edges induce no other edges),
- A lower bound when  $L$  has two strongly independent edges and is not a disjoint union of cliques, or
- An upper bound when  $L$  has no two strongly independent edges.

A similar question was studied for sub-hypercubes of a hypercube in [7].

Let  $f(n)$  denote the largest number of edges in a rainbow subgraph of a properly edge-colored complete graph on  $n$  vertices. Then the following was shown in [24].

**Theorem 51** ([24])

$$(2n)^{1/3} \leq f(n) \leq 8(n \ln n)^{1/3}.$$

Within the class of graphs in which each vertex is incident to many colors, the anti-Ramsey problem was studied in [19] under the title of local anti-Ramsey numbers. Many general bounds are presented in [19] while another specific number can be found in [76].

Haxell and Kohayakawa [102] considered an anti-Ramsey type problem for finding rainbow cycles in colored graphs with large girth.

Define the *size anti-Ramsey number* of a graph  $H$ , denoted  $AR_s(H)$ , to be the smallest number of edges in a graph  $G$  such that any proper edge-coloring of  $G$  contains a rainbow copy of  $H$ . The size anti-Ramsey number was originally defined in [20] along with several notions of online anti-Ramsey numbers. Relationships between these numbers as well as general bounds were also proven in [20]. The general behavior of  $AR_s(K_k)$  was settled in [4] with the following result, answering a question from [20].

**Theorem 52** ([4])

$$AR_s(K_k) = \Theta(k^6 / \log^2 k).$$

Also introduced in [4] is the concept of *degree anti-Ramsey number* of a graph  $H$ , denoted  $AR_d(H)$ , to be the minimum value of  $d$  such that there is a graph  $G$  with maximum degree at most  $d$  such that any proper edge-coloring of  $G$  contains a rainbow copy of  $H$ . Some observations about the function  $AR_d(H)$  are also included in [4].

In [159] and [117], the problem of vertex-coloring plane graphs avoiding rainbow faces is discussed. Other vertex colorings related to anti-Ramsey theory were discussed in, for example, [129, 154] and many others. In [146] the author discusses anti-Ramsey concepts for finding rainbow colored edge-cuts.

The anti-Ramsey problem has also been studied in a variety of other contexts. For hypergraphs, see [8, 132, 157]. For random graphs, see [37, 123, 124, 161]. For anti-Ramsey in groups, see [25, 167]. For directed graphs, see [28] among others. Concerning integers and rainbow arithmetic progressions, see [12, 118, 119, 120].

## 3 Rainbow Ramsey Theory

### 3.1 Classical Rainbow Ramsey Numbers

**Definition 2** For given two graphs  $G_1, G_2$ , the *rainbow Ramsey number* (also sometimes called the *constrained Ramsey number*)  $RR(G_1, G_2)$  is defined to be the minimum integer  $N$  such that any edge-coloring of the complete graph  $K_N$  using any number of colors must contain either a monochromatic copy of  $G_1$  or a rainbow copy of  $G_2$ .

Although commonly called constrained Ramsey numbers, we use the term rainbow Ramsey numbers to describe this concept in following the notation of [48]. In [104], the following is proven.

**Theorem 53 ([104])** *The rainbow Ramsey number  $RR(G_1, G_2)$  exists if and only if  $G_1$  is a star or  $G_2$  is a forest.*

As observed in [104],  $RR(G_1, K_{1,t+1})$  is equivalent to the  $t$ -local Ramsey number of  $G_1$ , introduced in [90, 92]. In [31], the following is proven.

**Theorem 54 ([31])** *For every positive integer  $n$ ,*

$$RR(nK_2, nK_2) = n(n - 1) + 2.$$

More generally, for  $G_1 = nK_2, G_2 = mK_2$ , the following natural conjecture is proposed in [65].

**Conjecture 4 ([65])** *For any two integers  $n, m$  with  $n \geq 3, m \geq 2$ ,*

$$RR(nK_2, mK_2) = m(n - 1) + 2.$$

We can easily see that  $m(n - 1) + 2 \leq RR(nK_2, mK_2) \leq 2(n - 1)m$ , when  $n \geq 2$ . For the lower bound, consider a coloring of the graph  $K_{m(n-1)+1}$  as follows. Color all of the edges of a subgraph isomorphic to  $K_{2n-1}$  with color 1. Choose  $n - 1$  additional vertices and color all of the edges among these vertices and between these vertices and those already colored with color 2. For each color  $i = 3, 4, \dots, m - 1$ , choose  $n - 1$  additional vertices and color the edges among those vertices and between those vertices and the part of the graph already colored with color  $i$ . The resulting graph has  $2n - 1 + (m - 2)(n - 1) = m(n - 1) + 1$  vertices and contains no set of  $n$  independent edges in the same color. Since only  $m - 1$  colors appear, it also can not contain a set of  $m$  independent edges in different colors.

For the upper bound, notice that it holds for  $n = 2$  and for  $m = 1$  provided  $n \geq 2$ . For any  $n \geq 3$  and  $m \geq 2$ , suppose  $RR(nK_2, (m - 1)K_2) \leq 2(n - 1)(m - 1)$  and  $RR((n - 1)K_2, mK_2) \leq 2(n - 2)m$ . Consider any edge-coloring of  $K_{2(n-1)m}$ . If the resulting graph does not contain a rainbow  $mK_2$ , then without loss of generality it must contain a monochromatic  $(n - 1)K_2$ . If we remove these  $2(n - 1)$  vertices, there are  $2(n - 1)(m - 1)$  vertices remaining. Thus, there is either a monochromatic  $nK_2$  or a rainbow  $(m - 1)K_2$  on the remaining vertices. Without loss of generality, we have a monochromatic  $(n - 1)K_2$ , say in color  $c$ , and a disjoint rainbow  $(m - 1)K_2$ . Either the rainbow  $(m - 1)K_2$  contains an edge in color  $c$  or it does not. If it contains an edge in color  $c$ , then this edge along with the monochromatic  $(n - 1)K_2$  form a monochromatic  $nK_2$ . Otherwise, an edge in color  $c$  from the  $(n - 1)K_2$  may be added to the rainbow  $(m - 1)K_2$  to produce a rainbow  $mK_2$ .

In attempts to prove this conjecture the following results are proven in [65].

**Theorem 55 ([65])** *For any two positive integers  $n, m$  with  $2 \leq m < n$ ,*

$$RR(nK_2, mK_2) = m(n - 1) + 2.$$

**Theorem 56 ([65])**

1.  $RR(3K_2, 4K_2) = 10$ .



2.  $RR(4K_2, 5K_2) = 17$ .

**Theorem 57** ([65]) For  $n > 5$  and  $2 \leq m \leq \frac{3}{2}(n-1)$ ,

$$RR(nK_2, mK_2) = m(n-1) + 2.$$

Next we deal with the case where  $G_1$  is a star and  $G_2$  is a matching. In [66], the following results are proven. First there is a general lower bound.

**Theorem 58** ([66]) For any positive integers  $n$  and  $m$ , provided that  $n$  is odd or  $m$  is even,

$$RR(K_{1,n}, mK_2) \geq (n-1)(m-1) + 2.$$

If  $n$  is even and  $m$  is odd, then

$$RR(K_{1,n}, mK_2) \geq (n-1)(m-1) + 1.$$

Next, the authors found the following upper bounds.

**Theorem 59** ([66]) For any positive integers  $n$  and  $m$ ,

$$RR(K_{1,n}, mK_2) \leq (n-1)(m-1) + 2 + \binom{m-n+3}{2}.$$

**Theorem 60** ([66]) For any positive integers  $n$  and  $m$ ,

$$RR(K_{1,n}, mK_2) \leq (n+1)(m-1) + 2.$$

If  $(n+1)(m-1) \geq 2m+1$  (for instance,  $n \geq 2$  and  $m \geq 4$  or  $n \geq 3$  and  $m \geq 3$ ), then we may improve the bound above to  $RR(K_{1,n}, mK_2) \leq (n+1)(m-1)$ .

More specifically, the following special case is proven.

**Theorem 61** ([66])  $RR(K_{1,3}, 3K_2) = 7$ .

On the other hand, when  $G_1 = nK_2, G_2 = K_{1,m}$ , the following upper and lower bounds are proven in [66].

**Theorem 62** ([66]) For any positive integers  $n \geq 2$  and  $m \geq 3$ ,

$$RR(nK_2, K_{1,m}) \geq (2m-3)(n-1) + 1.$$

**Theorem 63** ([66]) For any integers  $m$  and  $n$ , where  $m \geq 2$  and  $n \geq 2$ ,

$$RR(nK_2, K_{1,m}) \leq m(m-1)n - \frac{1}{2}(3m+1)(m-2).$$

Finally, we conclude this section by considering the case where  $G_2$  is a path. Let  $R(G, G), R(G, G, G)$  be the 2- and 3-coloring Ramsey numbers of  $G$  respectively. In [91], the authors proved the following general results which relate rainbow Ramsey numbers to classical graph Ramsey numbers.

**Theorem 64** ([91]) *For every graph  $G$  of order  $n \geq 5$ ,*

$$RR(G, P_4) = R(G, G).$$

**Theorem 65** ([91]) *For  $n \geq 3$ ,*

$$RR(P_n, P_5) = R(P_n, P_n, P_n).$$

**Theorem 66** ([91]) *If  $G$  is a connected non-bipartite graph then*

$$RR(G, P_5) = R(G, G, G).$$

**Theorem 67** ([91]) *For  $n \geq 3$ ,*

$$RR(C_n, P_5) = R(C_n, C_n, C_n).$$

Let  $T_5$  be the tree obtained from  $K_{1,3}$  by subdividing one edge.

**Theorem 68** ([91]) *If  $G = P_n$  or  $C_n$  ( $n \geq 3$ ) or  $G$  is non-bipartite and connected, then*

$$RR(G, T_5) = RR(G, P_5).$$

Alon, Jiang, Miller and Pritikin [6] provided general bounds on the rainbow Ramsey number for a star versus a complete graph.

**Theorem 69** ([6]) *For some absolute constants  $c_1$  and  $c_2$ , for  $t \geq 3$ ,*

$$\frac{c_1 mt^3}{\ln t} \leq RR(K_{1,m+1}, K_t) \leq \frac{c_2 mt^3}{\ln t}.$$

Some similar work has been done with rainbow Ramsey in algebraic structures in [69, 145, 137, 144] among many others.

### 3.2 Bipartite Rainbow Ramsey Numbers

Given two bipartite graphs  $G_1$  and  $G_2$ , the bipartite rainbow Ramsey number  $BRR(G_1, G_2)$  is the smallest integer  $N$  such that any coloring of the edges of  $K_{N,N}$  with any number of colors contains a monochromatic copy of  $G_1$  or a rainbow copy of  $G_2$ . For the existence of  $BRR(G_1, G_2)$ , the following is proved in [67]:

**Theorem 70** ([67]) *The bipartite rainbow Ramsey number  $BRR(G_1, G_2)$  exists if and only if  $G_1$  is a star or  $G_2$  is a star forest (i.e., a union of stars).*

Let  $\mathcal{S}_r$  denote any star forest with  $r$  components and let  $S_r, B_r, T_r$ , and  $\mathcal{F}_r$  be any star forest, bipartite graphs, tree or forest, respectively with  $r$  edges. In [67], the following general bounds for the bipartite rainbow Ramsey number are established. We first consider the case when the rainbow graphs of interest are general bipartite graphs.

**Theorem 71 ([67])** *Let  $G_n$  be any connected bipartite graph for which the largest partite set has  $n$  vertices. If  $BRR(G_n, B_m)$  exists, then*

$$BRR(G_n, B_m) \geq (n - 1)(m - 1) + 1.$$

This result easily implies the following two corollaries.

**Corollary 72 ([67])**

$$BRR(K_{1,n}, mK_2) \geq (n - 1)(m - 1) + 1.$$

**Corollary 73 ([67])**

$$BRR(T_n, S_m) \geq \left( \left\lceil \frac{n+1}{2} \right\rceil - 1 \right) (m - 1) + 1.$$

We next consider the case when the rainbow graph is a forest.

**Theorem 74 ([67])** *If  $G_n$  is any forest with  $n$  nontrivial components, then*

$$BRR(G_n, S_m) \geq (n - 1)(m - 1) + 1.$$

This result immediately implies the next two corollaries.

**Corollary 75 ([67])**

$$BRR(nK_2, K_{1,m}) \geq (n - 1)(m - 1) + 1.$$

**Corollary 76 ([67])**

$$BRR(nK_2, S_m) \geq (n - 1)(m - 1) + 1.$$

The next two corollaries take the previous bounds and provide exact results.

**Corollary 77 ([67])**

$$BRR(K_{1,n}, K_{1,m}) = (n - 1)(m - 1) + 1.$$

**Corollary 78 ([67])**

$$BRR(nK_2, mK_2) = (n - 1)(m - 1) + 1.$$

**Theorem 79 ([67])** *Suppose  $\mathcal{F}_m$  has no isolated vertices. Then*

$$\begin{aligned} BRR(K_{1,n}, \mathcal{F}_m) &= O(mn), \\ BRR(T_n, K_{1,m}) &= O(mn). \end{aligned}$$

**Corollary 80 ([67])**

$$BRR(\mathcal{F}_n, K_{1,m}) = O(mn).$$

Regarding matchings and stars, the following general results were proven in [67].

**Theorem 81** ([67]) *For any integers  $n, m \geq 2$ ,*

$$(n - 1)(m - 1) + 1 \leq BRR(K_{1,n}, mK_2) \leq n(m - 1) + 1.$$

Furthermore, if  $n > 2$ , the upper bound can be improved to  $n(m - 1)$ .

**Theorem 82** ([67]) *For positive integers  $n \geq 3$  and  $1 \leq m \leq n + 2$ ,*

$$BRR(K_{1,n}, mK_2) = (n - 1)(m - 1) + 1.$$

**Theorem 83** ([67]) *For any integers  $n, m \geq 2$ ,*

$$BRR(nK_2, K_{1,m}) \geq \max(2(n - 1)(m - 2) + 1, (n - 1)(m - 1) + 1).$$

**Theorem 84** ([67]) *For any integers  $n \geq 2$  and  $m \geq 3$ ,*

$$BRR(nK_2, K_{1,m}) \leq (3m - 5)(n - 1) + 1.$$

More specifically, for a small star or a small matching, the following cases were shown.

**Theorem 85** ([67]) *For any integer  $n \geq 2$ ,*

$$BRR(nK_2, K_{1,2}) = n.$$

**Theorem 86** ([67])

$$BRR(2K_2, K_{1,m}) = 2m - 3.$$

Concerning stars and paths, the following results were shown.

**Theorem 87** ([67]) *For integers  $n, m \geq 3$ ,*

$$2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)(m - 2) \leq BRR(P_{n+1}, K_{1,m}) \leq (n - 1)(m - 1).$$

**Theorem 88** ([67]) *For any integers  $n, m \geq 2$ ,*

$$\begin{aligned} (n - 1)(m - 1) + 1 &\leq BRR(K_{1,n}, P_{m+1}) \\ &\leq \max((m - 2)(n - 1) + \lceil \frac{m+2}{2} \rceil, (m - 1)(n - 1) + 1). \end{aligned}$$

This bound implies the following exact result for small paths.

**Corollary 89** ([67]) *For integers  $n, m \geq 2$  such that  $m \leq 2n - 3$ ,*

$$BRR(K_{1,n}, P_{m+1}) = (n - 1)(m - 1) + 1.$$

**Theorem 90** ([67]) *For any integer  $p \geq 1$ ,*

$$BRR(K_{1,2}, P_{4p-1}) \geq 4p + 1.$$

Also, in [26], the following is shown.

**Theorem 91** ([26])

$$BRR(K_{1,n}, K_{2,2}) = 3n - 2.$$

**Theorem 92** ([174]) For fixed  $t \geq 3$ ,  $s \geq (t - 1)! + 1$  and  $n$  large,

$$BRR(K_{s,t}, K_{1,n}) = \Theta(n^t).$$

**Theorem 93** ([174]) For  $n > t \geq 3$ ,

$$t^2(n - 1) + 1 \leq BRR(K_{1,n}, K_{t,t}) \leq t^3(n - 1) + t - 1.$$

**Theorem 94** ([174]) For  $m = 2, 3, 5$ , if  $n \rightarrow \infty$ , then

$$BRR(C_{2m}, K_{1,n}) \geq (1 - o(n))n^{m/(m-1)}.$$

**Theorem 95** ([174]) For any integers  $n, m \geq 2$ ,

$$(2m - 1)(n - 1) + 1 \leq BRR(K_{1,n}, C_{2m}) \leq 2m(n - 2) + \frac{1}{2}m(m - 1)(n - 1) + 2.$$

Let  $B_{s,t}$  denote the broom with  $s$  edges in the star part and  $t$  edges in the path part.

**Theorem 96** ([174]) For any integers  $n, s, t \geq 2$ ,

$$\max\{(n - 1)(s + \left\lceil \frac{t}{2} \right\rceil - 1), 2(n - 2)(\left\lceil \frac{t}{2} \right\rceil - 1)\} - 1 \leq BRR(B_{s,t}, K_{1,n}) \leq (2s + t - 3)(n - 1).$$

**Theorem 97** ([174]) For any integers  $n, s, t \geq 2$ ,

$$(n - 1)(s + t - 1) + 1 \leq BRR(K_{1,n}, B_{s,t}) \leq (n - 1)(s + t - 1) + s + \frac{t + 1}{2}.$$

### 3.3 Pattern Ramsey Theory

A color pattern is defined to be a graph with colored edges. A family of patterns  $\mathcal{F}$  is called a Ramsey family if there exists an integer  $n_0$  such that in every coloring of the edges of  $K_n$  with  $n \geq n_0$ , there exists some pattern in  $\mathcal{F}$ .

**Definition 3** The pattern Ramsey number for a Ramsey family  $\mathcal{F}$  of patterns is the smallest integer  $n_0$  such that in every coloring of  $K_n$  with  $n \geq n_0$ , there exists some pattern in  $\mathcal{F}$ .

Notice that this definition is closely related to the definition of the rainbow Ramsey number except, as opposed to restricting attention to monochromatic or rainbow graphs, one is allowed to choose any coloring. Also note another similarity in that the number of colors is unlimited.

In 1950, Erdős and Rado [59] classified which families of patterns are Ramsey. A coloring of a graph is said to be lexical if there exists an ordering of the vertices (left to right) such that two edges get the same color if and only if they share a right endpoint. For ease of notation, let  $H^{mono}$ ,  $H^{rain}$  and  $H^{lex}$  be the monochromatic, rainbow and lexical colorings of  $H$  respectively. Erdős and Rado showed the following.

**Theorem 98** ([59, 60]) *There is a constant  $C_p$  such that every coloring of  $E(K_n)$  for  $n > C_p$  contains a  $K_p$  that is monochromatic, rainbow or lexically colored.*

The result was actually proven for hypergraphs. This result was just the beginning of a very difficult problem, the problem of finding such pattern Ramsey numbers. In honor of Erdős and Rado, the pattern Ramsey number for finding a monochromatic, rainbow or lexical complete graph of order  $k$  is commonly denoted  $ER(k)$ . The original proof in [59] provides an upper bound and Galvin [63] (p. 30) noticed a lower bound on  $ER(k)$ . These bounds were improved by Lefmann and Rödl in [130] but in [131], Lefmann and Rödl provided a new proof of Theorem 98 and better bounds on  $ER(k)$  (again for hypergraphs).

**Theorem 99** ([131])

$$2^{c_1 k^2} \leq ER(k) \leq 2^{c_2 k^2 \log k}$$

for some constants  $c_1$  and  $c_2$ .

Similarly, in [130], Lefmann and Rödl considered finding ordered rainbow paths or monochromatic complete graphs in edge-colored totally ordered complete graphs. They proved that the necessary order is related to the classical Ramsey numbers for finding a path or a complete graph.

Concerning the more general problem of considering general patterns of colors, Jamison and West [105] considered a particular family of colorings (equipartitioned stars) while Axenovich and Jamison [16] studied another family ( $\mathcal{F} = \{K_n^{lex}, K_3^{rain}, H^{mono}\}$ ). Notice this is related to the rainbow triangle free work discussed in Section 4. The following 3 results were proven in this work. Define  $f(n, H)$  to be the smallest integer  $m$  such that every coloring of  $K_m$  contains either a  $K_n^{lex}$ , a  $K_3^{rain}$  or  $H^{mono}$ .

**Theorem 100** ([16])

$$f(n, H) \leq 3^{n|H|}.$$

**Theorem 101** ([16]) *For any connected graph  $H$  and any  $n$ , there is a constant  $c = c(n)$  such that  $f(n, H) \leq cR_{n-1}(H)$  (the classical  $n - 1$  color Ramsey number for  $H$ ).*

**Theorem 102** ([16])

$$5^{\lfloor n/2 \rfloor - 1} + 1 \leq f(n, K_3) \leq 5^{n/2}.$$

Similar coloring problems were considered in [132, 160] for finding colored subsets in a  $k$ -uniform hypergraph. Another bound on some such Ramsey numbers can be found in [108]. The authors of [29] also found Mixed Pattern Ramsey numbers for a rainbow or monochromatic triangle after exclusion of colored graphs  $H$  where  $H$  is any colored 4-cycle, almost any colored 4-clique and bounds when  $H$  is a monochromatic odd cycle or a star when the number of colors is sufficiently large.

## 4 Gallai-Ramsey Theory

### 4.1 Gallai Colorings

The avoidance of rainbow colored subgraphs began with Gallai in [80] where the author studied transitively orientable graphs. The results contained in [80] were reproduced in [95] where Gyárfás and Simonyi translated them to the terminology of graph coloring.

**Definition 4** *A coloring of a complete graph  $G$  is said to be a Gallai coloring if this coloring contains no rainbow triangle.*

Gyárfás and Simonyi restated the following theorem attributed to Gallai and also to Cameron and Edmonds in [44].

**Theorem 103 ([44, 80, 95])** *Any Gallai coloring can be obtained by substituting Gallai colored complete graphs into the vertices of a 2-colored complete graph.*

This theorem follows from the lemma below, which provides another useful description of Gallai colorings.

**Lemma 1 ([80, 95])** *Every Gallai coloring with at least three colors has a color which spans a disconnected graph.*

One may also note that Theorem 103 is equivalent to the following result.

**Theorem 104** *In any Gallai colored complete graph, there exists a partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on edges between each pair of parts.*

Using Theorem 103, Cameron, Edmonds and Lovász [45] proved the following extension of the Perfect Graph Theorem (see [138]).

**Theorem 105 ([45])** *Let  $G$  be a Gallai 3-coloring of a complete graph. If the graphs induced on two of the colors are perfect, then the graph induced on the third color is also perfect.*

In the same note, the authors go on to conjecture that if  $K_n$  is 3-colored so that no configuration from a given class occurs and two of the colors induce perfect graphs, then the third color also induces a perfect graph. This conjecture follows from the Strong Perfect Graph Theorem which was proven in [52] and is stated as follows.

**Theorem 106 (Strong Perfect Graph Theorem [52])** *A graph is perfect if and only if no induced subgraph is an odd cycle of length at least 5 or the complement of one.*

In honor of Berge who conjectured the above, the class of graphs having no such induced odd cycle or its complement have been called *Berge graphs*.

Also using Theorem 103, Gyárfás and Simonyi prove the following three results, each of which extends older results from 2-colorings to Gallai colorings. For the first result, a broom is a path with a star at one end. Although the following three results are also stated in [122], we present them here for the sake of completeness.

**Theorem 107 ([95])** *In every Gallai coloring of a complete graph, there exists a spanning monochromatic broom.*

This result generalizes a result of Burr [41] which states that any 2-colored complete graph contains a monochromatic spanning broom. Theorem 108 generalizes a result of Bialostocki, Dierker and Voxman [30] who proved the same result in the case of 2-colored complete graphs.

**Theorem 108 ([95])** *In every Gallai coloring, there is a monochromatic spanning tree with height at most two.*

Theorem 108 is proven, as in the following, using the structure provided by Theorem 103.

**Theorem 109 ([95])** *Any Gallai coloring of the complete graph  $K_n$  contains a monochromatic star  $S_t$  for some  $t \geq \frac{2n}{5}$ .*

Theorem 109 is sharp by the following construction. Consider 5 copies of  $K_{n/5}$  labeled as  $G_0, G_1, \dots, G_4$  each colored entirely in color 1. Color the edges between  $G_i$  and  $G_{i+1}$  with color 2 and color all edges between  $G_i$  and  $G_{i+2}$  with color 3 for all  $i$  (modulo 5). This graph contains the aforementioned monochromatic star but not a larger one.

Theorems 107, 108 and 109 answer questions posed by Bialostocki and Voxman in [33].

More recently, Gyárfás, Sárközy, Sebő and Selkow [93] provided even more monochromatic structure in Gallai colorings. A double star is defined to be a tree of diameter 3, or in other words, two disjoint stars with centers joined by an edge. In the first result, they extend Theorem 109 to a double star.

**Theorem 110 ([93])** *Every Gallai coloring of  $K_n$  contains a monochromatic double star with at least  $\frac{3n+1}{4}$  vertices. This is asymptotically best possible.*

Continuing in the tradition of extending 2-coloring results to Gallai colorings, the authors also extend results from [56] and [89] which find a monochromatic diameter 2 subgraph of a 2-colored complete graph.

**Theorem 111 ([93])** *In every Gallai coloring  $G$  of  $K_n$ , there is a monochromatic diameter 2 subgraph with at least  $\lceil \frac{3n}{4} \rceil$  vertices. This is best possible for every  $n$ .*

This result is best possible by the following construction. Consider a 2-coloring of  $K_4$  in which each color is isomorphic to  $P_4$ . Then substitute an equal (or as close to equal as possible) number of vertices for each vertex of the  $K_4$ . The coloring of these new blocks is arbitrary. This construction contains a monochromatic diameter 2 subgraph of order  $\lceil \frac{3n}{4} \rceil$  but no larger.

Others have studied *exact Gallai cliques* which are colorings of cliques in which every copy of a smaller clique has exactly a predetermined number of colors. In [53], an upper bound of approximately  $5^{k/2}$  was found for the number of vertices in an exact Gallai clique using  $k$  colors. As an extension of Theorem 103, a characterization of exact Gallai cliques was given by Ball, Pultr and Vojtěchovský [27].



More specifically, Chung and Graham [53] studied the function  $f(s, t; k)$  defined to be the largest value of  $m$  such that it is possible to  $k$ -color the edges of  $K_m$  so that every  $K_s \subseteq K_m$  has exactly  $t$  different colors. Using this notation, the aforementioned result can be restated as:

**Theorem 112** ([15, 53, 93])

$$f(3, 2; k) = \begin{cases} 5^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

In [15] and [93], the previous result was stated in terms of Gallai colorings and proven using the decomposition from Theorem 103. Axenovich and Iverson, in [15], even classified all rainbow colorings of a complete graph with no rainbow or monochromatic triangle. In [53], Chung and Graham also proved the following.

**Theorem 113** ([53]) For  $k \geq 4$ ,

$$f(4, 3; k) = k + 2.$$

Furthermore, Chung and Graham also stated, without proof, the following two results. The first extends the previous theorem for all values of  $s \leq k$  while the second demonstrates the extreme change in behavior if  $s$  is allowed to be larger than  $k$ .

**Theorem 114** ([53]) For  $5 \leq s \leq k$ ,

$$f(s, s - 1; k) = k + 1.$$

**Theorem 115** ([53])

$$(1 + o(1))k^2 \leq f(k + 1, k; k) \leq k^2 + k.$$

The idea of Gallai colorings has also been considered in the context of multigraphs. In another generalization of Theorem 103, the authors of [101] provided a construction of all finite Gallai multigraphs similar to that of Gallai for graphs. Concerning Gallai multigraphs, Diwan and Mubayi asked the following question.

**Question 2** ([54]) Let  $R$ ,  $G$ , and  $B$  be graphs on the same vertex set of size  $n$ . How large must  $\min\{e(R), e(G), e(B)\}$  be to guarantee that  $R \cup G \cup B$  contains a rainbow triangle?

Using the partition result from [101], Magnant [139] recently provided the following solution to the question of Diwan and Mubayi in the case where the graph is large and complete, meaning that between every pair of vertices, there is at least one edge.

Let  $G$  be a  $G$ -colored multigraph on  $n$  vertices using three colors and let  $m(G)$  be the minimum number of edges in a single color in  $G$ . Let  $M$  be limit of the maximum value of  $m(G)$  over all  $G$ -colored, complete, multigraphs  $G$  on  $n$  vertices as  $n \rightarrow \infty$ . With this notation, the main result is the following.

**Theorem 116** ([139])

$$M = \frac{26 - 2\sqrt{7}}{81}n^2 \sim 0.25566n^2.$$

## 4.2 General Gallai-Ramsey Theory for Rainbow Triangles

In [93], the authors introduced a restricted Ramsey number which they called  $RG(r, H)$  to be the minimum  $m$  such that in every Gallai coloring of  $K_m$  with  $r$  colors, there is a monochromatic copy of  $H$ . This concept naturally extends to any rainbow colored graph in the following sense.

**Definition 5** *Given two graphs  $G$  and  $H$ , the  $k$ -colored Gallai Ramsey number  $gr_k(G : H)$  is defined to be the minimum integer  $n$  such that every  $k$ -coloring (using all  $k$  colors) of the complete graph on  $n$  vertices contains either a rainbow copy of  $G$  or a monochromatic copy of  $H$ .*

Notice that this definition is similar to the rainbow (or constrained) Ramsey numbers (see Definition 2) except, in this case, the number of colors is fixed. This definition is also very closely related to the function  $MaxR(G, H)$  studied in [11, 15]. Essentially these functions are duals. Similarly, we define the following notation for when the number of colors used in the coloring is at most a fixed value  $k$ .

**Definition 6** *Given two graphs  $G$  and  $H$ , the  $k$ -colored upper Gallai Ramsey number  $gr'_k(G : H)$  is defined to be the minimum integer  $n$  such that every  $k$ -coloring (using at most  $k$  colors) of the complete graph on  $n$  vertices contains either a rainbow copy of  $G$  or a monochromatic copy of  $H$ .*

Note that if  $gr_k(H : G)$  is a monotone increasing function of  $k$  (on an interval  $a \leq k \leq b$ ), then these two functions will be equal (on the same interval). Somewhat surprisingly, this is not always the case (see Theorem 131 and Conjecture 5).

In particular, using this definition, Theorem 109 can be restated as follows, which was also noted in [143].

**Theorem 117 ([143])**

$$gr'_k(K_3 : S_t) = \begin{cases} \frac{5t-3}{2} & \text{for odd } t, \\ \frac{5t-6}{2} & \text{otherwise.} \end{cases}$$

Theorem 112 can also be restated in this notation. In [93], the authors provide the asymptotic behavior of  $gr_k(K_3 : H)$  for a general graph  $H$ .

**Theorem 118 ([93])** *Let  $H$  be a fixed graph with no isolated vertices. If  $H$  is not bipartite, then  $gr_k(K_3 : H)$  is exponential in  $k$ . If  $H$  is bipartite, then  $gr_k(K_3 : H)$  is linear in  $k$ .*

The lower bound for the case when  $H$  is not bipartite comes from the following inductive construction. Certainly there exists a small graph in one color containing no  $H$ . Suppose there exists  $G_k$  using  $k$  colors which contains no monochromatic copy of  $H$ . Then let  $G_{k+1}$  be two copies of  $G_k$  with all possible edges in between using the new color. The graph  $G_{k+1}$  also contains no monochromatic copy of  $H$ . For the lower bound when  $H$  is bipartite, the construction involves adding vertices to the graph with all edges in a single color. If  $H$  is a star, this result becomes more complicated in light of the difference between  $gr_k(\cdot : \cdot)$  and  $gr'_k(\cdot : \cdot)$ .

In [68], Faudree, Gould, Jacobson and Magnant proved the following specific Gallai Ramsey numbers

**Theorem 119** ([68])

1.  $gr_k(K_3 : C_4) = k + 4$  for  $k \geq 2$ .
2.  $gr_k(K_3 : P_4) = k + 3$  for  $k \geq 1$ .
3.  $gr_k(K_3 : P_5) = k + 4$  for  $k \geq 1$ .
4.  $gr_k(K_3 : P_6) = 2k + 4$  for  $k \geq 1$ .

The authors of [68] also found the Gallai Ramsey numbers for all trees of order at most 6. Regarding paths in general, the following represents the best known bounds.

**Theorem 120** ([68], [100]) *Given integers  $n \geq 3$  and  $k \geq 1$ ,*

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \leq gr_k(K_3 : P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3 \left\lceil \frac{n}{2} \right\rceil.$$

Regarding cycles, the following are the best known general bounds.

**Theorem 121** ([75], [100]) *Given integers  $n \geq 2$  and  $k \geq 1$ ,*

$$(n-1)k + n + 1 \leq gr_k(K_3 : C_{2n}) \leq (n-1)k + 3n.$$

**Theorem 122** ([75], [100]) *Given integers  $n \geq 2$  and  $k \geq 1$ ,*

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

For specific small cycles, Fujita and Magnant obtained the following.

**Theorem 123** ([75]) *For any positive integer  $k$ ,*

$$gr_k(K_3 : C_6) = 2k + 4.$$

**Theorem 124** ([75]) *For any positive integer  $k \geq 2$ ,*

$$gr_k(K_3 : C_5) = 2^{k+1} + 1.$$

For non-bipartite graphs, the picture is not clear. Given a graph  $H$ , call a graph  $H'$  a *reduction* of  $H$  if  $H'$  can be obtained from  $H$  by identifying sets of non-adjacent vertices (and removing any resulting repeated edges). Let  $\mathcal{H}$  be the set of all possible reductions of  $H$ . For the sake of the following main definition, let  $R_2(\mathcal{H})$  be the minimum integer  $n$  such that every 2-coloring of  $K_n$  contains a monochromatic copy of some graph in the set  $\mathcal{H}$ . Since this quantity is bounded above by the Ramsey number  $R(H, H)$ , its existence is obvious. Now a critical definition.

**Definition 7** ([140]) *If  $\mathcal{H}$  is the set of all reductions of a given graph  $H$ , define the function  $m(H)$  to be*

$$m(H) = R_2(\mathcal{H}).$$

Using this definition, a lower bound on the Gallai-Ramsey number for any non-bipartite graph  $H$  has been shown.

**Theorem 125** *[[140]] For a connected non-bipartite graph  $H$  and an integer  $k \geq 2$ , we have that  $gr_k(K_3 : H)$  is at least*

$$\begin{cases} (R(H, H) - 1) \cdot (m(H) - 1)^{(k-2)/2} + 1 & \text{if } k \text{ is even,} \\ (\chi(H) - 1) \cdot (R(H, H) - 1) \cdot (m(H) - 1)^{(k-3)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

In summary of the known sharp values of Gallai-Ramsey numbers concerning rainbow triangles, we present the following tables. For the tables, we describe some special trees as follows:

- $P_4^+$  is the graph consisting of a  $P_4$  with the addition of a pendant vertex adjacent to an interior vertex of the  $P_4$ ,
- $P_5^+$  is the graph consisting of a  $P_5$  with the addition of a pendant vertex adjacent to an interior vertex (but not the center) of the  $P_5$ ,
- $P_4^{++}$  is the graph consisting of a  $P_4$  with the addition of two pendant vertices adjacent to different interior vertices of the  $P_4$ ,
- $P_4^{+2}$  is the graph consisting of a  $P_4$  with the addition of two pendant vertices adjacent to a single interior vertex of the  $P_4$ ,
- $P_5^{+'}$  is the graph consisting of a  $P_5$  with the addition of a pendant vertex adjacent to the center vertex of the  $P_5$ ,
- $B_m$  is the book on  $m$  pages, or rather  $K_2 + \overline{K_m}$ .

Graph	$gr_k(K_3 : H)$	Cite	Graph	$gr_k(K_3 : H)$	Cite
$P_4$	$k + 3$	[68]	$C_4$	$k + 4$	[68]
$P_5$	$k + 4$	[68]	$C_5$	$2^{k+1} + 1$	[75]
$P_6$	$2k + 4$	[68]	$C_6$	$2k + 4$	[75]
$P_7$	$2k + 5$	[141]	$C_7$	$3 \cdot 2^k + 1$	[40]
$P_8$	$3k + 5$	[141]	$C_8$	$3k + 5$	[88]
$P_4^+$	$k + 4$	[68]	$C_9$	$4 \cdot 2^k + 1$	[38]
$P_5^+$	$k + 5$	[68]	$C_{11}$	$5 \cdot 2^k + 1$	[38]
$P_4^{++}$	$2k + 4$	[68]	$C_{13}$	$6 \cdot 2^k + 1$	[39]
$P_4^{+2}$	$2k + 4$	[68]	$C_{15}$	$7 \cdot 2^k + 1$	[39]
$P_5^{+'}$	$k + 5$	[68]			

Graph	$gr_k(K_3 : H)$	Cite
$K_3$	$\begin{cases} 5^{k/2} & k \text{ even} \\ 2 \cdot 5^{(k-1)/2} & k \text{ odd} \end{cases}$	[15, 53, 93]
$K_4$	$\begin{cases} 17^{k/2} + 1 & k \text{ even,} \\ 3 \cdot 17^{(k-1)/2} + 1 & k \text{ odd} \end{cases}$	[136]
$sK_3, (k-s)C_4$	$\begin{cases} (k-s+3) \cdot 2 \cdot 5^{(s-1)/2} + 1 & \text{if } s \text{ is odd and } k-s > 1, \\ (k-s+3) \cdot 5^{s/2} + 1 & \text{if } s \text{ is even and } k-s > 1, \\ 6 \cdot 5^{(s-1)/2} + 1 & \text{if } s \text{ is odd and } k-s = 1, \\ 3 \cdot 5^{s/2} + 1 & \text{if } s \text{ is even and } k-s = 1 \end{cases}$	[175]
$B_m \ (2 \leq m \leq 5)$	$\begin{cases} m+2 & \text{if } k=1, \\ (R(B_m, B_m) - 1) \cdot 5^{(k-2)/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot (R(B_m, B_m) - 1) \cdot 5^{(k-3)/2} + 1 & \text{otherwise} \end{cases}$	[176]

### 4.3 General Gallai-Ramsey Theory for Other Rainbow Graphs

In a similar work, Theorem 103 was extended by Fujita and Magnant as follows.

**Theorem 126 ([76])** *In any rainbow  $S_3^+$ -free coloring  $G$  of a complete graph, one of the following holds:*

1.  $V(G)$  can be partitioned such that there are at most 2 colors on the edges between the parts; or
2. There are three (different colored) monochromatic spanning trees, and moreover, there exists a partition of  $V(G)$  with exactly 3 colors on edges between parts and between each pair of parts, the edges have only one color.

Each conclusion of this result is best possible. In general, the following slightly weaker result also holds.

**Theorem 127 ([76])** *For  $k \geq 4$ , in any rainbow  $S_k^+$ -free coloring  $G$  of a complete graph, there exists a partition of  $V(G)$  such that between the parts, there are at most  $k$  colors. Furthermore, there exists a coloring with  $k$  colors between parts.*

Using Theorems 107 and 126, the authors also proved the following extension of Theorem 107. In light of Theorem 107, one may be inclined to ask whether there exists a monochromatic spanning broom in a rainbow  $S_3^+$ -free coloring. Unfortunately, this is not

the case by the following example. Consider  $G = G_1 \cup G_2 \cup G_3 \cup G_4$  where each  $G_i$  is a complete graph with all edges colored with color 1. The edges  $E(G_1, G_2)$  and  $E(G_3, G_4)$  are also colored with color 1 while the edges of  $E(G_1, G_3) \cup E(G_2, G_4)$  have color 2 and  $E(G_1, G_4) \cup E(G_2, G_3)$  have color 3 (where  $E(A, B)$  denotes the set of all edges between  $A$  and  $B$ ). This coloring contains no rainbow  $S_3^+$  and no monochromatic spanning structure.

**Theorem 128 ([76])** *In any rainbow  $S_3^+$ -free coloring of a complete graph, there exists a spanning 2 colored broom.*

Extending Theorem 119, Fujita and Magnant proved the following collection of results. The proof of Item 1 uses the decomposition from Theorem 126 whereas the proofs of Items 2 and 3 use techniques similar to those of Theorem 113 and Item 1 in Theorem 119, respectively.

**Theorem 129 ([76])** *For all  $k \geq 1$ ,*

1.  $gr_k(S_3^+ : P_4) = k + 3$ .
2.  $gr_k(S_3^+ : K_3) = \lambda(k)$  where  $\lambda(k) = 5^{k/2} + 1$  for  $k$  even and  $\lambda(k) = 2 \cdot 5^{(k-1)/2} + 1$  for  $k$  odd.
3.  $gr_k(S_3^+ : C_4) = k + 4$ .

As an extension of Theorem 118, Fujita and Magnant also proved the following for rainbow  $S_3^+$ -free colorings.

**Theorem 130 ([76])** *Let  $H$  be a fixed graph with no isolated vertices. If  $H$  is not bipartite, then  $gr_k(S_3^+, H)$  is exponential in  $k$ . If  $H$  is bipartite, then  $gr_k(S_3^+, H)$  is linear in  $k$ .*

Note that the results of Theorems 129 and 130 provide the same numbers as the rainbow triangle free cases but the proofs are more complicated due to the weaker structure from Theorem 126.

When we consider monochromatic stars, the picture becomes far more complicated.

**Conjecture 5 ([76])** *For all  $k \geq 4$ ,*

$$gr_k(S_3^+ : S_t) = 3t - 2k + 4.$$

This conjecture would be sharp by the following example. Given an integer  $k$ , let  $G = A_1 \cup A_2 \cup A_3 \cup H$  where  $H$  is a rainbow triangle free coloring of a complete graph on  $k - 2$  vertices where using colors  $4, \dots, k$ , and  $A_i$  is a complete graph of order  $\frac{n-k+2}{3}$  colored entirely with color  $i$  for each  $i = 1, 2, 3$ . The edges of  $E(A_1, A_2)$  have color 3,  $E(A_2, A_3)$  have color 1 and  $E(A_1, A_3)$  have color 2. Also,  $E(H, A_1)$  have color 3,  $E(H, A_2)$  have color 1 and  $E(H, A_3)$  have color 2. The graph  $G$  contains no rainbow  $S_3^+$  but contains a star of order  $\frac{n+2k-4}{3}$ .

Fortunately, the following result shows that if we allow the use of fewer colors on the edges, then things become much easier.

**Theorem 131** ([76]) *For all  $t \geq 1$ , and  $k \geq 3$ , we have  $gr'_k(S_3^+ : S_t) = 3t - 1$ .*

This problem was also considered in [105]. Using different notation, the authors of [72] consider the cases  $gr_k(P_4, K_{1,3})$ ,  $gr_k(K_{1,3}, P_4)$ ,  $gr_k(P_4, P_4)$ ,  $gr_k(K_{1,3}, K_3)$ ,  $gr_k(P_4, K_3)$ ,  $gr_k(K_3, P_4)$ , and show the following result.

**Theorem 132** ([72]) *For  $m \leq k$ ,*

$$gr_k(K_{1,m}, K_{1,n}) = (n - 1)(m - 1) + 2.$$

## 4.4 Gallai Colorings and Other Properties

In [49], Chen and Li considered using a color degree condition in Gallai colored complete graphs to find long rainbow paths. Recall that  $\delta^c$  denotes the minimum, over all vertices  $v \in V(G)$ , number of colors on the edges incident to  $v$ .

**Theorem 133** ([49]) *Any Gallai colored complete graph  $G$  has a rainbow path of length at least  $\delta^c(G)$ .*

For general graphs, Chen and Li also proved the following.

**Theorem 134** ([49]) *Any Gallai colored graph  $G$  with  $\delta^c(G) \geq k \geq 6$  has a rainbow path of length at least  $\frac{3k}{4}$ .*

Gallai colored non-complete graphs still have large monochromatic connected subgraphs.

**Theorem 135** ([94]) *Every Gallai colored graph  $G$  contains a monochromatic connected subgraph of order at least  $(\alpha(G)^2 + \alpha(G) + 1)^{-1}|G|$  vertices.*

In fact, Gallai colorings contain almost spanning highly connected subgraphs but more forbidden rainbow subgraphs have this property as well.

**Theorem 136** ([77]) *Let  $\mathcal{H}$  be the set of all graphs  $H$  such that if  $G$  is a colored  $K_n$  containing no rainbow copy of  $H$ , then  $G$  contains a monochromatic  $k$ -connected subgraph of order at least  $n - f(k, H)$  where  $f$  is a function not depending on  $n$ . Then  $\mathcal{H} = \{K_3, P_6, P_4^+\}$  (and the connected subgraphs of these graphs) where  $P_4^+$  is a path on 4 vertices with a pendant edge hanging off of one of the interior vertices.*

More generally, the following holds for hypergraphs.

**Theorem 137** ([94]) *If the edges of an  $r$ -uniform hypergraph  $H$  are colored so that there is no rainbow copy of a fixed  $F$ , then there is a monochromatic connected subhypergraph of order at least  $c|H|$  where  $c$  is a function only of  $F$ ,  $r$  and  $\alpha(H)$ .*

Another general graph result is the following. This result was actually proven by showing that for any integer  $\beta$ , there exists an integer  $h = h(\beta)$  such that if  $D$  is a multipartite digraph with no cyclic triangles and the largest independent set of vertices in different partite sets is  $\beta$ , then the smallest number of partite sets needed to dominate  $D$  is at most 4.

**Theorem 138** ([96]) *The vertices of a Gallai colored graph  $G$  can be covered by the vertices of at most  $k$  monochromatic components where  $k$  depends only on  $\alpha(G)$ .*

## 5 Other Generalizations

### 5.1 Sub-Ramsey Theory

**Definition 8** *Given a graph  $G$  and a positive integer  $k$ , the sub-Ramsey number  $sr(G, k)$  is said to be the minimum number  $n$  such that if the edges of  $K_n$  are colored with no color appearing more than  $k$  times, then the colored graph contains a rainbow  $G$ .*

When the edges of  $K_n$  are colored with no color appearing more than  $k$  times, define a new edge coloring with at most  $k$  colors with each new color class containing at most one edge from each original color class. In the new edge-coloring, if there exists a monochromatic  $G$ , then it corresponds to a rainbow  $G$  in the original edge-coloring. Therefore we know that  $sr(G, k) \leq r(G_1, G_2, \dots, G_k)$ , where  $G_i \approx G$  for all  $1 \leq i \leq k$ , so in general,  $sr(G, k)$  is finite for any graph  $G$  and any positive integer  $k$ .

Galvin [81] gave a result on the sub-Ramsey problem, that is,  $sr(K_3, k) = k + 2$ . For the complete graph, Hell and Montellano-Ballesteros [103] showed that  $cn^{3/2} \leq sr(K_n, k) \leq (2n - 3)(n - 2)(k - 1) + 3$  for some constant  $c$ , which improves upon a result due to Alspach, Gerson, Hahn and Hell [9].

The sub-Ramsey number of a cycle or a path have also been considered. Hahn and Thomassen [99] conjecture that there exists a linear function  $f$  such that  $sr(C_n, k) = n$  for  $k \leq f(n)$ . They showed that  $k$  could grow as fast as  $n^{1/3}$ , and this was improved by Frieze and Reed [71] to  $\frac{n}{\ln n}$  for sufficiently large  $n$ . Recently Albert, Frieze and Reed [2] settled the conjecture by Hahn and Thomassen; they showed that if  $n$  is sufficiently large and  $k \leq cn$  for  $c < \frac{1}{32}$ , then  $sr(C_n, k) = n$ .

For graphs other than the complete graph, cycle and path, Hahn [98] and Fraïsse, Hahn and Sotteau [70] studied the sub-Ramsey number of a star. On the other hand, sub-Ramsey number for arithmetic progressions are also studied [5, 22].

### 5.2 Monochromatic Degree

Let  $k$  and  $d$  be positive integers and  $n$  be a sufficiently large integer. An edge coloring of a graph  $G$  is called a  $(k, d)$ -coloring if it uses  $k$  colors and each vertex has degree at least  $d$  in each color. Given a graph  $F$  and  $k \geq E(F)$ , for any  $n > k$ , let  $d(n, F, k)$  denote the minimum integer  $d$  such that every  $(k, d)$ -coloring of  $K_n$  contains a rainbow copy of  $F$ . If there is no such  $d$ , we say  $d(n, F, k) = \infty$ .

This topic was first studied in [64], this topic was recently revisited by Tuza in [172] with the following problems.

**Problem 1 ([172])** *Given a graph  $F$  and  $k \geq E(F)$ , describe the behavior of  $d(n, f, k)$  as a function of  $n$ .*

**Problem 2 ([172])** *Does every  $(k, \lfloor (n-1)/k \rfloor)$ -coloring of  $K_n$  contain all graphs with fewer than  $k$  edges as rainbow subgraphs?*

Any counterexample to an affirmative answer to this claim must satisfy some necessary conditions [64].



**Problem 3 ([172])** Characterize those graphs  $F$  with the property that for  $k = |E(F)|$ , a rainbow copy of  $F$  occurs in every  $(k, d)$ -coloring of  $K_n$  for all sufficiently large  $n$  where

(i)  $d = \lfloor (n-1)/k \rfloor$ ;

(ii)  $d \leq (1-c)n/k$  for some constant  $c > 0$ ;

(iii)  $d = 1$ .

Some partial results were proven in [64]. In particular, there is the following case.

**Problem 4 ([172])** Is every tree in category (iii) of Problem 3?

### 5.3 Others

Let  $f(n)$  be the minimum number such that there is a proper edge coloring of  $K_n$  with  $f(n)$  colors with no path or cycle of four edge using one or two colors, Axenovich [10] proved that  $\frac{1+\sqrt{5}}{2}n - 3 \leq f(n) \leq 2n^{1+c/\sqrt{\log n}}$  for a positive constant  $c$ .

Based on Voloshin's definition in [173], several groups [23, 46, 109, 126, 169, 170] have worked on coloring the vertices of hypergraphs to avoid both monochromatic and rainbow hyper-edges.

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