Rainbow Hamilton cycles in uniform hypergraphs

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Abstract

Let $K_n^{(k)}$ be the complete k-uniform hypergraph, $k \geq 3$, and let ℓ be an integer such that $1 \leq \ell \leq k-1$ and $k-\ell$ divides n. An ℓ-overlapping Hamilton cycle in $K_n^{(k)}$ is a spanning subhypergraph C of $K_n^{(k)}$ with $n/(k-\ell)$ edges and such that for some cyclic ordering of the vertices each edge of C consists of k consecutive vertices and every pair of adjacent edges in C intersects in precisely ℓ vertices.

We show that, for some constant $c = c(k,\ell)$ and sufficiently large n, for every coloring (partition) of the edges of $K_n^{(k)}$ which uses arbitrarily many colors but no color appears more than $cn^{k-\ell}$ times, there exists a rainbow ℓ-overlapping Hamilton cycle C, that is every edge of C receives a different color. We also prove that, for some constant $c' = c'(k,\ell)$ and sufficiently large n, for every coloring of the edges of $K_n^{(k)}$ in which the maximum degree of the subhypergraph induced by any single color is bounded by $c'n^{k-\ell}$, there exists a properly colored ℓ-overlapping Hamilton cycle C, that is every two adjacent edges receive different colors. For $\ell = 1$, both results are (trivially) best possible up to the constants. It is an open question if our results are also optimal for $2 \leq \ell \leq k-1$.

The proofs rely on a version of the Lovász Local Lemma and incorporate some ideas from Albert, Frieze, and Reed.

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1 Introduction

By a *coloring* of a hypergraph H we mean any function $\phi : H \to \mathbb{N}$ assigning natural numbers (colors) to the edges of H. (In this paper we do not consider vertex colorings.) A hypergraph H together with a given coloring ϕ will be dubbed *a colored hypergraph*. A subhypergraph F of a colored hypergraph H is said to be *properly colored* if every two adjacent edges of F receive different colors. (Two different edges are *adjacent* if they share at least one vertex.) We say that a subhypergraph F of a colored hypergraph H is *rainbow* if every edge of F receives a different color, that is, when ϕ is injective on F.

In order to force the presence of properly colored or rainbow subhypergraphs one has to restrict the colorings ϕ, either globally or locally. A coloring ϕ is *r-bounded* if every color is used at most r times, that is, $|\phi^{-1}(i)| \leq r$ for all $i \in \mathbb{N}$. A coloring ϕ is *r-degree bounded* if the hypergraph induced by any single color has maximum degree bounded by r, that is, $\Delta(H[\phi^{-1}(i)]) \leq r$ for all $i \in \mathbb{N}$.

In this paper we study the existence of properly colored and rainbow Hamilton cycles in colored k-uniform complete hypergraphs, $k \geq 3$. (A hypergraph is *k-uniform* if every edge has size k; it is *complete* if all k-element subsets of the vertices form edges.) There is a broad literature on this subject for $k = 2$, that is, for graphs. Indeed, setting $r = cn$, Alon and Gutin proved in [2], improving upon earlier results from [5, 6, 16] that if $c < 1 - 1/\sqrt{2}$ then any r-degree bounded coloring of the edges of the complete graph K_n yields a properly colored Hamilton cycle (for the history of the problem, see [3]). It had been conjectured in [5] that the constant $1 - 1/\sqrt{2}$ can be replaced by $\frac{1}{2}$ which is the best possible.

Rainbow Hamilton cycles in r-bounded colorings of the complete graph have been studied in [1, 8, 10, 12]. Hahn and Thomassen conjectured that their existence is guaranteed if $r = cn$ for some $c > 0$. This was confirmed by Albert, Frieze, and Reed in [1] with $c = \frac{1}{64}$. Again, $c = 1/2$ seems to be a critical value here, since one can use each of $n - 1$ colors exactly $n/2$ times, making the presence of rainbow Hamilton cycles impossible. In striking contrast, there is literally nothing known on properly colored or rainbow Hamilton cycles in colored k-uniform hypergraphs for $k \geq 3$.

The notion of a hypergraph cycle can be ambiguous. In this paper we are *not* concerned with the Berge cycles as defined by Berge in [4] (see also [11]). Instead, following a recent trend in the literature ([7, 13, 15]), given an integer $1 \leq \ell < k$, we define an *ℓ-overlapping cycle* as a k-uniform hypergraph in which, for some cyclic ordering of its vertices, every edge consists of k consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. (See Fig. 1 for an example of a 2-overlapping and a 3-overlapping 5-uniform cycle.)

The two extreme cases of $\ell = 1$ and $\ell = k - 1$ are referred to as, respectively, *loose* and *tight* cycles. Note that the number of edges of an ℓ-overlapping cycle with s vertices is $s/(k - \ell)$. Note also that when $k - \ell$ divides s, every tight cycle on s vertices contains an ℓ-overlapping cycle on the same vertex set (with the same cyclic ordering).

Given a k-uniform hypergraph H on n vertices, where $k - \ell$ divides n, an ℓ-overlapping cycle contained in H is called *Hamilton* if it goes through every vertex of H, that is, if

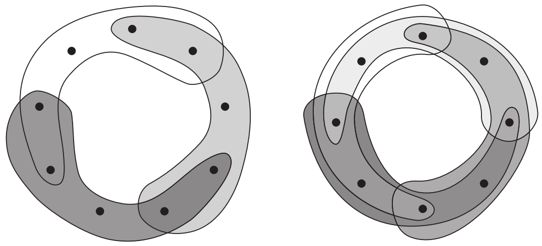

Figure 1: A 2-overlapping and a 3-overlapping 5-uniform cycle.

$s = n$. We denote such a Hamilton cycle by $C_n^{(k)}(\ell)$.

In this paper we prove the following two results. Let $K_n^{(k)}$ be the complete k-uniform hypergraph of order n.

Theorem 1.1 *For every $1 \leq \ell < k$ there is a constant $c = c(k, \ell)$ such that if n is sufficiently large and $k - \ell$ divides n then any $cn^{k-\ell}$-bounded coloring of $K_n^{(k)}$ yields a rainbow copy of $C_n^{(k)}(\ell)$.*

Theorem 1.2 *For every $1 \leq \ell < k$ there is a constant $c' = c'(k, \ell)$ such that if n is sufficiently large and $k - \ell$ divides n then any $c'n^{k-\ell}$-degree bounded coloring of $K_n^{(k)}$ yields a properly colored copy of $C_n^{(k)}(\ell)$.*

Note that for loose Hamilton cycles (i.e. $\ell = 1$) the above results are optimal up to the values of c and c'. Theorem 1.2 is trivially optimal, as the largest maximum degree can be at most $r = \binom{n-1}{k-1} \sim \frac{1}{(k-1)!}n^{k-1}$. To see that also Theorem 1.1 is optimal up to the constant for $\ell = 1$ consider any coloring of $K_n^{(k)}$ using each color precisely

$$r = \frac{\binom{n}{k}}{\frac{n}{k-1} - 1} \sim \frac{k-1}{k!}n^{k-1}$$

times, and thus using only $\frac{n}{k-1} - 1$ colors altogether. Such a coloring is r-bounded and, clearly, there is no rainbow copy of $C_n^{(k)}(1)$.

Problem 1.3 For all $k \geq 3$ and $\ell = 1$, determine $\sup c$ and $\sup c'$ over all values of c and, respectively, c' for which Theorems 1.1 and 1.2 hold.

We believe that Theorems 1.1 and 1.2 are optimal up to the constants also for $\ell \geq 2$, that is, we believe that the answer to the following question is positive.

Problem 1.4 For all $k \geq 3$ and $2 \leq \ell \leq k-1$, does there exist an r-bounded (r-degree bounded) coloring ϕ of $K_n^{(k)}$ such that $r = \Theta(n^{k-l})$ and no copy of $C_n^{(k)}(\ell)$ is rainbow (properly colored)?

As some evidence supporting our belief, consider the bipartite version of both problems for $k = 3$ and $\ell = 2$. Let $K_{n,2n}^{(3)} = (V_1, V_2, E)$, where $|V_1| = n$, $|V_2| = 2n$ and $E = \{e \subset V_1 \cup V_2 \ : \ |e \cap V_i| = i, \quad i = 1, 2\}$. To every edge e assign the pair $e \cap V_2$ as its color. Clearly, every color appears exactly n times and hence such a coloring is n-bounded (and thus n-degree bounded). Finally, note that every tight Hamilton cycle in $K^{(3)}_{n,2n}$ induces a cyclic sequence of vertices with a repeated pattern of two vertices from V_2 followed by one vertex from V_1. Hence, there is a pair of consecutive edges with the same color (actually, there are n such pairs), and so no copy of a properly colored (or rainbow) $C^{(3)}_n(2)$ exists.

2 The proofs

We will need a special version of the Lovász Local Lemma. A similar result was already established in [1, 9, 14]. Contrary to the above results, in our formulation of the lemma we avoid conditional probabilities so that we do not need to make a priori assumptions that certain events have positive probability.

Lemma 2.1 *Let $A_1, A_2, \ldots, A_m$ be events in an arbitrary probability space Ω. For each $1 \le i \le m$, let $[m] \setminus \{i\} = X_i \cup Y_i$ be a partition of the index set $[m] \setminus \{i\}$ and let*

$$d = \max\{|Y_i| : 1 \le i \le m\}. \tag{1}$$

If for each $1 \le i \le m$ and all $X \subseteq X_i$

$$\Pr\left(A_i \cap \bigcap_{j\in X} \overline{A_j}\right) \le \frac{1}{4(d+1)} \Pr\left(\bigcap_{j\in X} \overline{A_j}\right) \tag{2}$$

then $\Pr\left(\bigcap_{i=1}^m \overline{A_i}\right) > 0$. (We adopt the convention that $\bigcap_{j\in\emptyset} \overline{A_j} = \Omega$.)

Proof. We prove by induction on $t = 1, \ldots, m$, that for every $T \subseteq [m]$, $|T| = t$, and for every $i \in T$, setting $S = T \setminus \{i\}$, we have

$$\Pr\left(\bigcap_{i\in T} \overline{A_i}\right) > 0 \quad \text{and } \Pr\left(A_i \middle| \bigcap_{i\in S} \overline{A_i}\right) \le \frac{1}{2(d+1)}. \tag{3}$$

For $t = 1$ we apply (2) with $X = \emptyset$, obtaining for each i that $\Pr(A_i) \le \frac{1}{4(d+1)}$, equivalently $\Pr\left(\overline{A_i}\right) \ge 1 - \frac{1}{4(d+1)} > 0$, which confirms (3) for $t = 1$.

Now, assume truth for some t, $1 \le t \le m-1$, and consider a set $T = \{i\} \cup S$, where $i \notin S$ and $|S| = t$. Set $X = S \cap X_i$ and $Y = S \cap Y_i$, and observe that $S = X \cup Y$ and $|Y| \le |Y_i| \le d$. By the induction assumption $\Pr(\bigcap_{j\in S} \overline{A_j}) > 0$. If $Y = \emptyset$ (and thus $X = S$), by our assumption (2),

$$\Pr\left(A_i \middle| \bigcap_{j\in S} \overline{A_j}\right) \le \frac{1}{4(d+1)}.$$

Otherwise, $|X| < |S| = t$ and, again by (2) (in the numerator) and the induction assumption (in the denominator) we argue that

$$\Pr\left(A_i \middle| \bigcap_{j\in S} \overline{A_j}\right) = \frac{\Pr\left(A_i \cap \bigcap_{j\in Y} \overline{A_j} \middle| \bigcap_{j\in X} \overline{A_j}\right)}{\Pr\left(\bigcap_{j\in Y} \overline{A_j} \middle| \bigcap_{j\in X} \overline{A_j}\right)} \le \frac{\frac{1}{4(d+1)}}{1 - |Y|\frac{1}{2(d+1)}} \le \frac{1}{2(d+1)}.$$

Thus,

$$\Pr\left(\bigcap_{j\in T}\overline{A_j}\right) = \Pr\left(\overline{A_i}\Big|\bigcap_{j\in S}\overline{A_j}\right)\Pr(\bigcap_{j\in S}\overline{A_j}) > 0,$$

which completes the proof of Lemma 2.1. □

The proofs of Theorems 1.1 and 1.2 extend some ideas introduced by Albert, Frieze and Reed in [1].

Proof of Theorem 1.1. Fix $1 \le \ell < k$ and a $cn^{k-\ell}$-bounded coloring ϕ of $K_n^{(k)}$ for some constant $c > 0$ to be specified later. Define

$$M = \{\{e, f\} \ : \ e, f \in K_n^{(k)}, \quad |e \cap f| \le \ell \text{ and } \phi(e) = \phi(f)\}.$$

Moreover, for every pair $\{e, f\} \in M$ set

$$A_{e,f} = \{C \subset K_n^{(k)} \ : \ C \cong C_n^{(k)}(k-1) \text{ and } \{e, f\} \subset C\}.$$

In order to prove Theorem 1.1 it suffices to show that

$$\bigcap_{\{e,f\}\in M} \overline{A_{e,f}} \ne \emptyset. \tag{4}$$

Indeed, if (4) is true then there is a tight Hamilton cycle $C \cong C_n^{(k)}(k-1)$ such that for every pair of its edges e and f with $|e\cap f| \le \ell$ we have $\phi(e) \ne \phi(f)$. Since, by assumption, $k-\ell$ divides n, C contains a copy of $C_n^{(k)}(\ell)$ which is rainbow, as required.

To prove (4) we apply the probabilistic method and Lemma 2.1. To this end, for a given pair $\{e, f\} \in M$ let

$$Y_{e,f} = \{\{e', f'\} \in M : \{e', f'\} \ne \{e, f\} \text{ and } (e \cup f) \cap (e' \cup f') \ne \emptyset\}$$

and

$$X_{e,f} = M \setminus (Y_{e,f} \cup \{e, f\}).$$

To estimate d (cf. (1)), we bound from above the size of $Y_{e,f}$ as follows. For given edges e and f we can find at most $2kn^{k-1}$ edges e' sharing a vertex from $e \cup f$. For every such e' we have at most $cn^{k-\ell}$ candidates for f', since e' and f' must have the same color. Thus,

$$d = \max_{\{e,f\}\in M} |Y_{e,f}| \le 2ckn^{2k-\ell-1}. \tag{5}$$

Now, let us consider a uniform probability space consisting of all tight Hamilton cycles $C \cong C_n^{(k)}(k-1)$ in $K_n^{(k)}$. In order to prove (4), and thus finish the proof of Theorem 1.1, it suffices to show that

$$\Pr\left(\bigcap_{\{e,f\}\in M} \overline{A_{e,f}}\right) > 0. \tag{6}$$

Thus, it remains to verify assumption (2) of Lemma 2.1 with $m = |M|$, $A_i := A_{e,f}$, $X_i := X_{e,f}$, and $Y_i := Y_{e,f}$. Fix $\{e,f\} \in M$ and $X \subseteq X_{e,f}$ and set

$$\mathcal{C} = \bigcap_{\{e',f'\}\in X} \overline{A_{e',f'}}$$

and

$$\mathcal{C}_1 = A_{e,f} \cap \bigcap_{\{e',f'\}\in X} \overline{A_{e',f'}}.$$

In other words, $\mathcal{C}$ is the set of all copies C of $C_n^{(k)}(k-1)$ in $K_n^{(k)}$ such that $\{g,h\} \not\subset C$ for all $\{g,h\} \in X$, while $\mathcal{C}_1 = \{C \in \mathcal{C} \; : \; \{e,f\} \subset C\}$.

If $\mathcal{C}_1 = \emptyset$ then the R-H-S of (2) equals zero and there is nothing to prove. Otherwise, we rely on the following technical result the proof of which is postponed to the next section.

Proposition 2.2 *For all $1 \le \ell < k$ there exist constants $\delta = \delta(k,\ell)$, $0 < \delta < 1$, such that for every pair e,f of edges of $K_n^{(k)}$ with $|e \cap f| \le \ell$ and for every set X of pairs g,h of edges of $K_n^{(k)}$ satisfying $(g \cup h) \cap (e \cup f) = \emptyset$, if $\mathcal{C}_1 \ne \emptyset$, one can find a* disjoint *family $\{\mathcal{S}_C : C \in \mathcal{C}_1\}$ of sets of copies of $C_n^{(k)}(k-1)$ from $\mathcal{C}$ (indexed by the copies $C \in \mathcal{C}_1$) such that for all $C \in \mathcal{C}_1$ we have $|\mathcal{S}_C| \ge \delta n^{2k-\ell-1}$.*

We are now able to specify the constant c by setting

$$c = \frac{\delta}{10k}, \tag{7}$$

where $\delta = \delta(k,\ell)$ is the constant given by Proposition 2.2. Then by Proposition 2.2, (5), and (7)

$$\frac{\Pr\left(A_{e,f} \cap \bigcap_{\{e',f'\}\in X} \overline{A_{e',f'}}\right)}{\Pr\left(\bigcap_{\{e',f'\}\in X} \overline{A_{e',f'}}\right)} = \frac{|\mathcal{C}_1|}{|\mathcal{C}|} \le \frac{|\mathcal{C}_1|}{\sum_{C\in\mathcal{C}_1} |\mathcal{S}_C|} \le \frac{1}{\delta n^{2k-\ell-1}} \le \frac{1}{4(d+1)}.$$

Hence, (2) holds and we are in position to apply Lemma 2.1 and conclude that (6) and, consequently, (4) is true. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. This proof goes along the lines of the proof of Theorem 1.1. Let $c' = \frac{\delta}{10k^2}$, where $\delta = \delta(k,\ell)$ is the constant given by Proposition 2.2. Fix a $c'n^{k-\ell}$-degree bounded coloring of $K_n^{(k)}$. Here we slightly modify the definition of M. Let

$$M = \{\{e,f\} \; : \; e,f \in K_n^{(k)}, \quad 1 \le |e \cap f| \le \ell \text{ and } \phi(e) = \phi(f)\}.$$

As before,

$$A_{e,f} = \{C \subset K_n^{(k)} \; : \; C \cong C_n^{(k)}(k-1) \text{ and } \{e,f\} \subset C\}.$$

and in order to prove Theorem 1.2 it suffices to show that

$$\bigcap_{\{e,f\}\in M} \overline{A_{e,f}} \neq \emptyset. \tag{8}$$

Indeed, if (8) is true then there is a tight Hamilton cycle $C \cong C_n^{(k)}(k-1)$ such that for every pair of its edges e and f with $1 \leq |e \cap f| \leq \ell$ we have $\phi(e) \neq \phi(f)$. Since, by assumption, $k-\ell$ divides n, C contains a copy of $C_n^{(k)}(\ell)$ which is properly colored, as required.

We define sets $Y_{e,f}$ and $X_{e,f}$ as before and recalculate the upper bound on $|Y_{e,f}|$. For given edges e and f we can find at most $2kn^{k-1}$ edges e' sharing a vertex from $e \cup f$. For every such e' we have at most $c'kn^{k-\ell}$ candidates for f' since e' and f' intersect and have the same color. Thus,

$$|Y_{e,f}| \leq 2c'k^2n^{2k-\ell-1}.$$

The rest of the proof is identical to the proof of Theorem 1.1 and therefore is omitted. □

3 Proof of Proposition 2.2

Let e and f be given edges in $K_n^{(k)}$ such that $|e \cap f| \leq \ell$ and let $C \in \mathcal{C}_1$ be a tight Hamilton cycle containing e and f and missing at least one edge from each pair $\{g,h\} \in X$. We describe two constructions depending on the size of $e \cap f$.

Construction 1: for $2 \leq |e \cap f| \leq \ell$.
Let $|e \cap f| = a$ and let $e = (u_1, \ldots, u_k)$ and $f = (v_1, \ldots, v_k)$ be such that $u_{k-a+1} = v_1, u_{k-a+2} = v_2, \ldots, u_k = v_a$. This way we fix an orientation of C where e precedes f. Let $P = C \setminus \{e \cup f\}$ be the segment of C between f and e of length $n-2k+a$. We select arbitrarily $2k-a-1$ vertex disjoint edges $g_1, \ldots, g_{2k-a-1}$ from P, so that C is of the form $e \rightsquigarrow f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g_{2k-a-1} \rightsquigarrow e$, where the symbol $\rightsquigarrow$ indicates a path between the given edges. Clearly, we have $\Omega(n^{2k-a-1}) = \Omega(n^{2k-\ell-1})$ choices for the g_i's.

Let $g_i = (w_1^i, \ldots, w_k^i)$ for $1 \leq i \leq 2k-a-1$, where we list the vertices of g_i in the order of appearance on P. In order to create a cycle $\tilde{C} \in \mathcal{S}_C$, we remove all edges which contain at least one vertex from $(e \cup f) \setminus \{u_1, v_k\}$ and also all edges whose first vertex (in the order induced by C) is w_j^i for $1 \leq i \leq 2k-a-1$ and $j = 1, \ldots, k-1$. After this removal, the vertices in the set $\{u_2, \ldots, u_k, v_{a+1}, \ldots, v_{k-1}\}$ become isolated and the remains of the cycle C form a collection of vertex disjoint paths $v_k \rightsquigarrow w_{k-1}^1, w_k^1 \rightsquigarrow w_{k-1}^2, w_k^2 \rightsquigarrow w_{k-1}^3, \ldots, w_k^{2k-a-1} \rightsquigarrow u_1$.

To create $\tilde{C}$, we connect the above paths by absorbing the isolated vertices. Formally,

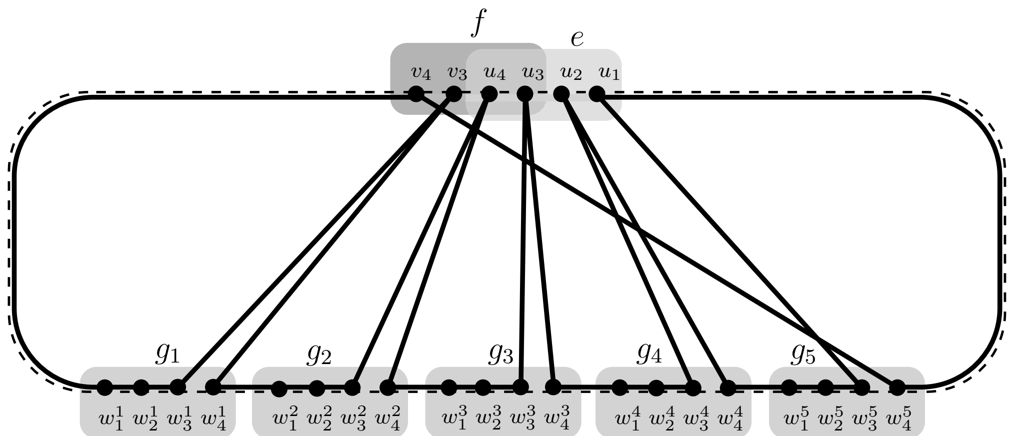

Figure 2: The first construction for $k = 4$ and $a = 2$. The dashed and solid lines denote C and $\tilde{C}$, respectively.

we define $\tilde{C}$ as the following sequence of vertices (see Fig. 2):

$$\begin{aligned} v_k &\rightsquigarrow w_1^1, w_2^1, \dots, w_{k-1}^1, v_{k-1}, w_k^1 \\ &\rightsquigarrow w_1^2, w_2^2, \dots, w_{k-1}^2, v_{k-2}, w_k^2 \\ &\rightsquigarrow \dots \\ &\rightsquigarrow w_1^{k-a-1}, w_2^{k-a-1}, \dots, w_{k-1}^{k-a-1}, v_{a+1}, w_k^{k-a-1} \\ &\rightsquigarrow w_1^{k-a}, w_2^{k-a}, \dots, w_{k-1}^{k-a}, u_k, w_k^{k-a} \\ &\rightsquigarrow w_1^{k-a+1}, w_2^{k-a+1}, \dots, w_{k-1}^{k-a+1}, u_{k-1}, w_k^{k-a+1} \\ &\rightsquigarrow \dots \\ &\rightsquigarrow w_1^{2k-a-2}, w_2^{2k-a-2}, \dots, w_{k-1}^{2k-a-2}, u_2, w_k^{2k-a-2} \\ &\qquad \rightsquigarrow w_1^{2k-a-1}, w_2^{2k-a-1}, \dots, w_{k-1}^{2k-a-1}, u_1 \rightsquigarrow w_k^{2k-a-1}, v_k. \end{aligned}$$

It is easy to check that every new edge intersects a vertex from $e \cup f$. Thus, $\tilde{C} \setminus C$ contains no edge from any pair of edges belonging to X. Moreover, note that different choices of g_i yield different cycles $\tilde{C}$. Thus, $|\mathcal{S}_C| = \Omega(n^{2k-\ell-1})$.

It remains to show that for any two tight Hamilton cycles $C \neq C' \in \mathcal{C}_1$ we have $\mathcal{S}_C \cap \mathcal{S}_{C'} = \emptyset$. In order to prove it, one can reverse the above procedure and uniquely determine C and the edges $g_1, g_2, \dots, g_{2k-a-1}$ from $\tilde{C}$. Note that we do not know the order in which the vertices of e and f are traversed by C.

To reconstruct C, we first find in $\tilde{C}$ a unique $e \rightsquigarrow f$ path Q with no endpoint in $e \cap f$ and exactly on vertex from e and f. From this we deduce that $u_1 = Q \cap e$, $v_k = Q \cap f$ and w_k^{2k-a-1} is the last vertex on Q before v_k. Now we start at v_k and follow $\tilde{C}$ in the direction opposite to w_k^{2k-a-1}. Before we reach u_1 we will intersect $f \cup e$ exactly $2k-a-2$ times. This way we restore the vertices $v_{k-1}, v_{k-2}, \dots, v_{a+1}, u_k, u_{k-1}, \dots, u_2$ (in the order of appearance on C). Note that every one of these vertices is adjacent to two vertices w_{k-1}^j and w_k^j for some $1 \le j \le 2k-a-2$. Consequently, we can uncover all edges g_i and hence C itself.

Construction 2: for $|e \cap f| \leq 1$.
Here we show a stronger result, namely, we construct a family $\mathcal{S}_C$ of size $\Omega(n^{2(k-1)})$. Let $e = (u_1, \ldots, u_k)$ and $f = (v_1, \ldots, v_k)$. Note that it might happen that $u_k = v_1$ if $|e \cap f| = 1$. Let P be a segment of vertices between e and f of size $\Omega(n)$. For given two vertices x and y denote by $d(x, y)$ the number of vertices on P in the segment between x and y. Now we select $2(k-1)$ vertex disjoint edges $g_1, \ldots, g_{2(k-1)}$ from P so that C is of the form $e \rightsquigarrow f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g_{2(k-1)} \rightsquigarrow e$ and

$$d(v_k, w^1_{k-1}) < d(w^1_k, w^2_{k-1}) + 1 < d(w^2_k, w^3_{k-1}) + 1 < \cdots < d(w^{k-2}_k, w^{k-1}_{k-1}) + 1, \qquad (9)$$

where $g_i = (w^i_1, \ldots, w^i_k)$ for each $1 \leq i \leq 2(k-1)$ and the vertices in g_i are listed in the order of appearance on P. This way we fix an orientation of C. (The above sequence of inequalities will be needed later to establish the orientation of C from $\tilde{C}$.) Clearly, we have $\Omega(n^{2(k-1)})$ choices for g_i's.

In order to create a cycle $\tilde{C} \in \mathcal{S}_C$, we remove all edges which contain at least one vertex from $(e \cup f) \setminus \{u_1, v_k\}$ and also all edges whose first vertex (in the order induced by C) is w^i_j for $1 \leq i \leq 2(k-1)$ and $j = 1, \ldots, k-1$. After this removal, the vertices in the set $\{u_2, \ldots, u_{k-1}, v_2, \ldots, v_{k-1}\}$ become isolated and the remains of the cycle C form a collection of vertex disjoint paths $v_k \rightsquigarrow w^1_{k-1}, w^1_k \rightsquigarrow w^2_{k-1}, w^2_k \rightsquigarrow w^3_{k-1}, \ldots, w^{2(k-1)}_k \rightsquigarrow u_1$, and $u_k \rightsquigarrow v_1$. (The latter may be degenerated to the set of isolated vertices.)

To create $\tilde{C}$, we connect the above paths by absorbing the isolated vertices. Formally, we define $\tilde{C}$ as the following sequence of vertices (see Fig. 3):

$$\begin{aligned}
v_1, w^1_{k-1}, &w^1_{k-2}, \ldots, w^1_1 \rightsquigarrow v_k, w^1_k \\
&\rightsquigarrow w^2_1, w^2_2, \ldots, w^2_{k-1}, v_{k-1}, w^2_k \\
&\rightsquigarrow w^3_1, w^3_2, \ldots, w^3_{k-1}, v_{k-2}, w^3_k \\
&\rightsquigarrow \ldots \\
&\rightsquigarrow w^{k-1}_1, w^{k-1}_2, \ldots, w^{k-1}_{k-1}, v_2, w^{k-1}_k \\
&\rightsquigarrow w^k_1, w^k_2, \ldots, w^k_{k-1}, u_{k-1}, w^k_k \\
&\rightsquigarrow w^{k+1}_1, w^{k+1}_2, \ldots, w^{k+1}_{k-1}, u_{k-2}, w^{k+1}_k \\
&\rightsquigarrow \ldots \\
&\rightsquigarrow w^{2k-3}_1, w^{2k-3}_2, \ldots, w^{2k-3}_{k-1}, u_2, w^{2k-3}_k \\
&\qquad\qquad \rightsquigarrow w^{2k-2}_1, w^{2k-2}_2, \ldots, w^{2k-2}_{k-1}, u_1 \rightsquigarrow w^{2k-2}_k, u_k \rightsquigarrow v_1.
\end{aligned}$$

It is easy to check that every new edge intersects a vertex from $e \cup f$. Thus, $\tilde{C} \setminus C$ contains no edge from any pair of edges belonging to X. Moreover, note that different choices of g_i yield different cycles $\tilde{C}$. Thus, $|\mathcal{S}_C| = \Omega(n^{2(k-1)})$.

It remains to show that for any two tight Hamilton cycles $C \neq C' \in \mathcal{C}_1$ we have $\mathcal{S}_C \cap \mathcal{S}_{C'} = \emptyset$. In order to prove it, one can reverse the above procedure and uniquely determine C and the edges $g_1, g_2, \ldots, g_{2(k-1)}$ from $\tilde{C}$. Note that we do not know the order in which the vertices of e and f are traversed by C.

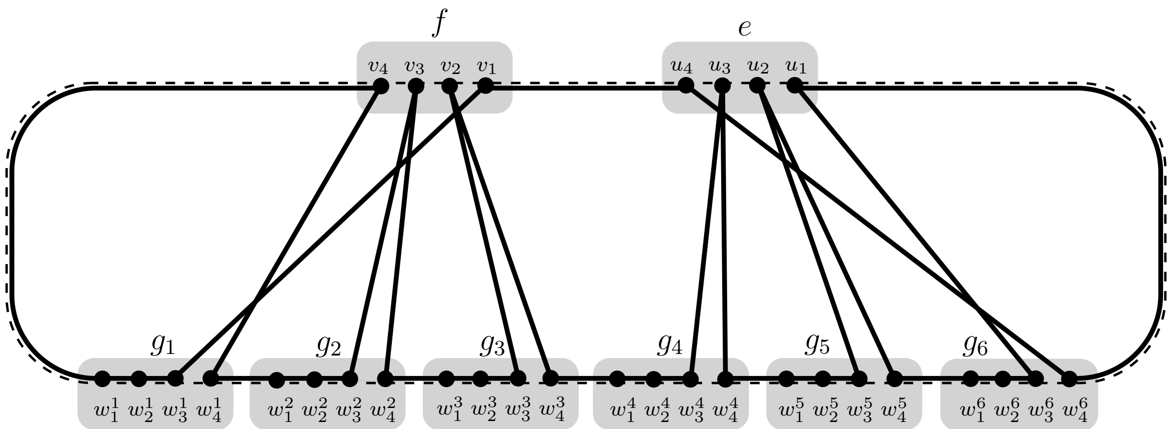

Figure 3: The second construction for $k = 4$. The dashed and solid lines denote C and $\tilde{C}$, respectively.

Note that there are exactly two shortest $e \rightsquigarrow f$ paths in $\tilde{C}$, say Q_1 and Q_2. One with u_k and v_1 as its endpoints and the second one with u_{k-1} and v_2 as the endpoints. Our goal is to determine vertex v_1. Once this is known then as in Construction 1 we can uncover all edges g_i and hence C itself.

We assume for a while that $v_1 = Q_1 \cap f$. Then we start at v_1 and follow $\tilde{C}$ in the direction opposite to the second endpoint of Q_1. Before we reach edge e we will intersect edge f exactly $k-1$ times. This way we pretend that we restore vertices $v_k, v_{k-1}, \ldots, v_2$ (in the order of appearance). Let $\tilde{d}(x, y)$ be the number of vertices on $\tilde{C}$ between vertices x and y. Note that $d(v_k, w_{k-1}^1) = \tilde{d}(v_1, v_k) - 1$ and $d(w_k^j, w_{k-1}^{j+1}) = \tilde{d}(v_{k-j+1}, v_{k-j}) - 2$ for $1 \leq j \leq k-2$. Now we check if (9) holds. If so then Q_1 is really the path with endpoints u_k and v_1; otherwise Q_2 is the one.

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