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# RAMANUJAN DUALS AND AUTOMORPHIC SPECTRUM

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ABSTRACT. We introduce the notion of the automorphic dual of a matrix algebraic group defined over $Q$. This is the part of the unitary dual that corresponds to arithmetic spectrum. Basic functorial properties of this set are derived and used both to deduce arithmetic vanishing theorems of "Ramanujan" type as well as to give a new construction of automorphic forms.

Let $G$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. In the arithmetic theory of automorphic forms the lattice $\Gamma = G(\mathbb{Z})$ and its congrunce subgroups

$$\Gamma(N) = \{\gamma \in G(\mathbb{Z}) \colon \gamma \equiv I(N)\}, \qquad N \in \mathbb{N}$$

play a central role. A basic problem is to understand the decomposition into irreducibles of the regular representation of $G(\mathbb{R})$ on $L^2(\Gamma(N)\backslash G(\mathbb{R}))$. In general this representation will not be a direct sum of irreducibles, and for our purposes of defining the spectrum, it is best to use the notions of weak containment and Fell topology on the unitary dual $\widehat{G}(\mathbb{R})$ of the Lie group $G(\mathbb{R})$. (See [D, 18.1].) For any closed subgroup $H$ of $G(\mathbb{R})$ we define the spectrum $\sigma(H\backslash G(R))$ to be the subset of $\widehat{G}(\mathbb{R})$ consisting of all $\pi \in \widehat{G}(\mathbb{R})$ that are weakly contained in $L^2(H\backslash G(\mathbb{R}))$. Furthermore, if $\widehat{G}^1(\mathbb{R})$ is the set of irreducible spherical representations, we set $\sigma^1(H\backslash G(\mathbb{R})) := \sigma(H\backslash G(\mathbb{R})) \cap \widehat{G}^1(\mathbb{R})$. When $H = \Gamma(N)$, $\sigma(\Gamma(N)\backslash G(\mathbb{R}))$ consists of all $\pi \in \widehat{G}(\mathbb{R})$ occurring as subrepresentations of $L^2(\Gamma(N)\backslash G(\mathbb{R}))$ as well as those $\pi$'s that are in wave packets of unitary Eisenstein series [La]. The latter occur only when $\Gamma\backslash G(\mathbb{R})$ is not compact. We now introduce the central object of this note.

**Definition.** The automorphic (resp. Ramanujan) dual of $G$ is defined by

$$\widehat{G}_{\text{Aut}} = \overline{\bigcup_{N=1}^{\infty} \sigma(\Gamma(N)\backslash G(\mathbb{R}))}, \tag{1}$$

$$\widehat{G}_{\text{Raman}} = \widehat{G}_{\text{Aut}} \cap \widehat{G}^1(\mathbb{R}).$$

Here closure is taken in the topological space $\hat{G}(\mathbb{R})$.

Thus, $\widehat{G}_{\text{Aut}}$ is the smallest closed set containing all the congruence spectrum. Here is an alternative description of $\widehat{G}_{\text{Raman}}$. Let $G(\mathbb{R}) = KAN$ be an Iwasawa decomposition of $G(\mathbb{R})$; then the theory of spherical functions identifies $\widehat{G}^1(\mathbb{R})$ with a subset of $\mathfrak{A}^*_{\mathbb{C}}/W$, where $\mathfrak{A} = \operatorname{Lie} A$, $W = \operatorname{Weyl}(G, A)$. Moreover, the Fell topology on $\widehat{G}^1(\mathbb{R})$ coincides with the topology of $\widehat{G}^1(\mathbb{R})$ viewed as a subset of $\mathfrak{A}^*_{\mathbb{C}}/W$. Let $\mathbb{D}$ be the ring of invariant differential operators on the associated symmetric space $X$. Then the duality theorem [GPS] shows that the spectrum

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of $\mathbb{D}$ in $L^2(\Gamma\backslash X)$, say $\mathrm{Sp}_\Gamma(\mathbb{D}) \subset \mathfrak{A}^*_{\mathbb{C}}/W$ is the image of $\sigma^1(\Gamma\backslash G(\mathbb{R}))$ in $\mathfrak{A}^*_{\mathbb{C}}/W$ under the above identification. In particular, $\widehat{G}_{\mathrm{Raman}}$ is identified with

$$\overline{\bigcup_{N=1}^{\infty} \mathrm{Sp}_{\Gamma(N)}(\mathbb{D})} \subset \mathfrak{A}^*_{\mathbb{C}}/W.$$

That there should be restrictions on $\widehat{G}_{\mathrm{Raman}}$ and $\widehat{G}_{\mathrm{Aut}}$ has its roots in the representation theoretic reinterpretation of the classical Ramanujan conjectures due to Satake [Sa]. Identifying the above sets may be viewed as the general Ramanujan conjectures. For example, Selberg's $1/4$-conjecture may be stated as follows: For $G = \mathrm{SL}_2$,

$$\widehat{G}_{\mathrm{Raman}} = \{\mathbf{1}\} \cup \widehat{G}^1(\mathbb{R})_{\mathrm{temp}}, \tag{2}$$

where, in general, $\widehat{G}(\mathbb{R})_{\mathrm{temp}} := \sigma(G(\mathbb{R}))$ is the set of tempered representations, and $\widehat{G}^1(\mathbb{R})_{\mathrm{temp}} = \widehat{G}(\mathbb{R})_{\mathrm{temp}} \cap \widehat{G}^1(\mathbb{R})$. (See [CHH] for equivalent definitions of temperedness.)

While the individual sets $\sigma(\Gamma(N)\backslash G(\mathbb{R}))$ are intractable, the set $\widehat{G}_{\mathrm{Aut}}$ (and $\widehat{G}_{\mathrm{Raman}}$) enjoy certain functorial properties.

**Theorem 1.** *Let $G$ be a connected semisimple linear algebraic group defined over $\mathbb{Q}$ and $H < G$ a $\mathbb{Q}$-subgroup*

(i) $\mathrm{Ind}_{H(\mathbb{R})}^{G(\mathbb{R})} \widehat{H}_{\mathrm{Aut}} \subset \widehat{G}_{\mathrm{Aut}}$.

(ii) *Assume that $H$ is semisimple; then*

$$\mathrm{Res}_{H(\mathbb{R})} \widehat{G}_{\mathrm{Aut}} \subset \widehat{H}_{\mathrm{Aut}}.$$

(iii) $\widehat{G}_{\mathrm{Aut}} \otimes \widehat{G}_{\mathrm{Aut}} \subset \widehat{G}_{\mathrm{Aut}}$.

A word about the meaning of these inclusions. Firstly, Ind denotes unitary induction and Res stands for restriction. By the inclusion, say in (i), we mean that if $\pi' \in \widehat{H}_{\mathrm{Aut}}$ and $\pi$ is weakly contained in $\mathrm{Ind}_{H(\mathbb{R})}^{G(\mathbb{R})} \pi'$ then $\pi \in \widehat{G}_{\mathrm{Aut}}$. (i) produces (after a local calculation) elements in $\widehat{G}_{\mathrm{Aut}}$ from ones in $\widehat{H}_{\mathrm{Aut}}$ and yields a new method for constructing automorphic representations. Observe also that if $\pi \in \widehat{G}(\mathbb{R})$ is an isolated point then $\pi \in \widehat{G}_{\mathrm{Aut}}$ implies that $\pi$ occurs as a subrepresentation in $L^2(\Gamma(N)\backslash G(\mathbb{R}))$ for some $N$. This fact will be used below to construct certain automorphic cohomological representations. (ii) allow one to transfer setwise upper bounds on $\widehat{H}_{\mathrm{Aut}}$ to $\widehat{G}_{\mathrm{Aut}}$ and for many $G$'s gives nontrivial approximations to the Ramanujan conjectures. (iii) exhibits a certain internal structure of the set $\widehat{G}_{\mathrm{Aut}}$. We illustrate the use of Theorem 1 with some examples.

**Example A.** If $H = \{e\}$ then (i) implies that

$$\widehat{G}_{\mathrm{Aut}} \supset \overline{\widehat{G(\mathbb{R})}_{\mathrm{temp}} \cup \{\mathbf{1}\}}. \tag{3}$$

When $G(\mathbb{Z})$ is cocompact this follows also from de George-Wallach [GW]. In comparison with (2) one might hope that (3) is an equality. However, using other $H$'s and (i) one finds typically that $\widehat{G}_{\mathrm{Aut}}$ contains nontrivial, nontempered spectrum. For $G = \mathrm{Sp}(4)$ the failure of the naive Ramanujan conjecture has been observed by Howe and Piatetski-Shapiro [HP-S] using theta liftings.

**Example B.** Let $k/\mathbb{Q}$ be a totally real field, $q$ a quadratic form over $k$ such that $q$ has signature $(n, 1)$ over $\mathbb{R}$, and all other conjugates are definite. Let $G = \mathrm{Res}_{k/\mathbb{Q}} \mathrm{SO}(q)$. Then $G(\mathbb{R})$ is of $\mathbb{R}$-rank one and the noncompact factor is $\mathrm{SO}(n, 1)$. We identify $\mathfrak{A}^*$ with $\mathbb{R}$ by sending $\rho$ to $(n-1)/2$. With this normalization $\widehat{G}(\mathbb{R})^1$ is identified with $i\mathbb{R} \cup [-\rho, \rho] \subset \mathbb{C}$ modulo$\{\pm 1\}$. [K]. We parametrize $\widehat{G}^1(\mathbb{R})$ by $s \in i\mathbb{R}^+ \cup [0, \rho]$ and denote the corresponding representation by $\pi_s$. Let $\varphi_0, \dots, \varphi_n$ be an orthogonal basis of $q$ such that $q(\varphi_i) > 0$, $0 \le i \le n-1$, and $q(\varphi_n) < 0$. Define $H = \mathrm{Res}_{k/\mathbb{Q}}\{g \in \mathrm{SO}(q)\colon\ g(\varphi_1) = \varphi_1\}$. Applying Theorem 1(i) to the trivial representation $\mathbf{1} \in \widehat{H}_{\mathrm{Aut}}$ we find that

$$\sigma(H(\mathbb{R})\backslash G(\mathbb{R})) \subset \widehat{G}_{\mathrm{Aut}}.$$

Now $\sigma^1(H(\mathbb{R})\backslash G(\mathbb{R}))$ has been computed ([F]), and we find

$$\widehat{G}_{\mathrm{Raman}} \supset \{\rho, \rho - 1, \rho - 2, \dots\} \cup i\mathbb{R}^+. \tag{4}$$

In particular, for $n \ge 4$ there are nontrivial nontempered spherical automorphic representations.

To find upper bounds on $\widehat{G}_{\mathrm{Raman}}$ one uses Theorem 1(ii) and

$$H = \mathrm{Res}_{k/\mathbb{Q}}\{g \in \mathrm{SO}(q)\colon\ g(\varphi_i) = \varphi_i,\ \ 1 \le i \le n-4\}.$$

Combining the Jacquet-Langlands correspondence [JL] with the Gelbart-Jacquet lift [GJ] one concludes that

$$\widehat{H}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup [0, \tfrac{1}{2}] \cup \{\mathbf{1}\}.$$

Applying (ii) it follows that

$$\widehat{G}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup [0, \rho - \tfrac{1}{2}] \cup \{\rho\}. \tag{5}$$

In the special case $k = \mathbb{Q}$, $n \ge 4$ this result has also been obtained by [EGM] and [LP-SS] using Poincaré series. Assuming the Ramanujan conjecture at $\infty$ for $\mathrm{GL}(2)$ one deduces

$$\widehat{G}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup [0, \rho - 1] \cup \{\rho\}. \tag{6}$$

(Compare with (4).) The natural conjecture arising from (4) and (6) is

$$\widehat{G}_{\mathrm{Raman}} = i\mathbb{R}^+ \cup \{\rho, \rho - 1, \dots\}.$$

This is apparently consistent with Arthur's conjectures [A].

**Example C.** Let $\mathbb{F}_{4(-20)}$ be the $\mathbb{R}$-rank one form of $\mathbb{F}_4$. Using a method of Borel [B], one may find $\mathbb{Q}$-groups $H < G$ such that $G(\mathbb{R})$, $H(\mathbb{R})$ both have rank one, the noncompact simple factors being $\mathbb{F}_{4(-20)}$ and $\mathrm{Spin}(8, 1)$ respectively. With notations similar to Example B, one may identify $\widehat{G}^1(\mathbb{R})$ with $i\mathbb{R}^+ \cup [0, 5] \cup \{11\}$, here $\rho = 11$. One may compute $\sigma^1(H(\mathbb{R})\backslash G(\mathbb{R}))$ and using Theorem 1(i) find that

$$\widehat{G}_{\mathrm{Raman}} \supset i\mathbb{R}^+ \cup \{3, 11\}.$$

**Example D.** Consider now $F_{4(4)}$, the split real form of $F_4$. The corresponding symmetric space has dimension 28. For any cocompact lattice $\Gamma \subset F_{4(4)}$ one knows from Vogan-Zuckerman [VZ] that the Betti numbers $\beta^m(\Gamma) = 0$ for $0 < m < 8$ or $20 < m < 28$, $m \ne 4$ or $24$, in these latter dimensions all the cohomology comes from parallel forms of the symmetric space. Nevertheless using Theorem 1(i) we have

**Theorem 2.** *For any cocompact lattice* $\Gamma$ *in* $F_{4(4)}$ *and* $N \geq 0$ *there exists* $\Gamma' \subseteq \Gamma$ *of finite index such that* $\beta^m(\Gamma')H \geq N$ *for* $m = 8, 20$.

The proof of Theorem 2 makes use of Matsushima's formula [BW] together with a recent result of Vogan ensuring that the unitary representation contributing to the above Betti numbers is isolated in the unitary dual of $F_{4(4)}$. The $\mathbb{Q}$-subgroup that we use in applying Theorem 1(i) has real points equal to Spin(5, 4) up to compact factors. By the well-known result of Oshima-Matsuki [MO], we conclude that the discrete series of the symmetric space $F_{4(4)}/\operatorname{Spin}(5, 4)$ contain a unitary representation with nonzero cohomology in degrees 8 and 20, which is isolated in the unitary dual. The fact that we have dealt with every lattice in $F_{4(4)}$ follows from Margulis's arithmeticity theorem [M], together with the classification of algebraic groups over number fields [T].

This method of constructing cohomology is rather general. If $\pi$ is isolated in $\widehat{G}(\mathbb{R})$ and is contained in the automorphic dual of $G$ then it occurs discretely in $L^2(\Gamma\backslash G(\mathbb{R}))$ for $\Gamma$ a congruence subgroup of deep enough level. David Vogan has recently obtained the necessary and sufficient conditions for a unitary representation with nonzero cohomology to be isolated, which implies that most of them do. Theorem 1 then allows us to obtain nonvanishing of cohomology in a large number of cases.

To end, we remark that these ideas extend in a natural way to $S$-arithmetic groups. The proof of Theorem 1(i) consists of approximating, in a suitable way, congruence subgroups of $H(\mathbb{Z})$ by congruence subgroups of $G(\mathbb{Z})$ and then applying criteria of weak containment. For the proofs of Theorem 1(ii), (iii) we refer the reader to [BS].

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