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RAMSEY'S THEOREM FOR n-PARAMETER SETS

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Dedicated to the memory of Jon Hal Folkman (1938–1969)

Abstract. Classes of objects called n-parameter sets are defined. A Ramsey theorem is proved to the effect that any partitioning into r classes of the k-parameter subsets of any sufficiently large n-parameter set must result in some l-parameter subset with all its k-parameter subsets in one class. Among the immediate corollaries are the lower dimensional cases of a Ramsey theorem for finite vector spaces (a conjecture of Rota), the theorem of van der Waerden on arithmetic progressions, a set theoretic generalization of a theorem of Schur, and Ramsey's Theorem itself.

1. **Introduction.** In 1930, F. P. Ramsey [10], [12] proved the following theorem:

THEOREM [RAMSEY]. *Let k, l, r be positive integers. Then there is a number $N=N(k, l, r)$, depending only on k, l and r, with the following property: If S is a set with at least N elements, and if all the subsets of S with k elements are divided into r classes in any way, then there is some subset of l elements with all of its subsets of k elements in a single class.*

Since this theorem appeared there has been interest in finding generalizations, applications and analogues of it. The work presented here was motivated by a conjecture made by Gian-Carlo Rota, a geometric analogue to Ramsey's Theorem, which can be stated as follows:

CONJECTURE [ROTA]. Let l, k, r be nonnegative integers, and F a field of q elements. Then there is a number $N=N(q, r, l, k)$ depending only on q, r, l and k with the following property: If V is a vector space over F of dimension at least N, and if all the k-dimensional subspaces of V are divided into r classes in any way, then there is some l-dimensional subspace with all of its k-dimensional subspaces in a single class.

This conjecture is obtained from the statement of Ramsey's Theorem essentially by replacing the notions of set and cardinality by those of vector space and dimension, respectively. If we replace the notion of vector space with that of affine space, then we obtain another conjecture. This conjecture is actually equivalent to Rota's conjecture [3], [11]. In this paper we prove another analogue to Ramsey's Theorem, in which we replace the notion of n-dimensional affine space by the

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notion of n-parameter set, which we define later. The n-parameter sets are similar to n-dimensional affine spaces in certain ways, and, in fact, by appropriate choice of certain variables we can obtain results for vector and affine spaces. In particular, the affine conjecture is shown to be true for the cases of $k=0$ and $k=1$, with any choice for l, r and q. This implies that Rota's conjecture is true for $k=1$ and $k=2$ [3], [11]. Some other interesting results which follow from the n-parameter set analogue are presented as corollaries to the main result.

All of these analogues to Ramsey's Theorem are just statements about some special kinds of subsets of certain sets and their inclusion relationships. Ramsey's Theorem itself can be thought of thus as a statement about the lattices of subsets of finite sets; Rota's conjecture refers to the lattices of subspaces of finite vector spaces; the affine analogue concerns the partially ordered sets of the subspaces of finite affine spaces. So also is the n-parameter set analogue a statement about partially ordered sets of special subsets of certain sets. We give here an informal description of n-parameter sets which may prove useful to the reader as motivation for the somewhat technical formal definition given in the next section.

Basically, just as n-dimensional affine space, as a set, consists of all q^n n-tuples of elements from $GF(q)$, so an n-parameter set essentially consists of all t^n n-tuples of elements of a set A with t elements, $A=\{a_1, \ldots, a_t\}$. Any 1-dimensional affine subspace of an affine n-space over $GF(q)$ consists of a set of q n-tuples which can be written in a column as

$$\begin{gathered}(x_{11}, \ldots, x_{1n}) \\ (x_{21}, \ldots, x_{2n}) \\ \vdots \\ (x_{q1}, \ldots, x_{qn})\end{gathered}$$

where for each i, $1 \leqq i \leqq n$, either $x_{1i}=x_{2i}=\cdots=x_{qi}$ or else $x_{1i}, \ldots, x_{qi}$ is a permutation of the elements $f_1, \ldots, f_q$ constituting $GF(q)$. The permutations obtainable in this way constitute a subset L of all the $q!$ possible permutations. In a similar way, then, we define a 1-parameter subset of A^n (the n-tuples of elements of A) as any set of t n-tuples which can be listed

$$\begin{gathered}(a_{11}, \ldots, a_{1n}) \\ \vdots \\ (a_{t1}, \ldots, a_{tn})\end{gathered}$$

such that for each i, $1 \leqq i \leqq n$, either $a_{1i}=\cdots=a_{ti} \in B \subseteq A$, or else $a_{1i}, \ldots, a_{ti}$ is one of a certain set L_H of permutations of $a_1, \ldots, a_t$ (the set of permutations considered is defined by a group H).

The general idea for k-parameter subsets can be illustrated by considering the case $k=2$. If A_2 is any 2-dimensional affine subspace of $GF(q)^n$, then $A_2=\{(x_1, \ldots, x_n)+\alpha(y_1, \ldots, y_n)+\beta(z_1, \ldots, z_n) : \alpha, \beta \in GF(q)\}$, where $x=(x_1, \ldots, x_n)$, $y=(y_1, \ldots, y_n)$, $z=(z_1, \ldots, z_n)$ are in $GF(q)^n$, $y, z \neq 0$, and addition and scalar multiplication are defined as usual. If it happens that $y_i z_i = 0$ for $i = 1, 2, \ldots, n$ (this is a relatively rare event), then we can partition the n coordinates into three disjoint sets: the coordinates i where $z_i = 0$ but $y_i \neq 0$, those where $z_i \neq 0$ but $y_i = 0$, and those where $z_i = y_i = 0$. Call these sets S_1, S_2 and S_0 respectively, and let

$$S_1 = \{i_1, \ldots, i_{n_1}\}, \qquad S_2 = \{j_1, \ldots, j_{n_2}\}, \qquad S_0 = \{k_1, \ldots, k_{n_0}\}.$$

If $v = (v_1, \ldots, v_n) \in A_2$, then there are only q possibilities for $(v_{i_1}, \ldots, v_{i_{n_1}})$, q possibilities for $(v_{j_1}, \ldots, v_{j_{n_2}})$, and one possibility for $(v_{k_1}, \ldots, v_{k_{n_0}})$. Hence A_2 can be formed precisely by listing the q values for each of S_1 and S_2 and the one from S_0 q times, and then selecting one from each of the lists in all q^2 possible ways:

$$\begin{array}{ccc} \overbrace{(x_{k_1}, \ldots, x_{k_{n_0}})}^{S_0} & \overbrace{(y_{1i_1}, \ldots, y_{1i_{n_1}})}^{S_1} & \overbrace{(z_{1j_1}, \ldots, z_{1j_{n_2}})}^{S_2} \\ \vdots & \vdots & \vdots \\ (x_{k_1}, \ldots, x_{k_{n_0}}) & (y_{qi_1}, \ldots, y_{qi_{n_1}}) & (z_{qj_1}, \ldots, z_{qj_{n_2}}) \end{array}$$

The possible columns $(y_{1i_1}, \ldots, y_{qi_1})$ and $(z_{1j_1}, \ldots, z_{qj_1})$ are just the same as the set L of permutations in the 1-dimensional case above.

2-parameter sets, then, are described in a similar way. For a set A and a subset L_H of the permutations of A, we form a 2-parameter subset of A^n as follows: First partition the set $\{1, \ldots, n\}$ into three disjoint subsets S_0, S_1, S_2, with S_1 and S_2 nonempty. Then write three lists

$$\begin{array}{ccc} \overbrace{(a, \ldots, b)}^{S_0} & \overbrace{(x, \ldots, x')}^{S_1} & \overbrace{(z, \ldots, z')}^{S_2} \\ \vdots & \vdots & \vdots \\ (a, \ldots, b) & (y, \ldots, y') & (w, \ldots, w') \end{array}$$

such that the columns under S_1 and S_2 are in L_H. Finally, all t^2 elements of the 2-parameter subset are obtained by taking one entry (row) from each list.

To get k-parameter subsets we do the same thing with partitions into $k+1$ subsets $S_0, \ldots, S_k$. For $k \geqq 2$, these correspond to special affine subspaces of $GF(q)^n$ but not to all of them. Thus the theorems we prove for n-parameter sets will not apply to all subspaces, as we would like, but only to some of them. For $k = 0$ and 1, however, we do prove results for all subspaces. These are considered later in the section on corollaries.

2. **Definition of k-parameter set.** In this section we formally define a k-parameter set. The reader might find it useful here to inspect the corollaries at the end of the paper. The examples of k-parameter sets there illustrate the definition.

Let $A = \{a_1, a_2, \ldots, a_t\}$ be a finite set with $t \geqq 2$. Let $H: A \to A$ be a permutation group acting on A. For $a \in A$, $\sigma \in H$, the action is denoted by $a \to a^\sigma$. Also, for $\sigma_1, \sigma_2 \in H$, $\sigma_1\sigma_2 \in H$ is defined by $a^{\sigma_1\sigma_2} = (a^{\sigma_1})^{\sigma_2}$ for all $a \in A$. For a nonempty subset $B \subseteq A$, let $\bar{B}=\{\bar{b} : b \in B\}$ be the set of constant maps of A into A given by $x^{\bar{b}}=b$ for $x \in A$, $\bar{b} \in \bar{B}$. A^t denotes the cartesian product $A \times A \times \cdots \times A$ (t factors), which is just $\{(x_1, \ldots, x_t) : x_i \in A, 1 \leqq i \leqq t\}$.

For $x=(x_1, \ldots, x_t) \in A^t$, $\sigma \in H$, we define an action of H: $A^t \to A^t$ by

$$x^\sigma = (x_1, \ldots, x_t)^\sigma = (x_1^\sigma, \ldots, x_t^\sigma) \in A^t.$$

Similarly $\bar{B}$ acts on A^t by

$$x^{\bar{b}} = (x_1, \ldots, x_t)^{\bar{b}} = (x_1^{\bar{b}}, \ldots, x_t^{\bar{b}}) = (b, \ldots, b) \in A^t$$

for $x \in A^t$, $\bar{b} \in \bar{B}$.

For fixed integers $n>0$ and $0 \leqq k \leqq n$, let $\Pi=\{S_0, S_1, \ldots, S_k\}$ be a partition of the set $I_n=\{1, 2, \ldots, n\}$ with $S_i \neq \varnothing$ for $1 \leqq i \leqq k$. $S_0=\varnothing$ is possible. Let f: $I_n \to H \cup \bar{B}$ be a mapping with the property:

$$\begin{aligned} &f(i) \in \bar{B} \quad \text{if } i \in S_0, \\ &f(i) \in H \quad \text{if } i \in I_n - S_0. \end{aligned}$$

The set $P(A, \bar{B}, H, \Pi, f, n, k)=P$ is defined by

$$P = \bigcup_{1 \leqq i_0, \ldots, i_k \leqq t} \{(x_1, \ldots, x_n) : x_j = a_{i_y}^{f(j)} \text{ if } j \in S_y\} \subseteq A^n.$$

Definition 1. A subset $P \subseteq A^n$ is said to be a *k-parameter set in A^n* if $P= P(A, \bar{B}, H, \Pi, f, n, k)$ for some meaningful choice of these variables.

Let us consider this definition in more detail. We can write Π symbolically as follows:

$$\underline{\quad S_0 \quad}\ \underline{\quad S_1 \quad}\ \cdots\ \underline{\quad S_k \quad}$$

We imagine that we have bunched together the elements in the blocks of the partition Π. With each $i \in I_n$ we associate an element $f(i) \in \bar{B} \cup H$. We can write this as

$$\frac{S_0}{[\bar{a} \cdots \bar{b}}\ \frac{S_1}{\pi_1 \cdots \delta_1}\ \cdots\ \frac{S_k}{\pi_k \cdots \delta_k]}$$

where $\bar{a}, \ldots, \bar{b} \in \bar{B}$, $\pi_1, \ldots, \delta_1, \ldots, \pi_k, \ldots, \delta_k \in H$. Define l_0 by

$$l_0 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} \in A^t.$$

We occasionally write elements of A^t as column vectors when this is useful for our purposes. The preceding is shorthand notation for

$$\frac{S_0}{[l_0^{\bar{a}} \cdots l_0^{\bar{b}}}\ \frac{S_1}{l_0^{\pi_1} \cdots l_0^{\delta_1}}\ \cdots\ \frac{S_k}{l_0^{\pi_k} \cdots l_0^{\delta_k}]},$$

which we can write as

$$\begin{array}{c}
\overbrace{\phantom{a_1^{\bar a} \cdots a_1^{\bar b}}}^{S_0} \;\; \overbrace{\phantom{a_1^{\pi_1} \cdots a_1^{\delta_1}}}^{S_1} \;\cdots\; \overbrace{\phantom{a_1^{\pi_k} \cdots a_1^{\delta_k}}}^{S_k} \\
\begin{bmatrix}
a_1^{\bar a} & \cdots & a_1^{\bar b} & a_1^{\pi_1} & \cdots & a_1^{\delta_1} & \cdots & a_1^{\pi_k} & \cdots & a_1^{\delta_k} \\
a_2^{\bar a} & \cdots & a_2^{\bar b} & a_2^{\pi_1} & \cdots & a_2^{\delta_1} & \cdots & a_2^{\pi_k} & \cdots & a_2^{\delta_k} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_t^{\bar a} & \cdots & a_t^{\bar b} & a_t^{\pi_1} & \cdots & a_t^{\delta_1} & \cdots & a_t^{\pi_k} & \cdots & a_t^{\delta_k}
\end{bmatrix}
\end{array}$$

which, of course, is just

$$\begin{array}{c}
\overbrace{}^{S_0} \;\; \overbrace{\phantom{a_1^{\pi_1} \cdots a_1^{\delta_1}}}^{S_1} \;\cdots\; \overbrace{\phantom{a_1^{\pi_k} \cdots a_1^{\delta_k}}}^{S_k} \\
\begin{bmatrix}
a & \cdots & b & a_1^{\pi_1} & \cdots & a_1^{\delta_1} & \cdots & a_1^{\pi_k} & \cdots & a_1^{\delta_k} \\
a & \cdots & b & a_2^{\pi_1} & \cdots & a_2^{\delta_1} & \cdots & a_2^{\pi_k} & \cdots & a_2^{\delta_k} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a & \cdots & b & a_t^{\pi_1} & \cdots & a_t^{\delta_1} & \cdots & a_t^{\pi_k} & \cdots & a_t^{\delta_k}
\end{bmatrix}
\end{array}.$$

Now consider an n-tuple $x=(x_1,\dots,x_n)\in A^n$ formed in the following way:

$$x = (\overbrace{a,\dots,b,}^{S_0}\ \overbrace{a_{i_1}^{\pi_1},\dots,a_{i_1}^{\delta_1},}^{S_1}\dots,\overbrace{a_{i_k}^{\pi_k},\dots,a_{i_k}^{\delta_k}}^{S_k}),$$

where $1\leqq i_1, i_2,\dots,i_k\leqq t$. In other words, for each i we select one of the rows in the array beneath S_i. Since each $\pi_i,\dots,\delta_i$ is a permutation on A, then $|P|=t^k$. It follows from the definition that P is a k-parameter set in A^n iff P can be generated by some expression of the form

$$(1)\qquad [\overbrace{\bar a \cdots \bar b}^{S_0}\ \overbrace{\pi_1\cdots\delta_1}^{S_1}\cdots\overbrace{\pi_k\cdots\delta_k}^{S_k}].$$

DEFINITION 2. If P_l is an l-parameter set in A^n, we say that P_k *is a k-parameter subset of* P_l if P_k is a k-parameter set in A^n and P_k is a subset of P_l (with the same $A, \bar B, H, n$).

We point out here that a set of t^k points of A^n may possibly have many representations of the form (1). It is a k-parameter set, however, iff there is at least one such representation.

For example, for *any* choice of $\sigma_1,\sigma_2,\dots,\sigma_n\in H$ the set denoted by

$$[\overbrace{\sigma_1}^{S_1}\ \overbrace{\sigma_2}^{S_2}\cdots\overbrace{\sigma_n}^{S_n}]$$

is just A^n, which is an n-parameter subset of itself.

Consider a k-parameter set P_k in A^n, say,

$$P_k = [\overbrace{\bar a \cdots \bar b}^{S_0}\ \overbrace{\pi_1\cdots\delta_1}^{S_1}\cdots\overbrace{\pi_k\cdots\delta_k}^{S_k}].$$

For a fixed i, $1 \leqq i \leqq k$, choose an element $\beta \in H$, and form the k-parameter set

$$P'_k = [\overbrace{\bar{a} \cdots \bar{b}}^{S_0}\ \overbrace{\pi_1 \cdots \delta_1}^{S_1} \cdots \overbrace{\beta\pi_i \cdots \beta\delta_i}^{S_i} \cdots \overbrace{\pi_k \cdots \delta_k}^{S_k}].$$

In other words, all the $f(j)$ for $j \in S_i$ have been replaced by $\beta f(j)$.

PROPOSITION 1. $P_k = P'_k$.

Proof. It is sufficient to show $P_k \supseteq P'_k$ since $|P_k| = |P'_k| = t^k$. Let $x \in P'_k$. Then

$$x = (\overbrace{a, \ldots, b}^{S_0},\ \overbrace{a_{j_1}^{\pi_1}, \ldots, a_{j_1}^{\delta_1}}^{S_1}, \ldots, \overbrace{a_{j_i}^{\beta\pi_i}, \ldots, a_{j_i}^{\beta\delta_i}}^{S_i}, \ldots, \overbrace{a_{j_k}^{\pi_k}, \ldots, a_{j_k}^{\delta_k}}^{S_k})$$

for some $1 \leqq j_1, j_2, \ldots, j_k \leqq t$. But $a_{j_i}^{\beta} = a_m$ for some m since $\beta \in H$. Also

$$\begin{aligned} a_{j_i}^{\beta\pi_i} &= (a_{j_i}^{\beta})^{\pi_i} = a_m^{\pi_i}, \\ &\vdots \\ a_{j_i}^{\beta\delta_i} &= (a_{j_i}^{\beta})^{\delta_i} = a_m^{\delta_i}. \end{aligned}$$

Hence,

$$x = (\overbrace{a, \ldots, b}^{S_0},\ \overbrace{a_{j_1}^{\pi_1}, \ldots, a_{j_1}^{\delta_1}}^{S_1}, \ldots, \overbrace{a_m^{\pi_i}, \ldots, a_m^{\delta_i}}^{S_i}, \ldots, \overbrace{a_{j_k}^{\pi_k}, \ldots, a_{j_k}^{\delta_k}}^{S_k}) \in P_k.$$

Therefore, $P'_k \subseteq P_k$ and the proof is complete.

If we premultiply by π_i^{-1} each $f(j)$ for $j \in S_i$, then P_k assumes the form

$$P_k = [\overbrace{\bar{a} \cdots \bar{b}}^{S_0}\ \overbrace{\pi_1 \cdots \delta_1}^{S_1} \cdots \overbrace{e \cdots \pi_i^{-1}\delta_i}^{S_i} \cdots \overbrace{\pi_k \cdots \delta_k}^{S_k}],$$

where e denotes the identity element of H. We may perform this premultiplication for each i, $1 \leqq i \leqq k$.

Further, assume that for each $i > 0$, the minimal element j of S_i has $f(j) = e$ and this entry is written as the leftmost entry under S_i. This brings P_k into the form

$$P_k = [\overbrace{\bar{a} \cdots \bar{b}}^{S_0}\ \overbrace{e \cdots \pi_1^{-1}\delta_1}^{S_1} \cdots \overbrace{e \cdots \pi_i^{-1}\delta_i}^{S_i} \cdots \overbrace{e \cdots \pi_k^{-1}\delta_k}^{S_k}].$$

This is a *canonical form* for k-parameter sets, in the sense of the following proposition.

PROPOSITION 2. *Let*

$$P_k = P(A, \bar{B}, H, \Pi, f, n, k) = [\overbrace{\bar{a} \cdots \bar{b}}^{S_0}\ \overbrace{e \cdots \gamma_1}^{S_1} \cdots \overbrace{e \cdots \gamma_k}^{S_k}],$$

$$P'_k = P(A, \bar{B}, H, \Pi', f', n, k) = [\overbrace{\overline{a'} \cdots \overline{b'}}^{S'_0}\ \overbrace{e \cdots \gamma'_1}^{S'_1} \cdots \overbrace{e \cdots \gamma'_k}^{S'_k}],$$

where these representations are in the form just described, and suppose $P_k = P'_k$. *Then* $\Pi = \Pi'$, *and* $f = f'$.

Proof. For all $x = (x_1, \ldots, x_n) \in P_k$, if $j \in S_0$ then x_j is *constant*, say c_j. Of course, the same is true for all $x' = (x'_1, \ldots, x'_n) \in P'_k$, i.e., $x'_j = c_j$. Thus, $S_0 = S'_0$ and $f(j) = f'(j)$ for all $j \in S_0 = S'_0$.

Now, suppose $j \in S_i \cap S'_{i'}$ and let $j' \notin S_i \cup S_0$. For $x = (x_1, \ldots, x_j, \ldots, x_{j'}, \ldots, x_n) \in P_k$, as x ranges over P_k, the pair $(x_j, x_{j'})$ ranges over all t^2 pairs (a_r, a_s), $a_r, a_s \in A$. Therefore j and j' must be in different blocks of the partition Π'. Thus, if j and j' are in different blocks of Π, then they must be in different blocks of Π'. By symmetry, this implies $\Pi = \Pi'$. By suitable relabelling, we get $S_i = S'_i$, $0 \leqq i \leqq k$.

For some fixed $i > 0$ consider $S_i = S'_i = \{j_1 < j_2 < \cdots < j_r\}$. By assumption, $f(j_1) = f'(j_1) = e$. If $j, j' \in S_i$ then for any $x = (x_1, \ldots, x_n) \in P_k$, *the value of* x_j *determines the value of* $x_{j'}$. For if $x_j = a$, then for exactly one q, $a_q^{f(j)} = a$,

$$(a_q^{f(j)})^{f(j)^{-1}} = a_q = a^{f(j)^{-1}}$$

and

$$x_{j'} = a_q^{f(j')} = (a^{f(j)^{-1}})^{f(j')} = a^{f(j)^{-1}f(j')}. \tag{2}$$

Since $S_i = S'_i$, if $j = j_1$, then by (2)

$$a^{f(j')} = x_{j'} = a^{(f'(j_1))^{-1}f'(j')} = a^{(e)^{-1}f'(j')} = a^{f'(j')}. \tag{3}$$

But as x ranges over all of P_k, $x_j\ (=a)$ ranges over all of A. (3) implies

$$a^{f'(j')} = a^{f(j')} \quad \text{for all } a \in A.$$

By the definition of a permutation group (in which any two elements with the same action are identified) we deduce $f'(j') = f(j')$. Finally, since j' was an arbitrary element of S_i, and $i > 0$ was arbitrary, then $f = f'$. This establishes the uniqueness of representation in canonical form up to labelling the blocks S_i of Π, $i > 0$, and completes the proof of Proposition 2.

3. ***k*-parameter subsets of an *l*-parameter set.** We describe here the structure of the k-parameter subsets of an l-parameter set. Let

$$P_l = [\overbrace{\bar{a} \cdots \bar{b}}^{S_0}\ \overbrace{e \cdots \gamma_1}^{S_1} \cdots \overbrace{e \cdots \gamma_l}^{S_l}] = P(A, \bar{B}, H, \Pi, f, n, l)$$

be an l-parameter set in A^n and let

$$P_k = [\overbrace{\overline{a'} \cdots \overline{b'}}^{S'_0}\ \overbrace{e \cdots \gamma'_1}^{S'_1} \cdots \overbrace{e \cdots \gamma'_k}^{S'_k}] = P(A, \bar{B}, H, \Pi', f', n, k)$$

be a k-parameter subset of P_l. (In general, when we write $P_k \subseteq P_l$, we mean that we are using the same A, $\bar{B}$, H and n.)

If $x=(x_1, \ldots, x_n) \in P_l$ and $j \in S_0$, then $x_j = b$ for some $b \in B$ (fixed for all $x \in P_l$). Thus $x'_j = b$ for all $x'=(x'_1, \ldots, x'_n) \in P_k \subseteq P_l$. Hence, $j \in S'_0$ and $S_0 \subseteq S'_0$.

Next, suppose $x'=(x'_1, \ldots, x'_n) \in P_k$ and $j' \in S'_0$, $j' \notin S_0$. Then $j' \in S_i$ for some $i>0$. Hence, for some $b \in B$, $x'_{j'} = b$ for all $x' \in P_k$. As stated in the proof of Proposition 2, this determines all the other values x'_m, $m \in S_i$. Thus, if $x'_{j'}$ is constant, then so is x'_m, and $S_i \subseteq S'_0$. We can write

$$P_l = [\cdots \quad \cdots \quad \overset{S_i}{\overline{e \cdots \gamma_i}} \quad \cdots],$$

and for some j

$$\begin{array}{lcccccccc}
 & \multicolumn{3}{c}{\overline{}^{\,S'_0}} & \overline{}^{\,S'_1} & & \overline{}^{\,S'_k} & & \\
P_k = [\cdots & \overset{S_i}{\overline{\overline{a_j^e} \cdots \overline{a_j^{\gamma_i}}}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots].
\end{array}$$

Note. Whenever nested boldface lines are used, it indicates that the subsets corresponding to the lower boldface lines are contained in the subsets corresponding to the boldface lines directly above, e.g., in the expression above, $S_i \subseteq S'_0$. In general, the uppermost level of boldface lines correspond to the blocks of the partition for the k-parameter subset.

PROPOSITION 3. *Suppose* $j_1 \in S_{i_1} \cap S'_{i_2}$ *and* $j_2 \in S_{i_1}$. *Then* $j_2 \in S'_{i_2}$.

Proof. Since j_1 and j_2 are in the same block of Π, then for any

$$x = (x_1, \ldots, x_{j_1}, \ldots, x_{j_2}, \ldots, x_n) \in P_l,$$

the value of x_{j_1} determines the value of x_{j_2}. But $P_k \subseteq P_l$, so this is true for all $x \in P_k$. Hence, j_1 and j_2 must be in the same block of Π', i.e., $j_2 \in S'_{i_2}$ as claimed.

We have shown that

$$S_{q_1} \cap S'_{q_2} \neq \varnothing \Rightarrow S_{q_1} \subseteq S'_{q_2}.$$

Thus, Π *is a refinement of* Π'.

Now, consider S_q, S_r for which $S_q, S_r \subseteq S'_j$, $q \neq r$. A typical point of P_k is

$$\begin{array}{lcccccc}
 & \overline{}^{\,S'_0} & \multicolumn{3}{c}{\overline{}^{\,S'_j}} & & \overline{}^{\,S'_k} \\
x' = (\ldots & \ldots & \overset{S_q}{\overline{\ldots, x_{j_1}, \ldots}} & \ldots & \overset{S_r}{\overline{\ldots, x_{j_2}, \ldots, x_{j_3}, \ldots}} & \ldots & \ldots)
\end{array}$$

where $j_1 \in S_q$, $j_2, j_3 \in S_r$. For x' in P_k, the value of x_{j_1} determines the value of x_{j_2}.

Of course, for any $x \in P_l$, the value of x_{j_2} determines the value of x_{j_3}. More precisely we have

$$P_l = [\cdots \quad \overbrace{e \cdots \gamma_q}^{S_q} \quad \cdots \quad \overbrace{e \cdots \gamma_r}^{S_r} \quad \cdots]$$

$$= \begin{array}{c} \begin{array}{ccccc} & \overbrace{\phantom{a_1^e \cdots a_1^{\gamma_q}}}^{S_q} & & \overbrace{\phantom{a_1^e \cdots a_1^{\gamma_r}}}^{S_r} & \end{array} \\ \begin{bmatrix} \cdots & a_1^e \cdots a_1^{\gamma_q} & \cdots & a_1^e \cdots a_1^{\gamma_r} & \cdots \\ & a_2^e \cdots a_2^{\gamma_q} & & a_2^e \cdots a_2^{\gamma_r} & \\ & \vdots \quad\;\; \vdots & & \vdots \quad\;\; \vdots & \\ & a_t^e \cdots a_t^{\gamma_q} & & a_t^e \cdots a_t^{\gamma_r} & \end{bmatrix} \end{array}$$

and, as x ranges over P_l,

$$x = (\ldots, \overbrace{a_u^e, \ldots, a_u^{\gamma_q}}^{S_q}, \ldots, \overbrace{a_v^e, \ldots, a_v^{\gamma_r}}^{S_r}, \ldots),$$

all t^2 possible choices of u and v will occur. On the other hand, in P_k, since any value under S_q determines the values under S_r, we must have

$$P_k = \begin{array}{c} \begin{array}{ccccc} & \multicolumn{3}{c}{\overbrace{\phantom{a_1^{\pi_q} \cdots a_1^{\delta_q} \cdots a_1^{\pi_r} \cdots a_1^{\delta_r}}}^{S_j'}} & \\ & \overbrace{\phantom{a_1^{\pi_q} \cdots a_1^{\delta_q}}}^{S_q} & & \overbrace{\phantom{a_1^{\pi_r} \cdots a_1^{\delta_r}}}^{S_r} & \end{array} \\ \begin{bmatrix} \cdots & a_1^{\pi_q} \cdots a_1^{\delta_q} & \cdots & a_1^{\pi_r} \cdots a_1^{\delta_r} & \cdots \\ & a_2^{\pi_q} \cdots a_2^{\delta_q} & & a_2^{\pi_r} \cdots a_2^{\delta_r} & \\ & \vdots \quad\;\; \vdots & & \vdots \quad\;\; \vdots & \\ & a_t^{\pi_q} \cdots a_t^{\delta_q} & & a_t^{\pi_r} \cdots a_t^{\delta_r} & \end{bmatrix} \end{array}$$

$$= [\cdots \quad \overbrace{\overbrace{\pi_q \cdots \delta_q}^{S_q} \quad \cdots \quad \overbrace{\pi_r \cdots \delta_r}^{S_r}}^{S_j'} \quad \cdots]$$

$$= [\cdots \quad \overbrace{\overbrace{e \cdots \pi_q^{-1}\delta_q}^{S_q} \quad \cdots \quad \overbrace{\pi_q^{-1}\pi_r \cdots \pi_q^{-1}\delta_r}^{S_r}}^{S_j'} \quad \cdots],$$

where we have premultiplied the entries under S_j' by π_q^{-1}. Since $P_k \subseteq P_l$ we must have $\pi_q^{-1}\delta_q = \gamma_q$ and $\pi_r^{-1}\delta_r = \gamma_r$. Hence, we can write P_k as

$$P_k = [\cdots \quad \overbrace{\overbrace{e \cdots \gamma_q}^{S_q} \quad \cdots \quad \overbrace{(\pi_q^{-1}\pi_r) \cdots (\pi_q^{-1}\pi_r\gamma_r)}^{S_r}}^{S_j'} \quad \cdots].$$

Thus, in forming P_k from P_l we are permitted to premultiply the entries $f(j)$, $j \in S_r$, by some arbitrary element of H as we form Π' from Π. Conversely, it is clear that this process of premultiplication and joining blocks of Π to form those of Π' always yields a k-parameter subset of P_l. We summarize this below.

Let $P = P(A, \bar{B}, H, \Pi, f, n, l)$ be an l-parameter set in A^n. The general k-parameter subset $P_k \subseteq P_l$ is formed as follows: Let Π' be a partition of which Π is a refinement, say, $\Pi' = \{S_0', S_1', \ldots, S_k'\}$ with $S_0 \subseteq S_0'$ and $S_i' \neq \varnothing$, $i > 0$. For each $S_i \subseteq S_0'$, $i > 0$, choose $\tau_i \in \bar{B}$; for each $S_i \nsubseteq S_0'$, choose $\tau_i \in H$. Define $f' \colon I_n \to H \cup \bar{B}$ by

$$f'(j) = \tau_i f(j), \qquad j \in S_i,\ i > 0,$$
$$f'(j) = f(j), \qquad j \in S_0.$$

Then $P_k = P(A, \bar{B}, H, \Pi', f', n, k)$ is a k-parameter set in A^n, $P_k \subseteq P_l$, and all k-parameter subsets of P_l can be obtained this way (though not necessarily in canonical form).

4. **Construction of *-sets.** We now give a new construction which will be essential in the remainder of the paper. What we do is replace A by the set of images $\{l_0^s : s \in \bar{A} \cup H\}$ and establish corresponding notation while retaining that of the preceding section. Define

$$L_{\bar{A}} = \{l_0^{\bar{a}} : a \in A\} = \{(a, \ldots, a) : a \in A\} \subseteq A^t,$$
$$L_{\bar{B}} = \{l_0^{\bar{b}} : b \in B\},$$
$$L_H = \{l_0^{\sigma} : \sigma \in H\},$$
$$L = L_{\bar{A}} \cup L_H = \{l_1, \ldots, l_u\} \subseteq A^t.$$

For $x = (x_1, \ldots, x_u) \in L^u$, $\sigma \in H$, we define an action of $H \colon L^u \to L^u$ by

$$x^{\sigma} = (x_1^{\sigma}, \ldots, x_u^{\sigma}).$$

Similarly, define $\bar{B} \colon L^u \to L^u$ by

$$x^{\bar{b}} = (x_1^{\bar{b}}, \ldots, x_u^{\bar{b}}).$$

For all l, $m \in L$, define the map $\bar{l} \colon L \to L$ by

$$m^{\bar{l}} = l.$$

This induces a map $\bar{l} \colon L^u \to L^u$ by

$$x^{\bar{l}} = (x_1^{\bar{l}}, \ldots, x_u^{\bar{l}}) = (l, \ldots, l) \in L^u.$$

Finally we make the following definitions:

$$l_0^* = (l_0^{\bar{a}_1}, \ldots, l_0^{\bar{a}_t}, l_0^{\sigma_1}, \ldots, l_0^{\sigma_h}) \in L^u,$$
$$C = L_H \cup L_{\bar{B}}, \qquad \bar{C} = \{\bar{c} : c \in L_H \cup L_{\bar{B}}\},$$
$$L_H^* = \{l_0^{*\sigma} : \sigma \in H\}, \qquad L_{\bar{C}}^* = \{l_0^{*\bar{c}} : \bar{c} \in \bar{C}\}.$$

As before, we have the notation of k-parameter sets in L^n. We note that the representation of H as a permutation group on L is faithful. For L^n we modify the notation slightly by writing a k-parameter set $P_k^* = P(L, \bar{C}, H, \Pi^*, g, n, k)$ as

$$[\overbrace{\overbrace{\overline{l_0^{\bar b}}\cdots\overline{l_0^{d}}}^{T_0^*}\;\overbrace{\overline{l_0^{\pi_0}}\cdots\overline{l_0^{\delta_0}}}^{V_0^*}}^{S_0^*}\;\overbrace{\pi_1\cdots\delta_1}^{S_1^*}\;\cdots\;\overbrace{\pi_k\cdots\delta_k}^{S_k^*}],$$

where $l_0^{\bar b}, \ldots, l_0^{d} \in L_B$ and $l_0^{\pi_0}, \ldots, l_0^{\delta_0} \in L_H$ (i.e., $\pi_0, \ldots, \delta_0 \in H$). Slightly expanded, this is

$$\begin{array}{cc} & \begin{array}{cccc} \overbrace{\phantom{l_0^{\bar b}\cdots l_0^{d}}}^{T_0^*} \overbrace{\phantom{l_0^{\pi_0}\cdots l_0^{\delta_0}}}^{V_0^*} \; S_0^* & S_1^* & & S_k^* \end{array} \\ (u \text{ rows}) & \begin{bmatrix} l_0^{\bar b}\cdots l_0^{d} & l_0^{\pi_0}\cdots l_0^{\delta_0} & (l_0^{\bar a_1})^{\pi_1}\cdots(l_0^{\bar a_1})^{\delta_1} & \cdots & (l_0^{\bar a_1})^{\pi_k}\cdots(l_0^{\bar a_1})^{\delta_k} \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots \\ l_0^{\bar b}\cdots l_0^{d} & l_0^{\pi_0}\cdots l_0^{\delta_0} & (l_0^{\bar a_t})^{\pi_1}\cdots(l_0^{\bar a_t})^{\delta_1} & \cdots & (l_0^{\bar a_t})^{\pi_k}\cdots(l_0^{\bar a_t})^{\delta_k} \\ l_0^{\bar b}\cdots l_0^{d} & l_0^{\pi_0}\cdots l_0^{\delta_0} & (l_0^{\sigma_1})^{\pi_1}\cdots(l_0^{\sigma_1})^{\delta_1} & \cdots & (l_0^{\sigma_1})^{\pi_k}\cdots(l_0^{\sigma_1})^{\delta_k} \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots \\ l_0^{\bar b}\cdots l_0^{d} & l_0^{\pi_0}\cdots l_0^{\delta_0} & (l_0^{\sigma_h})^{\pi_1}\cdots(l_0^{\sigma_h})^{\delta_1} & \cdots & (l_0^{\sigma_h})^{\pi_k}\cdots(l_0^{\sigma_h})^{\delta_k} \end{bmatrix}. \end{array}$$

5. **The map** M. We define a map $M: L^n \to 2^{A^n}$ as follows: For $x = (x_1, \ldots, x_n) \in L^n$, $x_i = (x_{i1}, \ldots, x_{it}) \in L \subseteq A^t$, $1 \leqq i \leqq n$, let

$$M(x) = \begin{Bmatrix} (x_{11}, x_{21}, \ldots, x_{n1}), \\ (x_{12}, x_{22}, \ldots, x_{n2}), \\ \vdots \quad\; \vdots \qquad\quad \vdots \\ (x_{1t}, x_{2t}, \ldots, x_{nt}) \end{Bmatrix} \subseteq A^n.$$

For $S \subseteq L^n$ we define $M(S)$ to be $\bigcup_{s \in S} M(s)$.

Suppose

$$P_k^* = [\overbrace{\overbrace{\overline{l_0^{\bar b}}\cdots\overline{l_0^{d}}}^{T_0^*}\;\overbrace{\overline{l_0^{\pi_0}}\cdots\overline{l_0^{\delta_0}}}^{V_0^*}}^{S_0^*}\;\overbrace{\pi_1\cdots\delta_1}^{S_1^*}\;\cdots\;\overbrace{\pi_k\cdots\delta_k}^{S_k^*}]$$

is a k-parameter set in L^n. Let us examine $M(P_k^*)$.

PROPOSITION 4. *If $V_0^* \neq \varnothing$, then $M(P_k^*)$ is the $(k+1)$-parameter set P_{k+1} in A^n given by*

$$P_{k+1} = [\overbrace{\bar b \cdots \bar d}^{S_0 = T_0^*}\;\overbrace{\pi_0 \cdots \delta_0}^{S_1 = V_0^*}\;\overbrace{\pi_1 \cdots \delta_1}^{S_2 = S_1^*}\;\cdots\;\overbrace{\pi_k \cdots \delta_k}^{S_{k+1} = S_k^*}]. \tag{4}$$

Proof. We first show $P_{k+1} \subseteq M(P_k^*)$.

Let $x \in P_{k+1}$. Then

$$x = (\overbrace{b \cdots d}^{T_0^*}\ \overbrace{a_{j_0}^{\pi_0} \cdots a_{j_0}^{\delta_0}}^{V_0^*}\ \overbrace{a_{j_1}^{\pi_1} \cdots a_{j_1}^{\delta_1}}^{S_1^*}\ \cdots\ \overbrace{a_{j_k}^{\pi_k} \cdots a_{j_k}^{\delta_k}}^{S_k^*}),$$

where we shall delete the commas between successive entries for notational convenience. In P_k^* we choose

$$x^* = (\overbrace{\overbrace{l_0^{\bar{b}} \cdots l_0^{\bar{d}}}^{T_0^*}\ \overbrace{l_0^{\pi_0} \cdots l_0^{\delta_0}}^{V_0^*}}^{S_0^*}\ \overbrace{(l_0^{\bar{a}_{j_1}})^{\pi_1} \cdots (l_0^{\bar{a}_{j_1}})^{\delta_1}}^{S_1^*}\ \cdots\ \overbrace{(l_0^{\bar{a}_{j_k}})^{\pi_k} \cdots (l_0^{\bar{a}_{j_k}})^{\delta_k}}^{S_k^*}).$$

Thus

$$\begin{array}{rl} & \begin{array}{ccccc} \overline{T_0^*} & \overline{V_0^*} & \overline{S_1^*} & & \overline{S_k^*} \end{array} \\ M(x^*) = & \left\{ \begin{array}{ccccc} (b \cdots d & a_1^{\pi_0} \cdots a_1^{\delta_0} & a_{j_1}^{\pi_1} \cdots a_{j_1}^{\delta_1} & \cdots & a_{j_k}^{\pi_k} \cdots a_{j_k}^{\delta_k}), \\ (b \cdots d & a_2^{\pi_0} \cdots a_2^{\delta_0} & a_{j_1}^{\pi_1} \cdots a_{j_1}^{\delta_1} & \cdots & a_{j_k}^{\pi_k} \cdots a_{j_k}^{\delta_k}), \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots \\ (b \cdots d & a_{j_0}^{\pi_0} \cdots a_{j_0}^{\delta_0} & a_{j_1}^{\pi_1} \cdots a_{j_1}^{\delta_1} & \cdots & a_{j_k}^{\pi_k} \cdots a_{j_k}^{\delta_k}), \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots \\ (b \cdots d & a_t^{\pi_0} \cdots a_t^{\delta_0} & a_{j_1}^{\pi_1} \cdots a_{j_1}^{\delta_1} & \cdots & a_{j_k}^{\pi_k} \cdots a_{j_k}^{\delta_k}) \end{array} \right\}. \end{array}$$

Therefore $x \in M(x^*)$ and $P_{k+1} \subseteq M(P_k^*)$.

We next show $P_{k+1} \supseteq M(P_k^*)$.

Let $y^* \in P_k^*$.

$$y^* = (\overbrace{\overbrace{l_0^{\bar{b}} \cdots l_0^{\bar{d}}}^{T_0^*}\ \overbrace{l_0^{\pi_0} \cdots l_0^{\delta_0}}^{V_0^*}}^{S_0^*}\ \cdots\ \overbrace{(l_0^{\bar{a}_{j_p}})^{\pi_p} \cdots (l_0^{\bar{a}_{j_p}})^{\delta_p}}^{S_p^*}\ \cdots\ \overbrace{(l_0^{\sigma})^{\pi_q} \cdots (l_0^{\sigma})^{\delta_q}}^{S_q^*}\ \cdots).$$

The entries under S_p^* and S_q^* represent the two possible forms which might occur. Thus,

$$\begin{array}{rl} & \begin{array}{ccccccc} \overline{T_0^*} & \overline{V_0^*} & & \overline{S_p^*} & & \overline{S_q^*} & \end{array} \\ M(y^*) = & \left\{ \begin{array}{ccccccc} (b \cdots d & a_1^{\pi_0} \cdots a_1^{\delta_0} & \cdots & a_{j_p}^{\pi_p} \cdots a_{j_p}^{\delta_p} & \cdots & a_1^{\sigma\pi_q} \cdots a_1^{\sigma\delta_q} & \cdots), \\ \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots & \\ (b \cdots d & a_j^{\pi_0} \cdots a_j^{\delta_0} & \cdots & a_{j_p}^{\pi_p} \cdots a_{j_p}^{\delta_p} & \cdots & a_j^{\sigma\pi_q} \cdots a_j^{\sigma\delta_q} & \cdots), \\ \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots & \\ (b \cdots d & a_t^{\pi_0} \cdots a_t^{\delta_0} & \cdots & a_{j_p}^{\pi_p} \cdots a_{j_p}^{\delta_p} & \cdots & a_t^{\sigma\pi_q} \cdots a_t^{\sigma\delta_q} & \cdots) \end{array} \right\}. \end{array}$$

A typical point of $M(y^*)$ is given by

$$y = (\overset{S_0 = T_0^*}{\overline{b \cdots d}}\ \overset{S_1 = V_0^*}{\overline{a_j^{\pi_0} \cdots a_j^{\delta_0}}}\ \cdots\ \overset{S_{p+1} = S_p^*}{\overline{a_{j_p}^{\pi_p} \cdots a_{j_p}^{\delta_p}}}\ \cdots\ \overset{S_{q+1} = S_q^*}{\overline{a_j^{\sigma\pi_q} \cdots a_j^{\sigma\delta_q}}}\ \cdots).$$

We see that $y \in P_{k+1}$ since the entries under each S_i are of the appropriate type. Therefore $M(P_k^*) \subseteq P_{k+1}$. This, together with the opposite inclusion establishes Proposition 4.

6. **The commutative diagram.** We come to the basic property of M. Suppose

$$P_l^* = [\overset{S_0^*}{\overline{\overset{T_0^*}{\overline{\bar{l}_0^{b} \cdots \bar{l}_0^{d}}}\ \overset{V_0^*}{\overline{\bar{l}_0^{\pi_0} \cdots \bar{l}_0^{\delta_0}}}}}\ \overset{S_1^*}{\overline{\pi_1 \cdots \delta_1}}\ \cdots\ \overset{S_l^*}{\overline{\pi_l \cdots \delta_l}}]$$

is an l-parameter set in L^n with $V_0^* \neq \varnothing$. Let

$$P_{l+1} = M(P_l^*) = [\overset{S_0 = T_0^*}{\overline{\bar{b} \cdots \bar{d}}}\ \overset{S_1 = V_0^*}{\overline{\pi_0 \cdots \delta_0}}\ \overset{S_2 = S_1^*}{\overline{\pi_1 \cdots \delta_1}}\ \cdots\ \overset{S_{l+1} = S_l^*}{\overline{\pi_l \cdots \delta_l}}]$$

denote the induced $(l+1)$-parameter set in A^n. Further, suppose P_{k+1} is a $(k+1)$-parameter subset of P_{l+1} in which T_0^* and V_0^* are not in the same block of the partition for P_{k+1}. We might write P_{k+1}, for example, as

$$P_{k+1} = [\overset{S'_0}{\overline{\overset{T_0^*}{\overline{\bar{b} \cdots \bar{d}}}\ \overset{S_1^*}{\overline{\overline{a_{j_1}^{\pi_1}} \cdots \overline{a_{j_1}^{\delta_1}}}} \cdots \overset{S_7^*}{\overline{\overline{a_{j_7}^{\pi_7}} \cdots \overline{a_{j_7}^{\delta_7}}}}}}\ \overset{S'_1}{\overline{\overset{V_0^*}{\overline{\pi_0 \cdots \delta_0}}\ \overset{S_3^*}{\overline{\sigma_3\pi_3 \cdots \sigma_3\delta_3}} \cdots \overset{S_{11}^*}{\overline{\sigma_{11}\pi_{11} \cdots \sigma_{11}\delta_{11}}}}}$$

$$\overset{S'_2}{\overline{\overset{S_9^*}{\overline{\sigma_9\pi_9 \cdots \sigma_9\delta_9}} \cdots \overset{S_{14}^*}{\overline{\sigma_{14}\pi_{14} \cdots \sigma_{14}\delta_{14}}}}} \cdots \overset{S'_{k+1}}{\overline{\overset{S_2^*}{\overline{\sigma_2\pi_2 \cdots \sigma_2\delta_2}} \cdots \overset{S_5^*}{\overline{\sigma_5\pi_5 \cdots \sigma_5\delta_5}}}}]$$

where we have adjusted the group elements in S'_1 by choosing the premultiplying factor of V_0^* to be the identity element e.

PROPOSITION 5. *There exists a k-parameter subset P_k^* of P_l^* such that the following diagram is commutative*:

$$\begin{tikzcd} P_k^* \arrow[r, phantom, "\subseteq"] \arrow[d, "M"'] & P_l^* \arrow[d, "M"] \\ P_{k+1} \arrow[r, phantom, "\subseteq"] & P_{l+1}. \end{tikzcd}$$

Proof. Our candidate for P_k^* is, of course,

$$P_k^* = [\overbrace{\overbrace{\overbrace{\overline{l_0^{b}}\cdots\overline{l_0^{d}}}^{T_0^*}\ \overbrace{\overline{\overline{l_0^{a_{j_1}^{\pi_1}}}}\cdots\overline{\overline{l_0^{a_{j_1}^{\delta_1}}}}}^{S_1^*}\cdots\overbrace{\overline{\overline{l_0^{a_{j_7}^{\pi_7}}}}\cdots\overline{\overline{l_0^{a_{j_7}^{\delta_7}}}}}^{S_7^*}}^{T_0'' = S_0'}\ \overbrace{\overbrace{\overline{l_0^{\pi_0}}\cdots\overline{l_0^{\delta_0}}}^{V_0^*}\ \overbrace{\overline{l_0^{\sigma_3\pi_3}}\cdots\overline{l_0^{\sigma_3\delta_3}}}^{S_3^*}\cdots\overbrace{\overline{l_0^{\sigma_{11}\pi_{11}}}\cdots\overline{l_0^{\sigma_{11}\delta_{11}}}}^{S_{11}^*}}^{V_0'' = S_1'}}^{S_0''}$$

$$\overbrace{\overbrace{\sigma_9\pi_9\cdots\sigma_9\delta_9}^{S_9^*}\cdots\overbrace{\sigma_{14}\pi_{14}\cdots\sigma_{14}\delta_{14}}^{S_{14}^*}}^{S_1'' = S_2'}\cdots\overbrace{\overbrace{\sigma_2\pi_2\cdots\sigma_2\delta_2}^{S_2^*}\cdots\overbrace{\sigma_5\pi_5\cdots\sigma_5\delta_5}^{S_5^*}}^{S_k'' = S_{k+1}'}].$$

We check:

(i) P_k^* is a k-parameter set in L^n since $S_2', \ldots, S_{k+1}'$ are all nonempty,

(ii) $M(P_k^*) = P_{k+1}$ is immediate by the construction of P_k^*,

(iii) $P_k^* \subseteq P_l^*$. This follows by inspection.

This completes the proof.

7. **The main result.** Before proceeding with the main result of the paper we make a remark on terminology.

DEFINITION 3. By an *r-coloring of a set X* we just mean a partition of X into r disjoint (possibly empty) classes.

Of course, the "r colors" correspond to the r classes into which X is partitioned. In general, we shall use this "chromatic" terminology in preference to that of partitions and classes.

THEOREM. *Given A, B, H and integers $k, r, t_1, \ldots, t_r$, there exists an $N = N(A, \bar{B}, H, k, r, t_1, \ldots, t_r)$ such that if $n \geqq N$ and $P_n = P(A, \bar{B}, H, \Pi, f, w, n)$ is any fixed n-parameter set in A^w, then for any r-coloring of the k-parameter subsets of P_n there exists an i, $1 \leqq i \leqq r$, such that there is some t_i-parameter subset of P_n with all its k-parameter subsets having color i.*

Proof. The proof will proceed basically by double induction on k and $t_1 + \cdots + t_r$. We defer the proof for $k = 0$ until later. For a fixed integer $k \geqq 0$ assume the theorem has been established for this k and all values of $r, t_1, \ldots, t_r$. We prove the theorem for $k+1$. Of course, the theorem is immediate for $r = 1$, and it is true vacuously for $t_1 + \cdots + t_r \leqq (k+1)r - 1$ (since in this case, for some i, $t_i < k+1$). Henceforth we assume that $r \geqq 2$, and $t_i \geqq k+1$, and furthermore that for some p the theorem holds when $t_1 + \cdots + t_r \leqq p$. We must now prove the theorem with $t_1 + \cdots + t_r = p+1$.

DEFINITION 4. Let $P_m = P(X, \bar{Y}, G, \Pi, f, w, m)$ be an m-parameter set in X^w, where Π is the partition $\{S_0, S_1, \ldots, S_m\}$. Then for $k \leqq m$ and $1 \leqq i \leqq m$, an *S_i-crossing* k-parameter subset of P_m is a k-parameter subset $P_k = P(X, \bar{Y}, G, \Pi', f', w, k)$ where the partition $\Pi' = \{S_0', S_1', \ldots, S_k'\}$, and $S_i \nsubseteq S_0'$.

We now prove two lemmas. The first says that for large enough m, we can extract from an $(m+1)$-parameter set an $(l+1)$-parameter set which is decomposed into disjoint "parallel hyperplanes," and such that the $(k+1)$-parameter subsets which "cut across" the hyperplanes (i.e., do not lie within any of them) all have the same color. The second lemma is the iteration of the first, and says that we can extract such a subset with many such decompositions (in different "directions") with monochromatic crossing subsets.

Let L, C and the map M be as before.

LEMMA 1. *Let* $P_{m+1}=P(A, \bar{B}, H, \Pi, f, w, m+1)$ *be an* $(m+1)$*-parameter set in* A^w *with partition* $\Pi=\{S_0, S_1, \ldots, S_{m+1}\}$. *Let* $l \geqq 0$ *be an integer. If* $m \geqq N(L, C, H, k, r, l, \ldots, l)$ (l *taken* r *times*), *which is meaningful by the induction hypothesis, then for any fixed* i, $1 \leqq i \leqq m+1$, *and for any* r*-coloring of the* $(k+1)$*-parameter subsets of* P_{m+1}, *there is an* S_i*-crossing* $(l+1)$*-parameter subset* P_{l+1} *of* P_{m+1} *such that for some* j, $1 \leqq j \leqq r$, *all the* S_i*-crossing* $(k+1)$*-parameter subsets of* P_{l+1} *have color* j.

Proof of Lemma 1. Let P_m^* denote the m-parameter set in L^w which has partition $\Pi^*=\{S_0^*=S_0 \cup S_i, S_1^*, \ldots, S_m^*\}$ and such that $M(P_m^*)=P_{m+1}$ (where $\{S_1^*, \ldots, S_m^*\}$ is some relabelling of $\{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_{m+1}\}$). We remark that if P_k^* is a k-parameter subset of P_m^*, then $M(P_k^*)$ is an S_i-crossing $(k+1)$-parameter subset of P_{m+1} by the definition of P_m^* and M.

The given r-coloring of the $(k+1)$-parameter subsets of P_{m+1} induces an r-coloring of the k-parameter subsets of P_m^* in the following way: P_k^* is given the same color as $P_{k+1}=M(P_k^*)$. By the remark above this is a well-defined r-coloring of the k-parameter subsets of P_m^*. By the choice of m, there exists an l-parameter subset $P_l^* \subseteq P_m^*$ such that all the k-parameter subsets of P_l^* have one color, say color j. But $P_{l+1}=M(P_l^*)$ is an S_i-crossing $(l+1)$-parameter subset of P_{m+1}. By Proposition 5, every S_i-crossing $(k+1)$-parameter subset of P_{l+1} is the image under M of a k-parameter subset of P_l^*. Thus, all these have color j and the lemma is proved.

At this point we find it convenient to assume that $B=A$. We proceed to prove the theorem for this case, and then, as a direct corollary (Lemma 3 below), we establish the general result. Thus, let $A=B$, and let C, H, L and M be as before.

Let $P_m^*=P(L, \bar{C}, H, \Pi^*, f, w, m)$ be an m-parameter set in L^w with partition $\Pi^*=\{T_0^* \cup V_0^*=S_0^*, S_1^*, \ldots, S_m^*\}$, $V_0^* \neq \varnothing$. Then $M(P_m^*)$ is an $(m+1)$-parameter set in A^w. Let P_{l+2} be a V_0^*-crossing $(l+2)$-parameter subset of $M(P_m^*)$,

$$P_{l+2} = [\overbrace{\bar{b} \cdots \bar{d}}^{S_0}\ \overbrace{\overbrace{\pi_0 \cdots \delta_0}^{V_0^*} \cdots \overbrace{\sigma_j\pi_j \cdots \sigma_j\delta_j}^{S_j^*}}^{S_1}\ \overbrace{\pi \cdots}^{S_2} \quad \cdots \quad \overbrace{\cdots \delta}^{S_{l+2}}].$$

Then P_{l+2} is the disjoint union of t $(l+1)$-parameter subsets P_{l+1}^i, $1 \leqq i \leqq t$, none of which are V_0^*-crossing subsets, defined by

$$\underline{\qquad\qquad S'_0 \qquad\qquad} \quad \underline{S'_1 = S_2} \quad \underline{S'_{l+2} = S_{l+2}}$$
$$\underline{S_0} \quad \underline{\qquad S_1 \qquad}$$
$$\underline{V_0^*} \quad \underline{S_j^*}$$
$$P^i_{l+1} = [\bar{b} \cdots \bar{d}\ \overline{a_i^{\pi_0}} \cdots \overline{a_i^{\delta_0}} \ \cdots \ \overline{a_i^{\sigma_j\pi_j}} \cdots \overline{a_i^{\sigma_j\delta_j}}\ \ \pi \cdots \quad \cdots \quad \cdots \delta].$$

DEFINITION 5. The P^i_{l+1} are called *V_0^*-translates* of each other in P_{l+2} (or just *translates* when no confusion arises).

REMARK 1. Let P_{l+2} be a V_0^*-crossing $(l+2)$-parameter subset of $M(P_m^*)$ with V_0^*-translates P^i_{l+1} as above, and let P_{k+2} be a V_0^*-crossing $(k+2)$-parameter subset of P_{l+2}. Then

$$\underline{S''_0} \quad \underline{\qquad\qquad S''_1 \qquad\qquad} \quad \underline{S''_2} \quad \underline{S''_{k+2}}$$
$$\underline{S_0} \quad \underline{\qquad S_1 \qquad} \quad \underline{S_q}$$
$$\underline{V_0^*} \quad \underline{S_j^*}$$
$$P_{k+2} = [\bar{b} \cdots \bar{d} \ \cdots \ \pi_0 \cdots \delta_0 \ \cdots \ \sigma_j\pi_j \cdots \sigma_j\delta_j \ \cdots \ \tau \cdots \eta \ \ \pi' \cdots \quad \cdots \quad \cdots \delta'].$$

P_{k+2} is the disjoint union of the t V_0^*-translates P^i_{k+1}, where

$$\underline{\qquad\qquad S'''_0 \qquad\qquad} \quad \underline{S'''_1 = S''_2} \quad \underline{S'''_{k+1} = S''_{k+1}}$$
$$\underline{S''_0} \quad \underline{\qquad\qquad S''_1 \qquad\qquad}$$
$$\underline{S_0} \quad \underline{\qquad S_1 \qquad} \quad \underline{S_q}$$
$$\underline{V_0^*} \quad \underline{S_j^*}$$
$$P^i_{k+1} = [\bar{b} \cdots \bar{d} \cdots \overline{a_i^{\pi_0}} \cdots \overline{a_i^{\delta_0}} \cdots \overline{a_i^{\sigma_j\pi_j}} \cdots \overline{a_i^{\sigma_j\delta_j}} \cdots \overline{a_i^{\tau}} \cdots \overline{a_i^{\eta}} \ \ \pi' \cdots \quad \cdots \quad \cdots \delta'].$$

We see that $P^i_{k+1} \subseteq P^i_{l+1} \cap P_{k+2}$ because $P^i_{k+1} \subseteq P^i_{l+1}$ and $P_{k+2} \subseteq P_{l+2}$. On the other hand, any point in $P^i_{l+1} \cap P_{k+2}$ must be in P^i_{k+1} as can be checked by verifying the inclusion properties of n-parameter sets. Thus, $P^i_{k+1} = P^i_{l+1} \cap P_{k+2}$.

REMARK 2. If P_{k+1} is any $(k+1)$-parameter subset of P^i_{l+1}, then there is some V_0^*-crossing $(k+2)$-parameter subset of P_{l+2} with $P_{k+2} = \bigcup_{j=1}^t P^j_{k+1}$, the P^j_{k+1} being V_0^*-translates, such that $P_{k+1} = P^i_{k+1}$. In particular, taking P^i_{l+1} to be as in Definition 5, $P_{k+1} \subseteq P^i_{l+1}$ must look like

$$\underline{\qquad\qquad\qquad S''_0 \qquad\qquad\qquad} \quad \underline{S''_1} \quad \underline{S''_{k+1}}$$
$$\underline{\qquad\qquad S'_0 \qquad\qquad} \quad \underline{S'_g}$$
$$\underline{S_0} \quad \underline{\qquad S_1 \qquad}$$
$$\underline{V_0^*} \quad \underline{S_j^*}$$
$$P_{k+1} = [\bar{b} \cdots \bar{d} \ \cdots \ \overline{a_i^{\pi_0}} \cdots \overline{a_i^{\delta_0}} \ \cdots \ \overline{a_i^{\sigma_j\pi_j}} \cdots \overline{a_i^{\sigma_j\delta_j}} \ \cdots \ \overline{a_{j_g}^{\tau}} \cdots \overline{a_{j_g}^{\eta}} \ \ \pi' \cdots \quad \cdots \quad \cdots \delta'].$$

Then we can take

$$P_{k+2} = [\overbrace{\overbrace{\bar{b} \cdots \bar{d}}^{S_0} \cdots \overbrace{\overline{a^{\tau}_{j_g}} \cdots \overline{a^{\eta}_{j_g}}}^{S'_g}}^{S_0''''} \; \overbrace{\overbrace{\pi_0 \cdots \delta_0}^{V_0^*} \cdots \overbrace{\sigma_j \pi_j \cdots \sigma_j \delta_j}^{S_j^*}}^{S_1'''' = S_1} \quad \overbrace{\pi' \cdots}^{S_2'''' = S_1''} \quad \cdots \quad \overbrace{\cdots \delta'}^{S_{k+2}'''' = S_{k+1}''}].$$

This choice of P_{k+2} is well defined. That is, $S_1''''=S_1$ is the smallest set we can choose from S_0'' to generate a $(k+2)$-parameter set which is V_0^*-crossing and is contained in P_{l+2} (since any such S_1'''' must contain S_1). We shall refer to this particular P_{k+2} as the *V_0^*-expansion of P_{k+1} in P_{l+2}*.

REMARK 3. It should be noted that if P_{k+1} is any $(k+1)$-parameter subset of P_{l+2}, then either P_{k+1} is a V_0^*-crossing $(k+1)$-parameter set or $P_{k+1} \subseteq P^i_{l+1}$ for some i. This follows from the way in which the $(k+1)$-parameter subsets of P_{l+2} must be formed.

DEFINITION 6. Let $A=B$, H be as above. Let P_{m+v} be an $(m+v)$-parameter subset of A^w with partition $\{S_0, S_1, \ldots, S_v, V_1, \ldots, V_m\}$. For each i, $1 \leqq i \leqq m$, P_{m+v} is the union of t disjoint $(m+v-1)$-parameter subsets $P^j_{(m+v-1),i}$, $1 \leqq j \leqq t$, which are V_i-translates of each other. Let P_{k+1} be a $(k+1)$-parameter subset of P_{m+v} which is V_i-crossing for at least one i. Let $l = m - \max\{i : P_{k+1} \text{ is } V_i\text{-crossing}\}$. Then we associate with P_{k+1} the $(l+1)$-tuple $(l; j_m, j_{m-1}, \ldots, j_{m-l+1})$, where for $m-l < i \leqq m$ we define j_i by $P_{k+1} \subseteq P^{j_i}_{m+v-1,i}$. (For $l=0$ we get merely (0).) We call this the *signature of P_{k+1} in P_{m+v} with respect to* $(V_1, V_2, \ldots, V_m)$. An r-coloring of the $(k+1)$-parameter subsets of P_{m+v} will be called a *$(V_1, V_2, \ldots, V_m)$-coloring* if the colors of all $(k+1)$-parameter subsets with the same signature are the same.

We next present an iterated form of Lemma 1. For arbitrary positive integers m and v, define the integers v_i, $1 \leqq i \leqq m$, as follows:

$$\begin{aligned} v_1 &= N(L, \bar{C}, H, k, r^{t^{m-1}}, v, \ldots, v), \\ v_2 &= N(L, \bar{C}, H, k, r^{t^{m-2}}, v_1+1, \ldots, v_1+1), \\ &\vdots \\ v_{i+1} &= N(L, \bar{C}, H, k, r^{t^{m-i-1}}, v_i+1, \ldots, v_i+1), \\ &\vdots \\ v_m &= N(L, \bar{C}, H, k, r^{t^0}, v_{m-1}+1, \ldots, v_{m-1}+1). \end{aligned}$$

LEMMA 2. *Let m and v be positive integers, let $A=B$. Let $P_x = P(A, \bar{B}, H, \Pi, f, w, x)$ be an x-parameter set in A^w with $x \geqq v_m$. Suppose the $(k+1)$-parameter subsets of P_x are r-colored. Then P_x contains an $(m+v)$-parameter subset P_{m+v}, with partition $\{S_0, S_1, \ldots, S_v, V_1, \ldots, V_m\}$, such that the r-coloring restricted to P_{m+v} is a $(V_m, V_{m-1}, \ldots, V_1)$-coloring.*

Proof. We remark first that if $m=1$, this lemma asserts that there is a P_{v+1} such that all of its V_1-crossing P_{k+1} have one color. This is just the conclusion of Lemma 1.

Assume, then, that Lemma 2 is true for $m-1$. We show that it is true for m. Let

$v'_{m-1}=v_m, \ldots, v'_1=v_2$, $v'=v_1+1$. Then, by induction, there is some $(m-1+v')$-parameter subset $P_{v_1+m}\subseteq P_x$, with partition $\{S'_0, S'_1, \ldots, S'_{v_1+1}, V'_1, \ldots, V'_{m-1}\}$, such that P_{v_1+m} is $(V'_{m-1}, \ldots, V'_1)$-colored.

Let $(P^j_{m+v_1-1,i})'$, $1\leqq j\leqq t$, be V'_i-translates in P_{v_1+m}. Let $P_{v_1+1}=\bigcap_{i=1}^{m-1}(P^1_{m+v_1-1,i})'$. This is a (v_1+1)-parameter subset of P_{v_1+m} with partition $\{S'_0\cup V'_1\cup\cdots\cup V'_{m-1}, S'_1, \ldots, S'_{v_1+1}\}$. Let P_{k+1} be a $(k+1)$-parameter subset of P_{v_1+1}, and let P_{k+m} be the V'_1-expansion of the V'_2-expansion of $\cdots$ of the V'_{m-1}-expansion of P_{k+1}. Then for each choice of $(j_1, \ldots, j_{m-1})$, $P_{k+m}\cap\bigcap_{i=1}^{m-1}(P^{j_i}_{m+v_1-1,i})'$ is a $(k+1)$-parameter subset. This $(k+1)$-parameter subset has some color. Thus, for each P_{k+1} in P_{v_1+1} there is a color associated with each of the t^{m-1} choices of the j_i's. Using this, we can recolor the $(k+1)$-parameter subsets of P_{v_1+1} by letting two of them have the same new color if and only if for each choice of the j_i's the associated (old) color is the same. This is an $r^{t^{m-1}}$-coloring of the $(k+1)$-parameter subsets of P_{v_1+1}.

By the choice of v_1, and by Lemma 1, there is some $(v+1)$-parameter subset $P_{v+1}\subseteq P_{v_1+1}$, with partition $\{S''_0, S_1, \ldots, S_v, V_1\}$, such that all V_1-crossing $(k+1)$-parameter subsets of P_{v+1} have the same new color. Let P_{m+v} be the V'_1-expansion of the V'_2-expansion of $\ldots$ of the V'_{m-1}-expansion of P_{v+1}. By iteration of Remark 2, every $(k+1)$-parameter subset of P_{m+v} which is not V'_i-crossing for any i, $1\leqq i\leqq m-1$, is in the V'_1-expansion of $\ldots$ of the V'_{m-1}-expansion of some $(k+1)$-parameter subset of P_{v+1}. By the definition of the new coloring, and the choice of P_{m+v}, any $(k+1)$-parameter subset of P_{m+v} which is V_1-crossing but not V'_i-crossing for any i, $1\leqq i\leqq m-1$, has its (old) color determined only by its corresponding j_i's (i.e., its signature with respect to $(V'_1, \ldots, V'_{m-1})$).

If $V_2=V'_1, \ldots, V_m=V'_{m-1}$, then this says that if P_{k+1} is a $(k+1)$-parameter subset of P_{m+v} which is V_1-crossing but not V_i-crossing for any $i>1$, then the (old) color of P_{k+1} is determined by its signature with respect to $(V_1, \ldots, V_m)$. On the other hand, since P_{v_1+m} is $(V'_1, \ldots, V'_{m-1})$-colored, any $P_{k+1}\subseteq P_{m+v}\subseteq P_{m+v_1}$, such that P_{k+1} is V_i-crossing for some $i>1$ (i.e., V'_{i-1}-crossing), has its (old) color determined only by its signature. Thus P_{v+m}, with partition $\{S_0, S_1, \ldots, S_v, V_1, V_2, \ldots, V_m\}$, is $(V_1, \ldots, V_m)$-colored, and the lemma is proved.

We are now ready to complete the proof of the induction step for the case of $B=A$.

Let $v=\max_{1\leqq i\leqq r} N(A, \bar{B}, H, k+1, r, t_1, \ldots, t_i-1, \ldots, t_r)$, $z=\binom{t^v}{t^{k+1}}$, $m=N(A, \bar{B}, H, 0, r^z, 1, 1, \ldots, 1)$, and let $v_1, v_2, \ldots, v_m$ be as previously defined. Then we assert that it is sufficient to choose $N(A, \bar{B}, H, k+1, r, t_1, \ldots, t_r)=v_m$.

To prove this, let $P_{v_m}\subseteq A^w$ be a v_m-parameter subset of A^w, and suppose all the $(k+1)$-parameter subsets of P_{v_m} are r-colored. By Lemma 2, there is an $(m+v)$-parameter subset of P_{v_m}, P_{m+v}

$$P_{m+v} = [\underbrace{\bar{b}\cdots\bar{c}}_{S_0}\ \underbrace{\pi_1\cdots\delta_1}_{S_1}\ \cdots\ \underbrace{\pi_v\cdots\delta_v}_{S_v}\ \underbrace{\tau_1\cdots\eta_1}_{V_1}\ \cdots\ \underbrace{\tau_m\cdots\eta_m}_{V_m}],$$

which is $(V_1, \ldots, V_m)$-colored.

We consider the v-parameter subsets of P_{m+v} defined by

$$P_v(i_1,\dots,i_m) = [\overbrace{\overbrace{\bar{b}\cdots\bar{c}}^{S_0}\ \overbrace{\overline{a_{i_1}^{\tau_1}}\cdots\overline{a_{i_1}^{\eta_1}}}^{V_1}\ \cdots\ \overbrace{\overline{a_{i_m}^{\tau_m}}\cdots\overline{a_{i_m}^{\eta_m}}}^{V_m}}^{S_0'}\ \overbrace{\pi_1\cdots\delta_1}^{S_1'=S_1}\ \cdots\ \overbrace{\pi_v\cdots\delta_v}^{S_v'=S_v}].$$

Let P_{k+1} and P'_{k+1} be $(k+1)$-parameter subsets with $P_{k+1}\subseteq P_v(i_1,\dots,i_m)$ and $P'_{k+1}\subseteq P_v(j_1,\dots,j_m)$. We say P_{k+1} and P'_{k+1} are *associated* with respect to $(V_1,\dots,V_m)$ if they are of the form

$$P_{k+1} = [\overbrace{\overbrace{\overbrace{\bar{b}\cdots\bar{c}}^{S_0}\ \overbrace{\overline{a_{i_1}^{\tau_1}}\cdots}^{V_1}\ \cdots\ \overbrace{\cdots\overline{a_{i_m}^{\eta_m}}}^{V_m}}^{S_0'}\ \overset{\cdots}{\cdots}}^{S_0''}\ \overbrace{\pi'\cdots}^{S_1''}\ \cdots\ \overbrace{\cdots\delta'}^{S_{k+1}''}]$$

and

$$P'_{k+1} = [\overbrace{\overbrace{\overbrace{\bar{b}\cdots\bar{c}}^{S_0}\ \overbrace{\overline{a_{j_1}^{\tau_1}}\cdots}^{V_1}\ \cdots\ \overbrace{\cdots\overline{a_{j_m}^{\eta_m}}}^{V_m}}^{S_0'}\ \overset{\cdots}{\cdots}}^{S_0''}\ \overbrace{\pi'\cdots}^{S_1''}\ \cdots\ \overbrace{\cdots\delta'}^{S_{k+1}''}]$$

(i.e., P'_{k+1} differs from P_{k+1} only in that $i_1,\dots,i_m$ have been replaced by $j_1,\dots,j_m$ respectively, and everything else is unchanged).

A v-parameter subset of P_{m+v} has some number of $(k+1)$-parameter subsets, which is at most $z=\binom{tv}{tk+1}$. The r-coloring of $(k+1)$-parameter subsets induces a coloring of the $P_v(i_1,\dots,i_m)$ (with at most r^z colors) as follows: two such sets $P_v(i_1,\dots,i_m)$ and $P_v(j_1,\dots,j_m)$ have the same color if and only if each pair of associated $(k+1)$-parameter subsets $P_{k+1}\subseteq P_v(i_1,\dots,i_m)$ and $P'_{k+1}\subseteq P_v(j_1,\dots,j_m)$ have the same color.

Now let P_m be the following m-parameter subset of P_{m+v}:

$$P_m = [\overbrace{\overbrace{\bar{b}\cdots\bar{c}}^{S_0}\ \overbrace{\overline{a_1^{\pi_1}}\cdots\overline{a_1^{\delta_1}}}^{S_1}\ \cdots\ \overbrace{\overline{a_1^{\pi_v}}\cdots\overline{a_1^{\delta_v}}}^{S_v}}^{S_0'''}\ \overbrace{\tau_1\cdots\eta_1}^{S_1'''=V_1}\ \cdots\ \overbrace{\tau_m\cdots\eta_m}^{S_m'''=V_m}].$$

Each of the t^m subsets $P_v(i_1,\dots,i_m)$ contains exactly one point of P_m, and this clearly exhausts the t^m points of P_m. Color the points of P_m according to the rule that $P_m\cap P_v(i_1,\dots,i_m)$ and $P_m\cap P_v(j_1,\dots,j_m)$ have the same color if and only if $P_v(i_1, \ldots, i_m)$ and $P_v(j_1, \ldots, i_m)$ have the same color. Thus, the 0-parameter sets of P_m are r^z-colored.

Now by the theorem for the case $k=0$, $t_1 = \cdots = t_r = 1$ (which we have not yet proved) and the choice of m, P_m contains a one-parameter set P_1,

$$P_1 = [\overbrace{\overbrace{\bar{b} \cdots \overline{a_1^{\delta_v}}}^{S_0'''}\ \overbrace{\overline{a_{i_l}^{\tau_l}} \cdots \overline{a_{i_l}^{\eta_l}}}^{V_l}\ \cdots}^{S_0''''}\ \overbrace{\overbrace{\sigma_j\tau_j \cdots \sigma_j\eta_j}^{V_j}\ \cdots}^{S_1''''}],$$

such that all of its 0-parameter subsets have the same color. Then by the construction of this coloring, the $(v+1)$-parameter subset P_{v+1},

$$P_{v+1} = [\overbrace{\overbrace{\bar{b} \cdots \bar{c}}^{S_0}\ \overbrace{\overline{a_{i_l}^{\tau_l}} \cdots \overline{a_{i_l}^{\eta_l}}}^{V_l}}^{S_0^{(5)}}\ \cdots\ \overbrace{\pi_1 \cdots \delta_1}^{S_1^{(5)} = S_1}\ \cdots\ \overbrace{\pi_v \cdots \delta_v}^{S_v^{(5)} = S_v}\ \overbrace{a_1^{\sigma_j\tau_j} \cdots a_1^{\sigma_j\eta_j}}^{S_{v+1}^{(5)} = S_1''''}],$$

has the property that all of the t $P_v(i_1, \ldots, i_m)$ contained in it have the same color. Let these be called $P_v^1, \ldots, P_v^t$. These are S_1''''-translates of each other.

By the definition of the coloring of the $P_v(i_1, \ldots, i_m)$, this means that if P_{k+1}^j is any $(k+1)$-parameter subset of P_v^j for some j, and if P_{k+2} is its S_1''''-expansion, then $P_{k+1}^i = P_{k+2} \cap P_v^i$ all have the same color, $1 \leqq i \leqq t$.

Since $P_{v+1} \subseteq P_{m+v}$, and all the S_1''''-crossing $(k+1)$-parameter subsets have the same signature with respect to $(V_1, \ldots, V_m)$, then all these $(k+1)$-parameter subsets have the same color, say color j. By choice of v, P_v^1 has either a t_1-parameter subset all of whose $(k+1)$-parameter subsets have color 1, or a t_2-parameter subset all of whose $(k+1)$-parameter subsets have color 2, or $\ldots$, or a $(t_j - 1)$-parameter subset all of whose $(k+1)$-parameter subsets have color j, or $\ldots$, or a t_r-parameter subset all of whose $(k+1)$-parameter subsets have color r.

Suppose $P_{t_j - 1}$ is a $(t_j - 1)$-parameter subset of P_v^1 with all its $(k+1)$-parameter subsets having color j. Let P_{t_j} be the S_1''''-expansion of $P_{t_j - 1}$. Then all the $(k+1)$-parameter subsets of $P_{t_j} \cap P_v^i$ have color j, $1 \leqq i \leqq t$. Since $P_{t_j} \subseteq P_{v+1}$, all the S_1''''-crossing $(k+1)$-parameter subsets also have color j. By Remark 3, this accounts for all $(k+1)$-parameter subsets of P_{t_j}. So P_{t_j} is a t_j-parameter subset of $P_{v+1} \subseteq P_{v+m}$ all of whose $(k+1)$-parameter subsets have color j. The alternative to this is the existence of a t_i-paramter subset of $P_v^1 \subseteq P_{v+1} \subseteq P_{m+v}$ all of whose $(k+1)$-parameter subsets have color i, $i \neq j$. This is precisely what we wished to obtain, and the induction step is completed for $B = A$.

LEMMA 3. *If the theorem is true for A, $B = A$, H and integers y, r, $t_1, \ldots, t_r$, then it is true for A, B, H, y, r, $t_1, \ldots, t_r$, where B is any nonempty subset of A.*

Proof. Let $\varnothing \neq B \subseteq A$. For each integer x we say that an x-parameter subset $P(A, \bar{A}, H, \Pi, f, n, x)$ is of type A, and that $P(A, \bar{B}, H, \Pi, f, n, x)$ is of type B. Let b be some fixed arbitrary element of B. Then if

$$P_x = [\overbrace{a \cdots d}^{S_0}\ \overbrace{\pi \cdots}^{S_1} \quad \cdots \quad \overbrace{\cdots \delta}^{S_x}]$$

is an x-parameter subset of type A, we can associate with it an x-parameter subset of type B, namely

$$P'_x = [\overbrace{b \cdots b}^{S_0}\ \overbrace{\pi \cdots}^{S_1} \quad \cdots \quad \overbrace{\cdots \delta}^{S_x}].$$

Now if all the y-parameter subsets of type B are r-colored, then this induces an r-coloring of the y-parameter subsets of type A by the rule that a subset P_y of type A gets the same color as P'_y, which is of type B.

If the theorem is true for subsets of type A, then for n sufficiently large we can find for some i a t_i-parameter subset (of type A), P_{t_i}, all of whose y-parameter subsets have color i. Then P'_{t_i} is a t_i-parameter subset of type B. All of its y-parameter subsets are of the form P'_y where P_y is a y-parameter subset of P_{t_i}. Thus P'_{t_i} is the desired subset. This proves the lemma. (See essentially the same argument in [11].)

With this lemma the induction step of the theorem is completed. The entire proof will be completed when we establish the case $k=0$. To do this some notation and a preliminary lemma are needed. We shall write elements $(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) \in A^n$ in the form $(a_{i_1}a_{i_2} \cdots a_{i_n})$, i.e., without commas. Further, we shall denote certain blocks of consecutive entries of an n-tuple by a single symbol, e.g., $(X_1 a_{j_1} X_2 a_{j_2} \cdots a_{j_s} X_{s+1})$, where each $X_k = x_{k1}x_{k2} \cdots x_{kn_k} \in A^{n_k}$ for some n_k (possibly $n_k=0$, in which case X_k is empty).

Lemma 4. *Let $A=\{a_1, \ldots, a_t\}$ be a finite set with $t \geqq 1$. Then for any positive integer r there exists an integer $N(r, t)$ such that if $n \geqq N(r, t)$ and the elements of A^n are r-colored, then we can find a set of t elements of A^n of the form*

$$X(i) = (X_1 a_i X_2 a_i \cdots a_i X_d), \qquad 1 \leqq i \leqq t,$$

where $d \geqq 2$ (i.e., the variable a_i occurs at least once in $X(i)$), all of which have the same color.

Proof. A proof of this result can be found in [5]. The proof we give is direct and more in the spirit of the preceding arguments. The proof proceeds by induction on t. The theorem holds for $t=1$ and any r by taking $N(r, 1)=1$. Assume that for some $t \geqq 2$ the lemma has been proved for all values of $|A| < t$. Let $A=\{a_1, \ldots, a_t\}$,

$A' = A - \{a_t\}$, and suppose the elements of A^n are r-colored where $n \geqq c_r + c_{r-1} + \cdots + c_1$ with

$$\begin{aligned} c_r &= N(r, t-1), \\ c_{r-1} &= N(r^{t^{c_r}}, t-1), \\ c_{r-2} &= N(r^{t^{c_r + c_{r-1}}}, t-1), \\ &\vdots \\ c_k &= N(r^{t^{c_r + \cdots + c_{k+1}}}, t-1), \\ &\vdots \\ c_1 &= N(r^{t^{c_r + \cdots + c_2}}, t-1). \end{aligned}$$

Write A^n as $A^{c_r + \cdots + c_2} \times A^{n-(c_r + \cdots + c_2)}$. The original r-coloring of A^n induces an $r^{t^{c_r + \cdots + c_2}}$-coloring of $A^{n-(c_r + \cdots + c_2)}$ as follows: For $x, y \in A^{n-(c_r + \cdots + c_2)}$, x and y have the same "new" color iff for each point $z \in A^{c_r + \cdots + c_2}$, $\{z\} \times \{x\}$ and $\{z\} \times \{y\}$ have the same original color. This in turn determines an $r^{t^{c_r + \cdots + c_2}}$-coloring of $(A')^{n-(c_r + \cdots + c_2)}$. Since

$$n - (c_r + \cdots + c_2) \geqq c_1 = N(r^{t^{c_r + \cdots + c_2}}, t-1)$$

then by the induction hypothesis there exist $t-1$ points of $(A')^{n-(c_r + \cdots + c_2)}$,

$$X_1(i) = (X_{11} a_i X_{12} a_i \cdots a_i X_{1d_1}), \qquad 1 \leqq i < t,$$

all of which have the same "new" color. By the definition of the "new" colors, for any choice of $Y \in A^{c_r + \cdots + c_2}$, all the $t-1$ points $Y \times X_1(i) \in A^n$, $1 \leqq i < t$, have the same original color.

Next, writing $A^{c_r + \cdots + c_2} \times \{X_1(1)\}$ as $A^{c_r + \cdots + c_3} \times A^{c_2} \times \{X_1(1)\}$, the original r-coloring of A^n induces an $r^{t^{c_3 + \cdots + c_r}}$-coloring of A^{c_2} as follows: For $x, y \in A^{c_2}$, x and y have the same "newer" color iff for each point $z \in A^{c_r + \cdots + c_3}$, $\{z\} \times \{x\} \times X_1(1)$ and $\{z\} \times \{y\} \times X_1(1)$ have the same original color. As before, this determines an $r^{t^{c_r + \cdots + c_3}}$-coloring of $(A')^{c_2} \subseteq A^{c_2}$. Since

$$c_2 = N(r^{t^{c_r + \cdots + c_3}}, t-1),$$

then, by the induction hypothesis, there exist $t-1$ points of $(A')^{c_2}$,

$$X_2(i) = (X_{21} a_i X_{22} a_i \cdots a_i X_{2d_2}), \qquad 1 \leqq i < t,$$

all of which have the same "newer" color. By the definition of the "newer" colors, for any choice of $Y \in A^{c_r + \cdots + c_3}$, all the $t-1$ points $Y \times X_2(i_2) \times X_1(1)$, $1 \leqq i_2 < t$, have the same original color. Hence, all the $(t-1)^2$ points $Y \times X_2(i_2) \times X_1(i_1)$, $1 \leqq i_1, i_2 < t$, have the same color.

In general, repeating this procedure, we obtain at the kth step

$$X_k(i) = (X_{k1} a_i X_{k2} a_i \cdots a_i X_{kd_k}), \qquad 1 \leqq i < t,$$

where $X_k(i) \in A^{c_k}$. For any choice of $Y \in A^{c_r+\cdots+c_{k+1}}$, all the $(t-1)^k$ points in A^n of the form

$$Y \times X_k(i_k) \times \cdots \times X_2(i_2) \times X_1(i_1), \qquad 1 \leqq i_1, i_2, \ldots, i_k < t,$$

have the same original color. Finally, taking $k=r$ (in which case Y is empty), we consider the t^r points of A^n,

$$X_r(j_r) \times \cdots \times X_2(j_2) \times X_1(j_1), \qquad 1 \leqq j_k \leqq t,\ 1 \leqq k \leqq r.$$

These have the property that for each u the original color of the point

$$X_r(j_r) \times \cdots \times X_{u+1}(j_{u+1}) \times X_u(i_u) \times \cdots \times X_1(i_1) \tag{5}$$

is *independent of the choice of* i_k *for* $1 \leqq i_k < t$. The set of $r+1$ points

$$X_u = X_r(t) \times \cdots \times X_{u+1}(t) \times X_u(1) \times \cdots \times X_1(1), \qquad 0 \leqq u \leqq r,$$

must contain a pair of points with the same color (by the pigeon-hole principle!), say X_h and $X_{h'}$, $h > h'$. Finally, consider the t points

$$X(i) = X_r(t) \times \cdots \times X_{h+1}(t) \times X_h(i) \times \cdots \times X_{h'+1}(i) \times X_{h'}(1) \times \cdots \times X_1(1), \qquad 1 \leqq i \leqq t.$$

For $1 \leqq i < t$, all the points $X(i)$ have the same color as that of X_h (by (5)). On the other hand, $X(t) = X_{h'}$ which by the choice of h' has the same color as that of X_h. Thus, all the points $X(i)$, $1 \leqq i \leqq t$, have the same color. We have shown that the lemma holds for the choice $N(r, t) = c_r + c_{r-1} + \cdots + c_1$. This completes the proof of the induction step and the lemma is proved.

We extend this special case to the complete statement of the theorem for $k=0$ in several steps, which follow.

Suppose now that $t \geqq 2$ and $l \geqq 1$. We can apply the preceding lemma to the set A^l instead of A in a straightforward manner to obtain the result that if $n \geqq lN(r, t^l)$ and the points of A^n are r-colored, then there exists a set of t^l points of the form

$$(X_1 a_{i_1} a_{i_2} \cdots a_{i_l} X_2 a_{i_1} a_{i_2} \cdots a_{i_l} \cdots a_{i_1} a_{i_2} \cdots a_{i_l} X_d) \in A^n,$$

$1 \leqq i_1, i_2, \ldots, i_l \leqq t$, all of which have the same color.

The reader will notice that this set of t^l points is nothing other than an l-parameter set $P_l = P(A, \bar{B}, H, \Pi, f, n, l)$ in A^n with $H = \{e\}$, $B = A$ (i.e., all constant maps are allowed) and Π and f appropriately defined. Further, the 0-parameter subsets of P_l are just the points of P_l, so that P_l has all its 0-parameter sets the same color.

We immediately extend the result to the case where B is not necessarily equal to A by invoking Lemma 3 with $y=0$.

Next, the extension to an arbitrary permutation group $H: A \to A$ (instead of $H=\{e\}$) is immediate since the choice of H does not affect the 0-parameter subsets of an l-parameter set (which always has just $|B|^l$ 0-parameter subsets).

Finally, we must consider the situation in which the initial n-parameter set A^n is replaced by a fixed arbitrary n-parameter set P_n in A^w (for some fixed w). This is immediate, however, since the obvious map from the points of P_n to the points of A^n induces a one-to-one map on their respective k-parameter subsets, for each k, and preserves inclusion both ways.

Thus, we have seen that if $n \geqq lN(r, t^l)$, and the 0-parameter subsets of an n-parameter set $P_n \subseteq A^w$ (for some fixed w) are r-colored, then there exists an l-parameter set P_l in P_n such that all the 0-parameter subsets of P_l have one color. This is just the statement of the case $k=0$, $t_1 = \cdots = t_r = l$, which, since l is arbitrary, clearly implies the theorem for $k=0$. With this fact, the proof of the theorem is completed.

8. **Consequences of the theorem.** In this section we present several corollaries to the theorem, the most well known of these being the theorems of van der Waerden (Corollary 8) and of Ramsey (Corollary 11). Other corollaries are new, in particular, the results for affine and vector spaces, which we present first.

COROLLARY 1. *Let l, r be positive integers, $F=GF(q)$ a finite field and $k=0$ or 1. Then there is an integer $N=N(q, r, l, k)$ depending only on q, r, l, and k, with the following property: If A is an affine space over F of dimension $n \geqq N$, and if all the k-dimensional affine subspaces of A are r-colored in any way, then there is some l-dimensional affine subspace of A with all of its k-dimensional affine subspaces having one color.*

Proof. We prove this by applying the theorem to the case in which $A=GF(q)=\{0, 1, a_3, \ldots, a_q\}$, $B=A$, $t_1=t_2=\cdots=t_r=l$, and

$$H = \{\sigma : \text{for some } a, b \in F, a \neq 0, \text{ and all } y \in F, \sigma: y \to ay+b\},$$

the affine group. All we need to show here is that all x-parameter subsets are x-dimensional affine subspaces of $A^n=F^n$, and that for $k=0$ or 1, *all* the k-dimensional affine subspaces are in fact k-parameter subsets. For once we know this, we can apply the theorem with $n \geqq N(A, \bar{B}, H, k, r, t_1, \ldots, t_r) = N(q, r, l, k)$ to deduce the desired result. Thus, if an l-parameter set has all its k-parameter sets one color, this is actually an l-dimensional affine subspace with all of its k-dimensional affine subspaces having one color, as required.

First, then, let

$$P_x = [\overbrace{\bar{a} \cdots \bar{b}}^{S_0} \; \overbrace{\pi_1 \cdots \delta_1}^{S_1} \; \cdots \; \overbrace{\pi_x \cdots \delta_x}^{S_x}].$$

Suppose that for all $y \in F$ we have

$$\begin{aligned} y^{\pi_1} &= c_1 y + a_1, \\ &\vdots \\ y^{\delta_1} &= d_1 y + b_1, \\ &\vdots \\ y^{\pi_x} &= c_x y + a_x, \\ &\vdots \\ y^{\delta_x} &= d_x y + b_x. \end{aligned}$$

Define $x+1$ vectors as follows:

$$v_0 = (\overbrace{a, \dots, b,}^{S_0}\ \overbrace{a_1, \dots, b_1,}^{S_1} \dots, \overbrace{a_x, \dots, b_x}^{S_x})$$

$$v_1 = (\overbrace{0, \dots, 0}^{S_0}\ \overbrace{c_1, \dots, d_1}^{S_1}\ \overbrace{0, \dots, 0}^{S_2}\ \cdots\ \overbrace{0, \dots, 0}^{S_x})$$

$$\vdots$$

$$v_x = (\overbrace{0, \dots, 0}^{S_0}\ \overbrace{0, \dots, 0}^{S_1}\ \cdots\ \overbrace{0, \dots, 0}^{S_{x-1}}\ \overbrace{c_x, \dots, d_x}^{S_x}).$$

Then

$$P_x = \{v_0 + \alpha_1 v_1 + \cdots + \alpha_x v_x : \alpha_1, \dots, \alpha_x \in F\},$$

an x-dimensional affine subspace of F^n.

Now any n-tuple, or point, of F^n is both a 0-dimensional affine subspace and a 0-parameter subset, since $B = A = F$ here. Thus all 0-dimensional affine subspaces are 0-parameter sets.

Finally, let A_1 be a 1-dimensional affine subspace of F^n. Then for some vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$, $A_1 = \{u + \alpha v : \alpha \in F\}$. Let $S_1 = \{i_1, \dots, i_g\} = \{i : v_i \neq 0\}$, and $S_0 = \{j_1, \dots, j_h\} = \{i : v_i = 0\}$. Then

$$A_1 = [\overbrace{u_{j_1} \cdots u_{j_h}}^{S_0}\ \overbrace{\pi_{i_1} \cdots \pi_{i_g}}^{S_1}]$$

where the maps π_i are defined, for $i \in S_1$, by $\pi_i : x \to v_i x + u_i$. Hence A_1 is a 1-parameter subset of F^n. Thus, all 1-dimensional affine subspaces are 1-parameter sets. This completes the proof of the corollary.

COROLLARY 2. *Let l, r be positive integers, $F = GF(q)$ a finite field and $k = 0$ or 1. Then there is a number $N' = N'(q, r, l, k)$, depending only on q, r, l, and k, with the following property: If V is an n-dimensional vector space over F with $n \geqq N'$, and if the k-dimensional vector subspaces of V are r-colored in any way, then there is an l-dimensional vector subspace of V with all of its k-dimensional vector subspaces having one color.*

Proof. We prove this by applying the theorem to the case where $A=F$, $B=\{0\}$, $t_1=t_2=\cdots=t_r=l$, and $H=\{\sigma\colon$ for some $a\neq 0$ in F, $\sigma=ay$ for all $y\in F\}$, the multiplicative group of F. Again, what we have to show is that any x-parameter set is an x-dimensional subspace, and that any 0- or 1-dimensional subspace is a 0- or 1-parameter set, respectively. As before, we can then apply the theorem with $n\geqq N(A,\bar{B},H,k,r,t_1,\ldots,t_r)=N'(q,r,l,k)$ to obtain the required result. Let P_x be an x-parameter set. Then

$$P_x = [\overbrace{0\,0\cdots 0}^{S_0}\ \overbrace{\pi_1\cdots\delta_1}^{S_1}\ \cdots\ \overbrace{\pi_x\cdots\delta_x}^{S_x}].$$

Suppose

$$\begin{aligned} y^{\pi_1} &= c_1 y,\\ &\vdots\\ y^{\delta_1} &= d_1 y,\\ &\vdots\\ y^{\pi_x} &= c_x y,\\ &\vdots\\ y^{\delta_x} &= d_x y. \end{aligned}$$

Let x vectors be defined by

$$v_1 = (\overbrace{0,\ldots,0,}^{S_0}\ \overbrace{c_1,\ldots,d_1,}^{S_1}\ \overbrace{0,\ldots,0,}^{S_2}\ldots,\overbrace{0,\ldots,0}^{S_x})$$

$$\vdots$$

$$v_x = (\overbrace{0,\ldots,0,}^{S_0}\ \overbrace{0,\ldots,0,}^{S_1}\ldots,\overbrace{0,\ldots,0,}^{S_{x-1}}\ \overbrace{c_x,\ldots,d_x}^{S_x}).$$

Then $P_x=\{\alpha_1v_1+\cdots+\alpha_xv_x : \alpha_1,\ldots,\alpha_x\in F\}$. So P_x is an x-dimensional vector subspace.

There is only one 0-dimensional subspace of V, namely $\{(0,0,\ldots,0)\}$, and this is a 0-parameter subset. If V_1 is a 1-dimensional subspace, then for some vector $(v_1,\ldots,v_n)$, $V_1=\{\alpha(v_1,\ldots,v_n) : \alpha\in F\}$. Let $S_1=\{i_1,\ldots,i_g\}=\{i : v_i\neq 0\}$, and $S_0=\{1,2,\ldots,n\}-S_1$. Then

$$V_1 = [\overbrace{00\cdots 0}^{S_0}\ \overbrace{\pi_{i_1}\cdots\pi_{i_g}}^{S_1}],$$

$\pi_i\colon x\to v_ix$ for all $x\in F$, and V_1 is a 1-parameter set. Thus all 1-dimensional subspaces are 1-parameter sets, and the corollary is proved.

Remark. The last corollary (Rota's conjecture for $k=0$, 1) is also true for $k=2$. This result is not a direct corollary of the theorem, but follows from Corollary 1 by an inductive argument which can be found in [3], [11]. That argument, in fact, shows that if the affine statement is true for some fixed k, and all q, r, l, then Rota's conjecture is true for $k+1$, and all q, r, l.

We have established the affine analogue to Ramsey's Theorem for the 0- and 1-dimensional cases ($k=0, 1$) in Corollary 1 above by choosing objects A, B and H appropriately and applying the theorem to the resulting n-parameter sets. These same choices, however, do not yield the corresponding higher dimensional cases ($k \geqq 2$) of the affine analogue. What we obtain instead is a theorem about some but not all of the affine subspaces of an affine space. We illustrate with an example.

Let A be the field of two elements, $B=A$, and H the affine group defined in the proof of Corollary 1. Then, as we observed in the proof of Corollary 1, the 0-parameter subsets of A^n are precisely the 0-dimensional affine subspaces of A^n, and the 1-parameter subsets of A^n are precisely the 1-dimensional affine subspaces of A^n. Furthermore, all the k-parameter subsets of A^n, even for $k \geqq 2$, are k-dimensional affine subspaces of A^n. The difficulty in extending the results arises from the fact that not all of the k-dimensional affine subspaces, $k \geqq 2$, are k-parameter subsets.

Consider, for example, the 2-dimensional affine subspace of A^n defined by $S=\{\alpha(1, 1, 0, \ldots, 0)+\beta(0, 1, 1, 0, \ldots, 0) : \alpha, \beta \in A\}$. This has four points in it: $(0, 1, 1, 0, \ldots, 0)$, $(1, 1, 0, 0, \ldots, 0)$, $(1, 0, 1, 0, \ldots, 0)$, $(0, 0, 0, 0, \ldots, 0)$. It is clear that there is no way to partition the coordinates so that these four points can be represented in the usual way as a 2-parameter subset.

The trouble in the 2-dimensional case illustrated by this example is common to all the higher dimensional cases over all fields. Namely, our concept of k-parameter set requires a partitioning of the coordinates of A^n into $k+1$ disjoint subsets, whereas a basis for a k-dimensional subspace need not arise from such a partition. This problem also arises in the projective analogue. The disjointness of the coordinates in the "parameters," S_i, was essential in the induction step of the proof of the theorem. Any overlapping of the S_i would require some sort of rule for combining the overlapping entries, which in turn would have to be consistent with a similar rule in the *-sets, where overlapping would also occur.

COROLLARY 3. *Given integers l and r, there exists an integer $N(l, r)$ such that if S is a finite set with $|S| \geqq N(l, r)$ and the subsets of S are r-colored, then there exist l disjoint nonempty subsets $S_1, \ldots, S_l$ of S such that all 2^l-1 unions $\bigcup_{j\in J} S_j$, $\varnothing \neq J \subseteq \{1, 2, \ldots, l\}$, have one color.*

Proof. In the theorem, let $A=\{0, 1\}$, $B=\{0\}$, $H=\{e\}$, $k=1$, $t_1=t_2=\cdots=t_r=l$ and $P_n=A^n$. Then we conclude that if $n \geqq N(A, \bar{B}, H, k, r, t_1, \ldots, t_r)=N(l, r)$, and if the 1-parameter sets in A^n are r-colored, then there exists an l-parameter set P_l in A^n all of whose 1-parameter subsets have one color. Let $S=\{1, 2, \ldots, n\}$, and with each nonempty subset $X \subseteq S$ associate an element $h(X)=(a_1, \ldots, a_n) \in A^n$ in the following way:

$$\begin{aligned} a_i &= 1 \quad \text{if } i \in X, \\ a_i &= 0 \quad \text{otherwise.} \end{aligned}$$

Note that not all the a_i are 0. However, with each nonzero point $(a_1, \ldots, a_n) \in A^n$ we can associate the 1-parameter set $\{(0, 0, \ldots, 0), (a_1, a_2, \ldots, a_n)\}$ in A^n. Hence, any r-coloring of the nonempty subsets of S induces a natural r-coloring of the 1-parameter subsets of A^n. Since $n \geqq N(l, r)$ then, as mentioned at the beginning of the proof, there exists an l-parameter set P_l in A^n all of whose 1-parameter sets have one color. Let $\Pi = \{S_0, S_1, \ldots, S_l\}$ be the partition of $\{1, 2, \ldots, n\}$ associated with P_l. The important fact to notice here is that not only is $h(S_i) \in P_l$ for any $i > 0$, but, in fact, by the definition of an l-parameter set, $h(X) \in P_l$ for any $X = \bigcup_{j\in J} S_j$, $\varnothing \neq J \subseteq \{1, 2, \ldots, l\}$. Thus, all $2^l - 1$ of the subsets $\bigcup_{j\in J} S_j$, $\varnothing \neq J \subseteq \{1, 2, \ldots, l\}$, correspond to the 1-parameter subsets of P_l which by the conclusion of the theorem all have one color. Finally, since the color of any 1-parameter set in A^n was just that of its associated subset of S, then all the subsets $\bigcup_{j\in J} S_j$, $\varnothing \neq J \subseteq \{1, 2, \ldots, l\}$, have the same color. This proves the corollary.

COROLLARY 4 (J. FOLKMAN [1], R. RADO [9], J. SANDERS [13]). *Given integers l and r, there exists an integer $N'(l, r)$ such that if $n \geqq N'(l, r)$ and the positive integers $\leqq n$ are r-colored then there exist l integers $a_1, \ldots, a_l$ such that all the sums $\{\sum_{i=1}^{l} \varepsilon_i a_i : \varepsilon_i = 0$ or 1, not all $\varepsilon_i = 0\}$ have one color.*

Proof. Let h map the binary n-tuples $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$ into the integers by $h(x) = \sum_{i=1}^{n} x_i 2^{i-1}$. A direct application of Corollary 3 with $n \geqq N'(l, r) = 2^{N(l,r)}$ shows that for any r-coloring of the binary n-tuples (i.e., integers $< 2^n$) we can find l binary n-tuples (i.e., l integers) $x^{(1)}, \ldots, x^{(l)}$ such that $x_k^{(i)} \cdot x_k^{(j)} = 0$ for all i, j and k (i.e., the powers of 2 used in the dyadic expansions of $h(x^{(1)}), \ldots, h(x^{(l)})$ are all distinct) and all $2^l - 1$ componentwise sums $\{\sum_{j\in J} x^{(j)} : \varnothing \neq J \subseteq \{1, 2, \ldots, l\}\}$ (i.e., all $2^l - 1$ sums $\{\sum_{i=1}^{l} \varepsilon_i h(x^{(i)}) : \varepsilon_i = 0$ or 1, not all $\varepsilon_i = 0\}$) have the same color. This proves the corollary.

The case $l = 2$ of Corollary 4 was first proved by Schur [14]. Corollary 4 is actually a special case of Corollary 6 below.

COROLLARY 5. *Given integers l, r, there exists an integer $N''(l, r)$ such that if G is any group with $|G| \geqq N''(l, r)$, and if the elements of G are r-colored, then there exist l elements $a_1, \ldots, a_l$ in G such that all the products $a_{i_1} a_{i_2} \cdots a_{i_j}$ have one color for all $j \geqq 1$ and all choices of distinct $i_1, \ldots, i_j$ in $\{1, 2, \ldots, l\}$.*

Proof. For each finite group G let $A(G)$ be the size of the largest abelian subgroup of G. Let $m(n) = \min_{|G|=n} A(G)$. Then it is known [6] that $m(n) \to \infty$ as $n \to \infty$. That is, every large group has a large abelian subgroup. Thus it is sufficient to establish Corollary 5 for abelian groups.

Let A be an abelian group of order at least $(N'(l, r) - 1)^{(N(l,r)-1)} + 1$, where $N(l, r)$ is the number guaranteed in Corollary 3 above, and $N'(l, r)$ is from Corollary 4. Let the elements of A be r-colored. Since A is the product of cyclic groups, say $A = Z_{i_1} \times \cdots \times Z_{i_M}$, where i_j is the order of Z_{i_j}, then either $M \geqq N(l, r)$ or $i_j \geqq N'(l, r)$ for some i_j. In this latter case we can apply Corollary 4 to the cyclic group Z_{i_j} of order i_j and obtain l elements $a_1, \ldots, a_l$, satisfying the conclusion of Corollary 5.

On the other hand, suppose $M \geqq N(l, r)$. Let $g_1, \ldots, g_M$ be the generators respectively of the cyclic subgroups $Z_{i_1}, \ldots, Z_{i_M}$. We associate with each subset of $\{g_i : 1 \leqq i \leqq M\}$ the color of the product of its members. By Corollary 3 there must be l disjoint subsets whose unions all have the same color. This means that there are l products h_j, $1 \leqq j \leqq l$, of the g_i, no two with a common factor, such that all the products $h_{j_1} \cdots h_{j_k}$, for $1 \leqq k \leqq l$ and for any choice of the $j_1, \ldots, j_k$, have the same color. This completes the proof of Corollary 5.

It is interesting to note that the corresponding result for finite semigroups is false. For consider the semigroup S with n elements, including 0, such that $ab=0$ for all $a, b \in S$. Then if we color 0 one color and all the other elements of S another color, we clearly cannot find even two elements a, b such that a, b and ab are all the same color.

COROLLARY 6. *Let $\mathscr{L}=L_i(x_1, \ldots, x_m)$, $1 \leqq i \leqq h$, be a system of homogeneous linear equations with real coefficients with the property that for each j, $1 \leqq j \leqq m$, there exists a solution $(\varepsilon_1, \ldots, \varepsilon_m)$ to the system $\mathscr{L}$ with $\varepsilon_i=0$ or 1 and $\varepsilon_j=1$. Then given an integer r there exists an integer $N(r)$ such that if $n \geqq N(r)$ and the positive integers $<n$ are r-colored, then $\mathscr{L}$ can be solved with integers having one color.*

Proof. Let $E_i=(\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{im})$, $1 \leqq i \leqq m$, be solutions to the system $\mathscr{L}$ with $\varepsilon_{ij}=0$ or 1 and $\varepsilon_{ii}=1$. As in Corollary 4 we choose $N(r)=2^{N(m,r)}$. For $n \geqq N(r)$ any r-coloring of the positive integers $<n$ induces a coloring of the (nonzero) binary $N(m, r)$-tuples of $\{0, 1\}^{N(m,r)}$, which, by the arguments of the preceding corollaries and the choice of n, implies that there exists an m-parameter set P_m with $A=\{0, 1\}$, $B=\{0\}$, $H=\{e\}$ and such that *all* 2^m-1 *nonzero points of* P_m *have one color*. Thus, the points c_i given by

$$c_i = (\overbrace{0, \ldots, 0,}^{S_0}\ \overbrace{\varepsilon_{1i}, \ldots, \varepsilon_{1i},}^{S_1} \ldots, \overbrace{\varepsilon_{ji}, \ldots, \varepsilon_{ji},}^{S_j} \ldots, \overbrace{\varepsilon_{mi}, \ldots, \varepsilon_{mi}}^{S_m})$$

for $1 \leqq i \leqq m$, all have the same color. As before, reinterpreting these n-tuples as integers written to the base 2, the hypothesis that $\mathscr{L}$ is homogeneous and linear together with the definition of the ε_{ij} show that $(c_1, \ldots, c_m)$ is a monochromatic solution of $\mathscr{L}$. This proves the corollary.

Corollary 6 is similar to the important results of R. Rado [8].

By a *multigrade of order* m we mean two disjoint sets of integers $\{c_i : 1 \leqq i \leqq n\}$, $\{d_i : 1 \leqq i \leqq n\}$ such that

$$\sum_{i=1}^{n} c_i^k = \sum_{i=1}^{n} d_i^k \quad \text{for } k = 1, 2, \ldots, m.$$

This is denoted by

$$c_1, \ldots, c_n \overset{m}{=} d_1, \ldots, d_n.$$

Since $\{ac_i+b : 1 \leqq i \leqq n\}$, $\{ad_i+b : 1 \leqq i \leqq n\}$ is a multigrade of order m if $\{c_i : 1 \leqq i \leqq n\}$, $\{d_i : 1 \leqq i \leqq n\}$ is, then a straightforward application of the theorem along the lines used in the preceding corollaries yields

COROLLARY 7. *If the multigrade equations*

$$(*) \qquad x_1, \ldots, x_n \overset{m}{=} y_1, \ldots, y_n$$

have any integer solution (which always happens, for example, if $n \geqq 2^{m-1}$), then for any r-coloring of the positive integers, $(*)$ always has a solution in integers having one color.

COROLLARY 8 (VAN DER WAERDEN [6], [14]). *Given integers t and r, there exists an integer $M(t, r)$ such that if $n \geqq M(t, r)$ and the nonnegative integers $<n$ are arbitrarily r-colored, then there must exist a monochromatic arithmetic progression of length t.*

Proof. We apply the theorem to the case $A=\{0, 1, \ldots, t-1\}$, $B=A$, $H=\{e\}$, $k=0$, $t_1=\cdots=t_r=1$ and $P_n=A^n$. Let $N=N(A, \bar{B}, H, k, r, t_1, \ldots, t_r)$, let $M(t, r)=t^N$ and choose $n \geqq M(t, r)$. By writing any integer j, $0 \leqq j < M(t, r)$ in the form $j=\sum_{i=0}^{n-1} c_{ji}t^i$, $0 \leqq c_{ji} < t$ (i.e., to the base t), we have a one-to-one correspondence between the integers j, $0 \leqq j < M(t, r)$, and elements of A^N given by $j \leftrightarrow (c_{j0}, \ldots, c_{j\,n-1})$. Hence, an r-coloring of the integers $\{0, 1, \ldots, n-1\}$ induces an r-coloring of the elements of A^N (where we ignore the integers $\geqq M(t, r)$). Since all these elements of A^N are 0-parameter sets of A^N then by the choice of N, the theorem guarantees the existence of a 1-parameter set

$$P_1 = [\overbrace{\bar{a}\cdots\bar{b}}^{S_0}\ \overbrace{\pi_1\cdots\delta_1}^{S_1}] = [\overbrace{\bar{a}\cdots\bar{b}}^{S_0}\ \overbrace{e\cdots e}^{S_1}]$$

$$= \begin{array}{c} \begin{array}{cc} \overline{\quad S_0 \quad} & \overline{\qquad S_1 \qquad} \end{array} \\ \begin{bmatrix} a\cdots b & 0 & \cdots & 0 \\ a\cdots b & 1 & \cdots & 1 \\ \vdots\ \ \vdots & \vdots & & \vdots \\ a\cdots b & t-1 & \cdots & t-1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} \overline{\quad S_0 \quad} & \overline{\qquad S_1 \qquad} \end{array} \\ \left\{\begin{matrix} (a, \ldots, b, & 0, \ldots, & 0), \\ (a, \ldots, b, & 1, \ldots, & 1), \\ \vdots \quad \vdots & \vdots & \vdots \\ (a, \ldots, b, & t-1, \ldots, & t-1) \end{matrix}\right\} \end{array}$$

all of whose 0-parameter sets (=points) have one color. But the t points of P_1 (shown above) certainly correspond to t integers which lie in an arithmetic progression (since $S_1 \neq \varnothing$). This proves the corollary.

This result is implied by the stronger

COROLLARY 9 (HALES-JEWETT [5]). *Let $A=\{a_1, \ldots, a_t\}$ be a finite set. Given an integer r there exists an integer $N(r, t)$ such that if $n \geqq N(r, t)$ and the set A^n is r-colored then there exists a set of t elements of A^n of the form*

$$X_i = (x_{11}, \ldots, x_{1u}, a_i, x_{21}, \ldots, x_{2v}, a_i, \ldots, a_i, x_{d1}, \ldots, x_{dz}) \in A^n, \qquad 1 \leqq i \leqq t,$$

all of which have the same color.

Proof. This result is a special case of the theorem in which $A=\{a_1, \ldots, a_t\}$, $B=A$, $H=\{e\}$, $k=0$ and $t_1=\cdots=t_r=1$ (also, see Lemma 3).

We remark that the elegant derivation of van der Waerden's Theorem from Corollary 9 given in [5] is essentially different from the one given here.

The next corollary is a Ramsey theorem for partitions of a finite set with the ordering on the partitions inverted from the usual ordering. For the usual ordering ($\Pi \leqq \Pi'$ if Π is a refinement of Π') a Ramsey theorem is trivially true:

For integers k, l, r, and any r-coloring of the partitions of any sufficiently large set S, $|S|=n$, there is a partition Π with $n-l$ blocks with all partitions $\Pi' \leqq \Pi$ with $n-k$ blocks having the same color.

The proof is simply the observation that the lattice of refinements of the partition Π: $\{1, 2\}, \{3, 4\}, \ldots, \{2m-1, 2m\}, \{2m+1\}, \{2m+2\}, \ldots, \{n\}$ is isomorphic to the lattice of subsets of a set of m elements, and is a lower ideal in the lattice of partitions of S. Then Ramsey's Theorem (for subsets) can be invoked.

For the inverted ordering, we define $\Pi' \gtreqless \Pi$ if Π is a refinement of Π'.

Corollary 10. *Given integers k, l, r, there exists an integer $M(k, l, r)$ such that if $n \geqq M(k, l, r)$, and the partitions of a set of n elements into k blocks are r-colored, then there is a partition into l blocks,* Π, *with all partitions* $\Pi' \gtreqless \Pi$ *with k blocks having the same color.*

Proof. Let $A=\{0, 1\}$, $B=\{0\}$, $H=\{e\}$. Let $S_0=\{1\}$, $S_1=\{2\}, \ldots, S_{n-1}=\{n\}$, and let

$$\begin{array}{rcccccc} & & \underline{S_0} & \underline{S_1} & & \underline{S_{n-1}} & \\ P_{n-1} & = & [0 & e & \cdots & e]. & \end{array}$$

By the choice of A, B and H, the x-parameter subsets of P_{n-1} are determined exactly by their corresponding partitions Π. The subset P_x, with partition Π, is contained in the subset P_y, with partition Π' if and only if $\Pi \gtreqless \Pi'$. Thus, applying the theorem to this case produces the desired result. We just let

$$M(k, l, r) = N(A, \bar{B}, H, k-1, r, l-1, \ldots, l-1)+1.$$

We remark that these results on partitions of sets have analogues for partitions of integers which can be derived from the above by associating each set with its cardinality.

Corollary 11 (Ramsey's Theorem). *Given positive integers k, l, r there exists an integer $N_1=N_1(k, l, r)$ such that if $n \geqq N_1$ and the k-subsets of an n-set M_n are r-colored, then all the k-subsets of some l-set $M_l \subseteq M_n$ have the same color.*

Proof. As in Corollary 10, let $A=\{0, 1\}$, $B=\{0\}$ and $H=\{e\}$. Let $N_1=N_1(k, l, r)$ $=N(A, \bar{B}, H, k, r, l, \ldots, l)$ of the theorem. It is sufficient to establish the result for the set $X=\{1, 2, \ldots, N_1\}$. Assume the k-subsets of X have been r-colored. This induces an r-coloring of the k-parameter subsets of the N_1-parameter set A^{N_1} as follows: For a k-parameter subset $P_k \subseteq A^{N_1}$ with partition $\Pi = \{S_0, S_1, \ldots, S_k\}$ let m_i denote the minimal element of S_i, $1 \leqq i \leqq k$, and let $M_k = \{m_1, m_2, \ldots, m_k\}$; assign to P_k the color of the k-set M_k. This is a well-defined coloring of all the k-parameter subsets of A^{N_1}. By the definition of N_1, there exists an l-parameter set P_l with all its k-parameter subsets having one color. In particular, if the partition for P_l is $\Pi' = \{T_0, T_1, \ldots, T_l\}$ and $M_l = \{m'_1, \ldots, m'_l\}$ where m'_j is the minimal element of T_j, then for any k-subset $M_k = \{m'_{i_1}, \ldots, m'_{i_k}\} \subseteq M_l$, the color of M_k is the same as the color of the k-parameter subset $P_k \subseteq P_l$ which has partition $\Pi = \{T_0^*, T_{i_1}, \ldots, T_{i_k}\}$ with $T_0^* = \{1, 2, \ldots, N_1\} - \bigcup_{j=1}^{k} T_{i_j}$. Since all of these P_k have the same color, then all k-subsets of M_l have the same color and the corollary is proved.

We conclude with a final (stronger) application of the theorem.

Let $C_n = \{(x_1, \ldots, x_n) : x_i = 0 \text{ or } 1\}$ be the set of 2^n vertices of a unit n-cube in $\boldsymbol{R}^n$. Let us call a subset $Q_k \subseteq C_n$ a *k-subspace* of C_n if $|Q_k| = 2^k$ and Q_k is contained in some k-dimensional euclidean subspace of $\boldsymbol{R}^n$.

COROLLARY 12. *Given integers k, l, r, there exists an integer $N(k, l, r)$ such that if $n \geqq N(k, l, r)$ and the k-subspaces of C_n are r-colored, then there exists an l-subspace of C_n all of whose k-subspaces have one color.*

Proof. We first establish a preliminary result. Let P_k denote a k-dimensional (euclidean) subspace of $\boldsymbol{R}^n$ and let $T_k = P_k \cap C_n$. Then we assert

$$|T_k| \leqq 2^k \tag{6}$$

and if $|T_k| = 2^k$, then T_k is a k-parameter subset of C_n with $A = B = \{0, 1\}$, and $H = \{e, \pi\}$ = the group of order 2. To prove this, write P_k as

$$P_k = \{\alpha_1 X_1 + \cdots + \alpha_k X_k + X_0 : \alpha_i \in \boldsymbol{R}\}$$

where the $X_1, \ldots, X_k$ are linearly independent vectors in $\boldsymbol{R}^n$, and $X_0 \in \boldsymbol{R}^n$.

Consider the jth component of a point of T_k. It is either 0 or 1. Thus one of the following two equations must hold:

$$\alpha_1 x_{1j} + \alpha_2 x_{2j} + \cdots + \alpha_k x_{kj} + x_{0j} = 0,$$
$$\alpha_1 x_{1j} + \alpha_2 x_{2j} + \cdots + \alpha_k x_{kj} + x_{0j} = 1.$$

Hence, the only possible α_i's for T_k must lie on one of the two parallel hyperplanes determined by these equations. We have such a pair of equations for each $j = 1, 2, \ldots, n$. The hyperplanes have directions (in pairs) respectively:

$$\begin{gathered} x_{11}, \ldots, x_{k1}, \\ x_{12}, \ldots, x_{k2}, \\ \vdots \\ x_{1n}, \ldots, x_{kn}. \end{gathered}$$

But by assumption, the columns $X_1, \ldots, X_k$ are independent.

Therefore we can find k independent rows, say, for example, rows $1, 2, \ldots, k$, and consequently the corresponding matrix

$$\begin{pmatrix} x_{11}, \ldots, x_{k1} \\ \vdots \\ x_{1k}, \ldots, x_{kk} \end{pmatrix}$$

is nonsingular. Thus, for each set of equations

$$\begin{aligned} x_{11}\alpha_1 + \cdots + x_{k1}\alpha_k &= \varepsilon_1 - x_{01}, \\ &\vdots \\ x_{k1}\alpha_1 + \cdots + x_{kk}\alpha_k &= \varepsilon_k - x_{0k}, \qquad \varepsilon_i = 0 \text{ or } 1, \end{aligned}$$

there is exactly one choice for the α_i's satisfying them. Since the α_i's determine the points of T_k, and since there are at most 2^k possible choices for the ε_i, we have at most 2^k possibilities for the α_i's. Furthermore, the only way we get all 2^k is when all 2^k possibilities for the ε_i's occur. In this case ($|T_k| = 2^k$), we have 2^{k-1} solutions with $\varepsilon_1 = 0$, and 2^{k-1} solutions with $\varepsilon_1 = 1$. If $\varepsilon_2, \ldots, \varepsilon_k$ are fixed, and we look at the two solutions from $\varepsilon_1 = 0$ and $\varepsilon_1 = 1$, then these two solutions differ by a vector $v = (v_1, \ldots, v_k)$ which is independent of $\varepsilon_2, \ldots, \varepsilon_k$. In particular, $(v_1, \ldots, v_k)$ must satisfy

$$\begin{aligned} x_{11}v_1 + \cdots + x_{k1}v_k &= 1, \\ x_{12}v_1 + \cdots + x_{k2}v_k &= 0, \\ \vdots \qquad & \quad \vdots \\ x_{1k}v_1 + \cdots + x_{kk}v_k &= 0. \end{aligned}$$

v is thus uniquely determined by the x's independent of the ε_i's. Certainly, if $\alpha = (\alpha_1, \ldots, a_k)$ is a solution for $\varepsilon_1 = 0$ and some $\varepsilon_2, \ldots, \varepsilon_k$, then $\alpha + v$ is a solution for the same $\varepsilon_2, \ldots, \varepsilon_k$ with $\varepsilon_1 = 1$.

This means that for each point p in T_k with $\varepsilon_1 = 0$, there is a point q in T_k with $\varepsilon_1 = 1$ such that

$$q = p + (v_1 X_1 + \cdots + v_k X_k) = p + U_1.$$

Since q and p have all entries 0 and 1, U_1 must have all entries 0, 1 and -1. In fact, repeating this argument with $\varepsilon_2, \ldots, \varepsilon_k$ replacing ε_1, we obtain a set of vectors $U_1, U_2, \ldots, U_k$, with entries 0, 1, -1, and the point P_0 with $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 0$ such that

$$T_k = \{P_0 + \varepsilon_1 U_1 + \cdots + \varepsilon_k U_k : \varepsilon_i = 0, 1\}.$$

No two of the U_i can have a nonzero entry in the same coordinate, or else there would be three values occurring there, violating the fact that all points of T_k have only entries of 0 and 1.

If U_i has a -1 entry in, say, the hth position, then U_j has a 0 in the hth position for $j \neq i$, and P_0 must have a $+1$ in the hth position, in order to insure entries of 0 and 1 in T_k.

T_k is a k-parameter set, then, with $A=B=\{0,1\}$ and $H=\{e,\pi\}=$ the group of order 2. We can write T_k as

$$T_k = [\underbrace{a \cdots b}_{S_0}\ \underbrace{\pi_1 \cdots \delta_1}_{S_1}\ \cdots\ \underbrace{\pi_k \cdots \delta_k}_{S_k}]$$

$$= \begin{bmatrix} \overbrace{a \cdots b}^{S_0} & \overbrace{00 \cdots 11}^{S_1} & \cdots & \overbrace{00 \cdots 11}^{S_k} \\ a \cdots b & 11 \cdots 00 & \cdots & 11 \cdots 00 \end{bmatrix}.$$

S_0 consists of those coordinates j for which every U_i is 0; the value $f(j)$ (where f is the function required in the definition of a k-parameter set) for $j \in S_0$ is 0 or 1 according to the corresponding entry in P_0. Each S_i, $i>0$, consists of those j for which U_i has a nonzero jth component; the value $f(j)$ for $j \in S_i$ is e if the component is 1 and π if the component is -1. This proves (6) and the assertion which follows it. The proof of the corollary now follows at once from the theorem by choosing $A=B=\{0,1\}$, $H=\{e,\pi\}$, $t_1=\cdots=t_r=l$, and $N(k,l,r)=N(A,\bar{B},H,k,r,t_1,\ldots,t_r)$.

We point out that even though the techniques of the proof of the theorem are constructive so that upper bounds on the various N's of the corollaries can be given, these bounds are usually enormous, to say the least. To illustrate this, we consider the first nontrivial case of Corollary 12, the determination of an upper bound on $N(1,2,2)$. We recall that by definition $N(1,2,2)$ is an integer such that if $n \geqq N(1,2,2)$ and the $\binom{2^n}{2}$ straight line segments joining all possible pairs of vertices of a unit n-cube are arbitrarily 2-colored, then there always exists a set of four *coplanar* vertices which determines six line segments of the same color. Let N^* denote the least possible value $N(1,2,2)$ can assume. We introduce a calibration function $F(m,n)$ with which we may compare our estimate of N^*. This is defined recursively as follows:

$$F(1,n) = 2^n, \qquad F(m,2) = 4, \qquad m \geqq 1,\ n \geqq 2,$$
$$F(m,n) = F(m-1, F(m,n-1)), \qquad m \geqq 2,\ n \geqq 3.$$

It is recommended that the reader calculate a few small values of F to get a feeling for its rate of growth, e.g., $F(5,5)$ or $F(10,3)$.

If the bounds generated by the recursive constructions needed for the proof of Corollary 12 are explicitly tabulated, the best estimate for N^* we obtain this way is roughly

$$N^* \leqq F(F(F(F(F(F(F(12,3),3),3),3),3),3),3).$$

On the other hand, it is known only that $N^* \geqq 6$. Clearly, there is some room for improvement here.

9. **Concluding remarks.** We conclude with several questions.

(i) In the corollaries of the theorem listed, we never really make much use of the freedom we have in choosing B and H. What are some interesting applications for some less trivial choices of B and H?

(ii) Are the various infinite versions of certain of the corollaries valid? A specific simple case would be: If the positive integers are 2-colored, is it true that there always exists an infinite subset A such that all sums $\sum_{b\in B} b$, $\varnothing \neq B \subseteq A$, B-finite, have one color?

(iii) With respect to the corollaries, the upper bounds given by the theorem on the various N's are rather crude, as has been pointed out. Is it possible to improve significantly the estimates of these numbers? For example, in Corollary 12, the upper bound on $N(1, 2, 2)$ given by the theorem is truly enormous, where, in fact, the exact bound is probably < 10.

(iv) It was suggested by M. Simonovits that perhaps it would be possible to give an intrinsic definition of k-parameter sets, i.e., one which does not depend on coordinates. If this is possible then conceivably the corresponding proofs might become simpler.

(v) Our particular definition of a k-parameter set was chosen, to a certain extent, because a Ramsey theorem for them could be proved. What other definitions will have this property? In particular, can a suitable one be found which will establish Rota's original conjecture for k-subspaces of finite vector space, $k \geqq 3$?

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